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## The natural-conformation tensor constitutive laws

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<b>Abstract:</b>	In this work we revisit the seminal work of Renardy [M. Renardy, J. Non-Newtonian Fluid Mech., 52(1) (1994), 91-95] on the reformulation of the stress tensor in its "natural" basis, and present a generic framework for the natural -conformation tensor for a large class of differential constitutive models. We show that the proposed dyadic transformation can be equated as an orthogonal transformation of the conformation tensor into a streamlined orthonormal basis given by a rotation tensor expressed in terms of the unit velocity vectors. We also show that the natural -conformation tensor formulation is a particular sub-case of the kernel -conformation tensor transformation [A.M. Afonso, F.T. Pinho, M.A. Alves, J. Non-Newtonian Fluid Mech. 167-168(2012) 30-37] with the kernel function acting on the rotation of the eigenvectors rather than on the magnitude of the extension of the conformation tensor.
<b>Opposed Reviewers:</b>	

Prof. Robert J. Poole  
Prof. Ian Frigaard  
Editors-in-Chief  
JNNFM

Porto, February 27, 2025

**Ref.:** “The *natural*-conformation tensor constitutive laws”, by A.M. Afonso, J.D. Evans and I.L. Palhares Junior.

Dear Ian and Rob,

We are (re)submitting this short note on "The *natural*-conformation tensor constitutive laws" to be considered to *Journal of Non-Newtonian Fluid Mechanics*.

In this work we revisit the seminal work of Renardy [M. Renardy. “*How to integrate the upper convected Maxwell (UCM) stresses near a singularity (and maybe elsewhere, too)*”, *Journal of Non-Newtonian Fluid Mechanics* 52 (Apr. 1994), pp. 91–95.] on the reformulation of the stress tensor in its *natural* basis, and present the *natural*-conformation tensor constitutive laws for a large class of differential constitutive models.

The current version was extended to address the observations of the previous reviewers and also the JNNFM editors.

We hope to hear from you in due course.

With best regards,



Alexandre Afonso

# The *natural*-conformation tensor constitutive laws

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## Abstract

In this work we revisit the seminal work of Renardy [M. Renardy, J. Non-Newtonian Fluid Mech., 52(1) (1994), 91-95] on the reformulation of the stress tensor in its “natural” basis, and present a generic framework for the *natural*-conformation tensor for a large class of differential constitutive models. We show that the proposed dyadic transformation can be equated as an orthogonal transformation of the conformation tensor into a streamlined orthonormal basis given by a rotation tensor expressed in terms of the unit velocity vectors. We also show that the *natural*-conformation tensor formulation is a particular sub-case of the *kernel*-conformation tensor transformation [A.M. Afonso, F.T. Pinho, M.A. Alves, J. Non-Newtonian Fluid Mech. 167-168(2012) 30-37] with the *kernel* function acting on the rotation of the eigenvectors rather than on the magnitude of the extension of the conformation tensor.

*Keywords:* Natural-conformation tensor transformation, “natural” basis reformulation, computational rheology

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## 1. The natural basis tensor reformulation

More than 30 years ago, Renardy [12] proposed a stress reformulation by representing the extra-stress tensor in its “natural” basis, rather than in Cartesian or polar coordinates. It was proposed that this “natural” basis in planar flow be aligned with the streamlines. The idea was based on the key evidence that in a flow near a re-entrant corner, where the upper convected derivative dominates, the stresses are of the type  $h(\psi)\mathbf{u}\mathbf{u}^T$ , for an arbitrary  $h(\psi)$  function of the stream function  $\psi$  and  $\mathbf{u}$  the velocity field. Hinch [10] had earlier used this exact solution to derive a local similarity solution for the Oldroyd-B model.

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Following this simple observation, Renardy proposed the extra-stress tensor in its "natural" basis, aligned with streamlines. As such, the extra stress tensor  $\mathbf{T}$  is represented relative to a basis formed with the dyadic products of  $\mathbf{u}$  and  $\mathbf{v}$ ,

$$\mathbf{T} = \lambda \mathbf{u} \mathbf{u}^T + \mu (\mathbf{u} \mathbf{v}^T + \mathbf{v} \mathbf{u}^T) + \nu \mathbf{v} \mathbf{v}^T, \quad (1)$$

where  $\lambda, \mu, \nu$  were termed the natural stress variables. The vector  $\mathbf{v}$  is taken perpendicular to the velocity  $\mathbf{u}$  such that the determinant formed from their components is unity. Explicitly, for 2D flow,

$$\mathbf{u} = (u, v), \quad \mathbf{v} = \left( -\frac{v}{u^2 + v^2}, \frac{u}{u^2 + v^2} \right). \quad (2)$$

For models such as UCM, a significant decoupling in the component equations for the natural stress variables takes place in steady flow. In fact, the equation for  $\nu$  is decoupled from the rest. This is significant, as it is the term  $\mathbf{v} \mathbf{v}^T$  which grows unboundedly at solid boundaries where the velocity vanishes. By keeping the equation for  $\nu$  separate, it avoids the excitation of numerical discretization errors which were found to amplify at the downstream wall of re-entrant corners [15].

