FIRST-ORDER HYPERBOLIC FORMULATION OF THE TELEPARALLEL GRAVITY THEORY

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ABSTRACT

Driven by the need for numerical solutions to the Einstein field equations, we derive a first-order reduction of the second-order ~~$f(T)$-teleparallel gravity~~ partial differential equations of the teleparallel equivalent of general relativity (TEGR) in the pure-tetrad formulation (no spin connection). We then ~~restrict our attention to the teleparallel equivalent of general relativity (TEGR) and~~ propose a 3+1 decomposition of these equations suitable for computational implementation. We demonstrate that in vacuum (matter-free spacetime) the obtained system of first-order equations is equivalent to the tetrad reformulation of general relativity by Estabrook, Robinson, Wahlquist, and Buchman and Bardeen, and therefore also admits a symmetric hyperbolic formulation. However, the question of hyperbolicity of the 3+1 TEGR equations for arbitrary spacetimes remains unaddressed so far. Furthermore, the structure of the 3+1 equations resembles a lot of similarities with the equations of relativistic electrodynamics and the recently proposed dGREM tetrad-reformulation of general relativity.

1 INTRODUCTION

The class of teleparallel gravity theories is one of the alternative reformulations of Einstein's general relativity (GR) [33, 3, 17]. While for GR gravitational interaction is a manifestation of the curvature of a torsion-free spacetime, for the teleparallel framework it is realized as a curvature-free linear connection with non-zero torsion (or/and non-zero nonmetricity which is not considered in this paper, e.g. [1]). Although different variables can be taken as the main dynamical fields in GR (tetrad fields, soldering forms, etc.), the most extended choice is the metric tensor accompanied by the Levi-Civita connection. In contrast, for teleparallel theory the metric is trivial and the main dynamical field is usually the space-time tetrad (or frame) field.

Despite the teleparallel geometries are considered as an alternative framework[1] to Einstein's gravity with several promising features missing in GR, e.g. see the discussion in [3, Sec.18] and [17], in this paper, we are interested in the teleparallel gravity only from a pure computational viewpoint and our goal is to use its mathematical structure to develop an efficient computational framework for numerical relativity. Thus, the main goal of this paper is to derive a 3+1-split of the so-called *teleparallel equivalent of general relativity* (TEGR) [3, 39] which is known to pass all standard tests of GR. Up to now, not many attempts have been done to obtain a $3+1$ formulation of the TEGR equations, e.g. ~~[18, 49]~~[43, 18, 49]. A Hamiltonian formulation of TEGR was used in ~~[49]~~ [43, 49] to obtain evolution equations for the tetrads and conjugate momenta. In [18], the spatial tetrad and their first order Lie derivative along the normal vector to the foliations were chosen as the state variables. ~~Here,~~ Somewhat similar to the Hamiltonian formalism in systematic use of conjugate variables and Legendre transforms, here we explore another line of deriving a 3+1 formulation of TEGR ~~which is rather~~ that is aligned with the relativistic electrodynamics. In particular, the key difference from the Hamiltonian formulations [43, 49] is in the promotion of the torsion to the dynamical variable with its own evolution equation. As the result, the obtained ~~equations are complitelly~~ 3+1 equations are completely different from the mentioned papers and have a rather elegant structure similar to Maxwell's equations.

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[1]Yet, it is guaranteed that simplest realizations of teleparallel geometries, such as TEGR for example, produce all the classical results of general relativity [3, 6].

A second objective of this paper is to develop a 3+1 formulation entirely independent of the Einstein's theory, i.e. our development avoids foundational GR tools like the Levi-Civita connection and the Hilbert-Einstein action. Instead, we start from an arbitrary Lagrangian density which is a function of the torsion scalar and obtain corresponding Euler-Lagrange equations coupled with some constraints (differential identities). The obtained equations are then reduced to the first-order form and the 3+1 decomposition is performed. However, despite this independent route, we demonstrate that the resulting 3+1 equations are equivalent to tetrad reformulations of the GR such as the one by Estabrook-Robinson-Wahlquist [26] and Buchman-Bardeen [14] (the ERWBB formulation for brevity), and recent dGREM formulation of GR [47].

From the numerical view point, any 3+1 formulation has to have the well-posed initial value problem (i.e. a solution exists, the solution is unique and changes continuously with changes in the initial data) in order to compute stable evolution of the numerical solution. In other words, the system of governing equations has to be hyperbolic[2]. Therefore, the third objective of the paper is to test if the proposed 3+1 formulation of TEGR is hyperbolic. As is the case with other first-order reductions of the Einstein equations [8], the question of hyperbolicity of 3+1 TEGR equations considered here is not trivial and depends on the delicate use of multiple involution constraints (stationary identities), e.g. see [22]. In particular, for a vacuum space-time, we have found that the proposed 3+1 TEGR equations (if written in the tetrad frame) are equivalent to ERWBB [26, 14] which is known to by symmetric hyperbolic for a certain choice of gauge conditions.

We note that we do not consider the most general TEGR formulation as for example presented in [3]. The linear connection of TEGR is the sum of the two parts: the Weitzenböck connection (which is the historical connection of TEGR) and the spin connection (parametrized by Lorentz matrices) representing the inertial content of the tetrad. The spin connection is necessary to separate the inertial effect of a chosen frame from its gravity content as well as for establishing the full covariance of the theory [3, 30, 39], i.e. with respect to both the diffeomorphisms of the spacetime and the Lorentz transformations of tangent spaces. However, being important from the theoretical viewpoint and for extensions of teleparallel gravity [30], the spin connection of TEGR introduces extra degrees of freedom which do not have evolution equations and therefore can be treated as parameters (not state variables) of the theory. Since we are interested in developing a computational framework for GR, the consideration of this paper is, therefore, restricted to the frames for which the spin connection is set to zero globally (Weitzenböck gauge). We thus consider TEGR in its historical, or *pure tetrad*, formulation [30].

Interestingly that another motivation to study the mathematical structure of the teleparallel gravity is coming from the continuum fluid and solid mechanics. In particular, the role of the torsion to describe defects in solids has been known for decades now, e.g. see [36, 34, 56, 45, 11, 42]. Moreover, the material tetrad field (called also the distortion field in our papers) is the key field for the unified hyperbolic model of fluid and solid mechanics [51, 24]. In such a theory, the concept of torsion can be connected with the inertial effect of small-scale eddies in turbulent flows and with dispersion effects in heterogeneous solids (e.g. acoustic metamaterials) as discussed in [52]. Interestingly, that the ~~3 + 1~~ 3+1 equations we obtain in this paper resemble very closely the structure of the equations for continuum fluid and solid mechanics with torsion [52]. Furthermore, the unified theory of fluids and solids has been also extended in the general relativistic settings [54] and therefore, as being a tetrad theory by its nature, it can be straightforwardly coupled with the 3+1 TEGR equations discussed in this paper.

This work is organized as follows. We start with recalling important definitions in Section 2. In Section 3, we recall the Euler-Lagrange equations for a general Lagrangian defined as an arbitrary function of the tetrad field and its first gradients. Then, in Section 5, we replace the second-order Euler-Lagrange equations by an extended system of first-order partial differential equations. In Section 7, we rewrite the obtained system in a form similar to the relativistic electrodynamics. Section 8 discusses the details of the 3+1 split of the TEGR equations, which is the main result of this paper. The summary of the 3+1 equations is given in Section 9. In Section 10, we discuss the question of hyperbolicity of the 3+1 vacuum TEGR equations via demonstrating its equivalence to the symmetric hyperbolic ERWBB tetrad formulation of GR [26, 14]. Finally, we outline possible directions for future research in Section 13.

[2] Note that in the numerical relativity the term "strong hyperbolicity" is used emphasizing that not only eigenvalues must be real but that the full basis of eigenvectors must exist

2 DEFINITIONS

2.1 Non-holonomic frame field

Throughout this paper, we use the following index convention. Greek letters $\alpha, \beta\,\gamma, \ldots, \lambda, \mu, \nu, ... = 0,1,2,3$ are used to index quantities related to the spacetime manifold, the Latin letters from the beginning of the alphabet $a, b, c, ... = \hat{0}, \hat{1}, \hat{2}, \hat{3}$ are used to index quantities related to the tangent Minkowski space. In the 3+1 split, the letters $i, j, k, \ldots = 1,2,3$ from the middle of the Latin alphabet are used to denote spatial components of the spacetime tensors, and the upper case Latin letters $A, B, C, \ldots = \hat{1}, \hat{2}, \hat{3}$ index spatial components of the tensors written in a chosen frame of the tangent space.

We shall consider a spacetime manifold M equipped with a general Riemannian metric $g_{\mu\nu}$ and a coordinate system x^μ. At each point of the spacetime, there is the tangent space $T_x M$ spanned by the frame (or tetrad) $\boldsymbol{\partial}_\mu$ which is the standard coordinate basis. There is also the cotangent space $T_x^* M$ spanned by the coframe field $\mathbf{d}\boldsymbol{x}^\mu$. Recall that the frames $\boldsymbol{\partial}_\mu$ and $\mathbf{d}\boldsymbol{x}^\mu$ are *holonomic* [3].

In addition to $T_x M$, we assume that at each point of M, there is a soldered tangent space which is a Minkowski space spanned by an orthonormal frame (tetrad) $\boldsymbol{\eta}_a$ and equipped with the Minkowski metric

$$\kappa_{ab} = \mathrm{diag}(-1,1,1,1). \tag{1}$$

Similarly, there is the corresponding cotangent space spanned by the co-frames $\boldsymbol{h}^a$. It is assumed that the frames $\boldsymbol{h}^a$ and $\boldsymbol{\eta}_a$ are independent of the coordinates x^μ and, therefore, are non-holonomic in general.

The components of the non-holonomic frames $\boldsymbol{h}^a$ and $\boldsymbol{\eta}_a$ in the holonomic frames $\mathbf{d}\boldsymbol{x}^\mu$ and $\boldsymbol{\partial}_\mu$ are denoted by $h^a{}_\mu$ and $\eta^\mu{}_a$, i.e.

$$\boldsymbol{h}^a = h^a{}_\mu \mathbf{d}\boldsymbol{x}^\mu, \qquad \text{or} \qquad \boldsymbol{\eta}_a = \eta^\mu{}_a \boldsymbol{\partial}_\mu \tag{2}$$

with $h^a{}_\mu$ being the inverse of $\eta^\mu{}_a$, i.e.

$$h^a{}_\mu \eta^\mu{}_b = \delta^a_b, \qquad \eta^\nu{}_a h^a{}_\mu = \delta^\nu_\mu, \tag{3}$$

where δ^a_b and δ^ν_μ are the Kronecker deltas.

The soldering of the spacetime and the tangent Minkowski space means that the metrics $g_{\mu\nu}$ and κ_{ab} are related by

$$g_{\mu\nu} = \kappa_{ab} h^a{}_\mu h^b{}_\nu. \tag{4}$$

Note that

$$h := \det(h^a{}_\mu) = \sqrt{-g}, \tag{5}$$

if $g = \det(g_{\mu\nu})$.

2.2 Observer's 4-velocity

Observer's 4-velocity is associated with the 0-*th* vector of the tetrad basis

$$u^\mu := \eta^\mu{}_{\hat{0}}, \qquad u_\mu = g_{\mu\nu} u^\nu. \tag{6}$$

Also, due to

$$u_\mu = g_{\mu\nu} u^\nu = \kappa_{ab} h^a{}_\mu h^b{}_\nu \eta^\nu{}_{\hat{0}} = \kappa_{a\hat{0}} h^a{}_\mu = -h^{\hat{0}}{}_\mu, \tag{7}$$

the covariant components of the 4-velocity equal to entries of the 0-*th* vector of the co-basis with the opposite sign

$$u_\mu = -h^{\hat{0}}{}_\mu. \tag{8}$$

2.3 Connection and torsion

Because we work in the framework of the pure-tetrad formulation of TEGR (Weitzenböck gauge), the linear connection is set to the pure Weitzenböck connection[3] [3, 38]:

$$W^a{}_{\mu\nu} := \partial_\mu h^a{}_\nu, \qquad \text{or} \qquad W^\lambda{}_{\mu\nu} := \eta^\lambda{}_a \partial_\mu h^a{}_\nu. \tag{9}$$

The torsion is then defined as

$$T^a{}_{\mu\nu} := \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu = W^a{}_{\mu\nu} - W^a{}_{\nu\mu}. \tag{10}$$

Note that while the spacetime derivatives commute

$$\partial_\mu(\partial_\nu V^\lambda) - \partial_\nu(\partial_\mu V^\lambda) = 0, \tag{11}$$

$$\partial_\mu(\partial_\nu V^a) - \partial_\nu(\partial_\mu V^a) = 0, \tag{12}$$

their tangent space counterparts $\partial_a = \eta^\mu{}_a \partial_\mu$ do not (for non-vanishing torsion)

$$\partial_b(\partial_c V^a) - \partial_c(\partial_b V^a) = -T^d{}_{bc} \partial_d V^a, \tag{13}$$

where $T^d{}_{bc} = T^d{}_{\mu\nu} \eta^\mu{}_b \eta^\nu{}_c$.

[3] Note that we use a different convention on the positioning of the lower indices of the Weitzenböck connection $W^a{}_{\mu\nu} = \partial_\mu h^a{}_\nu$ than in [3]. Precisely, the derivative index goes first.

2.4 Levi-Civita symbol (tensor density)

We shall also need the Levi-Civita symbol (tensor-density[4] of weight $+1$)

$$\in^{\lambda\mu\nu\rho} = \begin{cases} +1, & \text{if } \lambda\mu\nu\rho \text{ is an even permutation of 0123,} \\ -1, & \text{if } \lambda\mu\nu\rho \text{ is an odd permutation of 0123,} \\ 0, & \text{otehrwise.} \end{cases} \tag{14}$$

Its covariant components $\in_{\lambda\mu\nu\rho}$ define a tensor density of weight -1 with the reference value $\in_{0123} = -1$. One could define an absolute Levi-Civita contravariant $\varepsilon^{\lambda\mu\nu\rho} = h^{-1} \in^{\lambda\mu\nu\rho}$ and covariant $\varepsilon_{\lambda\mu\nu\rho} = h \in_{\lambda\mu\nu\rho}$ ordinary tensors but for our further considerations (see Sec. 5), it is important that the derivatives $\partial_\sigma \in^{\lambda\mu\nu\rho}$ vanish:

$$\partial_\sigma \in^{\lambda\mu\nu\rho} = 0, \tag{15}$$

whereas for $\varepsilon^{\lambda\mu\nu\rho}$ one has

$$\partial_\sigma \varepsilon^{\lambda\mu\nu\rho} = \partial_\sigma (h^{-1} \in^{\lambda\mu\nu\rho}) = \in^{\lambda\mu\nu\rho} \partial_\sigma h^{-1} = \\ - \in^{\lambda\mu\nu\rho} h^{-1} \eta^\eta{}_a \partial_\sigma h^a{}_\eta = -\varepsilon^{\lambda\mu\nu\rho} W^\eta{}_{\sigma\eta}, \tag{16}$$

which is not zero in general.

