Complexity of presenting cohesive powers

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Reminder: cohesive sets

Let

$$\vec{A} = (A_0, A_1, A_2, \dots)$$

be a countable sequence of subsets of $\mathbb{N}$.

Then there is an **infinite** set $C \subseteq \mathbb{N}$ such that for every n:

$$\text{either } C \subseteq^* A_n$$
$$\text{or } C \subseteq^* \mathbb{N} \setminus A_n.$$

C is called **cohesive** for $\vec{A}$, or simply $\vec{A}$-**cohesive**.

If $\vec{A}$ is the sequence of r.e. sets, then C is called **cohesive**.

Cohesive powers

Dimitrov (2009):
Let $\mathcal{A}$ be a computable structure.
(i.e., $\mathcal{A}$ has domain $\mathbb{N}$ and recursive functions and relations.)

Let C be cohesive. Form the **cohesive power** $\prod_C \mathcal{A}$ of $\mathcal{A}$ over C:

Consider partial recursive $\varphi, \psi \colon \mathbb{N} \to \mathbb{N}$ with $C \subseteq^* \mathrm{dom}(\varphi)$. Define:

$$\begin{array}{lcl} \varphi =_C \psi & \text{if} & C \subseteq^* \{n : \varphi(n) = \psi(n)\} \\ R(\psi_0, \dots, \psi_{k-1}) & \text{if} & C \subseteq^* \{n : R(\psi_0(n), \dots, \psi_{k-1}(n))\} \\ F(\psi_0, \dots, \psi_{k-1})(n) & = & F(\psi_0(n), \dots, \psi_{k-1}(n)) \end{array}$$

Let $[\varphi]$ denote the $=_C$-equivalence class of φ.

Let $\prod_C \mathcal{A}$ be the structure with domain $\{[\varphi] : C \subseteq^* \mathrm{dom}(\varphi)\}$ and

$$\begin{array}{lcl} R([\psi_0], \dots, [\psi_{k-1}]) & \text{if} & R(\psi_0, \dots, \psi_{k-1}) \\ F([\psi_0], \dots, [\psi_{k-1}]) & = & [F(\psi_0, \dots, \psi_{k-1})]. \end{array}$$

Decidability, n-decidability, and a little Łoś

A computable structure $\mathcal{A}$ is:

- **decidable** if its elementary diagram is recursive
- **n-decidable** if its Σ_n-elementary diagram is recursive.

The following is due to Dimitrov, Harizanov, Morozov, (S), A. Soskova, and Vatev, building on work of Dimitrov.

Theorem

Let $\mathcal{A}$ be a computable structure, C be cohesive, $\Phi(v)$ a first-order formula, and $[\varphi]$ an element of $\prod_C \mathcal{A}$.

- *If $\mathcal{A}$ is n-decidable and Φ is Π_{n+2}, then*

$$\forall^\infty i \in C \;\; \mathcal{A} \models \Phi(\varphi(i)) \quad \textit{implies} \quad \prod_C \mathcal{A} \models \Phi([\varphi]).$$

- *If $\mathcal{A}$ is decidable, then*

$$\forall^\infty i \in C \;\; \mathcal{A} \models \Phi(\varphi(i)) \quad \textit{if and only if} \quad \prod_C \mathcal{A} \models \Phi([\varphi]).$$

Cohesive powers and saturation

A structure is:

- **recursively saturated** if it realizes every recursive type
- **Σ_n-recursively saturated** if it realizes every recursive type of Σ_n formulas.

Let $\mathcal{A}$ be a computable structure and C be cohesive.