## 2. The natural stress formulation

In general flows, it is advantageous to consider the conformation tensor  $\mathbf{A}$  instead of the extra-stress tensor, using the same "natural" basis alignment. This is apparent when considering High Weissenberg number boundary layers [14, 13, 18], for example. Specifically,

$$\mathbf{A} = \lambda \mathbf{u} \mathbf{u}^T + \mu (\mathbf{u} \mathbf{v}^T + \mathbf{v} \mathbf{u}^T) + \nu \mathbf{v} \mathbf{v}^T. \quad (3)$$

The two representations are related through  $\mathbf{T} = \frac{\eta_p}{\tau_p} (\mathbf{A} - \mathbf{I})$ , where  $\eta_p$  is the polymer viscosity and  $\tau_p$  its relaxation time. Evans and Oishi [6] proposed to term (3) the Natural Stress Formulation. The use of the conformation tensor is motivated by several reasons. Primarily it must remain positive definite, due to its representation as a variance-covariance matrix in the microstructure model. Also, for simple models such as UCM, the range of stresses are restricted due to the extra-stress being dependent on the relative strain Finger tensor. This again manifests itself in the conformation tensor having to remain positive definite. This condition is also suffice to ensure that the UCM equations never become Hadamard unstable i.e. catastrophically unstable to short waves [11]. It is also beneficial in viscoelastic models where the identity term plays a significant role in the boundary layer equations at solid surfaces. This is seen in the Giesekus model for general flows at high Weissenberg numbers [9] and specifically at re-entrant corners [5]. Alves et al. [2] observed that the numerical use of the natural stress formulation can alleviate numerical diffusion problems when using upwind-based schemes since the transported quantity is advected along the streamlines.

The time and spatial evolution of this conformation tensor obeys an evolution equation, represented by the following generic differential constitutive model,

$$\frac{D\mathbf{A}}{Dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{A} = \mathbf{A} (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \mathbf{A} + \frac{\mathcal{F}(\mathbf{A})}{Wi} \mathcal{H}(\mathbf{A}) \quad (4)$$

where  $\mathcal{F}(\mathbf{A})$  is a scalar and  $\mathcal{H}(\mathbf{A})$  a tensor, and both of these functions allow the definition of a specific constitutive model, such as the Oldroyd-B (UCM), PTT or FENE-type models. The Weissenberg number is defined as  $Wi = \tau_p U/L$ , where  $U$  and  $L$  are characteristic velocity and length scales, respectively. It is a non-dimensional parameter that measures the elasticity of a fluid, indicating the degree of anisotropy or orientation generated by the deformation.

For computational purposes, Evans and Oishi [6] suggested a form of (3) using rescaled natural stress variables, namely

$$\mathbf{A} = \hat{\lambda} \hat{\mathbf{u}} \hat{\mathbf{u}}^T + \hat{\mu} (\hat{\mathbf{u}} \hat{\mathbf{v}}^T + \hat{\mathbf{v}} \hat{\mathbf{u}}^T) + \hat{\nu} \hat{\mathbf{v}} \hat{\mathbf{v}}^T, \quad (5)$$

where

$$\hat{\lambda} = |\mathbf{u}|^2 \lambda, \quad \hat{\mu} = \mu, \quad \hat{\nu} = \frac{\nu}{|\mathbf{u}|^2}, \quad (6)$$

and

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{|\mathbf{u}|} \begin{pmatrix} u \\ v \end{pmatrix} \quad \hat{\mathbf{v}} = |\mathbf{u}| \mathbf{v} = \frac{1}{|\mathbf{u}|} \begin{pmatrix} -v \\ u \end{pmatrix}, \quad (7)$$

are unit vectors.

### 3. The *natural*-conformation tensor formulation

In this short note we present a generalization for the *natural*-conformation tensor formulation for a family of differential constitutive equations, following the key idea that the *natural*-conformation tensor,  $\mathbf{N}$ , can be obtained by equating the proper relation with the conformation tensor,  $\mathbf{A}$ . Therefore, based on the original idea proposed by Renardy [12], we can define the *natural*-conformation tensor,  $\mathbf{N}$ , as the orthogonal transformation of the conformation tensor,

$$\mathbf{N} = \mathbf{R}^T \mathbf{A} \mathbf{R}, \quad (8)$$

where the *natural* basis rotation tensor,  $\mathbf{R}$ , is a proper rotation tensor transforming into a streamlined orthonormal basis defined by three unitary vectors,  $\hat{\mathbf{u}}$ ,  $\hat{\mathbf{v}}$  and  $\hat{\mathbf{w}}$ , with  $|\hat{\mathbf{u}}| = |\hat{\mathbf{v}}| = |\hat{\mathbf{w}}| = 1$ ,  $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = 0$  and  $\hat{\mathbf{u}} \times \hat{\mathbf{v}} = \hat{\mathbf{w}}$ . Denoting by  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$  and  $\hat{\mathbf{k}}$  the unit vectors in fixed Cartesian  $x$ ,  $y$  and  $z$  directions, we have

$$\mathbf{R} = \hat{\mathbf{u}} \hat{\mathbf{i}}^T + \hat{\mathbf{v}} \hat{\mathbf{j}}^T + \hat{\mathbf{w}} \hat{\mathbf{k}}^T. \quad (9)$$