3 VARIATIONAL FORMULATION

We consider a general Lagrangian (scalar-density) $\Lambda(h^a{}_\mu, \partial_\lambda h^a{}_\nu)$ of the teleparallel gravity which is a function of the frame field $h^a{}_\mu$ and its first gradients $W^a{}_{\lambda\nu} = \partial_\lambda h^a{}_\nu$. In what follows, we shall not explicitly split Λ into the gravity (g) and matter (m) parts (unless it is explicitly mentioned otherwise), i.e.

$$\Lambda = \Lambda^{(\mathrm{m})}(h^a{}_\mu, \partial_\lambda h^a{}_\mu, \ldots) + \Lambda^{(\mathrm{g})}(h^a{}_\mu, \partial_\lambda h^a{}_\nu) \tag{17}$$

but the derivation will be performed for the total unspecified Lagrangian Λ, so that in principle ~~the~~ some elements of our derivation can be adopted for the extensions of the teleparallel gravity such as $f(\mathcal{T})$-teleparallel gravity ~~, where $f(\mathcal{T})$ is~~ [41, 40, 28, 10], in which the Lagrangian $\Lambda = f(\mathcal{T})$ is defined as some function of the torsion scalar $\mathcal{T}$, see Section 12. We shall utilize the explicit form of the TEGR Lagrangian only in the last part of the paper~~.—~~, Section 8. Yet, we emphasize that the $f(\mathcal{T})$-type teleparallel theories remain beyond the scope of this paper.

Varying the action of the teleparallel gravity $\int \Lambda(h^a{}_\mu, \partial_\lambda h^a{}_\mu) \mathbf{d}\boldsymbol{x}$ with respect to the tetrad, one obtains the Euler-Lagrange equations

$$\frac{\delta\Lambda}{\delta h^a{}_\mu} = \partial_\lambda(\Lambda_{\partial_\lambda h^a{}_\mu}) - \Lambda_{h^a{}_\mu} = 0, \tag{18}$$

where $\Lambda_{\partial_\lambda h^a{}_\mu} = \frac{\partial\Lambda}{\partial(\partial_\lambda h^a{}_\mu)}$ and $\Lambda_{h^a{}_\mu} = \frac{\partial\Lambda}{\partial h^a{}_\mu}$. Equations (18) form a system of 16 *second-order* partial differential equations for 16 unknowns $h^a{}_\mu$. Our goal is to replace this second-order system by an equivalent but larger system of only first-order partial differential equations.

4 EQUIVALENCE TO GR

Before writing system (18) as a system of first-order equations let us first make a comment on the equivalence of the GR and TEGR formulations.

The Lagrangian density of TEGR (and its extensions) is formed from the torsion scalar $\mathcal{T}$, see (109). As it is known, e.g. see [3, Eq.(9.30)], the torsion scalar can be written as

$$h\mathcal{T} = -\sqrt{-g}R - \partial_\mu(2hT^\nu{}_{\lambda\nu} g^{\lambda\mu}), \tag{19}$$

where R is the Ricci scalar and $T^\nu{}_{\lambda\mu} = \eta^\nu{}_a T^a{}_{\lambda\mu}$. In other words, the Lagrangians of TEGR and GR differ by the four-divergence term (surface term). The latter does not affect the Euler-Lagrange equations if there are no boundaries which is implied in this paper. Therefore, Euler-Lagrange equations of TEGR (18) in vacuum is nothing else but the Euler-Lagrange equations of GR written in terms of the tetrads and hence, their physical solutions must be equivalent because the information about the physical interaction is contained in the Euler-Lagrange equations of a theory. What is different in GR and TEGR is the way one interprets the tetrads and their first derivatives, i.e. the way one defines the linear connection of the spacetime from the gradients of tetrads, e.g. torsion-free Levi-Civita connection of GR and curvature-free Weitzenböck connection of TEGR. These different geometrical interpretations then define extra evolution equations (*compatibility constraints/identities*, e.g. see (23)) that are merely consequences of the geometrical definitions but do not define the physics of the gravitational interaction. The critical point for the numerical relativity, though, is that these extra evolution equations must be solved simultaneously with the Euler-Lagrange equations and may affect the

[4] We use the sign convention for the tensor density weight according to [55, 32], i.e. under a general coordinate change $x^\mu \to x^{\mu'}$, the determinant $\det(h^a{}_\mu) = h$ transforms as $h' = \det\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)^W \cdot h$ with $W = +1$. Therefore, the tetrad's determinant h and the square root of the metric determinant $h = \sqrt{-g}$ have weights $+1$, as well as the Lagrangian density in the action integral.

mathematical regularity (well-posedness of the Cauchy problem) of the resulting system.

5 FIRST-ORDER EXTENSION

Our first goal is to replace second-order system (18) by a larger but first-order system. This is achieved in this section.

From now on, we shall treat the frame field $h^a{}_\mu$ and its gradients (the Weitzenböck connection) $\partial_\lambda h^a{}_\mu$ formally as independent variables and in what follows, we shall rewrite system of second-order PDEs (18) as a larger system of first-order PDEs for the extended set of unknowns $\{h^a{}_\mu, \partial_\lambda h^a{}_\mu\}$, or actually, for their equivalents.

In the setting of ~~the~~ teleparallel gravity, Λ is not a function of a general combination of the gradients $\partial_\lambda h^a{}_\nu$ but of their special combination, that is torsion. Yet, we shall employ not the torsion directly but its Hodge dual, i.e. we assume that

$$\Lambda(h^a{}_\mu, \partial_\lambda h^a{}_\nu) = L(h^a{}_\mu, \overset{\star}{T}{}^{a\mu\nu}), \tag{20}$$

where $\overset{\star}{T}{}^{a\mu\nu}$ is the Hodge dual to the torsion, i.e.

$$\overset{\star}{T}{}^{a\mu\nu} := \frac{1}{2} \in^{\mu\nu\rho\sigma} T^a{}_{\rho\sigma} = \in^{\mu\nu\rho\sigma} \partial_\rho h^a{}_\sigma, \tag{21a}$$

$$T^a{}_{\mu\nu} = -\frac{1}{2} \in_{\mu\nu\rho\sigma} \overset{\star}{T}{}^{a\rho\sigma}. \tag{21b}$$

It is important to emphasize that we deliberately chose to define the Hodge dual using the Levi-Civita symbol $\in^{\lambda\mu\nu\rho}$ and *not* the Levi-Civita tensor $\varepsilon^{\lambda\mu\nu\rho} = h^{-1} \in^{\lambda\mu\nu\rho}$ that will be important later for the so-called integrability condition (23). Remark that according to definition (21a), $\overset{\star}{T}{}^{a\mu\nu}$ is a *tensor-density* of weight $+1$.

In terms of the Lagrangian density $L(h^a{}_\mu, \overset{\star}{T}{}^{a\mu\nu})$, using notations (20) and definitions (21a), we can instead rewrite Euler-Lagrange equations (18) as

$$\partial_\nu(\in^{\mu\nu\lambda\rho} L_{\overset{\star}{T}{}^{a\lambda\rho}}) = -L_{h^a{}_\mu}. \tag{22}$$

The latter has to be supplemented by the integrability condition

$$\partial_\nu \overset{\star}{T}{}^{a\mu\nu} = 0, \tag{23}$$

which is a trivial consequence of the definition of the Hodge dual (21a), i.e. of the commutativity property of the standard spacetime derivative ∂_μ, and the identity (15). We note that if the Hodge dual was defined using the Levi-Civita tensor $\varepsilon^{\mu\nu\rho\sigma}$ instead of the Levi-Civita symbol, then one would have that $\partial_\mu \overset{\star}{T}{}^{a\mu\nu} \neq 0$.

Another consequence of the commutative property of ∂_μ and the definition of the Hodge dual (based on the Levi-Civita symbol) is that the Noether energy-momentum current density

$$J^\mu{}_a := L_{h^a{}_\mu} \tag{24}$$

is conserved in the ordinary sense:

$$\partial_\mu J^\mu{}_a = 0. \tag{25}$$

If equations (22), (23) and (25) are accompanied with the torsion definition

$$\partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu = T^a{}_{\mu\nu}, \tag{26a}$$

they form the following system of *first-order* partial differential equations (only first-order derivatives are involved)

$$\partial_\nu(\in^{\mu\nu\lambda\rho} L_{\overset{\star}{T}{}^{a\lambda\rho}}) = -L_{h^a{}_\mu}, \tag{27a}$$

$$\partial_\nu \overset{\star}{T}{}^{a\mu\nu} = 0, \tag{27b}$$

$$\partial_\mu L_{h^a{}_\mu} = 0, \tag{27c}$$

$$\partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu = T^a{}_{\mu\nu}, \tag{27d}$$

for the unknowns $\{h^a{}_\mu, \overset{\star}{T}{}^{a\mu\nu}\}$.

This system forms a base on which we shall build our 3+1-split of TEGR in Sections 7 and 8.

6 ENERGY-MOMENTUM BALANCE LAWS

Any conservation law written as a 4-*ordinary* divergence is a true conservation law, meaning that it yields a time-conserved "charge" [3]. Hence, the Noether current $J^\mu{}_a = L_{h^a{}_\mu}$ is a conserved charge in the ordinary sense, see (25), (27c). It expresses the conservation of the total[5] energy-momentum current density.

However, its spacetime counterpart

$$\sigma^\mu{}_\nu := h^a{}_\nu L_{h^a{}_\mu}, \tag{28}$$

which can be called the total energy-momentum tensor density, does not conserved in the ordinary sense nor in the covariant one.

[5] The "total" here means the gravitational + matter/electromagnetic energy-momentum current, i.e. $J^a{}_\mu = L^{(\mathrm{g})}_{h^a{}_\mu} + L^{(\mathrm{m})}_{h^a{}_\mu}$.

Indeed, after contracting with $h^a{}_\nu$ and adding to it $0 \equiv L_{h^a{}_\mu}\partial_\mu h^a{}_\nu - L_{h^a{}_\mu}\partial_\mu h^a{}_\nu = L_{h^a{}_\mu}\partial_\mu h^a{}_\nu - L_{h^a{}_\mu} h^a{}_\lambda W^\lambda{}_{\mu\nu}$, Noether current conservation law (25) can be rewritten in a pure spacetime form:

$$\partial_\mu \sigma^\mu{}_\nu = \sigma^\mu{}_\lambda W^\lambda{}_{\mu\nu}, \tag{29}$$

which has a production term on the right hand-side and, therefore, $\sigma^\mu{}_\nu$ is not a conserved quantity in the ordinary sense. Both forms (27c) and (29) will be put into a 3+1 form in Section 8.

On the other hand, if

$$\mathcal{D}_\lambda V^\mu = \partial_\lambda V^\mu + V^\rho W^\mu{}_{\lambda\rho}, \tag{30a}$$

$$\mathcal{D}_\lambda V_\mu = \partial_\lambda V_\mu - V_\rho W^\rho{}_{\lambda\mu} \tag{30b}$$

is the covariant derivative of the Weitzenböck connection, and keeping in mind that $h^a{}_\nu L_{h^a{}_\mu}$ is a tensor density of weight $+1$, balance law (29) can be rewritten as a covariant divergence with a production term

$$\mathcal{D}_\mu \sigma^\mu{}_\nu = -\sigma^\mu{}_\nu T^\rho{}_{\mu\rho},$$

and hence, $\sigma^\mu{}_\nu$ does not conserved also in the covariant sense.

For later needs, the following expression of the energy momentum $\sigma^\mu{}_\nu = h^a{}_\nu L_{h^a{}_\mu}$ is required

$$\sigma^\mu{}_\nu = 2\overset{\star}{T}{}^{a\lambda\mu} L_{\overset{\star}{T}{}^{a\lambda\nu}} - (\overset{\star}{T}{}^{a\lambda\rho} L_{\overset{\star}{T}{}^{a\lambda\rho}} - L)\delta^\mu_\nu. \tag{31}$$

which is valid for the TEGR Lagrangian discussed in Sec. 12. This formula will be used later in the 3+1-split and is analogous to [3, Eq.(10.13)].

7 PRELIMINARIES FOR THE 3+1 SPLIT

7.1 Transformation of the torsion equations

Before performing a 3+1-split [2] of system (27), we need to do some preliminary transformations of every equation in (27).

Similar to electromagnetism, we introduce the "electric" and "magnetic" fields:

$$E^a{}_\mu := T^a{}_{\mu\nu} u^\nu, \qquad B^{a\mu} := \overset{\star}{T}{}^{a\mu\nu} u_\nu \tag{32}$$

Note that $E^a{}_\mu$ is a tensor, while $B^{a\mu}$ is a tensor-density.