- If $\mathcal{A}$ is decidable, then $\prod_C \mathcal{A}$ is recursively saturated. (Essentially Nelson).
- If $\mathcal{A}$ is n-decidable for $n \geq 1$, then $\prod_C \mathcal{A}$ is Σ_n-recursively saturated. (Dimitrov, Harizanov, Morozov, (S), A. Soskova, and Vatev).
- If $\mathcal{A}$ is n-decidable and C is Π_1, then $\prod_C \mathcal{A}$ is Σ_{n+1}-recursively saturated. (Dimitrov, Harizanov, Morozov, (S), A. Soskova, and Vatev).
- **If $\mathcal{A}$ is n-decidable and C is Δ_2, then $\prod_C \mathcal{A}$ is Σ_{n+1}-recursively saturated.** ((S), building on the above).

The point: If C is Δ_2, then we get one more level of saturation (and also $n = 0$).

Cohesive powers of recursive presentations of ω

Previous work of Dimitrov, Harizanov, Morozov, (S), A. Soskova, and Vatev and of (S) focused on cohesive powers of different recursive presentations of ω.

For example: For the standard presentation $(\mathbb{N}; <)$ and any cohesive C:

$$\prod_C (\mathbb{N}; <) \;\cong\; \omega + \zeta\eta \quad \text{(i.e., } \omega \text{ plus dense copies of the integers).}$$

But also:

Theorem (S)

Let $X \subseteq \mathbb{N} \setminus \{0\}$ be a Boolean combination of Σ_2 sets. There is a recursive copy $\mathcal{L}$ of ω such that for every Δ_2 cohesive C:

$$\prod_C \mathcal{L} \;\cong\; \omega + \boldsymbol{\sigma}\big(X \cup \{\omega + \zeta\eta + \omega^*\}\big).$$

Moreover, if X is finite, then $\omega + \zeta\eta + \omega^*$ can be removed.

Here X represents a collection of finite linear orders, and $\boldsymbol{\sigma}$ denotes **shuffle sum**.

Are non-recursive order-types possible?

From the previous slide:
If $X \subseteq \mathbb{N} \setminus \{0\}$ is a Boolean combination of Σ_2 sets, then

$$\omega + \boldsymbol{\sigma}\big(X \cup \{\omega + \zeta\eta + \omega^*\}\big)$$

appears as a cohesive power of a recursive copy of ω.

However, by **Ash, Jockusch, Knight**:
If X is Σ_3, then $\omega + \boldsymbol{\sigma}\big(X \cup \{\omega + \zeta\eta + \omega^*\}\big)$ is a recursive order-type.

On the other hand:

- If $\mathcal{L}$ is a recursive linear order, then $\{n : \mathcal{L} \text{ contains a block of size } n\}$ is Σ_3.
- Hence if X is (say) Π_3 but not Σ_3, then $\omega + \boldsymbol{\sigma}\big(X \cup \{\omega + \zeta\eta + \omega^*\}\big)$ is **not** a recursive order-type.

Question:
If $X \subseteq \mathbb{N} \setminus \{0\}$ is Π_3, does $\omega + \boldsymbol{\sigma}\big(X \cup \{\omega + \zeta\eta + \omega^*\}\big)$ appear as a cohesive power of a recursive copy of ω?

Are non-recursive order-types possible?

Indeed, is it possible to achieve non-recursive order-types at all?

Questions:

- Is there a recursive copy $\mathcal{L}$ of ω such that $\prod_C \mathcal{L}$ has non-recursive order-type for every cohesive C? For every Δ_2 cohesive C? For some cohesive C?
- Is there a recursive linear order $\mathcal{L}$ such that $\prod_C \mathcal{L}$ has non-recursive order-type for every cohesive C? For every Δ_2 cohesive C? For some cohesive C?
- Is there uniformly recursive sequence of linear orders $(\mathcal{L}_n)_n$ such that $\prod_C \mathcal{L}_n$ has non-recursive order-type for every cohesive C? For every Δ_2 cohesive C? For some cohesive C?

Gonzalez & (S):
The answer to the last question is **yes**. We will come back to this.

How complicated are cohesive powers anyway?

If $\mathcal{A}$ is a computable structure and C is cohesive, then how complicated is $\prod_C \mathcal{A}$?