The *natural* basis rotation tensor,  $\mathbf{R}$ , is a real, orthogonal matrix, with columns represented by unit vectors, it belongs to the special orthogonal group with dimensions three, such as

$\mathbf{R}^T = \mathbf{R}^{-1}$ ,  $\mathbf{R}^T \mathbf{R} = \mathbf{R} \mathbf{R}^T = \mathbf{I}$  and  $\det \mathbf{R} = +1$ . We can use this property and perform the material derivative on both sides of  $\mathbf{R}^T \mathbf{R} = \mathbf{I}$ , obtaining  $\frac{D\mathbf{R}^T}{Dt} \mathbf{R} + \mathbf{R}^T \frac{D\mathbf{R}}{Dt} = \mathbf{0}$ , which indicates that  $\mathbf{\Omega} = \frac{D\mathbf{R}^T}{Dt} \mathbf{R}$  is a skew-symmetric tensor satisfying  $\mathbf{\Omega} + \mathbf{\Omega}^T = \mathbf{0}$ .

The *natural* conformation tensor differential constitutive model can be derived by equating the material derivative on relation (8) and using the generic differential constitutive model for the conformation tensor of Equation (4), to obtain:

$$\frac{D\mathbf{N}}{Dt} = \mathbf{N} \widetilde{\nabla \mathbf{u}} + \widetilde{\nabla \mathbf{u}}^T \mathbf{N} + \frac{\mathcal{F}(\mathbf{N})}{W_i} \mathcal{H}(\mathbf{N}), \quad (10)$$

where the *natural* velocity gradient tensor is defined by  $\widetilde{\nabla \mathbf{u}} = \mathbf{R}^T \nabla \mathbf{u} \mathbf{R} - \mathbf{\Omega}$  and  $\widetilde{\nabla \mathbf{u}}^T = \mathbf{R}^T \nabla \mathbf{u}^T \mathbf{R} + \mathbf{\Omega}$ .

The *natural* velocity gradient tensor can be related to a symmetric *natural* rate of strain tensor,  $\widetilde{\mathbf{D}} = \frac{1}{2} [\widetilde{\nabla \mathbf{u}} + \widetilde{\nabla \mathbf{u}}^T] = \mathbf{R}^T \frac{1}{2} [\nabla \mathbf{u} + \nabla \mathbf{u}^T] \mathbf{R} = \mathbf{R}^T \mathbf{D} \mathbf{R}$ , and an anti-symmetric *natural* vorticity tensor  $\widetilde{\mathbf{W}} = \frac{1}{2} [\widetilde{\nabla \mathbf{u}} - \widetilde{\nabla \mathbf{u}}^T] = \mathbf{R}^T \frac{1}{2} [\nabla \mathbf{u} - \nabla \mathbf{u}^T] \mathbf{R} - \mathbf{\Omega} = \mathbf{R}^T \mathbf{W} \mathbf{R} - \mathbf{\Omega}$ . This *natural* vorticity tensor can be also described as a form of the relative-rate-of-rotation tensor proposed by Astarita [3], measuring the rate of rotation of the material as measured by an observer who is fixed, not into the principal directions of the rate of strain as proposed by Astarita [3], but rather into the principal streamline direction. In fact, the right hand side of Equation (10) is obtained by means of an orthogonal transformation (operated by the rotation  $\mathbf{R}$  tensor) of a Jaumann corotational derivative of the conformation tensor (operated by the spin tensor,  $\mathbf{\Omega}$ ). Other pairs of *rotation/spin* corotational operators have been proved as objective tensors by Goddard and Miller [8] and Dafalias [4], for example.