It is known that for any skew-symmetric tensor, its Hodge dual, and a time-like vector u^μ the following decompositions hold

$$\overset{\star}{T}{}^{a\mu\nu} = u^\mu B^{a\nu} - u^\nu B^{a\mu} + \epsilon^{\mu\nu\lambda\rho}\, u_\lambda E^a{}_\rho, \tag{33a}$$

$$T^a{}_{\mu\nu} = u_\mu E^a{}_\nu - u_\nu E^a{}_\mu - \epsilon_{\mu\nu\lambda\rho}\, u^\lambda B^{a\rho}. \tag{33b}$$

Furthermore, we assume that the Lagrangian density can be *equivalently* expressed in different sets of variables, i.e.

$$L(h^a{}_\mu, \overset{\star}{T}{}^{a\mu\nu}) = \mathfrak{L}(h^a{}_\mu, T^a{}_{\mu\nu}) = \mathcal{L}(h^a{}_\mu, B^{a\mu}, E^a{}_\nu). \tag{34}$$

It then can be shown that the derivatives of these different representations of the Lagrangian are related as

$$L_{\overset{\star}{T}{}^{a\mu\nu}} u^\nu = -\frac{1}{2}\left(\mathcal{L}_{B^{a\mu}} + u_\mu \mathcal{L}_{B^{a\lambda}} u^\lambda\right), \tag{35a}$$

$$\mathfrak{L}_{T^a{}_{\mu\nu}} u_\nu = -\frac{1}{2}\left(\mathcal{L}_{E^a{}_\mu} + u^\mu \mathcal{L}_{E^a{}_\lambda} u_\lambda\right), \tag{35b}$$

and hence, (27a) and (27b) can be written as (see Appendix (C))

$$\partial_\nu(u^\mu \mathcal{L}_{E^a{}_\nu} - u^\nu \mathcal{L}_{E^a{}_\mu} + \epsilon^{\mu\nu\lambda\rho}\, u_\lambda \mathcal{L}_{B^{a\rho}}) = J^\mu{}_a \tag{36a}$$

$$\partial_\nu(u^\mu B^{a\nu} - u^\nu B^{a\mu} + \epsilon^{\mu\nu\lambda\rho}\, u_\lambda E^a{}_\rho) = 0, \tag{36b}$$

where the source $J^\mu{}_a = L_{h^a{}_\mu}$ has yet to be developed.

Let us now introduce a new potential $U(h^a{}_\mu, B^{a\mu}, D^\mu{}_a)$ as a partial Legendre transform of the Lagrangian $\mathcal{L}$

$$U(h^a{}_\mu, B^{a\mu}, D^\mu{}_a) := E^a{}_\lambda \mathcal{L}_{E^a{}_\lambda} - \mathcal{L}. \tag{37}$$

By abusing a little bit notations for $B^{a\mu}$ (we shall use the same letter for $B^{a\mu}$ and $-B^{a\mu}$, this is an intermediate change of variables and will not appear in the final formulation), we introduce the new state variables

$$D^\mu{}_a := \mathcal{L}_{E^a{}_\mu}, \qquad B^{a\mu} := -B^{a\mu}, \qquad h^a{}_\mu := h^a{}_\mu. \tag{38}$$

Note that both $D^\mu{}_a$ and $B^{a\mu}$ are *tensor-densities*. For derivatives of the new potential, we have the following relations

$$U_{D^\mu{}_a} = E^a{}_\mu, \quad U_{B^{a\mu}} = \mathcal{L}_{B^{a\mu}}, \quad U_{h^a{}_\mu} = -\mathcal{L}_{h^a{}_\mu}. \tag{39}$$

This allows us to rewrite equations (36) in the form similar to the non-linear electrodynamics of moving media [46, 25, 35]

$$\partial_\nu(u^\mu D^\nu{}_a - u^\nu D^\mu{}_a + \epsilon^{\mu\nu\lambda\rho}\, u_\lambda U_{B^{a\rho}}) = J^\mu{}_a, \tag{40a}$$

$$\partial_\nu(u^\mu B^{a\nu} - u^\nu B^{a\mu} - \epsilon^{\mu\nu\lambda\rho}\, u_\lambda U_{D^\rho{}_a}) = 0, \tag{40b}$$

with $B^{a\mu}$ and $D^{\mu}{}_{a}$ being the analogs of the magnetic and electric displacement fields, accordingly.

Finally, we need to express also the Noether current $J^{\mu}{}_{a} = L_{h^{a}{}_{\mu}}$ in terms of the new potential U and the fields $D^{\mu}{}_{a}$ and $B^{a\mu}$. One has (see details in Appendix A)

$$\begin{aligned} J^{\mu}{}_{\hat{0}} = -U_{h^{\hat{0}}{}_{\mu}} + u^{\lambda} B^{b\mu} U_{B^{b\lambda}} - u^{\mu} B^{b\lambda} U_{B^{b\lambda}} \\ - u^{\mu} D^{\lambda}{}_{b} U_{D^{\lambda}{}_{b}} + \epsilon^{\mu\lambda\rho\sigma} u_{\rho} U_{B^{b\lambda}} U_{D^{\sigma}{}_{b}}, \end{aligned} \tag{41a}$$

$$\begin{aligned} J^{\mu}{}_{A} = -U_{h^{A}{}_{\mu}} \\ - \eta^{\nu}{}_{A} u^{\mu} \left(u_{\lambda} D^{\lambda}{}_{b} U_{D^{\nu}{}_{b}} - \epsilon_{\nu\lambda\rho\sigma} u^{\rho} B^{b\sigma} D^{\lambda}{}_{b} \right). \end{aligned} \tag{41b}$$

7.2 Transformation of the tetrad equations

Contracting (27d) with the 4-velocity u^{μ}, and then after change of variables (38) and (39), the resulting equation reads as

$$(\partial_{\mu} h^{a}{}_{\nu} - \partial_{\nu} h^{a}{}_{\mu}) u^{\nu} = U_{D^{\mu}{}_{a}}. \tag{42}$$

Furthermore, using the identity $\eta^{\mu}{}_{b} \partial_{\nu} h^{a}{}_{\nu} = -h^{a}{}_{\nu} \partial_{\nu} \eta^{\mu}{}_{b}$ and the definition $u^{\mu} = \eta^{\mu}{}_{\hat{0}}$, the latter equation can be rewritten as

$$u^{\nu} \partial_{\nu} h^{a}{}_{\mu} + h^{a}{}_{\nu} \partial_{\mu} u^{\nu} = -U_{D^{\mu}{}_{a}}, \tag{43}$$

that later will be used in the 3+1-split.

7.3 Transformation of the energy-momentum

Finally, we express the gravitational part of the energy-momentum tensor $\sigma^{\mu}{}_{\nu}$ (31) in terms of new variables (38) and the potential $U(h^{a}{}_{\mu}, B^{a\mu}, D^{\mu}{}_{a})$, while we keep energy-momentum equation (29) unchanged. It reads

$$\begin{aligned} \sigma^{\mu}{}_{\nu} = & - B^{a\mu} U_{B^{a\nu}} - D^{\mu}{}_{a} U_{D^{\nu}{}_{a}} \\ & - u^{\lambda} u_{\nu} B^{a\mu} U_{B^{a\lambda}} - u^{\mu} u_{\lambda} D^{\lambda}{}_{a} U_{D^{\nu}{}_{a}} \\ & + u^{\mu} u_{\nu} B^{a\lambda} U_{B^{a\lambda}} + u^{\mu} u_{\nu} D^{\lambda}{}_{a} U_{D^{\lambda}{}_{a}} \\ & + \epsilon_{\nu\sigma\lambda\rho} u^{\mu} u^{\sigma} B^{a\lambda} D^{\rho}{}_{a} + \epsilon^{\mu\sigma\lambda\rho} u_{\nu} u_{\sigma} U_{B^{a\lambda}} U_{D^{\rho}{}_{a}} \\ & + (B^{a\lambda} U_{B^{a\lambda}} + D^{\lambda}{}_{a} U_{D^{\lambda}{}_{a}} - U) \delta^{\mu}{}_{\nu}. \end{aligned} \tag{44}$$

We shall need this expression for $\sigma^{\mu}{}_{\nu}$ in the last part of the derivation of the $3+1$ equations.

Therefore, the TEGR system in its intermediate form for the unknowns $\{h^{a}{}_{\mu}, D^{\mu}{}_{a}, B^{a\mu}\}$ reads

$$\partial_{\nu}(u^{\mu} D^{\nu}{}_{a} - u^{\nu} D^{\mu}{}_{a} + \epsilon^{\mu\nu\lambda\rho} u_{\lambda} U_{B^{a\rho}}) = J^{\mu}{}_{a}, \tag{45a}$$

$$\partial_{\nu}(u^{\mu} B^{a\nu} - u^{\nu} B^{a\mu} - \epsilon^{\mu\nu\lambda\rho} u_{\lambda} U_{D^{\rho}{}_{a}}) = 0, \tag{45b}$$

$$\partial_{\mu} J^{\mu}{}_{a} = 0, \tag{45c}$$

$$u^{\nu} \partial_{\nu} h^{a}{}_{\mu} + h^{a}{}_{\nu} \partial_{\mu} u^{\nu} = -U_{D^{\mu}{}_{a}}, \tag{45d}$$

with $J^{\mu}{}_{a}$ given by (41). After we introduce a particular choice of the observer's 4-velocity u^{μ} at the beginning of the next section, we shall finalize transformation of system (45) to present the final 3+1 equations of TEGR.

8 3+1 SPLIT OF THE TEGR EQUATIONS

In this section, we derive a 3+1 version of system (45) that can be used in a computational code for numerical relativity.

We first recall that Latin indices from the middle of the alphabet $i, j, k, \ldots = 1, 2, 3$ are used to denote the spatial components of the spacetime tensors, and Latin indices $A, B, C, \ldots = \hat{1}, \hat{2}, \hat{3}$ to denote the spatial directions in the tangent Minkowski space. Additionally, we use the hat on top of a number, e.g. $\hat{0}$, for the indices marking the time and space direction in the tangent space in order to distinguish them from the indices of the spacetime tensors. Also recall that observer's 4-velocity u^{μ} is associated with the $\hat{0}$-th column of the inverse tetrad $\eta^{\mu}{}_{a}$, while the covariant components u_{μ} of the 4-velocity with the $\hat{0}$-th row of the frame field. For u^{μ} and u_{μ} we standardly assume [2, 53]:

$$u^{\mu} = \eta^{\mu}{}_{\hat{0}} = \alpha^{-1}(1, -\beta^{i}), \tag{46a}$$

$$u_{\mu} = -h^{\hat{0}}{}_{\mu} = (-\alpha, 0, 0, 0), \tag{46b}$$

with α being the *lapse function*, and β^{i} being the *shift vector*. One can write down $h^{a}{}_{\mu}$ and $\eta^{\mu}{}_{a}$ explicitly :

$$\boldsymbol{h} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ \beta^{\hat{1}} & & & \\ \beta^{\hat{2}} & & h^{A}{}_{i} & \\ \beta^{\hat{3}} & & & \end{pmatrix}, \tag{47a}$$

$$\boldsymbol{h}^{-1} = \boldsymbol{\eta} = \begin{pmatrix} 1/\alpha & 0 & 0 & 0 \\ -\beta^{1}/\alpha & & & \\ -\beta^{2}/\alpha & & (h^{A}{}_{i})^{-1} & \\ -\beta^{3}/\alpha & & & \end{pmatrix}, \tag{47b}$$

where $\beta^A = h^A{}_i\beta^i$. The metric tensor and its inverse are (e.g. see [31])

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix}, \tag{48a}$$

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^i/\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{pmatrix}, \tag{48b}$$

where $\gamma_{ij} = \kappa_{AB}h^A{}_ih^B{}_j$, $\gamma^{ij} = (\gamma_{ij})^{-1}$, and $\beta_i = \gamma_{ij}\beta^j$.

In the rest of the paper, h_3 stands for $\det(h^A{}_i)$ so that

$$h := \det(h^a{}_\mu) = \alpha h_3, \tag{49}$$

and we use the following convention for the three-dimensional Levi-Civita symbol

$$\in^{0ijk} = \in^{ijk}, \qquad \in_{0ijk} = - \in_{ijk}. \tag{50}$$

8.1 3+1 split of the torsion PDEs

CASE $\mu = i = 1,2,3$. For our choice of observer's velocity (46), equations (45a) and (45b) read

$$\partial_t(\alpha^{-1}D^i{}_a) + \partial_k(\beta^i\alpha^{-1}D^k{}_a - \beta^k\alpha^{-1}D^i{}_a - \in^{ikj}\alpha\, U_{B^{aj}}) = -J^i{}_a, \tag{51a}$$

$$-\partial_t(\alpha^{-1}B^{ai}) + \partial_k(\beta^k\alpha^{-1}B^{ai} - \beta^i\alpha^{-1}B^{ak} + \in^{ikj}\alpha\, U_{D^j{}_a}) = 0. \tag{51b}$$

Because of the factor α^{-1} everywhere in these equations, it is convenient to rescale the variables:

$$\mathcal{D}^\mu{}_a := \alpha^{-1}D^\mu{}_a, \qquad \mathcal{B}^{a\mu} := -\alpha^{-1}B^{a\mu} \tag{52}$$

so that the derivatives of the potential $\mathcal{U}(h^a{}_\mu, \mathcal{D}^\mu{}_a, \mathcal{B}^{a\mu}) := U(h^a{}_\mu, D^\mu{}_a, B^{a\mu})$ transform as

$$\mathcal{U}_{\mathcal{D}^\mu{}_a} = \alpha\, U_{D^\mu{}_a}, \qquad \mathcal{U}_{\mathcal{B}^{a\mu}} = -\alpha\, U_{B^{a\mu}}. \tag{53}$$

Hence, (51) can be rewritten as

$$\partial_t\mathcal{D}^i{}_a + \partial_k(\beta^i\mathcal{D}^k{}_a - \beta^k\mathcal{D}^i{}_a - \in^{ikj}\mathcal{U}_{\mathcal{B}^{aj}}) = -J^i{}_a, \tag{54a}$$

$$\partial_t\mathcal{B}^{ai} + \partial_k(\beta^i\mathcal{B}^{ak} - \beta^k\mathcal{B}^{ai} + \in^{ikj}\mathcal{U}_{\mathcal{D}^j{}_a}) = 0. \tag{54b}$$

Finally, extending formally the shift vector as $\boldsymbol{\beta} = (-1, \beta^k)$ and introducing the change of variables

$$\mathtt{D}^\mu{}_a := \mathcal{D}^\mu{}_a + \beta^\mu\mathcal{D}^0{}_a, \qquad \mathtt{B}^{a\mu} := \mathcal{B}^{a\mu} \tag{55}$$

which are essentially 3-by-3 matrices ($\mathtt{D}^0{}_a = 0$, $\mathtt{B}^{a0} = \mathtt{B}^{\hat{0}\mu} = 0$, and $\mathtt{D}^k{}_{\hat{0}}$ are given by (87a)), we arrive at the final form of the 3+1 equations for the fields $\mathtt{D}^i{}_A$ and $\mathtt{B}^{Ai}$

$$\partial_t\mathtt{D}^i{}_A + \partial_k(\beta^i\mathtt{D}^k{}_A - \beta^k\mathtt{D}^i{}_A - \in^{ikj}\mathtt{H}_{jA}) = -J^i{}_A, \tag{56a}$$

$$\partial_t\mathtt{B}^{Ai} + \partial_k(\beta^i\mathtt{B}^{Ak} - \beta^k\mathtt{B}^{Ai} + \in^{ikj}\mathtt{E}^A{}_j) = 0. \tag{56b}$$

where

$$\mathtt{E}^A{}_j = \mathcal{U}_{\mathcal{D}^j{}_A} = \alpha U_{D^j{}_A} = \alpha E^A{}_j, \tag{57a}$$

$$\mathtt{H}_{jA} = \mathcal{U}_{\mathcal{B}^{Aj}} = -\alpha U_{B^{Aj}}. \tag{57b}$$