Potentially this depends on the complexity of C.

So we stick to Δ_2 cohesive sets for a moment.

Note that there are differences between **powers over $\boldsymbol{\Delta_2}$ cohesive sets** and **powers over $\boldsymbol{\Pi_2}$ cohesive sets**:

Example (S):

- There is a computable copy $\mathcal{L}$ of ω such that $\prod_C \mathcal{L} \cong \omega + \eta$ for every Δ_2 cohesive C.
- For every computable copy $\mathcal{L}$ of ω, there is a Π_2 cohesive C such that $\prod_C \mathcal{L} \ncong \omega + \eta$.

Presenting cohesive powers over Δ_2 cohesive sets

Fact:
If $\mathcal{A}$ is a computable structure and C is a Δ_2 cohesive set, then $\prod_C \mathcal{A}$ has a Δ_3 presentation.

(The next slides have the calculation, but we'll skip it.)

Presenting cohesive powers over Δ_2 cohesive sets

Represent elements of $\prod_C \mathcal{A}$ by pairs $\langle e, N\rangle$ where

$$\underbrace{\forall n > N \, \big(n \in C \;\rightarrow\; \varphi_e(n)\downarrow\big)}_{\Pi_2 \text{ formula } D(\langle e, N\rangle)}$$

We need to identify when $\langle e, N\rangle$ and $\langle i, M\rangle$ represent the same element. Define:

$$\langle e, N\rangle \sim \langle i, M\rangle \;\Leftrightarrow$$
$$\underbrace{D(\langle e, N\rangle) \;\wedge\; D(\langle i, M\rangle) \;\wedge\; \exists K \, \forall n > K \, \big(n \in C \rightarrow \varphi_e(n) = \varphi_i(n)\big)}_{\Sigma_3 \text{ property}}$$

By cohesiveness:

$$\langle e, N\rangle \nsim \langle i, M\rangle \;\Leftrightarrow$$
$$\underbrace{\neg D(\langle e, N\rangle) \;\vee\; \neg D(\langle i, M\rangle) \;\vee\; \exists K \, \forall n > K \, \big(n \in C \rightarrow \varphi_e(n) \neq \varphi_i(n)\big)}_{\Sigma_3 \text{ property}}$$

Presenting cohesive powers over Δ_2 cohesive sets

Thus $\langle e, N\rangle \sim \langle i, M\rangle$ is a Δ_3 relation. So the set X of least representatives is Δ_3:

$$X = \big\{\langle e, N\rangle \;:\; D(\langle e, N\rangle) \;\wedge\; \forall\langle i, M\rangle < \langle e, N\rangle \,\big(\langle e, N\rangle \nsim \langle i, M\rangle\big)\big\}.$$

For simplicity, let's say $\mathcal{A}$ has one binary relation R.

By reasoning as above, the following relation S is Δ_3:

$$S(\langle e, N\rangle, \langle i, M\rangle) \;\Leftrightarrow$$
$$D(\langle e, N\rangle) \;\wedge\; D(\langle i, M\rangle) \;\wedge\; \exists K\, \forall n > K\, \big(n \in C \;\to\; R(\varphi_e(n), \varphi_i(n))\big).$$

Then $(X, X^2 \cap S)$ is a Δ_3 presentation of $\prod_C \mathcal{A}$.

Achieving the maximum complexity

Theorem (Gonzalez & S)

There is a recursive graph $\mathcal{G}$ such that for every cohesive set C, every presentation of $\prod_C \mathcal{G}$ computes $0''$.

So if we restrict to Δ_2 cohesive sets C:

- every $\prod_C \mathcal{G}$ has a $0''$-recursive presentation, and
- every presentation of every $\prod_C \mathcal{G}$ computes $0''$.