As final remark, if we use other orthogonal basis in Equation (8), for instance, the set of eigenvectors of the conformation tensor it self, meaning a change in the pair *rotation/spin* defined as  $\mathbf{R}|\mathbf{\Omega} \rightarrow \mathbf{O}|\mathbf{\mathcal{O}}$ , we obtain, by Equation (10), the *eigendecomposed* version of Vaithianathan and Collins [17], for the diagonal tensor,  $\mathbf{\Lambda} = \mathbf{O}^T \mathbf{A} \mathbf{O}$ ,

$$\frac{D\mathbf{\Lambda}}{Dt} = \mathbf{\Lambda} \overline{\nabla \mathbf{u}} + \overline{\nabla \mathbf{u}}^T \mathbf{\Lambda} + \frac{\mathcal{F}(\mathbf{\Lambda})}{W_i} \mathcal{H}(\mathbf{\Lambda}) \quad (11)$$

where  $\overline{\nabla \mathbf{u}} = \mathbf{O}^T \nabla \mathbf{u} \mathbf{O} - \mathbf{\mathcal{O}}$ , and  $\mathbf{\mathcal{O}} = \frac{D\mathbf{O}^T}{Dt} \mathbf{O}$  is the corresponding spin tensor. This means that the *natural*-conformation tensor reformulation is an orthogonal operation on the eigenvectors of the conformation tensor, and therefore can be devised as a particular case of the *kernel*-conformation tensor transformation [1]. In the *natural* conformation tensor differential constitutive model the *kernel* operation acts directly in the orientation of the principal directions of the eigenvectors rather than on the magnitude of the extension of the conformation tensor.

**Example for 3D case** The extension to three dimensions of the *natural*-conformation tensor formulation is also simpler, and here we present the demonstration for a 3D case for the generalization for the natural conformation tensor formulation. There are infinite possibilities to

define the *natural* basis rotation tensor,  $\mathbf{R}$ , in a 3D flow. One possibility to define the *natural* basis rotation tensor is to use a Frenet-Serret type of representation [16] for the streamlined orthonormal basis defined by three unitary vectors,  $\left\{ \hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}, \hat{\mathbf{v}} = \frac{d\hat{\mathbf{u}}/ds}{|d\hat{\mathbf{u}}/ds|}, \hat{\mathbf{w}} = \hat{\mathbf{u}} \times \hat{\mathbf{v}} \right\}$ , defined as the unit tangent vector, the normal vector, and the binormal vector, respectively. Here, we propose a different streamlined orthonormal basis in which the plane of rotation is formed by  $\hat{\mathbf{u}}$  and  $\hat{\mathbf{v}} = \frac{-v}{\sqrt{u^2+v^2}}\hat{\mathbf{i}} + \frac{u}{\sqrt{u^2+v^2}}\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$ , meaning that the axis of rotation is along a unit vector  $\hat{\mathbf{w}} = \frac{\hat{\mathbf{u}} \times \hat{\mathbf{v}}}{|\hat{\mathbf{u}} \times \hat{\mathbf{v}}|}$ , forming the orthonormal basis:  $\left\{ \hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|}, \hat{\mathbf{v}}, \hat{\mathbf{w}} = \frac{\hat{\mathbf{u}} \times \hat{\mathbf{v}}}{|\hat{\mathbf{u}} \times \hat{\mathbf{v}}|} \right\}$ . This representation preserves the original idea from Renardy [12], preserving the same definition for both the angle and axis of rotation,  $\cos \theta = \frac{u}{|\mathbf{u}|}$  and  $\hat{\mathbf{w}}$  (or  $\hat{\mathbf{k}}$  in a 2D case), respectively. Using this representation in dyadic products (9),  $\mathbf{R}$  takes the form of,

$$\mathbf{R} = \begin{bmatrix} \frac{u}{|\mathbf{u}|} & \frac{-v}{\sqrt{u^2+v^2}} & \frac{-uw}{\sqrt{(uw)^2+(vw)^2+(u^2+v^2)^2}} \\ \frac{v}{|\mathbf{u}|} & \frac{u}{\sqrt{u^2+v^2}} & \frac{-vw}{\sqrt{(uw)^2+(vw)^2+(u^2+v^2)^2}} \\ \frac{w}{|\mathbf{u}|} & 0 & \frac{u^2+v^2}{\sqrt{(uw)^2+(vw)^2+(u^2+v^2)^2}} \end{bmatrix}. \quad (12)$$

The conformation tensor can be defined in the dyadic decomposition, as:

$$\begin{aligned} \mathbf{A} &= \mathbf{R}\mathbf{N}\mathbf{R}^T = \mathbf{R} \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{12} & n_{22} & n_{23} \\ n_{13} & n_{23} & n_{33} \end{bmatrix} \mathbf{R}^T \\ &= \begin{bmatrix} \frac{u}{|\mathbf{u}|} & \frac{-v}{\sqrt{u^2+v^2}} & \frac{-uw}{\sqrt{(uw)^2+(vw)^2+(u^2+v^2)^2}} \\ \frac{v}{|\mathbf{u}|} & \frac{u}{\sqrt{u^2+v^2}} & \frac{-vw}{\sqrt{(uw)^2+(vw)^2+(u^2+v^2)^2}} \\ \frac{w}{|\mathbf{u}|} & 0 & \frac{u^2+v^2}{\sqrt{(uw)^2+(vw)^2+(u^2+v^2)^2}} \end{bmatrix} \mathbf{N} \begin{bmatrix} \frac{u}{|\mathbf{u}|} & \frac{v}{\sqrt{u^2+v^2}} & \frac{w}{|\mathbf{u}|} \\ \frac{-v}{\sqrt{u^2+v^2}} & \frac{u}{\sqrt{u^2+v^2}} & 0 \\ \frac{-uw}{\sqrt{(uw)^2+(vw)^2+(u^2+v^2)^2}} & \frac{-vw}{\sqrt{(uw)^2+(vw)^2+(u^2+v^2)^2}} & \frac{u^2+v^2}{\sqrt{(uw)^2+(vw)^2+(u^2+v^2)^2}} \end{bmatrix} \\ &= n_{11}\hat{\mathbf{u}}\hat{\mathbf{u}}^T + n_{22}\hat{\mathbf{v}}\hat{\mathbf{v}}^T + n_{33}\hat{\mathbf{w}}\hat{\mathbf{w}}^T + n_{12}(\hat{\mathbf{u}}\hat{\mathbf{v}}^T + \hat{\mathbf{v}}\hat{\mathbf{u}}^T) + n_{13}(\hat{\mathbf{u}}\hat{\mathbf{w}}^T + \hat{\mathbf{w}}\hat{\mathbf{u}}^T) + n_{23}(\hat{\mathbf{w}}\hat{\mathbf{v}}^T + \hat{\mathbf{v}}\hat{\mathbf{w}}^T). \end{aligned} \quad (13)$$

For the 2D case, the natural basis rotation tensor obtained from Equation (12), takes the form

$$\mathbf{R} = \frac{1}{|\mathbf{u}|} \begin{bmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & |\mathbf{u}| \end{bmatrix}, \quad (14)$$

and the conformation tensor can be expressed by,

$$\begin{aligned}
\mathbf{A} &= \mathbf{R} \begin{bmatrix} n_{11} & n_{12} & 0 \\ n_{12} & n_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{R}^T = \mathbf{R} \begin{bmatrix} \hat{\lambda} & \hat{\mu} & 0 \\ \hat{\mu} & \hat{\nu} & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{R}^T \\
&= \frac{1}{|\mathbf{u}|^2} \begin{bmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & |\mathbf{u}| \end{bmatrix} \begin{bmatrix} \hat{\lambda} & \hat{\mu} & 0 \\ \hat{\mu} & \hat{\nu} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & |\mathbf{u}| \end{bmatrix} \\
&= \frac{1}{|\mathbf{u}|^2} \begin{bmatrix} \hat{\lambda} u^2 - 2\hat{\mu} uv + \hat{\nu} v^2 & \hat{\lambda} uv + \hat{\mu} (u^2 - v^2) - \hat{\nu} uv & 0 \\ \hat{\lambda} uv + \hat{\mu} (u^2 - v^2) - \hat{\nu} uv & \hat{\lambda} v^2 + 2\hat{\mu} uv + \hat{\nu} u^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (15) \\
&= \frac{\hat{\lambda}}{|\mathbf{u}|^2} \begin{bmatrix} u^2 & uv & 0 \\ uv & v^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{\hat{\mu}}{|\mathbf{u}|^2} \left( \begin{bmatrix} -uv & u^2 & 0 \\ -v^2 & uv & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -uv & -v^2 & 0 \\ u^2 & uv & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) + \frac{\hat{\nu}}{|\mathbf{u}|^2} \begin{bmatrix} v^2 & -uv & 0 \\ -uv & u^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \hat{\lambda} \hat{\mathbf{u}} \hat{\mathbf{u}}^T + \hat{\mu} (\hat{\mathbf{u}} \hat{\mathbf{v}}^T + \hat{\mathbf{v}} \hat{\mathbf{u}}^T) + \hat{\nu} \hat{\mathbf{v}} \hat{\mathbf{v}}^T \quad (16)
\end{aligned}$$

which recovers the relations proposed by Evans and Oishi [6] in (5). We can also observe that  $\hat{\mathbf{u}}$  and  $\hat{\mathbf{v}}$  are eigenvectors of both the first and fourth dyadic products in (5), whilst separately  $\hat{\mathbf{u}}$  the third dyadic term. Consequently we have the following diagonalisations

$$\hat{\mathbf{u}} \hat{\mathbf{u}}^T = \mathbf{R} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{R}^T, \quad \hat{\mathbf{v}} \hat{\mathbf{v}}^T = \mathbf{R} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{R}^T, \quad (17)$$

and further

$$\hat{\mathbf{u}} \hat{\mathbf{v}}^T = \mathbf{R} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{R}^T, \quad \hat{\mathbf{v}} \hat{\mathbf{u}}^T = \mathbf{R} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{R}^T. \quad (18)$$

so that expression (5) can be written succinctly in the matrix form  $\mathbf{R} \mathbf{N} \mathbf{R}^T$ .