In terms of $\{\mathtt{B}^{Ak}, \mathtt{D}^k{}_A\}$ the potential $\mathcal{U}(h^a{}_\mu, \mathcal{D}^\mu{}_a, \mathcal{B}^{a\mu})$ will be denoted as $\mathtt{U}(h^a{}_\mu, \mathtt{B}^{a\mu}, \mathtt{D}^\mu{}_a)$ and, in the case of TEGR, it reads (see (113))

$$\mathtt{U}(h^A{}_k, \mathtt{B}^{Ak}, \mathtt{D}^k{}_A) = -\frac{\alpha}{2h_3}\left(\varkappa\left(\mathtt{D}^A{}_B\mathtt{D}^B{}_A - \frac{1}{2}(\mathtt{D}^A{}_A)^2\right) + \frac{1}{\varkappa}\left(\mathtt{B}^A{}_B\mathtt{B}^B{}_A - \frac{1}{2}(\mathtt{B}^A{}_A)^2\right)\right), \tag{58}$$

Alternatively, it can be computed as

$$\mathtt{U} = \frac{1}{2}(\mathtt{D}^k{}_a\mathtt{E}^a{}_k + \mathtt{B}^{ak}\mathtt{H}_{ka}). \tag{59}$$

CASE $\mu = 0$. The 0-th equations (45a) and (45b) are actually not time-evolution equations but pure spatial (stationary) constraints

$$\partial_k\mathtt{D}^k{}_a = J^0{}_a, \qquad \partial_k\mathtt{B}^{ak} = 0. \tag{60}$$

Their possible violation during the numerical integration of the time-evolution equations (56) is a well-known problem in the numerical analysis of hyperbolic equations with involution constraints. Different strategies to preserve such stationary constraints are known, e.g. constraint-cleaning approach [44, 19, 20] or constraint-preserving integrators [47, 13, 48].

8.2 Tetrad PDE

Because of our choice of the tetrad (47), we are only interested in the evolution equations for the components $h^A{}_k$. Thus, using the definition of the 4-velocity (46), equation (45d) can be written as

$$\partial_t h^A{}_k - \beta^i\partial_i h^A{}_k - h^A{}_i\partial_k\beta^i = -\mathtt{E}^A{}_k. \tag{61}$$

This equation can be also written in a slightly different form. After adding $0 \equiv -\beta^i\partial_k h^A{}_i + \beta^i\partial_k h^A{}_i$ to the left hand-side of (61), one has

$$\partial_t h^A{}_k - \partial_k(\beta^i h^A{}_i) - \beta^i(\partial_i h^A{}_k - \partial_k h^A{}_i) = -\mathtt{E}^A{}_k. \tag{62}$$

Finally, using the definition of $B^{A\mu}$ and u_μ, we have that

$$B^{A\mu} = \overset{\star}{T}{}^{A\mu\nu} u_\nu = u_\nu \in^{\mu\nu\alpha\beta} \partial_\alpha h^A{}_\beta = \\ - u_\nu \in^{\nu\mu\alpha\beta} \partial_\alpha h^A{}_\beta = \alpha \in^{0\mu\alpha\beta} \partial_\alpha h^A{}_\beta \quad (63)$$

and hence (use that $\in^{0ijk} = \in^{ijk}$)

$$\mathtt{B}^{Ak} = \in^{kij} \partial_i h^A{}_j, \quad (64)$$

or

$$-\beta^i(\partial_i h^A{}_k - \partial_k h^A{}_i) = \in_{klj} \beta^l \mathtt{B}^{Aj}. \quad (65)$$

Therefore, (62) can be also rewritten as

$$\partial_t h^A{}_k - \partial_k(\beta^i h^A{}_i) = -\mathtt{E}^A{}_k - \in_{klj} \beta^l \mathtt{B}^{Aj}. \quad (66)$$

It is not clear yet which one of these equivalent forms (61), (62), or (66) at the continuous level is more suited for the numerical discretization. However, for a structure-preserving integrator, e.g. [13, 47, 48], which is able to preserve (64) up to the machine precision, all these forms are equivalent.

8.3 Energy-momentum PDE

We now turn to the energy-momentum evolution equation

$$\partial_\mu \sigma^\mu{}_\nu = \sigma^\mu{}_\lambda W^\lambda{}_{\mu\nu} \quad (67)$$

and provide its 3+1 version. Because the Lagrangian density in (20) represents the sum of the gravitational and matter Lagrangian, the energy-momentum tensor is also assumed to be the sum of the gravity and matter parts: $\sigma^\mu{}_\nu = {}^{(\mathrm{m})}\sigma^\mu{}_\nu + {}^{(\mathrm{g})}\sigma^\mu{}_\nu$. However, in what follows, we shall omit the matter part keeping in mind that the energy and momentum equations discussed below are equations for the total (matter+gravity) energy-momentum.

We first, explicitly split (67) into the three momentum and one energy equation:

$$\partial_t \sigma^0{}_i + \partial_k \sigma^k{}_i = \sigma^\mu{}_\lambda W^\lambda{}_{\mu i}, \quad (68a)$$

$$\partial_t \sigma^0{}_0 + \partial_k \sigma^k{}_0 = \sigma^\mu{}_\lambda W^\lambda{}_{\mu 0}. \quad (68b)$$

Applying the change of variables (52), (53), and (55), expression (44) for the energy-momentum reduces to the following expressions

- for the total momentum density, $\sigma^0{}_i = \rho_i$ (the matter part is left unspecified and omitted in this paper):

$$\rho_i := \in_{ijl} \mathtt{D}^j{}_a \mathtt{B}^{al}, \quad (69)$$

- for the momentum flux, $\sigma^k{}_i$:

$$\sigma^k{}_i = -\mathtt{B}^{ak}\mathtt{H}_{ia} - \mathtt{D}^k{}_a \mathtt{E}^a{}_i - \in_{ijl} \beta^k \mathtt{D}^j{}_a \mathtt{B}^{al} \\ + (\mathtt{B}^{aj}\mathtt{H}_{ja} + \mathtt{D}^j{}_a \mathtt{E}^a{}_j - \mathtt{U})\delta^k{}_i, \quad (70)$$

- for the energy density:

$$\rho_0 := \sigma^0{}_0 = \in_{ijl} \beta^i \mathtt{D}^j{}_a \mathtt{B}^{al} - \mathtt{U} = \beta^i \rho_i - \mathtt{U}, \quad (71)$$

- and for the energy flux, $\sigma^k{}_0$:

$$\sigma^k{}_0 = -\beta^j(\mathtt{B}^{ak}\mathtt{H}_{ja} + \mathtt{D}^k{}_a \mathtt{E}^a{}_j) \\ + \beta^k(\mathtt{B}^{aj}\mathtt{H}_{ja} + \mathtt{D}^j{}_a \mathtt{E}^a{}_j) \\ - \in_{ijl} \beta^k \beta^i \mathtt{D}^j{}_a \mathtt{B}^{al} - \in^{kjl} \mathtt{E}^a{}_j \mathtt{H}_{la}. \quad (72)$$

It is convenient to split the momentum flux $\sigma^k{}_i$ and energy flux $\sigma^k{}_0$ in (68) into advective and constitutive parts, so that the finale form of the 3+1 equations for the total energy-momentum (68) reads

$$\partial_t \rho_i + \partial_k(-\beta^k \rho_i + s^k{}_i) = f_i, \quad (73a)$$

$$\partial_t \rho_0 + \partial_k(-\beta^k \rho_0 + \beta^i s^k{}_i - \in^{kjl} \mathtt{E}^a{}_j \mathtt{H}_{la}) = f_0, \quad (73b)$$

where the non-matter (gravity + inertia) part of $s^k{}_i$ is given by

$$s^k{}_i = -\mathtt{B}^{ak}\mathtt{H}_{ia} - \mathtt{D}^k{}_a \mathtt{E}^a{}_i + (\mathtt{B}^{aj}\mathtt{H}_{ja} + \mathtt{D}^j{}_a \mathtt{E}^a{}_j - \mathtt{U})\delta^k{}_i. \quad (73c)$$

Drawing parallels between electrodynamics and TEGR, one can note the presence of the Poynting vector in two forms in (73): $\mathtt{D} \times \mathtt{B}$ in the momentum density in (73a) and $\mathtt{E} \times \mathtt{H}$ in the energy flux in (73b).

The source terms in (73a) and (73b) are given by

$$f_i = -\rho_k \eta^k{}_A \mathtt{E}^A{}_i + \rho_j \partial_i \beta^j + s^k{}_j W^j{}_{ki}, \quad (73d)$$

$$f_0 = -\Big(\partial_t \ln(\alpha) - \beta^k \partial_k \ln(\alpha)\Big)\mathtt{U} \\ + \Big(\partial_t \beta^A - \beta^k \partial_k \beta^A\Big)\rho_j \eta^j{}_A \\ - \in^{kjl} \mathtt{E}^a{}_j \mathtt{H}_{la} \partial_k \ln(\alpha) + s^k{}_j \eta^j{}_A \partial_k \beta^A. \quad (73e)$$

8.3.1 *Noether current*

Because the energy-momentum $\sigma^k{}_i$ was defined as $\sigma^k{}_i = h^a{}_i J^k{}_a = h^A{}_i J^k{}_A$, we can use the expression for $\sigma^k{}_i$ to have an explicit formula for the Noether current. Thus, we have

$$J^k{}_A = \eta^i{}_A \sigma^k{}_i = \eta^i{}_A(-\beta^k \rho_i + s^k{}_i) = -\beta^k \rho_A + s^k{}_A. \quad (74)$$

In the same way we can find $J^0{}_a$ necessary in (60): $J^0{}_a = \eta^\lambda{}_a\sigma^0{}_\lambda = \eta^0{}_a\sigma^0{}_0 + \eta^i{}_a\sigma^0{}_i$, and hence

$$J^0{}_a = \begin{cases} -\alpha^{-1}\mathtt{U} = \rho_{\hat{0}}, & a = \hat{0}, \\ \eta^i{}_A\rho_i = \rho_A, & A = \hat{1}, \hat{2}, \hat{3}. \end{cases} \tag{75}$$

8.3.2 *Alternative form of the energy-momentum equations*

In (67), we deliberately use the energy-momentum tensor with both spacetime indices because we would like to use the SHTC and Hamiltonian structure [50] of these equations for designing numerical schemes in the future, e.g. [16, 37]. However, the resulting PDEs (73) do not have a fully conservative form preferable for example when dealing with the shock waves in the matter fields. Therefore, one may want to use the true conservation law (25) for the total (matter+gravity) Noether current $J^\mu{}_a$ instead of (73). Thus, in notations (74), (75), four conservation laws $\partial_\mu J^\mu{}_a = 0$ now read

$$\partial_t\rho_A + \partial_k(-\beta^k\rho_A + s^k{}_A) = 0, \tag{76a}$$

$$\partial_t\rho_{\hat{0}} + \partial_k(-\beta^k\rho_{\hat{0}} - \alpha^{-1}\in^{kjl}\mathtt{E}^a{}_j\mathtt{H}_{la}) = 0. \tag{76b}$$

8.3.3 *Evolution of the space volume*

As in the computational Newtonian mechanics [24, 13], the evolution of the tetrad field at the discrete level has to be performed consistently with the volume/-mass conservation law. In TEGR, the equivalent to the volume conservation in the Newtonian mechanics is

$$\partial_\mu(hu^\mu) = -hE^\mu{}_\mu \tag{77}$$

which can be obtained after contracting (45d) with $\partial h/\partial h^a{}_\mu = h\eta^\mu{}_a$, and where $E^\mu{}_\mu = \eta^\mu{}_a E^a{}_\mu$.

After using (46), (47), and (80b), this balance law becomes

$$\partial_t h_3 - \partial_k(h_3\beta^k) = -\frac{\alpha\varkappa}{2}\mathtt{D}^i{}_i, \tag{78}$$

where $h_3 = \det(h^A{}_k) = \sqrt{\det(\gamma_{ij})}$.

9 SUMMARY OF THE 3+1 TEGR EQUATIONS

Here, we summarize the 3+1 TEGR equations and give explicit expressions for the constitutive fluxes $\mathtt{E}^A{}_k$ and $\mathtt{H}_{kA}$ which then close the specification of the entire system.

9.1 Evolution equations

The system of 3+1 TEGR governing equations reads

$$\partial_t\mathtt{D}^i{}_A + \partial_k(\beta^i\mathtt{D}^k{}_A - \beta^k\mathtt{D}^i{}_A - \in^{ikj}\mathtt{H}_{jA}) = \beta^i\rho_A - s^i{}_A, \tag{79a}$$

$$\partial_t\mathtt{B}^{Ai} + \partial_k(\beta^i\mathtt{B}^{Ak} - \beta^k\mathtt{B}^{Ai} + \in^{ikj}\mathtt{E}^A{}_j) = 0, \tag{79b}$$

$$\partial_t\rho_A + \partial_k(-\beta^k\rho_A + s^k{}_A) = 0, \tag{79c}$$

$$\partial_t\rho_{\hat{0}} + \partial_k(-\beta^k\rho_{\hat{0}} - \alpha^{-1}\in^{kjl}\mathtt{E}^a{}_j\mathtt{H}_{la}) = 0, \tag{79d}$$

$$\partial_t h^A{}_k - \beta^i\partial_i h^A{}_k - h^A{}_i\partial_k\beta^i = -\mathtt{E}^A{}_k, \tag{79e}$$

with the total (matter+gravity) momentum $\rho_A = \eta^i{}_A\rho_i$ and the total energy density $\rho_{\hat{0}} = -\alpha^{-1}\mathtt{U} = J^0{}_{\hat{0}}$ computed from (75), and the gravitational part of the momentum flux $s^A_i = \eta^k{}_A s^k{}_i$ computed from (73c).