Idea:

- Code Σ_3 facts about arithmetic into Σ_1 facts about $\prod_C \mathcal{G}$.
- Then both $k \in 0''$ and $k \notin 0''$ can be coded into Σ_1 facts about $\prod_C \mathcal{G}$. That is, $0'' \oplus \overline{0''}$ becomes r.e. in all presentations of $\prod_C \mathcal{G}$.
- So every presentation of $\prod_C \mathcal{G}$ computes $0''$.

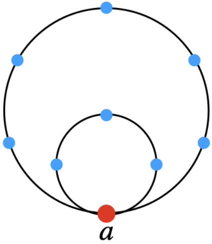
Achieving the maximum complexity

The plan:

Let $\Phi(k)$ be a Σ_3 formula.

Arrange for $\prod_C \mathcal{G}$ to have a vertex $\boldsymbol{a}$ such that for all k:

$$\Phi(k) \quad \Leftrightarrow \quad \boldsymbol{a} \text{ lies on a } (\langle k, \ell \rangle + 3)\text{-cycle for some } \ell.$$



The lengths of the cycles at $\boldsymbol{a}$ determine the k for which $\Phi(k)$ holds.

Back to linear orders

We can compute a sequence of linear orders $(\mathcal{L}_n)_n$ whose cohesive products $\prod_C \mathcal{L}_n$ never have recursive presentations.

Theorem (Gonzalez & S)

There is a uniformly recursive sequence $(\mathcal{L}_n)_n$ of linear orders such that for every cohesive set C, the cohesive product $\prod_C \mathcal{L}_n$ is not elementarily equivalent to any recursive linear order.

Idea:
Adapt the diagonalization strategy of **Jockusch & Soare**.

The diagonalization strategy

For each k, let $\mathcal{S}_k = (k+2) + \mathbb{Q} + (k+2) + \mathbb{Q} + (k+2)$.

Compute $(\mathcal{L}_n)_n$ so that:

$$\prod_C \mathcal{L}_n \;\cong\; (\mathcal{S}_0 + \mathcal{A}_0 + \mathcal{S}_1 + \mathcal{A}_1 + \mathcal{S}_2 + \cdots) + J$$

Where

- each $\mathcal{A}_k$ is infinite and every non-max element has an immediate successor;
- J (for 'junk') does not have finite blocks of size ≥ 2.

Then $\mathcal{S}_k$ is the only interval of its type in $\prod_C \mathcal{L}_n$.

Diagonalization:
If φ_e computes an infinite l.o. $\mathcal{O}_e$ with unique intervals like $\mathcal{S}_e$ and $\mathcal{S}_{e+1}$, then:

the interval between $\mathcal{S}_e$ and $\mathcal{S}_{e+1}$ in $\mathcal{O}_e$ has a maximum element

$\Leftrightarrow$

$\mathcal{A}_e$ has no maximum element.

The diagonalization strategy

Recall: $\mathcal{S}_k = (k+2) + \mathbb{Q} + (k+2) + \mathbb{Q} + (k+2)$.

Compute $(\mathcal{L}_n)_n$ so that:

$$\prod_C \mathcal{L}_n \;\cong\; (\mathcal{S}_0 + \mathcal{A}_0 + \mathcal{S}_1 + \mathcal{A}_1 + \mathcal{S}_2 + \cdots) + J$$

Diagonalization:
If φ_e computes an infinite l.o. $\mathcal{O}_e$ with unique intervals like $\mathcal{S}_e$ and $\mathcal{S}_{e+1}$, then:

the interval between $\mathcal{S}_e$ and $\mathcal{S}_{e+1}$ in $\mathcal{O}_e$ has a maximum element

$\Leftrightarrow$

$\mathcal{A}_e$ has no maximum element.

Then $\prod_C \mathcal{L}_n$ and $\mathcal{O}_e$ differ on the sentence that says:

There are unique intervals like $\mathcal{S}_e$ and $\mathcal{S}_{e+1}$, and there is a maximum element between those intervals.