**The limit for zero velocity** Other advantage of the notation used to describe the natural-conformation tensor on Equation (8) is that it clarifies the possible divergences issues for the limiting behavior when approaching points with zero velocity magnitudes ( $|\mathbf{u}| \rightarrow 0$ ), such as internal stagnation points [2].

Since the *natural* basis rotation tensor,  $\mathbf{R}$ , is a proper rotation tensor, it can also be represented to the 2D case (complementing equation (14)), as

$$\mathbf{R} = e^{\theta \mathbf{K}}, \quad (19)$$

with the angle of rotation  $\theta = \arccos \frac{u}{|\mathbf{u}|}$  and  $\mathbf{K} = \left[ \hat{\mathbf{k}} \right]_x$  the skew-symmetric tensor obtained by the cross product matrix of axis of rotation unit vector,

$$\mathbf{K} = \left[ \hat{\mathbf{k}} \right]_x = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (20)$$



This representation allows the physical interpretation that even when the velocity magnitude goes to zero it goes at a finite and fixed angle of rotation  $\theta = \alpha$ , leading to a finite rotation  $\mathbf{R} = e^{\alpha \mathbf{K}}$ . When  $\alpha \mathbf{R} = \mathbf{I}$  and  $\mathbf{N} = \mathbf{A}$ .

#### 4. Numerical results

In this section, we present numerical results for the 4:1 contraction benchmark flow using the Oldroyd-B viscoelastic model. For this, we set  $\mathcal{F}(\mathbf{A}) = 1$  and  $\mathcal{H}(\mathbf{A}) = (\mathbf{A} - \mathbf{I})$  in Eq. (4). The same substitutions are applied to Eq. (10), but using  $\mathbf{N}$  instead of  $\mathbf{A}$ . The *natural*-conformation tensor formulation was implemented in an in-house code using a finite difference scheme. For more details about the code, refer to [7]. The simulations were conducted with the following parameters: Reynolds number  $Re = 0.01$ , Weissenberg number  $Wi = 1$ , and solvent viscosity ratio  $\beta = 0.5$ . A non-uniform Cartesian mesh, with a minimum element size of  $\Delta x_{min} = \Delta y_{min} = 4 \times 10^{-3}$ , was employed to discretize the domain, with a finer mesh near the corners.

Figure 1 illustrates the slope of the  $u$ -velocity component and pressure obtained with both the *natural*-conformation and the Cartesian formulation, compared to the expected asymptotic results. The  $u$ -velocity component aligns reasonably well with the inclination line, and the *natural*-conformation formulation shows better convergence behavior. Although the pressure results do not fully capture the expected theoretical behavior, a slight improvement is observed with the use of the *natural*-conformation formulation.

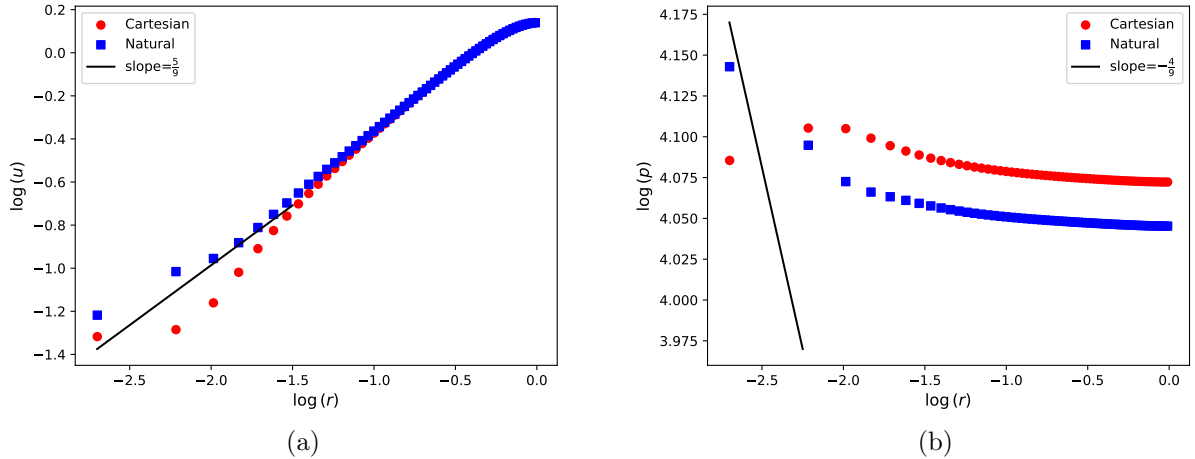


Figure 1: Numerical verification of the asymptotic behavior for the  $u$ -velocity component and pressure  $p$  using both Cartesian and *natural*-conformation formulations.