In particular, the structure of this system resembles very much the structure of the nonlinear electrodynamics of moving media already solved numerically in [25] for example, as well as the structure of the continuum mechanics equations with torsion [52]. Moreover, despite deep conceptual differences, it is identical to the new dGREM formulation of the Einstein equations recently pushed forward in [47]. However, it is important to mention that the Lagrangian approach adopted here in principle permits to generalize the formulation to other $f(\mathcal{T})$ theories. A detailed comparison of these two formulations will be a subject of a future paper.

It is also important to note that the structure of system (79) remains the same independently of the Lagrangian in use as it can be seen from system (45) where all the constitutive parts are defined through the derivatives of the potential (37). If the Lagrangian changes, then only the constitutive functions $\mathtt{E}^a{}_\mu$ and $\mathtt{H}_{\mu a}$ have to be recomputed.

9.2 Constitutive relations

For the TEGR Lagrangian (109), $\mathtt{E}^A{}_k$ and $\mathtt{H}_{kA}$ read

$$\mathtt{H}_{kA} := -\frac{\alpha}{\varkappa h_3}\kappa_{AB}\kappa_{CD}\left(h^D{}_k h^B{}_j - \frac{1}{2}h^D{}_j h^B{}_k\right)\mathtt{B}^{Cj} - \frac{1}{\varkappa h_3}\kappa_{A,B}\in^{ijl}\gamma_{ik}h^B{}_j\mathtt{E}^{\hat{0}}{}_l \tag{80a}$$

$$\mathtt{E}^A{}_k := -\frac{\alpha\varkappa}{h_3}\left(h^A{}_j h^B{}_k - \frac{1}{2}h^A{}_k h^B{}_j\right)\mathtt{D}^j{}_B - \frac{\varkappa}{h_3}\in^{ijl}\gamma_{ik}h^A{}_j\mathtt{H}_{l\hat{0}}. \tag{80b}$$

Note that these relations can also be written as

$$\mathtt{H}_{kA} = \frac{\partial \mathtt{U}}{\partial \mathtt{B}^{Ak}} - \frac{1}{\varkappa} h_3 \in_{kjl} \eta^j{}_A A^l, \tag{81}$$

$$\mathtt{E}^A{}_k = \frac{\partial \mathtt{U}}{\partial \mathtt{D}^k{}_A} - \varkappa h_3 \in_{kjl} \eta^j{}_B \Omega^l \kappa^{AB}, \tag{82}$$

where $A_k = \mathtt{E}^{\hat{0}}{}_k$ and $\Omega_k = \mathtt{H}_{k\hat{0}}$, and with the potential $\mathtt{U}$ given by (58).

9.3 Stationary differential constraints

System (79) is supplemented by several differential constraints. Thus, as already was mentioned, the 0-*th* equations ($\mu = 0$) of (45a) are not actually time-evolution equations but reduce to the so-called Hamiltonian and momentum stationary divergence-type constraints that must hold on the solution to (79) at every time instant:

$$\partial_k \mathtt{D}^k{}_{\hat{0}} = \rho_{\hat{0}}, \qquad \partial_k \mathtt{D}^k{}_A = \rho_A, \tag{83}$$

with $\rho_{\hat{0}} = -\alpha^{-1}\mathtt{U} = J^0{}_{\hat{0}}$.

Accordingly, the 0-*th* equation of (45b) gives the divergence constraint on the $\mathtt{B}^{Ai}$ field

$$\partial_k \mathtt{B}^{Ak} = 0, \tag{84}$$

Finally, from the definition of $\mathtt{B}^{Ai}$, we also have a curl-type constraint on the spatial components of the tetrad field:

$$\in^{kij} \partial_i h^A{}_j = \mathtt{B}^{Ak}. \tag{85}$$

9.4 Algebraic constraints

As a consequence of our choice of observer's reference frame (46), and the fact that $E^a{}_\mu u^\mu = 0$ and $B^{a\mu} u_\mu = 0$, we have the following algebraic constraints

$$\mathtt{B}^{\hat{0}\mu} = 0, \qquad \mathtt{B}^{a0} = 0, \tag{86a}$$

$$\mathtt{E}^{\hat{0}}{}_0 = \beta^k \mathtt{E}^{\hat{0}}{}_k, \qquad \mathtt{E}^{\hat{0}}{}_k = \partial_k \alpha, \qquad \mathtt{E}^A{}_0 = \beta^j \mathtt{E}^A{}_j, \tag{86b}$$

$$\mathtt{D}^0{}_a = 0, \qquad \mathtt{D}^k{}_{\hat{0}} = -\frac{1}{\varkappa h_3} \in^{kli} \mathtt{B}_{li}, \qquad \mathtt{D}_{ik} = \mathtt{D}_{ki}, \tag{87a}$$

$$\mathtt{H}_{0A} = \beta^k \mathtt{H}_{kA}, \tag{87b}$$

where $\mathtt{D}_{ik} = \gamma_{ij} \mathtt{D}^j{}_A h^A{}_k$. As already noted in [3], the gravitational part of the energy-momentum $\sigma^\mu{}_\nu$ is trace-free:

$$\sigma^\mu{}_\mu = 0. \tag{88}$$

10 HYPERBOLICITY OF THE VACUUM 3+1 TEGR EQUATIONS

~~The 3+1 TEGR system (79) is an overdetermined system due~~ Due to the presence of the total (gravity + matter) momentum and energy equations (79c) and (79d)~~. The question~~ , analysis of hyperbolicity of ~~(79), therefore, requires considering~~ the 3+1 TEGR system (79) requires consideration of a model for matter, which however goes beyond the scope of this paper. ~~However~~Nevertheless, we can still analyze ~~the~~ hyperbolicity of the vacuum ~~equations (79) without~~ TEGR equations (79) for which the momentum and energy equations ~~which might be considered as a starting point of further research. In particular~~can be omitted. This would be still an important result since in the empty space the field equations must be causal and have well-posed initial value problem.

In empty space, the 3+1 TEGR system reduces to the evolution equations (79a), (79b), and (79e) for the gravitational variables $\{\mathtt{D}^i{}_A, \mathtt{B}^{Ai}, h^A{}_i\}$, subject to the constraints (83) and (84) and appropriate gauge conditions on $\mathtt{E}^{\hat{0}}{}_i$ and $\mathtt{H}_{i\hat{0}}$. In fact, as we shall see, the choice of the gauge conditions for $\mathtt{E}^{\hat{0}}{}_i$ and $\mathtt{H}_{i\hat{0}}$ plays an important role in our hyperbolicity analysis. Moreover, we shall ~~show that the vacuum~~ assume that the lapse α and shift β^i are not dynamical but some prescribed functions of space and time.

We will show how the 3+1 ~~TEGR equations are equivalent to the~~ vacuum TEGR equations, when restricted to solutions satisfying the Hamiltonian constraint (83), can be transformed into an equivalent first-order system. This system has the principal part of the differential operator identical to that of the first-order tetrad reformulation of GR ~~by Estabrook-Robinson-Wahlquist [26] and Buchman-Bardeen [14] which~~ developed by Estabrook, Robinson, and Wahlquist [26] and by Buchman and Bardeen [26]. This specific formulation, which we will refer to as the ERWBB formulation, is known to be ~~symmetric hyperbolic, and will be referenced to as~~ *symmetric hyperbolic* if subjected to a certain type of gauge conditions. This indirectly demonstrates the hyperbolicity of the vacuum TEGR equations. However, this result should be taken with care, especially in the context of numerical relativity since some differential terms will be turned into algebraic terms using the Hamiltonian constraint. This means that the differential operators of the original 3+1 TEGR equations (79a), (79b), and (79e) and the new system having the form of the ERWBB formulation ~~. We~~ are not exactly equivalent. In other words, in the context of numerical relativity, this may require a constraint-compatible discretization, e.g. see [47, Sec.VI.G] or [48].

Moreover, we could not confirm or disprove the equivalence of the TEGR equations written in the ERWBB form and the tetrad ERWBB formulation itself. Although their differential parts are the same, we were not able to get the algebraic parts of the equations to match. Consequently, we do not claim an exact equivalence between the 3+1 TEGR equations and the ERWBB tetrad formulation of GR. Our hyperbolicity analysis remains valid, however, since it only depends on the principal part of the differential operator.

Let us also remind that solely hyperbolicity is not enough for well-posedness of the initial value problem for a general quasi-linear system of first-order equations. At least, to the best of our knowledge, there is no an existence and uniqueness theorem for such a class of equations in multiple dimensions[6]. In contrary, such a theorem exists for symmetric hyperbolic systems, e.g. see [9].

Let us first introduce the main elements of the ERWBB formulation which relies on the Ricci rotation coefficients

$$\begin{aligned}\mathcal{R}_{abc} :=& \boldsymbol{\eta}_a \cdot \nabla_c \boldsymbol{\eta}_b \\ =& \eta^\mu{}_a g_{\mu\nu} \eta^\lambda{}_c \partial_\lambda \eta^\nu{}_b + \eta^\mu{}_a g_{\mu\nu} \eta^\lambda{}_c \Gamma^\nu{}_{\lambda\rho} \eta^\rho{}_b \end{aligned} \tag{89}$$

where $\nabla_c := \eta^\lambda{}_c \nabla_\lambda$ and ∇_λ is the standard covariant derivative of GR ~~associated with~~ of the symmetric Levi-Civita connection, and $\Gamma^\mu{}_{\nu\lambda}$ are the Christoffel symbols of the Levi-Civita connection.

If $T^a{}_{bc} = T^a{}_{\mu\nu} \eta^\mu{}_b \eta^\nu{}_c$ and $T_{abc} = \kappa_{ad} T^d{}_{bc}$ is the torsion coefficients if it is written in the tetrad basis, then the relation between the Ricci rotation ~~coefficient~~ coefficients and torsion can be expressed by the formula

$$\mathcal{R}_{abc} = \frac{1}{2}\left(T_{abc} + T_{bca} - T_{cab}\right), \tag{90}$$

which also shows that the Ricci rotation coefficients equal exactly to the so-called contortion tensor with the opposite sign $\mathcal{R}_{abc} = -K_{abc}$, e.g. see [3, Eq.(1.63)].

Then, 24 independent entries of $\mathcal{R}_{abc}$ are organized into the following state variables

$$\mathtt{K}_{CA} := \mathcal{R}_{A\hat{0}C}, \qquad \mathtt{N}_B{}^A := \frac{1}{2}\varepsilon^{ACD}\mathcal{R}_{CDB} \tag{91}$$

[6] Despite this, for a system describing causal propagation of signals, the hyperbolicity is considered as a minimum requirement for numerical discretization.

and

$$a_A = \mathcal{R}_{A\hat{0}\hat{0}}, \qquad \omega^A = \frac{1}{2}\varepsilon^{ABC}\mathcal{R}_{CB\hat{0}} \tag{92}$$

Here, $\varepsilon_{ABC} = h_3 \in_{ijk} \eta^i{}_A \eta^j{}_B \eta^k{}_C$.

Additionally, introducing the vector

$$n^A = \frac{1}{2}\varepsilon^{ABC}\kappa_{CD}\mathtt{N}_B{}^D, \tag{93}$$

the Hamiltonian constraint in terms of ERWBB takes the form (see [14, eq.(A1)])

$$\begin{aligned}\partial_A n^A =& \frac{1}{2}\mathtt{N}_{AB}\mathtt{N}^{AB} + \frac{1}{4}\left(\mathtt{K}^{AB}\mathtt{K}_{AB} - \mathtt{N}_{AB}\mathtt{N}^{BA}\right) \\ & - \frac{1}{4}\left((\mathtt{K}^A{}_A)^2 + (\mathtt{N}_A{}^A)^2\right), \end{aligned} \tag{94}$$

where $\partial_A = \eta^k{}_A \partial_k$.

It can be shown that the following one-to-one relations between the ERWBB and TEGR state variables hold

$$\mathtt{K}_{AB} = \frac{\varkappa}{h_3}\left(\mathtt{D}_{AB} - \frac{1}{2}\mathtt{D}^C{}_C \kappa_{AB}\right), \tag{95a}$$

$$-\mathtt{N}_A{}^B = \frac{1}{h_3}\left(\mathtt{B}_A{}^B - \frac{1}{2}\mathtt{B}^C{}_C \delta^B{}_A\right) \cancel{.}\,, \tag{95b}$$

where $\mathtt{D}_{AB} = \kappa_{AC} h^B{}_i \mathtt{D}^i{}_B$ and $\mathtt{B}_A{}^B = \kappa_{AC} h^B{}_i \mathtt{B}^{Ci}$. We remark that due to our choice of the 3+1 split (47), i.e. that observer's time vector, $\boldsymbol{\eta}_{\hat{0}}$, is ~~align~~ aligned with the normal vector to the spatial hypersurfaces, the matrices $\mathtt{K}_{AB}$ and $\mathtt{D}_{AB}$ are *symmetric*.

Additionally, we have the following relations between the ERWBB vectors a_A, ω^A and the TEGR vectors $\cancel{A_k = \mathtt{E}^{\hat{0}}{}_k = \partial_k \alpha}$ ~~and~~ $\cancel{\Omega_k = \mathtt{H}_{k\hat{0}}}$ $A_A = \eta^k{}_A \mathtt{E}^{\hat{0}}{}_k = \partial_k \alpha$ and $\Omega_A = \eta^k{}_A \mathtt{H}_{k\hat{0}}$

$$a_A = \alpha^{-1} A_A, \qquad \omega^A = -\varkappa\alpha^{-1}\kappa^{AB}\Omega_B. \tag{96}$$

Finally, expression of the constitutive fluxes $\mathtt{E}^A{}_k$ and $\mathtt{H}_{kA}$ in terms of $\{\mathtt{K}_{AB}, \mathtt{N}_{AB}, a_A, \omega_A\}$ read

$$\mathtt{E}_{AB} = -\alpha\,\mathtt{K}_{AB} + \alpha\varepsilon_{ABC}\omega^C, \tag{97a}$$

$$\mathtt{H}_{AB} = \frac{\alpha}{\varkappa}\mathtt{N}_{AB} - \frac{\alpha}{\varkappa}\varepsilon_{ABC}a^C, \tag{97b}$$

where $\mathtt{E}_{AB} = \kappa_{AC}\eta^k{}_B \mathtt{E}^C{}_k$ and $\mathtt{H}_{BA} = \eta^k{}_A \mathtt{H}_{Bk}$.