The set-up

Compute each $\mathcal{L}_n$ as an ω-sum

$$\mathcal{L}_n = \mathcal{M}_0^n + \mathcal{M}_1^n + \mathcal{M}_2^n + \cdots .$$

Then:

$$\prod_C \mathcal{L}_n \;=\; \prod_C \sum_{m\in\mathbb{N}} \mathcal{M}_m^n \;\cong\; \sum_{[\theta]\in\prod_C(\mathbb{N};<)} \prod_C \mathcal{M}_{\theta(n)}^n$$

$$= \left(\underbrace{\prod_C \mathcal{M}_0^n}_{\mathcal{S}_0} + \underbrace{\prod_C \mathcal{M}_1^n}_{\mathcal{A}_0} + \cdots \right) + \underbrace{\sum_{\substack{[\theta]\in\prod_C(\mathbb{N};<)\\ [\theta]\text{ nonstd}}} \prod_C \mathcal{M}_{\theta(n)}^n}_{\mathcal{J}}$$

The set-up

Again remember: $\mathcal{S}_k = (k+2) + \mathbb{Q} + (k+2) + \mathbb{Q} + (k+2)$.

It's not hard to show that $\prod_C \mathcal{S}_k \;\cong\; \mathcal{S}_k$.

So set $\mathcal{M}^n_{2m} = \mathcal{S}_m$ for all m.

Compute each $\mathcal{M}^n_{2m+1}$ to have either order-type:

- $\omega\boldsymbol{\ell}$ for some $\ell > 0$ or
- $\omega\boldsymbol{\ell} + \boldsymbol{q}$ for some $\ell > 0$ and $q > m$

with uniformly recursive successor relation.

This suffices to make

$$\mathcal{J} = \sum_{\substack{[\theta] \in \prod_C (\mathbb{N};<) \\ [\theta] \text{ nonstd}}} \prod\nolimits_C \mathcal{M}^n_{\theta(n)}$$

have no finite blocks of size ≥ 2.

Diagonalizing

Recall: $\mathcal{O}_e$ is the linear order computed by φ_e (if total).

Goal:
Compute $(\mathcal{M}^n_{2e+1})_n$ to diagonalize $\mathcal{A}_e = \prod_C \mathcal{M}^n_{2e+1}$ against $\mathcal{O}_e$.

That is:
If $\mathcal{O}_e$ has unique intervals like $\mathcal{S}_e$ and $\mathcal{S}_{e+1}$, then:

$$\text{the interval between } \mathcal{S}_e \text{ and } \mathcal{S}_{e+1} \text{ in } \mathcal{O}_e \text{ has a maximum element}$$
$$\Leftrightarrow$$
$$\mathcal{A}_e \text{ has no maximum element.}$$

Diagonalizing

Guess where the finite blocks of copies of $\mathcal{S}_e$ and $\mathcal{S}_{e+1}$ in $\mathcal{O}_e$ might be.

Order the guesses by priority. Verifying a guess is Π_2.

Collect evidence that guesses are correct.

When a guess of the locations of $\mathcal{S}_e$ and $\mathcal{S}_{e+1}$ in $\mathcal{O}_e$ gets evidence, check if the $\mathcal{O}_e$-interval between them has a bigger max element since the last time we checked.

- If so, it looks like the $\mathcal{O}_e$-interval has **no** max element. Add an $e+1$-sequence to the top of $\mathcal{M}^n_{2e+1}$ for the next low priority n to try to make $\prod_C \mathcal{M}^n_{2e+1}$ **have** a max element.
- If not, it looks like the $\mathcal{O}_e$-interval **has** a max element. Add an ω-sequence to the top of $\mathcal{M}^n_{2e+1}$ for the next low priority n to try to make $\prod_C \mathcal{M}^n_{2e+1}$ have **no** max element.

The highest priority correct guess wins!

Thank you!

Thank you for coming to my talk!
Do you have a question about it?