Finally, Figure 2 displays the results for the stress and natural components, comparing both the Cartesian and *natural*-conformation formulations. Once again, the *natural*-conformation formulation accurately captures the expected asymptotic behavior. However, the Cartesian stress results are less satisfactory, indicating a lack of resolution.

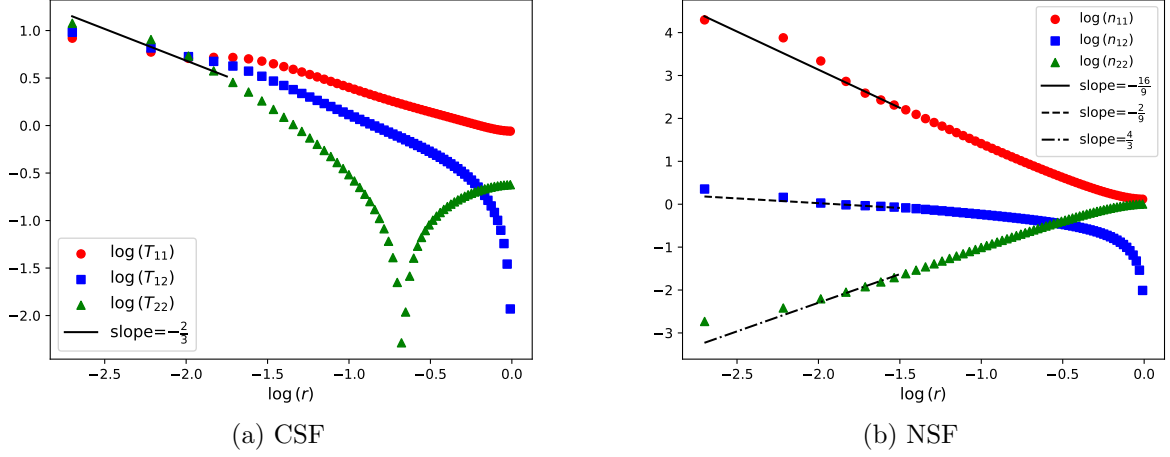


Figure 2: Numerical verification of the asymptotic behavior for the polymer and natural stress components using Cartesian and *natural*-conformation formulations, respectively.

## 5. Discussion

In this work, we have a generic framework for the *natural*-conformation tensor for a large class of differential constitutive models. We show that the proposed dyadic transformation proposed by Renardy [12] can be equated as a proper rotation of the conformation tensor into a streamlined orthonormal basis given by a rotation tensor expressed in terms of the unit velocity vectors. We also show that the *natural*-conformation tensor formulation is a particular sub-case of the *kernel*-conformation tensor transformation [1] with the *kernel* function acting on the rotation of the eigenvalues rather than on the magnitude of the eigenvalues.

The main contributions of this short communication can be summarized as follows:

- A simple relation between the *natural* basis tensor reformulation proposed by Renardy [12] and the *natural*-conformation tensor was presented;
- The *natural*-conformation tensor formulation for a family of differential constitutive laws for two and three dimensional unsteady flows have been presented.

In the extension of this work, we will present further details regarding the stabilization methods for the numerical method for the *natural*-conformation tensor in 3D and different strategies for the transient term in the *natural* velocity gradient tensor.