First of all, let us note that, in the tetrad frame, the tetrad equation (79e) reads

$$\partial_0 \eta^k{}_A = \frac{1}{\alpha}\left(\mathtt{E}^A{}_i \eta^i{}_B \eta^k{}_A - \partial_A \beta^k\right), \tag{98}$$

where $\partial_{\hat{0}} = \eta^{\mu}{}_{\hat{0}}\partial_{\mu} = u^{\mu}\partial_{\mu} = \alpha^{-1}\partial_t - \alpha^{-1}\beta^k\partial_k$, and $\partial_A = \eta^k{}_A\partial_k$. For a non-dynamical shift β^k, this is simply an ordinary differential equation for the frame field $\eta^k{}_A$ along the observer's trajectories and hence, it does not affect the hyperbolicity analysis.

Now, rewriting the 3+1 TEGR equations on $\mathtt{D}^i{}_A$ and $\mathtt{B}^{Ai}$ (eqs. (79a) and (79b)) in terms of $\mathtt{D}^B{}_A = h^B{}_i\mathtt{D}^i{}_A$ and $\mathtt{B}^{AB} = h^B{}_i\mathtt{B}^{Ai}$, then applying transformations (95) to these equations and using relations (95)–(97), after a lengthy but rather straightforward sequence of transformations, one obtains the following equations

$$\frac{1}{\alpha}\partial_{\hat{0}}\mathtt{K}_{AB} - \alpha\kappa_{AC}\varepsilon^{CDE}\partial_D\mathtt{N}_{EB} - \alpha\partial_A a_B + \kappa_{AB}\partial_C\eta^C = \text{l.o.t.} \tag{99a}$$

$$\frac{1}{\alpha}\partial_{\hat{0}}\mathtt{N}_{AB} + \alpha\kappa_{BC}\varepsilon^{CDE}\partial_D\mathtt{K}_{EA} + \alpha\partial_A\omega_B = \text{l.o.t.} \tag{99b}$$

where 'l.o.t.' stands for 'low-order terms' (i.e. algebraic terms that do not contain space and time derivatives)~~,~~ ~~$\partial_{\hat{0}} = \eta^{\mu}{}_{\hat{0}}\partial_{\mu} = u^{\mu}\partial_{\mu} = \alpha^{-1}\partial_t - \alpha^{-1}\beta^k\partial_k$,~~.

Finally, using the Hamiltonian constraint (94), the divergence term $\partial_C\eta^C$ can be transformed into algebraic terms, and moved to the right-hand side of the equations. Thus, the 3+1 TEGR equations for $\mathtt{D}^i{}_A$ and ~~$\partial_A = \eta^k{}_A\partial_k$.~~ $\mathtt{B}^{Ai}$ become

$$\partial_{\hat{0}}\mathtt{K}_{AB} - \alpha\kappa_{AC}\varepsilon^{CDE}\partial_D\mathtt{N}_{EB} - \alpha\partial_A a_B = \text{l.o.t.} \tag{100a}$$

$$\partial_{\hat{0}}\mathtt{N}_{AB} + \alpha\kappa_{BC}\varepsilon^{CDE}\partial_D\mathtt{K}_{EA} + \alpha\partial_A\omega_B = \text{l.o.t.} \tag{100b}$$

We note the opposite order of the subscripts A and B in (100b) in the second term with respect to [14, eq.(40)]. This ~~however~~ is solely conditioned by our initial choice of ordering the spacetime and tetrad indices in $\mathtt{B}^{Ai}$ and it does not affect the symmetric hyperbolicity of the system discussed ~~in what follows.~~ below.

Similar to A_k and Ω_k in TEGR, the vectors a_B and ω_A in the ERWBB formulation of GR are not provided with particular evolution equations following from the variational formulation, and therefore they are considered as gauge conditions. Hence, one could try to choose their evolution equations in such a way as to guaranty the well-posedness of the enlarged system for the unknowns $\{\mathtt{K}_{AB}, \mathtt{N}_{AB}, a_B, \omega_B\}$. Thus, as shown in [26, 14], if coupled with the ~~following~~ Nester of Lorentz gauge conditions on a_A and ω_A having the form:

$$\partial_{\hat{0}}a_A - \alpha\kappa^{BC}\partial_B\mathtt{K}_{CA} = \cancel{0}\,\text{l.o.t.}, \tag{101a}$$

$$\partial_{\hat{0}}\omega_A + \alpha\kappa^{BC}\partial_B\mathtt{N}_{CA} = \cancel{0}\,\text{l.o.t.}, \tag{101b}$$

the resulting system (100)–(101) is symmetric hyperbolic, i.e. it can be written in a quasi-linear form

$$\partial_t\boldsymbol{Q} + \boldsymbol{M}^k\partial_k\boldsymbol{Q} = \text{l.o.t.} \tag{102}$$

with matrices $\boldsymbol{M}^k = \boldsymbol{M}^k(\alpha, \eta^i{}_A)$ being symmetric for arbitrary $\eta^i{}_A$. To see this, one needs to order the entries of $\mathtt{K}^A{}_B$ and $\mathtt{N}_{AB}$ in the following way $\boldsymbol{Q} = \{\mathtt{K}_{\hat{1}A}, \mathtt{K}_{\hat{2}A}, \mathtt{K}_{\hat{3}A}, \mathtt{N}_{A\hat{1}}, \mathtt{N}_{A\hat{2}}, \mathtt{N}_{A\hat{3}}, a_A, \omega_A\}$.

In summary, the 3+1 TEGR equations in vacuum, eqs. (79a) and (79b), if coupled with ~~the~~ proper gauge conditions on A_k and Ω_k, are equivalent to ~~a~~ symmetric-hyperbolic system (100)–(101) ~~.~~ ~~However, this is a very preliminary result for further investigation of the well-posedness of the entire 3+1 TEGR system because even the time evolution on the tetrad field $h^A{}_i$ was excluded from the consideration.~~ on solutions satisfying the Hamiltonian constraint. This demonstrates that these equations are causal and have well-posed initial value problem under mentioned conditions.

11 RELATION BETWEEN THE TORSION AND EXTRINSIC CURVATURE

It is useful to relate the state variables of TEGR to the conventional quantities used in numerical general relativity [5, 8, 31]. In what follows, we relate the spatial extrinsic curvature of GR to the $\mathtt{D}^k{}_A$ field. We remark that the two fields are conceptually different due to the different geometry interpretations in GR and TEGR. Therefore, the obtained relation is possible only due to the equivalence of TEGR and GR.

In GR, the evolution equation of the spatial metric γ_{ij} is (see [53, Eq.(7.64)])

$$\partial_t\gamma_{ij} - \beta^l\partial_l\gamma_{ij} - \gamma_{il}\partial_j\beta^l - \gamma_{jl}\partial_i\beta^l = -2\alpha K_{ij}, \tag{103}$$

where K_{ij} is the spatial extrinsic curvature. On the other hand, in TEGR, contracting (61) with $\kappa_{AB}h^B{}_j$, one obtains the following evolution equation for the spatial metric:

$$\partial_t\gamma_{ij} - \beta^l\partial_l\gamma_{ij} - \gamma_{il}\partial_j\beta^l - \gamma_{jl}\partial_i\beta^l = - \kappa_{AB}(h^A{}_i\mathtt{E}^B{}_j + h^A{}_j\mathtt{E}^B{}_i). \tag{104}$$

Therefore, one can deduce an expression for the extrinsic curvature in terms of $\mathtt{E}^A{}_i$:

$$K_{ij} = \frac{1}{2\alpha}\kappa_{AB}(h^A{}_i\mathtt{E}^B{}_j + h^A{}_j\mathtt{E}^B{}_i). \tag{105}$$

To obtain another expression for K_{ij} in terms of the primary state variable $\mathtt{D}^i{}_A$, one needs to use the the constitutive relationship (80b) to deduce

$$K_{ij} = -\frac{\varkappa}{h_3}\left(\mathtt{D}_{ij} - \frac{1}{2}\mathtt{D}^k{}_k\gamma_{ij}\right), \tag{106}$$

which, if contracted, gives the relationship for the traces $K^i{}_i = \gamma^{ij}K_{ji}$ and $\mathtt{D}^i{}_i = \mathtt{D}^i{}_A h^A{}_i$

$$K^i{}_i = \frac{\varkappa}{2h_3}\mathtt{D}^i{}_i. \tag{107}$$

Remark that if written in the tetrad frame $K_{AB} = \eta^i{}_A\eta^j{}_A K_{ij}$ then the extrinsic curvature is exactly $\mathtt{K}_{AB}$ introduced in (95) apart from the opposite sign

$$K_{AB} = -\mathtt{K}_{AB}. \tag{108}$$

12 TORSION SCALAR

In TEGR and its $f(\mathcal{T})$-extensions, the Lagrangian density is a function of the torsion scalar $\mathcal{T}$, e.g. in TEGR, the Lagrangian density is

$$\mathfrak{L}(h^a{}_\mu, T^a{}_{\mu\nu}) = \frac{h}{2\varkappa}\mathcal{T}, \tag{109a}$$

$$\mathcal{T}(h^a{}_\mu, T^a{}_{\mu\nu}) := \tag{109b}$$
$$\frac{1}{4}g^{\beta\lambda}g^{\mu\gamma}g_{\alpha\eta}T^\alpha{}_{\lambda\gamma}T^\eta{}_{\beta\mu} \tag{109c}$$
$$+\frac{1}{2}g^{\mu\gamma}T^\lambda{}_{\mu\beta}T^\beta{}_{\gamma\lambda} \tag{109d}$$
$$-g^{\mu\lambda}T^\rho{}_{\mu\rho}T^\gamma{}_{\lambda\gamma}, \tag{109e}$$

where $\varkappa = 8\pi Gc^{-4}$ is the Einstein gravitational constant, G is the gravitational constant, c is the speed of light in vacuum.

However, to close system (79), we need not the Lagrangian $\mathfrak{L}(h^a{}_\mu, T^a{}_{\mu\nu})$ directly but we need to perform a sequence of variable and potential changes: $\mathfrak{L}(h^a{}_\mu, T^a{}_{\mu\nu}) = \mathcal{L}(h^a{}_\mu, E^a{}_\mu, B^{a\mu}) \longrightarrow E^a{}_\mu D^\mu{}_a - \mathcal{L} = U(h^a{}_\mu, D^\mu{}_a, B^{a\mu}) = \mathtt{U}(h^a{}_\mu, \mathtt{D}^\mu{}_a, \mathtt{B}^{a\mu})$. Thus, we have

$$\begin{aligned}&\mathcal{L}(h^a{}_\mu, E^a{}_\mu, B^{a\mu}) = \\ &\frac{h}{2\varkappa}\Big(-\frac{1}{2}(E^{\hat{0}\lambda}E^{\hat{0}}{}_\lambda - 2E^\lambda{}_\lambda E^\beta{}_\beta + E_\beta{}^\lambda E^\beta{}_\lambda + E^\lambda{}_\beta E^\beta{}_\lambda) \\ &+\epsilon_{\lambda\gamma\eta\rho}\,u^\eta(E^{\lambda\gamma}B^{\hat{0}\rho} + 2E^{\hat{0}\lambda}B^{\gamma\rho}) \\ &-\frac{1}{2}h^{-2}(B^{\hat{0}\lambda}B^{\hat{0}}{}_\lambda + B^\lambda{}_\lambda B^\beta{}_\beta - 2B^\lambda{}_\beta B^\beta{}_\lambda)\Big).\end{aligned} \tag{110}$$

In turn, if we perform the Legendre transform $U(h^A{}_k, D^k{}_A, B^{Ak}) = E^a{}_\mu D^\mu{}_a - \mathcal{L}$ then the new potential U reads

$$\begin{aligned}&U(h^A{}_k, D^k{}_A, B^{Ak}) = \\ &\frac{1}{4h}\Big(\varkappa\left(D^i{}_iD^k{}_k - 2D^k{}_iD^i{}_k\right) \\ &+\frac{1}{\varkappa}\left(B^i{}_iB^k{}_k - 2B^i{}_kB^k{}_i\right) \\ &+\varkappa(\beta^jD^0{}_j + 2\beta^jD^0{}_j\mathtt{D}^k{}_k - 4\beta^jD^0{}_k\mathtt{D}^k{}_j)\Big),\end{aligned} \tag{111}$$

where $D^i{}_k = D^i{}_A h^A{}_k$ and $B^k{}_i = B^{Aj}\eta^k{}_A\gamma_{ji}$, and in the last two terms, one should pay attention that the new field $\mathtt{D}^k{}_A$ introduced in (55) appears there.

Apparently, apart from the last terms in (111) depending on $D^0{}_k$, the potential U is more symmetric in the variables $D^\mu{}_a$ and $B^{a\mu}$ rather than the Lagrangian $\mathcal{L}$ in $E^a{}_\mu$ and $B^{a\mu}$. This in particular, justifies the introduction of the new and final variables (55)

$$\mathtt{D}^k{}_A = D^k{}_A + \beta^kD^0{}_A, \qquad \mathtt{B}^{Ak} = B^{Ak}, \tag{112}$$

so that the potential $U(h^A{}_k, D^k{}_A, B^{Ak}) = \mathtt{U}(h^A{}_k, \mathtt{D}^k{}_A, \mathtt{B}^{Ak})$ becomes just

$$\begin{aligned}&\mathtt{U}(h^A{}_k, \mathtt{D}^k{}_A, \mathtt{B}^{Ak}) = \\ &-\frac{\alpha}{2h_3}\Big(\varkappa\left(\mathtt{D}^A{}_B\mathtt{D}^B{}_A - \frac{1}{2}(\mathtt{D}^A{}_A)^2\right) \\ &+\frac{1}{\varkappa}\left(\mathtt{B}^A{}_B\mathtt{B}^B{}_A - \frac{1}{2}(\mathtt{B}^A{}_A)^2\right)\Big).\end{aligned} \tag{113}$$

13 CONCLUSION AND DISCUSSION

We have presented a 3+1-split of the TEGR equations in their historical pure tetrad version, i.e. with the spin connection set to zero. To the best of our knowledge, there were not many attempts to obtain a 3+1-split of the TEGR equations, e.g. ~~[18, 49]~~[43, 18, 49], and we are not aware of any attempts to solve numerically the full TEGR system of equations for general spacetimes. This paper, therefore, may help to cover this gap. However, first attempts to solve similar equations for the dGREM equations [47] in vacuum were done recently [48] with a structure compatible discretization.

Our derivation started from the action integral ~~of TEGR with an arbitrary Lagrangian denisty~~ with a Lagrangian denisty as an arbitrary function of the tetrad fields and their first gradietns, and therefore some elements of our derivation might be used for deriving 3+1 equations of extensions of TEGR such as

$f(\mathcal{T})$-teleparallel theories ~~also can be covered~~ in the future. After separating the spatial and time components of the torsion, the 3+1 governing partial differential equations have appeared to have ~~the same structure as~~ similar structure to the equations of nonlinear electrodynamics [25] and equations for continuum fluid and solid mechanics with torsion [52] if, howeveer, the gauge vectors A_k and Ω_k are set to 0. For more general gauge conditions on A_k and Ω_k, the 3+1 equations have a structure of the Maxwell equations coupled with the acoustic-type equations, e.g. see [23]. Moreover, it has appeared that the derived equations are equivalent to the recently proposed dGREM tetrad formulation of GR [47]. However, we emphasize that that the starting point of [47] was different. Here, we started from the variational principle while the fundamental elements of the dGREM formulation are the frame field, their exterior derivatives, and the Nester-Witten and Sparling forms.

The derived 3+1 TEGR equations are not immediately hyperbolic as usually the case for many first-order reductions of the Einstein equations. We demonstrated that for the ~~vacuum equations~~ empty space, the differential operator of the 3+1 vacuum TEGR equations can be transformed into a different but equivalent ~~form which is~~ quasi-linear form, provided that the Hamiltonian constraint is fulfilled. This form is equivalent to the symmetric hyperbolic tetrad reformulation of GR by Estabrook-Robinson-Wahlquist [26] and Buchman-Bardeen [14] if certain type of gauge conditions is imposed on the vectors $\mathrm{E}^{\hat{0}}{}_{k}$ and $\mathrm{H}_{k\hat{0}}$. The question of hyperbolicity of the full 3+1 TEGR equations coupled with matter is still open and requires further investigation.

Despite it is argued that TEGR is fully equivalent to Einstein's general relativity, the proposed 3+1 TEGR equations have yet to be carefully tested in a computational code and have yet to be proved to pass all the standard benchmark tests of GR. Therefore, further research will concern implementation of the TEGR equations in a high-order discontinuous Galerkin code [21, 15], with a possibility of constraint cleaning [20] and well-balancing [29]. This in particular would allow a direct comparison of the TEGR with other 3+1 equations of GR, such as Z4 formulations [4] forwarded by Bona *et al* in [12], and FO-CCZ4 by Dumbser *et al* [22], a strongly hyperbolic formulations of GR, within the same computational code. Another numerical strategy would be to use staggered grids and to develop a structure-preserving discretization [13, 47, 27, 48] that should allow to keep errors of div and curl-type involution constraints of TEGR at the machine precision. Finally, from the theoretical perspective, it would be important to extend the presented technique to covariant teleparallel geometries (non-zero spin connection) and $f(\mathcal{T})$ teleparallel theories and compare the 3+1 equations with their Hamiltonian formulations [41, 40, 28, 10, 7].

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A TRANSFORMATION OF NOETHER'S CURRENT $J^{\mu}{}_{a}$

Here, we express Noether's current $J^{\mu}{}_{a} = L_{h^{a}{}_{\mu}}$ for the gravitational part of the Lagrangian (i.e. the matter part is ignored in this section) in terms of the potential U and new variables $D^{\mu}{}_{a}$ and $B^{a\mu}$.

Thus, for the parametrization $L(h^{a}{}_{\mu}, \overset{\star}{T}{}^{a\mu\nu}) = \mathcal{L}(h^{a}{}_{\mu}, B^{a\mu}, E^{a}{}_{\mu})$, one has

$$L_{h^{a}{}_{\mu}} = \mathcal{L}_{h^{a}{}_{\mu}} + \mathcal{L}_{B^{b\lambda}} \frac{\partial B^{b\lambda}}{\partial h^{a}{}_{\mu}} + \mathcal{L}_{E^{b}{}_{\lambda}} \frac{\partial E^{b}{}_{\lambda}}{\partial h^{a}{}_{\mu}}. \tag{114}$$

Then, using the definitions of the frame 4-velocity $u^{\nu} = \eta^{\nu}{}_{\hat{0}}$ and $u_{\nu} = -h^{\hat{0}}{}_{\nu}$ and the torsion fields $B^{b\lambda} = \overset{\star}{T}{}^{b\lambda\nu} u_{\nu} = -\overset{\star}{T}{}^{b\lambda\nu} h^{\hat{0}}{}_{\nu}$ and $E^{b}{}_{\lambda} = T^{b}{}_{\lambda\nu} u^{\nu} = -\frac{1}{2} \in_{\lambda\nu\alpha\beta} \overset{\star}{T}{}^{b\alpha\beta} \eta^{\nu}{}_{\hat{0}}$, and the fact that $\partial \eta^{\lambda}{}_{b} / \partial h^{a}{}_{\mu} = -\eta^{\lambda}{}_{a} \eta^{\mu}{}_{b}$, we can rewrite (114) as

$$\begin{aligned} L_{h^{a}{}_{\mu}} = \mathcal{L}_{h^{a}{}_{\mu}} + \mathcal{L}_{B^{b\lambda}} \frac{\partial B^{b\lambda}}{\partial h^{a}{}_{\mu}} + \mathcal{L}_{E^{b}{}_{\lambda}} \frac{\partial E^{b}{}_{\lambda}}{\partial h^{a}{}_{\mu}} = \\ \mathcal{L}_{h^{a}{}_{\mu}} - \mathcal{L}_{B^{b\lambda}} \delta^{\hat{0}}{}_{a} (u^{\lambda} B^{b\mu} - u^{\mu} B^{b\lambda} + \in^{\lambda\mu\alpha\beta} u_{\alpha} E^{b}{}_{\beta}) \\ - \mathcal{L}_{E^{b}{}_{\lambda}} (u_{\lambda} E^{b}{}_{\nu} - u_{\nu} E^{b}{}_{\lambda} - \in_{\lambda\nu\alpha\rho} u^{\alpha} B^{b\rho}) \eta^{\nu}{}_{a} u^{\mu}. \end{aligned} \tag{115}$$

Using the definitions (38) and (39), the latter can be rewritten as

$$\begin{aligned} & L_{h^{a}{}_{\mu}} = -U_{h^{a}{}_{\mu}} \\ & - U_{B^{b\lambda}} \delta^{\hat{0}}{}_{a} (-u^{\lambda} B^{b\mu} + u^{\mu} B^{b\lambda} + \in^{\lambda\mu\alpha\beta} u_{\alpha} U_{D^{\beta}{}_{b}}) \\ & - D^{\lambda}{}_{b} \eta^{\nu}{}_{a} u^{\mu} (u_{\lambda} U_{D^{\nu}{}_{b}} - u_{\nu} U_{D^{\lambda}{}_{b}} + \in_{\lambda\nu\alpha\rho} u^{\alpha} B^{b\rho}), \end{aligned} \tag{116}$$

which, after some term rearrangements, reads

$$\begin{aligned} & L_{h^{a}{}_{\mu}} = -U_{h^{a}{}_{\mu}} \\ & + \delta^{\hat{0}}{}_{a} \Big(u^{\lambda} B^{b\mu} U_{B^{b\lambda}} - u^{\mu} B^{b\lambda} U_{B^{b\lambda}} \\ & - u^{\mu} D^{\lambda}{}_{b} U_{D^{\lambda}{}_{b}} + \in^{\mu\lambda\rho\sigma} u_{\rho} U_{B^{b\lambda}} U_{D^{\sigma}{}_{b}} \Big) \\ & - \eta^{\nu}{}_{a} u^{\mu} \left(u_{\lambda} D^{\lambda}{}_{b} U_{D^{\nu}{}_{b}} - \in_{\nu\lambda\rho\sigma} u^{\rho} B^{b\sigma} D^{\lambda}{}_{b} \right), \end{aligned} \tag{117}$$

and exactly is (41).

This formula, in particular, can be used to get the following expression for the energy-momentum $\sigma^{\mu}{}_{\nu} = h^{a}{}_{\nu} J^{\mu}{}_{a}$:

$$\begin{aligned} \sigma^{\mu}{}_{\nu} = & - h^{a}{}_{\nu} U_{h^{a}{}_{\mu}} \\ & - u^{\lambda} u_{\nu} B^{a\mu} U_{B^{a\lambda}} - u^{\mu} u_{\lambda} D^{\lambda}{}_{a} U_{D^{\nu}{}_{a}} \\ & + u^{\mu} u_{\nu} B^{a\lambda} U_{B^{a\lambda}} + u^{\mu} u_{\nu} D^{\lambda}{}_{a} U_{D^{\lambda}{}_{a}} \\ & + \in_{\nu\sigma\lambda\rho} u^{\mu} u^{\sigma} B^{a\lambda} D^{\rho}{}_{a} \\ & + \in^{\mu\sigma\lambda\rho} u_{\nu} u_{\sigma} U_{B^{a\lambda}} U_{D^{\rho}{}_{a}}. \end{aligned} \tag{118}$$

B EXPRESSION FOR THE ENERGY-MOMENTUM

In this section, we derive expression (44) for the gravitational part fof the energy-momentum

$$\sigma^{\mu}{}_{\nu} = 2 \overset{\star}{T}{}^{a\lambda\mu} L_{\overset{\star}{T}{}^{a\lambda\nu}} - (\overset{\star}{T}{}^{a\lambda\rho} L_{\overset{\star}{T}{}^{a\lambda\rho}} - L) \delta^{\mu}{}_{\nu} \tag{119}$$

Because $\overset{\star}{T}{}^{a\mu\nu}$ is antisymmetric tensor, to compute the derivative $L_{\overset{\star}{T}{}^{a\lambda\nu}}$ one needs to use its parametrization via the Weitzenböck connection, which is not symmetric, i.e. $\overset{\star}{T}{}^{a\mu\nu} = \in^{\mu\nu\rho\sigma} W^{a}{}_{\rho\sigma}$. Thus, for Lagrangians $\Lambda(h^{a}{}_{\mu}, W^{a}{}_{\mu\nu}) = L(h^{a}{}_{\mu}, \overset{\star}{T}{}^{a\mu\nu})$ one can write

$$\Lambda_{W^{a}{}_{\lambda\mu}} = \in^{\lambda\mu\rho\sigma} L_{\overset{\star}{T}{}^{a\rho\sigma}}, \tag{120}$$

or using the identity $\in_{\lambda\mu\alpha\beta} \in^{\lambda\mu\rho\sigma} = -2(\delta^{\rho}{}_{\alpha} \delta^{\sigma}{}_{\beta} - \delta^{\rho}{}_{\beta} \delta^{\sigma}{}_{\alpha})$,

$$\in_{\alpha\beta\rho\sigma} \Lambda_{W^{a}{}_{\lambda\mu}} = -4 L_{\overset{\star}{T}{}^{a\rho\sigma}}. \tag{121}$$

On the other hand, using the definitions $E^{a}{}_{\mu} = (W^{a}{}_{\mu\nu} - W^{a}{}_{\nu\mu}) u^{\nu}$, $B^{a\mu} = \in^{\mu\nu\rho\sigma} W^{a}{}_{\rho\sigma} u_{\nu}$, and the parametrization $\Lambda(h^{a}{}_{\mu}, W^{a}{}_{\mu\nu}) = \mathcal{L}(h^{a}{}_{\mu}, B^{a\mu}, E^{a}{}_{\mu})$, one can write

$$\begin{aligned} \Lambda_{W^{b}{}_{\lambda\gamma}} & = \mathcal{L}_{E^{a}{}_{\mu}} \frac{\partial E^{a}{}_{\mu}}{\partial W^{b}{}_{\lambda\gamma}} + \mathcal{L}_{B^{a\mu}} \frac{\partial B^{a\mu}}{\partial W^{b}{}_{\lambda\gamma}} \\ & = u^{\gamma} \mathcal{L}_{E^{b}{}_{\lambda}} - u^{\lambda} \mathcal{L}_{E^{b}{}_{\gamma}} - \in^{\lambda\gamma\nu\mu} u_{\nu} \mathcal{L}_{B^{a\mu}}, \end{aligned} \tag{122}$$

and hence, from (121) and (122), one can deduce

$$\begin{aligned} & L_{\overset{\star}{T}{}^{a\lambda\nu}} = \\ & \quad - \frac{1}{2} \left(u_{\lambda} \mathcal{L}_{B^{a\nu}} - u_{\nu} \mathcal{L}_{B^{a\lambda}} - \in_{\lambda\nu\rho\sigma} u^{\rho} \mathcal{L}_{E^{a}{}_{\sigma}} \right). \end{aligned} \tag{123}$$

Then, after contracting the later equation with $\overset{\star}{T}{}^{a\lambda\mu} = u^{\lambda} B^{a\mu} - u^{\mu} B^{a\lambda} + \in^{\lambda\mu\alpha\beta} u_{\alpha} E^{a}{}_{\beta}$, one obtains

$$\begin{aligned} & 2 \overset{\star}{T}{}^{a\lambda\mu} L_{\overset{\star}{T}{}^{a\lambda\nu}} = \\ & \quad B^{a\mu} \mathcal{L}_{B^{a\nu}} - E^{a}{}_{\nu} \mathcal{L}_{E^{a}{}_{\mu}} \\ & \quad + u^{\lambda} u_{\nu} B^{a\mu} \mathcal{L}_{B^{a\lambda}} - u^{\mu} u_{\nu} B^{a\lambda} \mathcal{L}_{B^{a\lambda}} \\ & \quad - \in_{\nu\lambda\rho\sigma} u^{\mu} u^{\sigma} B^{a\lambda} \mathcal{L}_{E^{a}{}_{\rho}} \\ & \quad - \in^{\mu\lambda\rho\sigma} u_{\nu} u_{\rho} E^{a}{}_{\sigma} \mathcal{L}_{B^{a\lambda}} \\ & \quad + (u^{\mu} u_{\nu} + \delta^{\mu}{}_{\nu}) E^{a}{}_{\lambda} \mathcal{L}_{E^{a}{}_{\lambda}} - u^{\mu} u_{\lambda} E^{a}{}_{\nu} \mathcal{L}_{E^{a}{}_{\lambda}}. \end{aligned} \tag{124}$$

This can be used to demonstrate that the full contraction $\overset{\star}{T}{}^{a\lambda\rho} L_{\overset{\star}{T}{}^{a\lambda\rho}}$ results in

$$\overset{\star}{T}{}^{a\lambda\rho} L_{\overset{\star}{T}{}^{a\lambda\rho}} = B^{a\lambda} \mathcal{L}_{B^{a\lambda}} + E^{a}{}_{\lambda} \mathcal{L}_{E^{a}{}_{\lambda}}. \tag{125}$$

Collecting together (124) and (125) and using the change of variables and potential (37)–(39), we arrive at

$$
\begin{aligned}
2\overset{\star}{T}^{a\lambda\mu}L_{\overset{\star}{T}^{a\lambda\nu}} - (\overset{\star}{T}^{a\lambda\rho}L_{\overset{\star}{T}^{a\lambda\rho}} - L)\delta^{\mu}_{\nu} = \\
&- B^{a\mu}U_{B^{a\nu}} - D^{\nu}{}_{a}U_{D^{\mu}{}_{a}} \\
&- u^{\lambda}u_{\nu}B^{a\mu}U_{B^{a\lambda}} - u^{\mu}u_{\lambda}D^{\lambda}{}_{a}U_{D^{\nu}{}_{a}} \\
&+ u^{\mu}u_{\nu}B^{a\lambda}U_{B^{a\lambda}} + u^{\mu}u_{\nu}D^{\lambda}{}_{a}U_{D^{\lambda}{}_{a}} \\
&+ \in_{\nu\sigma\lambda\rho}\, u^{\mu}u^{\sigma}B^{a\lambda}D^{\rho}{}_{a} \\
&+ \in^{\mu\sigma\lambda\rho}\, u_{\nu}u_{\sigma}U_{B^{a\lambda}}U_{D^{\rho}{}_{a}} \\
&+ (B^{a\lambda}U_{B^{a\lambda}} + D^{\lambda}{}_{a}U_{D^{\lambda}{}_{a}} - U)\delta^{\mu}_{\nu}. \qquad (126)
\end{aligned}
$$

C TRANSFORMATION OF THE TORSION PDE

In this appendix, we demonstrate how the Euler-Lagrange equation (35a)

$$\partial_{\nu}(\in^{\mu\nu\lambda\rho}\, L_{\overset{\star}{T}^{a\lambda\rho}}) = -L_{h^{a}{}_{\mu}} \qquad (127)$$

can be transformed to the form (36a).

Based on the different parametrization of the Lagrangian (we omit for the moment dependence of the Lagrangian on the tetrad field) $\Lambda(W^{a}{}_{\mu}) = L(\overset{\star}{T}^{a\mu\nu}) = \mathfrak{L}(T^{a}{}_{\mu\nu})$, one can obtain

$$\Lambda_{W^{a}{}_{\lambda\mu}} = \in^{\lambda\mu\rho\sigma}\, L_{\overset{\star}{T}^{a\rho\sigma}}, \quad \Lambda_{W^{a}{}_{\lambda\mu}} = \mathfrak{L}_{T^{a}{}_{\lambda\mu}} - \mathfrak{L}_{T^{a}{}_{\mu\lambda}}, \qquad (128)$$

from which it follows that the objects

$$\overset{\star}{\mathbb{T}}_{a\mu\nu} := L_{\overset{\star}{T}^{a\mu\nu}}, \qquad \mathbb{T}_{a}{}^{\mu\nu} := \frac{1}{2}\left(\mathfrak{L}_{T^{a}{}_{\mu\nu}} - \mathfrak{L}_{T^{a}{}_{\nu\mu}}\right) \qquad (129)$$

are Hodge duals of each others:

$$\overset{\star}{\mathbb{T}}_{a\mu\nu} := -\frac{1}{2}\in_{\mu\nu\rho\sigma}\,\mathbb{T}_{a}{}^{\rho\sigma}, \qquad (130a)$$

$$\mathbb{T}_{a}{}^{\mu\nu} := \frac{1}{2}\in^{\mu\nu\rho\sigma}\,\overset{\star}{\mathbb{T}}_{a\rho\sigma}. \qquad (130b)$$

Therefore, one can write the following identities

$$\overset{\star}{\mathbb{T}}_{a\mu\nu} = u_{\mu}\mathbb{H}_{a\nu} - u_{\nu}\mathbb{H}_{a\mu} - \in_{\mu\nu\rho\sigma}\, u^{\rho}\mathbb{D}_{a}{}^{\sigma}, \qquad (131a)$$

$$\mathbb{T}_{a}{}^{\mu\nu} = u^{\mu}\mathbb{D}_{a}{}^{\nu} - u^{\nu}\mathbb{D}_{a}{}^{\mu} + \in^{\mu\nu\rho\sigma}\, u_{\rho}\mathbb{H}_{a\sigma}, \qquad (131b)$$

where

$$\mathbb{H}_{a\mu} := \overset{\star}{\mathbb{T}}_{a\mu\nu}u^{\nu}, \qquad \mathbb{D}_{a}{}^{\mu} := \mathbb{T}_{a}{}^{\mu\nu}u_{\nu}, \qquad (132)$$

Hence, the Euler-Lagrange equation (127) now reads

$$\partial_{\nu}\mathbb{T}_{a}{}^{\mu\nu} = -\frac{1}{2}L_{h^{a}{}_{\mu}}, \qquad (133)$$

or, according to (131b), it can be written as

$$\partial_{\nu}(u^{\mu}\mathbb{D}_{a}{}^{\nu} - u^{\nu}\mathbb{D}_{a}{}^{\mu} + \in^{\mu\nu\alpha\beta}\, u_{\alpha}\mathbb{H}_{a\beta}) = -\frac{1}{2}L_{h^{a}{}_{\mu}}. \qquad (134)$$

It remains to express $\mathbb{D}_{a}{}^{\mu}$ and $\mathbb{H}_{a\mu}$ in terms of $\mathcal{L}(B^{a\mu}, E^{a}{}_{\mu})$. Thus, using the fact that $\Lambda_{W^{a}{}_{\mu\nu}} = 2\mathbb{T}_{a}{}^{\mu\nu}$ and $\Lambda_{W^{a}{}_{\mu\nu}} = \in_{\mu\nu\rho\sigma}\,\overset{\star}{\mathbb{T}}_{a\rho\sigma}$, and the expression (122), one can derive that

$$\mathbb{H}_{a\mu} = -\frac{1}{2}\left(\mathcal{L}_{B^{a\mu}} + u^{\lambda}\mathcal{L}_{B^{a\lambda}}u_{\mu}\right), \qquad (135a)$$

$$\mathbb{D}_{a}{}^{\mu} = -\frac{1}{2}\left(\mathcal{L}_{E^{a}{}_{\mu}} + u_{\lambda}\mathcal{L}_{E^{a}{}_{\lambda}}u^{\mu}\right). \qquad (135b)$$

Finally, plugging these expressions in (134), one obtains the desired result

$$\partial_{\nu}(u^{\mu}\mathcal{L}_{E^{a}{}_{\nu}} - u^{\nu}\mathcal{L}_{E^{a}{}_{\mu}} + \in^{\mu\nu\rho\sigma}\, u_{\rho}\mathcal{L}_{B^{a\sigma}}) = L_{h^{a}{}_{\mu}}. \qquad (136)$$

D SOME EXPRESSIONS OF THE TORSION SCALAR

Denoting the scalars in the right-hand side of (109b) as ($\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$)

$$\mathcal{T}_1 = \frac{1}{4}\kappa_{ab}g^{\beta\lambda}g^{\mu\gamma}T^{a}{}_{\lambda\gamma}T^{b}{}_{\beta\mu}, \qquad (137a)$$

$$\mathcal{T}_2 = \frac{1}{2}g^{\mu\gamma}\eta^{\lambda}{}_{a}\eta^{\beta}{}_{b}T^{a}{}_{\mu\beta}T^{b}{}_{\gamma\lambda}, \qquad (137b)$$

$$\mathcal{T}_3 = -g^{\mu\lambda}\eta^{\rho}{}_{a}\eta^{\gamma}{}_{b}T^{a}{}_{\mu\rho}T^{b}{}_{\lambda\gamma}, \qquad (137c)$$

we can also write

$$\mathcal{T}_1 = Q_3 + \frac{1}{2}C_4, \qquad (138a)$$

$$\mathcal{T}_2 = -Q_1 + Q_4 + L_1 - C_1 + C_2 + \frac{1}{2}C_4, \qquad (138b)$$

$$\mathcal{T}_3 = 2Q_1 + Q_2 + L_2 + 2C_1 + C_3 - C_4, \qquad (138c)$$

where the scalars Q, L, and C are scalars made of $E^{a}{}_{\mu}$ and $B^{a\mu}$ as follows.

Quadratic in $E^a{}_\mu$:

$$Q_1 = -\frac{1}{2} E^{\hat{0}}{}_\alpha g^{\alpha\beta} E^{\hat{0}}{}_\beta, \tag{139a}$$

$$Q_2 = \eta^\alpha{}_a E^a{}_\alpha \eta^\beta{}_b E^b{}_\beta = E^\alpha{}_\alpha E^\beta{}_\beta, \tag{139b}$$

$$Q_3 = -\frac{1}{2} \kappa_{ab} E^a{}_\alpha g^{\alpha\mu} E^b{}_\mu, \tag{139c}$$

$$Q_4 = -\frac{1}{2} \eta^\lambda{}_a E^a{}_\beta \eta^\beta{}_b E^b{}_\lambda = E^\lambda{}_\beta E^\beta{}_\lambda. \tag{139d}$$

Mixed scalars (linear in $E^a{}_\mu$):

$$L_1 = \in_{\lambda\tau\varphi\gamma} u^\varphi g^{\tau\beta} \eta^\lambda{}_a E^a{}_\beta B^{\hat{0}\gamma}, \tag{139e}$$

$$L_2 = 2 \in_{\lambda\tau\varphi\gamma} u^\varphi g^{\lambda\beta} \eta^\tau{}_a E^{\hat{0}}{}_\beta B^{a\gamma}. \tag{139f}$$

Quadratic in $B^{a\mu}$ (constant in $E^a{}_\mu$)

$$C_1 = -\frac{1}{2} h^{-2} g_{\lambda\rho} B^{\hat{0}\rho} B^{\hat{0}\lambda}, \tag{139g}$$

$$C_2 = -\frac{1}{2} h^{-2} g_{\rho\varphi} \eta^\varphi{}_a B^{a\rho} \eta^\beta{}_b B^{b\lambda} g_{\lambda\beta}, \tag{139h}$$

$$C_3 = h^{-2} \eta^\varphi{}_a B^{a\sigma} g_{\varphi\beta} \eta^\lambda{}_b B^{b\beta} g_{\lambda\sigma}, \tag{139i}$$

$$C_4 = h^{-2} \eta^\varphi{}_a B^{a\sigma} g_{\varphi\lambda} \eta^\lambda{}_b B^{b\beta} g_{\beta\sigma}. \tag{139j}$$

In terms of the Hodge dual $\overset{\star}{T}{}^{a\mu\nu}$, the torsion scalar $\mathcal{T}$ can be rewritten as

$$\mathcal{T} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4, \tag{140a}$$

where

$$\mathcal{H}_1 = \frac{1}{2h^2} \kappa_{ac} \kappa_{bd} h^c{}_\lambda h^d{}_\sigma g_{\tau\rho} \overset{\star}{T}{}^{a\rho\sigma} \overset{\star}{T}{}^{b\lambda\tau}, \tag{140b}$$

$$\mathcal{H}_2 = \frac{1}{2h^2} \kappa_{ac} \kappa_{bd} h^c{}_\tau h^d{}_\rho g_{\lambda\sigma} \overset{\star}{T}{}^{a\rho\sigma} \overset{\star}{T}{}^{b\lambda\tau}, \tag{140c}$$

$$\mathcal{H}_3 = -\frac{1}{4h^2} \kappa_{ac} \kappa_{bd} h^c{}_\sigma h^d{}_\lambda g_{\tau\rho} \overset{\star}{T}{}^{a\rho\sigma} \overset{\star}{T}{}^{b\lambda\tau}, \tag{140d}$$

$$\mathcal{H}_4 = -\frac{1}{4h^2} \kappa_{ac} \kappa_{bd} h^c{}_\rho h^d{}_\tau g_{\lambda\sigma} \overset{\star}{T}{}^{a\rho\sigma} \overset{\star}{T}{}^{b\lambda\tau}. \tag{140e}$$

Finally, in terms of the Weitzenböck connection, the torsion scalar reads:

$$\mathcal{T} = \sum_{n=1}^{8} \mathcal{W}_n \tag{141a}$$

$$\mathcal{W}_1 = \frac{1}{2} \kappa_{ab} g^{\beta\lambda} g^{\mu\gamma} W^a{}_{\lambda\gamma} W^b{}_{\beta\mu}, \tag{141b}$$

$$\mathcal{W}_2 = -\frac{1}{2} \kappa_{ab} g^{\beta\lambda} g^{\mu\gamma} W^a{}_{\gamma\lambda} W^b{}_{\beta\mu}, \tag{141c}$$

$$\mathcal{W}_3 = \frac{1}{2} g^{\mu\gamma} \eta^\beta{}_a W^a{}_{\lambda\gamma} \eta^\lambda{}_b W^b{}_{\beta\mu}, \tag{141d}$$

$$\mathcal{W}_4 = \frac{1}{2} g^{\mu\gamma} \eta^\beta{}_a W^a{}_{\gamma\lambda} \eta^\lambda{}_b W^b{}_{\mu\beta}, \tag{141e}$$

$$\mathcal{W}_5 = -g^{\mu\gamma} \eta^\beta{}_a W^a{}_{\gamma\lambda} \eta^\lambda{}_b W^b{}_{\beta\mu}, \tag{141f}$$

$$\mathcal{W}_6 = -g^{\mu\lambda} w^1_\lambda w^1_\mu, \tag{141g}$$

$$\mathcal{W}_7 = -g^{\mu\lambda} w^2_\lambda w^2_\mu, \tag{141h}$$

$$\mathcal{W}_8 = 2 g^{\mu\lambda} w^1_\mu w^2_\lambda, \tag{141i}$$

where $w^1_\mu = \eta^\lambda{}_a W^a{}_{\mu\lambda}$, $w^2_\mu = \eta^\lambda{}_a W^a{}_{\lambda\mu}$.

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