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## References

- [1] AM Afonso, FT Pinho, and MA Alves. “The kernel-conformation constitutive laws”. In: *Journal of Non-Newtonian Fluid Mechanics* 167 (2012), pp. 30–37. ISSN: 0377-0257. DOI: 10.1016/j.jnnfm.2011.09.008. URL: <https://dx.doi.org/10.1016/j.jnnfm.2011.09.008>.
- [2] M.A. Alves, P.J. Oliveira, and F.T. Pinho. “Numerical Methods for Viscoelastic Fluid Flows”. In: *Annual Review of Fluid Mechanics* 53, Volume 53, 2021 (2021), pp. 509–541. ISSN: 1545-4479. DOI: <https://doi.org/10.1146/annurev-fluid-010719-060107>. URL: <https://www.annualreviews.org/content/journals/10.1146/annurev-fluid-010719-060107>.
- [3] Gianni Astarita. “Objective and generally applicable criteria for flow classification”. In: *Journal of Non-Newtonian Fluid Mechanics* 6.1 (1979), pp. 69–76.
- [4] Yannis F Dafalias. “Corotational integrals in constitutive formulations”. In: *Journal of Engineering Mechanics* 113.12 (1987), pp. 1967–1973.
- [5] J.D. Evans. “Re-entrant corner behaviour of the Giesekus fluid with a solvent viscosity”. In: *Journal of Non-Newtonian Fluid Mechanics* 165 (May 2010), pp. 538–543. ISSN: 0377-0257. DOI: 10.1016/j.jnnfm.2010.01.010. URL: <https://dx.doi.org/10.1016/j.jnnfm.2010.01.010>.
- [6] J.D. Evans and C.M. Oishi. “Transient computations using the natural stress formulation for solving sharp corner flows”. In: *Journal of Non-Newtonian Fluid Mechanics* 249 (Nov. 2017), pp. 48–52. ISSN: 0377-0257. DOI: 10.1016/j.jnnfm.2017.08.012. URL: <https://dx.doi.org/10.1016/j.jnnfm.2017.08.012>.
- [7] Jonathan D Evans, Hugo L França, and Cassio M Oishi. “Application of the natural stress formulation for solving unsteady viscoelastic contraction flows”. In: *Journal of Computational Physics* 388 (2019), pp. 462–489.
- [8] JD Goddard and Chester Miller. “An inverse for the Jaumann derivative and some applications to the rheology of viscoelastic fluids”. In: *Rheologica Acta* 5.3 (1966), pp. 177–184.
- [9] T. Hagen and M. Renardy. “Boundary layer analysis of the Phan-Thien-Tanner and Giesekus model in high Weissenberg number flow”. In: *Journal of Non-Newtonian Fluid Mechanics* 73 (Nov. 1997), pp. 181–189. ISSN: 0377-0257. DOI: 10.1016/s0377-0257(97)00035-9. URL: [https://dx.doi.org/10.1016/s0377-0257\(97\)00035-9](https://dx.doi.org/10.1016/s0377-0257(97)00035-9).
- [10] E.J. Hinch. “The flow of an Oldroyd fluid around a sharp corner”. In: *Journal of Non-Newtonian Fluid Mechanics* 50 (Dec. 1993), pp. 161–171. ISSN: 0377-0257. DOI: 10.1016/0377-0257(93)80029-b. URL: [https://dx.doi.org/10.1016/0377-0257\(93\)80029-b](https://dx.doi.org/10.1016/0377-0257(93)80029-b).
- [11] Daniel D. Joseph. “Fluid Dynamics of Viscoelastic Liquids”. In: *Applied Mathematical Sciences* (1990). ISSN: 0066-5452, 2196-968X. DOI: 10.1007/978-1-4612-4462-2. URL: <https://dx.doi.org/10.1007/978-1-4612-4462-2>.
- [12] M. Renardy. “How to integrate the upper convected Maxwell (UCM) stresses near a singularity (and maybe elsewhere, too)”. In: *Journal of Non-Newtonian Fluid Mechanics* 52 (Apr. 1994), pp. 91–95. ISSN: 0377-0257. DOI: 10.1016/0377-0257(94)85060-7. URL: [https://dx.doi.org/10.1016/0377-0257\(94\)85060-7](https://dx.doi.org/10.1016/0377-0257(94)85060-7).

- [13] Michael Renardy. “Asymptotic structure of the stress field in flow past a cylinder at high Weissenberg number”. In: *Journal of Non-Newtonian Fluid Mechanics* 90 (Apr. 2000), pp. 13–23. ISSN: 0377-0257. DOI: 10.1016/S0377-0257(99)00050-6. URL: [https://dx.doi.org/10.1016/S0377-0257\(99\)00050-6](https://dx.doi.org/10.1016/S0377-0257(99)00050-6).
- [14] Michael Renardy. “High Weissenberg number boundary layers for the upper convected Maxwell fluid”. In: *Journal of Non-Newtonian Fluid Mechanics* 68 (Jan. 1997), pp. 125–132. ISSN: 0377-0257. DOI: 10.1016/S0377-0257(96)01491-7. URL: [https://dx.doi.org/10.1016/S0377-0257\(96\)01491-7](https://dx.doi.org/10.1016/S0377-0257(96)01491-7).
- [15] Michael Renardy. “The stresses of an upper convected Maxwell fluid in a Newtonian velocity field near a re-entrant corner”. In: *Journal of Non-Newtonian Fluid Mechanics* 50 (Dec. 1993), pp. 127–134. ISSN: 0377-0257. DOI: 10.1016/0377-0257(93)80027-9. URL: [https://dx.doi.org/10.1016/0377-0257\(93\)80027-9](https://dx.doi.org/10.1016/0377-0257(93)80027-9).
- [16] Huan-Chang Tseng. “A revisitation of generalized Newtonian fluids”. In: *Journal of Rheology* 64.3 (2020), pp. 493–504.
- [17] T Vaithianathan and Lance R Collins. “Numerical approach to simulating turbulent flow of a viscoelastic polymer solution”. In: *Journal of Computational Physics* 187.1 (2003), pp. 1–21.
- [18] P. Wapperom and M. Renardy. “Numerical prediction of the boundary layers in the flow around a cylinder using a fixed velocity field”. In: *Journal of Non-Newtonian Fluid Mechanics* 125 (Jan. 2005), pp. 35–48. ISSN: 0377-0257. DOI: 10.1016/j.jnnfm.2004.09.005. URL: <https://dx.doi.org/10.1016/j.jnnfm.2004.09.005>.

**Declaration of interests**

☒The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: