The *kernel-natural stress formulation* constitutive laws

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Abstract

This work presents a kernel transformation for the natural stress formulation. The stabilization procedure follows the idea of applying a generic kernel function [A.M. Afonso, F.T. Pinho, M.A. Alves, J. Non-Newtonian Fluid Mech., 167-168 (2012) 30-37] on the tensor equation of the natural stress formulation []

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1. Introduction

2. Governing equations

The dimenionless governing equations in the present work are the mass, momentum and the constitutive equations given, respectively, by

$$\begin{aligned}
&\nabla.\mathbf{v} = 0, \qquad \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\nabla) = -\nabla p + \frac{\beta}{Re}\nabla^2\mathbf{v} + \frac{1}{Re}\nabla.\mathbf{T}^p,\\
&\mathbf{T}^p + Wi\left(+\frac{\kappa}{1-\beta}\mathbf{g}\left(\mathbf{T}^p\right)\right) = 2\left(1-\beta\right)\mathbf{D},
\end{aligned} \tag{1}$$

GovEq

with

$$\mathbf{g}\left(\mathbf{T}^p\right) = \begin{cases} \mathbf{0}, & \text{Oldroyd-B,}\\ tr\left(\mathbf{T}^p\right)\mathbf{T}^p, & \text{PTT,}\\ \left(\mathbf{T}^p\right)^2, & \text{Giesekus,}\end{cases} \tag{2}$$

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where $\mathbf{v}$ is the fluid velocity, p is the pressure, $\mathbf{T}^p$ is the elastic extra-stress. The dimensionless parameters are the Reynolds number Re, the Weissenberg number $Wi = \lambda_p \frac{U}{L}$ (the dimensionless relaxation time) and retardation parameter $\beta \in [0, 1]$. The upper-convected stress derivative is defined as

$$= \frac{\partial \mathbf{T}^p}{\partial t} + (\mathbf{v}.\nabla)\,\mathbf{T}^p - (\nabla \mathbf{v})\,\mathbf{T}^p - \mathbf{T}^p\,(\nabla \mathbf{v})^T . \tag{3}$$

An alternative stress formulation introduces the conformation tensor $\mathbf{A}$ through the transformation

$$\mathbf{T}^p = \frac{1-\beta}{Wi}\,(\mathbf{A} - \mathbf{I})\,, \tag{4}$$

so that the constitutive equation (1) becomes

$$Wi\mathbf{A} + (\mathbf{A} - \mathbf{I}) + \kappa \mathbf{g}\,(\mathbf{A} - \mathbf{I}) = \mathbf{0}. \tag{5}$$

The conformation tensor is a variance-covariance, symmetric positive definite tensor (SPD), and it may be decomposed using the natural stress basis as

$$\mathbf{A} = \lambda \mathbf{v}\mathbf{v}^T + \mu\left(\mathbf{v}\mathbf{w}^T + \mathbf{w}\mathbf{v}^T\right) + \nu \mathbf{w}\mathbf{w}^T, \tag{6}$$

where

$$\mathbf{v} = (u, v)^T\,, \qquad \mathbf{w} = \frac{1}{|\mathbf{v}|^2}\,(-v, u)^T\,. \tag{7}$$

This is referred to as the natural stress formulation (NSF) and its components satisfy the following system of equation

$$\begin{aligned}
\frac{D\lambda}{Dt} &= -\frac{2}{|\mathbf{v}|^2}\left[\frac{\partial u}{\partial t}\left(\lambda u + \mu\frac{v}{|\mathbf{v}|^2}\right) + \frac{\partial v}{\partial t}\left(\lambda v - \mu\frac{u}{|\mathbf{v}|^2}\right)\right] - 2\mu\nabla.\mathbf{w} + \frac{1}{Wi}\left(\frac{1}{|\mathbf{v}|^2} - \lambda\right) + \frac{\kappa}{Wi}g_\lambda \\
\frac{D\mu}{Dt} &= \frac{\partial u}{\partial t}\left(\lambda v - \nu\frac{v}{|\mathbf{v}|^4}\right) + \frac{\partial v}{\partial t}\left(-u\lambda + \nu\frac{u}{|\mathbf{v}|^4}\right) - \nu\nabla.\mathbf{w} - \frac{\mu}{Wi} + \frac{\kappa}{Wi}g_\mu \\
\frac{D\nu}{Dt} &= 2\left[\frac{\partial u}{\partial t}\left(\mu v + \nu\frac{u}{|\mathbf{v}|^2}\right) + \frac{\partial v}{\partial t}\left(\nu\frac{v}{|\mathbf{v}|^2} - u\mu\right)\right] + \frac{1}{Wi}\left(|\mathbf{v}|^2 - \nu\right) + \frac{\kappa}{Wi}g_\nu,
\end{aligned} \tag{8}$$

NSFeq

with

$$g_\lambda = \begin{cases} 0, & \text{Oldroyd-B,} \\ \left(\lambda|\mathbf{v}|^2 - 2 + \frac{\nu}{|\mathbf{v}|^2}\right)\left(\frac{1}{|\mathbf{v}|^2} - \lambda\right), & \text{PTT,} \\ -\left(\frac{1}{|\mathbf{v}|^2} - \lambda\right)^2|\mathbf{v}|^2 - \frac{\mu^2}{|\mathbf{v}|^2}, & \text{Giesekus,} \end{cases} \qquad g_\mu = \begin{cases} 0, & \text{Oldroyd-B,} \\ -\left(\lambda|\mathbf{v}|^2 - 2 + \frac{\nu}{|\mathbf{v}|^2}\right)\mu, & \text{PTT/Giesekus,} \end{cases}$$

$$g_\nu = \begin{cases} 0, & \text{Oldroyd-B,} \\ \left(\lambda|\mathbf{v}|^2 - 2 + \frac{\nu}{|\mathbf{v}|^2}\right)\left(|\mathbf{v}|^2 - \nu\right), & \text{PTT,} \\ -\left(\nu - |\mathbf{v}|^2\right)^2\frac{1}{|\mathbf{v}|^2} - \mu^2|\mathbf{v}|^2, & \text{Giesekus} \end{cases} \tag{9}$$

and

$$\nabla.\mathbf{w} = \frac{1}{|\mathbf{v}|^4}\left[\left(v^2 - u^2\right)\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + 4uv\frac{\partial u}{\partial x}\right]. \tag{10}$$

Numerically, it is more interesting to work with the following format of Eq. (8)

$$
\begin{aligned}
\frac{\partial \hat{\lambda}}{\partial t} &= -|\mathbf{v}|^2 \left(\mathbf{v}.\nabla\right) \left(\frac{\hat{\lambda}}{|\mathbf{v}|^2}\right) + 2\frac{\hat{\mu}}{|\mathbf{v}|^2}\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right) - 2\hat{\mu}|\mathbf{v}|^2\nabla.\mathbf{w} + \frac{1}{Wi}\left(1-\hat{\lambda}\right) + \frac{\kappa}{Wi}g_{\hat{\lambda}},\\
\frac{\partial \hat{\mu}}{\partial t} &= -\left(\mathbf{v}.\nabla\right)\hat{\mu} + \frac{\hat{\nu}-\hat{\lambda}}{|\mathbf{v}|^2}\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right) - \nu|\mathbf{v}|^2\nabla.\mathbf{w} - \frac{\hat{\mu}}{Wi} + \frac{\kappa}{Wi}g_{\hat{\mu}},\\
\frac{\partial \hat{\nu}}{\partial t} &= -\frac{1}{|\mathbf{v}|^2}\left(\mathbf{v}.\nabla\right)\left(\hat{\nu}|\mathbf{v}|^2\right) - 2\frac{\hat{\mu}}{|\mathbf{v}|^2}\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right) + \frac{1}{Wi}\left(1-\hat{\nu}\right) + \frac{\kappa}{Wi}g_{\hat{\nu}},
\end{aligned}
\tag{11}
$$

or, in terms of the total derivative, as

$$
\begin{aligned}
\frac{D\hat{\lambda}}{Dt} &= \frac{\hat{\lambda}}{|\mathbf{v}|^2}\left(\mathbf{v}.\nabla\right)|\mathbf{v}|^2 + 2\frac{\hat{\mu}}{|\mathbf{v}|^2}\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right) - 2\hat{\mu}|\mathbf{v}|^2\nabla.\mathbf{w} + \frac{1}{Wi}\left(1-\hat{\lambda}\right) + \frac{\kappa}{Wi}g_{\hat{\lambda}},\\
\frac{D\hat{\mu}}{Dt} &= \frac{\hat{\nu}-\hat{\lambda}}{|\mathbf{v}|^2}\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right) - \nu|\mathbf{v}|^2\nabla.\mathbf{w} - \frac{\hat{\mu}}{Wi} + \frac{\kappa}{Wi}g_{\hat{\mu}},\\
\frac{D\hat{\nu}}{Dt} &= -\frac{\hat{\nu}}{|\mathbf{v}|^2}\left(\mathbf{v}.\nabla\right)|\mathbf{v}|^2 - 2\frac{\hat{\mu}}{|\mathbf{v}|^2}\left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right) + \frac{1}{Wi}\left(1-\hat{\nu}\right) + \frac{\kappa}{Wi}g_{\hat{\nu}},
\end{aligned}
\tag{12}
$$

NSFeqHat

where

$$
\hat{\lambda} = \lambda|\lambda|^2, \quad \hat{\mu} = \mu, \quad \hat{\nu} = \frac{\nu}{|\mathbf{v}|^2},
\tag{13}
$$

$$
\begin{aligned}
g_{\hat{\lambda}} &= \begin{cases} 0, & \text{Oldroyd-B},\\ \left(\hat{\lambda}-2+\hat{\nu}\right)\left(1-\hat{\lambda}\right), & \text{PTT},\\ -\left(1-\hat{\lambda}\right)^2 - \hat{\mu}^2, & \text{Giesekus}, \end{cases}
\qquad
g_{\hat{\mu}} = \begin{cases} 0, & \text{Oldroyd-B},\\ -\left(\hat{\lambda}-2+\hat{\nu}\right)\hat{\mu}, & \text{PTT/Giesekus}, \end{cases}\\
g_{\hat{\nu}} &= \begin{cases} 0, & \text{Oldroyd-B},\\ \left(\hat{\lambda}-2+\hat{\nu}\right)\left(1-\hat{\nu}\right), & \text{PTT},\\ -\left(1-\hat{\nu}\right)^2 - \hat{\mu}^2, & \text{Giesekus} \end{cases}
\end{aligned}
\tag{14}
$$

3. The kernel-natural stress formulation

Defining the tensors

$$
\mathbf{S} = \begin{pmatrix} \hat{\lambda} & \hat{\mu} \\ \hat{\mu} & \hat{\nu} \end{pmatrix}, \quad \mathbf{R} = \frac{1}{|\mathbf{v}|}\begin{pmatrix} u & -v \\ v & u \end{pmatrix},
\tag{15}
$$

then we have the following relations between the conformation tensor $\mathbf{A}$ and the natural tensor $\mathbf{S}$:

$$
\mathbf{A} = \mathbf{R}\mathbf{S}\mathbf{R}^T \Rightarrow \mathbf{S} = \mathbf{R}^T\mathbf{A}\mathbf{R}.
\tag{16}
$$

RelationA

The equation for the conformation tensor is giving by

$$
\frac{D\mathbf{A}}{Dt} = \nabla\mathbf{v}\mathbf{A} + \mathbf{A}\nabla\mathbf{v}^T + \frac{1}{Wi}\left(\mathbf{I}-\mathbf{A}\right) + \frac{\kappa}{Wi}\mathbf{g}\left(\mathbf{A}\right),
\tag{17}
$$

where

$$
\mathbf{g}\left(\mathbf{A}\right) = \begin{cases} 0, & \text{Oldroyd-B},\\ tr\left(\mathbf{A}-\mathbf{I}\right)\left(\mathbf{I}-\mathbf{A}\right), & \text{PTT},\\ \left(\mathbf{A}-\mathbf{I}\right)\left(\mathbf{I}-\mathbf{A}\right), & \text{Giesekus}. \end{cases}
\tag{18}
$$

Then, the equation for $\mathbf{S}$ can be constructed by applying the total derivative in the relation (16), i.e.,

$$\frac{D\mathbf{S}}{Dt} = \frac{D\mathbf{R}^T}{Dt}\mathbf{A}\mathbf{R} + \mathbf{R}^T\frac{D\mathbf{A}}{Dt}\mathbf{R} + \mathbf{R}^T\mathbf{A}\frac{\mathbf{R}}{Dt}, \tag{19}$$

which results in

$$\frac{D\mathbf{S}}{Dt} = \frac{D\mathbf{R}^T}{Dt}\mathbf{R}\mathbf{S} + \mathbf{R}^T\left[\nabla\mathbf{v}\mathbf{A} + \mathbf{A}\nabla\mathbf{v}^T + \frac{1}{Wi}\left(\mathbf{I}-\mathbf{A}\right) + \frac{\kappa}{Wi}\mathbf{g}\left(\mathbf{A}\right)\right]\mathbf{R} + \mathbf{S}\mathbf{R}^T\frac{D\mathbf{R}}{Dt}, \tag{20}$$

and finally

$$\frac{D\mathbf{S}}{Dt} = \left(\mathbf{S}\mathbf{H} - \mathbf{H}\mathbf{S}\right) + \left(\overline{\nabla}\mathbf{v}\mathbf{S} + \mathbf{S}\overline{\nabla}\mathbf{v}^T\right) + \frac{1}{Wi}\left(\mathbf{I}-\mathbf{S}\right) + \frac{\kappa}{Wi}\mathbf{g}\left(\mathbf{S}\right), \tag{21} \qquad \boxed{\texttt{MainEq}}$$

where

$$\mathbf{H} = \mathbf{R}^T\frac{D\mathbf{R}}{Dt}, \quad \overline{\nabla}\mathbf{v} = \mathbf{R}^T\nabla\mathbf{v}\mathbf{R}, \tag{22}$$

$$\mathbf{g}\left(\mathbf{S}\right)\begin{cases} 0, & \text{Oldroyd-B} \\ tr\left(\mathbf{S}-\mathbf{I}\right)\left(\mathbf{I}-\mathbf{S}\right), & \text{PTT,} \\ \left(\mathbf{S}-\mathbf{I}\right)\left(\mathbf{I}-\mathbf{S}\right), & \text{Giesekus.} \end{cases} \tag{23}$$

Eq. (21) can be written in 2D as

$$\begin{aligned}
\frac{DS_{11}}{Dt} &= -2S_{12}H_{12} + 2\left(S_{11}\frac{\overline{\partial u}}{\partial x} + S_{12}\frac{\overline{\partial u}}{\partial y}\right) + \frac{1}{Wi}\left(1-S_{11}\right) + \frac{\kappa}{Wi}g\left(\mathbf{S}\right)_{11}, \\
\frac{DS_{12}}{Dt} &= H_{12}\left(S_{11}-S_{22}\right) + S_{11}\frac{\overline{\partial v}}{\partial x} + S_{22}\frac{\overline{\partial u}}{\partial y} + \frac{1}{Wi}\left(-S_{12}\right) + \frac{\kappa}{Wi}g\left(\mathbf{S}\right)_{12}, \\
\frac{DS_{22}}{Dt} &= 2S_{12}H_{12} + 2\left(S_{12}\frac{\overline{\partial v}}{\partial x} + S_{22}\frac{\overline{\partial v}}{\partial y}\right) + \frac{1}{Wi}\left(1-S_{22}\right) + \frac{\kappa}{Wi}g\left(\mathbf{S}\right)_{22}
\end{aligned} \tag{24} \qquad \boxed{\texttt{2DEq}}$$

with

$$\begin{cases}
\mathbf{H} = \frac{D\mathbf{R}}{Dt}\mathbf{R}^T = \begin{pmatrix} 0 & \frac{1}{|\mathbf{v}|^2}\left(v\frac{Du}{Dt} - u\frac{Dv}{Dt}\right) \\ -\frac{1}{|\mathbf{v}|^2}\left(v\frac{Du}{Dt} - u\frac{Dv}{Dt}\right) & 0 \end{pmatrix}, \\
\overline{\nabla}\mathbf{v} = \mathbf{R}^T\nabla\mathbf{v}\mathbf{R} = \frac{1}{|\mathbf{v}|^2}\begin{pmatrix} u^2\frac{\partial u}{\partial x} + uv\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + v^2\frac{\partial v}{\partial y} & u^2\frac{\partial u}{\partial y} + uv\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) - v^2\frac{\partial v}{\partial x} \\ u^2\frac{\partial v}{\partial x} + uv\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) - v^2\frac{\partial u}{\partial y} & v^2\frac{\partial u}{\partial x} - uv\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + u^2\frac{\partial v}{\partial y} \end{pmatrix}.
\end{cases} \tag{25}$$

$$g\left(\mathbf{S}\right)_{11} = \begin{cases} 0, & \text{Oldroyd-B,} \\ \left(S_{11}+S_{22}-2\right)\left(1-S_{11}\right), & \text{PTT,} \\ -\left(1-2S_{11}+S_{11}^2+S_{12}^2\right), & \text{Giesekus,} \end{cases} \quad g\left(\mathbf{S}\right)_{12} = \begin{cases} 0, & \text{Oldroyd-B,} \\ -S_{12}\left(S_{11}+S_{22}-2\right), & \text{PTT/Giesekus,} \end{cases}$$

$$g\left(\mathbf{S}\right)_{22} = \begin{cases} 0, & \text{Oldroyd-B,} \\ \left(S_{11}+S_{22}-2\right)\left(1-S_{22}\right), & \text{PTT,} \\ -\left(1-2S_{22}+S_{22}^2+S_{12}^2\right), & \text{Giesekus.} \end{cases} \tag{26}$$

Note that, $\mathbf{H}$ can also be written as

$$\mathbf{H} = \begin{pmatrix} R_{11}\frac{DR_{11}}{Dt} + R_{21}\frac{DR_{21}}{Dt} & R_{11}\frac{DR_{12}}{Dt} + R_{21}\frac{DR_{22}}{Dt} \\ R_{12}\frac{DR_{11}}{Dt} + R_{22}\frac{DR_{21}}{Dt} & R_{12}\frac{DR_{12}}{Dt} + R_{22}\frac{DR_{22}}{Dt} \end{pmatrix} \tag{27}$$

where

$$\begin{cases} R_{11}\dfrac{DR_{11}}{Dt} = \dfrac{uv}{|\mathbf{v}|^4}\left(v\dfrac{Du}{Dt} - u\dfrac{Dv}{Dt}\right), \\ R_{21}\dfrac{DR_{21}}{Dt} = -\dfrac{uv}{|\mathbf{v}|^4}\left(v\dfrac{Du}{Dt} - u\dfrac{Dv}{Dt}\right), \\ R_{11}\dfrac{DR_{12}}{Dt} = \dfrac{u^2}{|\mathbf{v}|^4}\left(v\dfrac{Du}{Dt} - u\dfrac{Dv}{Dt}\right), \\ R_{21}\dfrac{DR_{22}}{Dt} = \dfrac{v^2}{|\mathbf{v}|^4}\left(v\dfrac{Du}{Dt} - u\dfrac{Dv}{Dt}\right), \end{cases} \tag{28}$$

Observe in Eq. (24) that

$$\begin{cases} -\left(\mathbf{v}.\nabla\right)S_{11} + 2S_{11}\dfrac{\overline{\partial u}}{\partial x} = -|\mathbf{v}|^2\left(\mathbf{v}.\nabla\right)\left(\dfrac{\hat{\lambda}}{|\mathbf{v}|^2}\right) = -\left(\mathbf{v}.\nabla\right)\hat{\lambda} + \dfrac{\hat{\lambda}}{|\mathbf{v}|^2}\left(\mathbf{v}.\nabla\right)|\mathbf{v}|^2, \\ -2S_{12}H_{12} + 2S_{12}\dfrac{\overline{\partial u}}{\partial y} = \dfrac{2\hat{\mu}}{|\mathbf{v}|^2}\left(u\dfrac{\partial v}{\partial t} - v\dfrac{\partial u}{\partial t}\right) - 2\hat{\mu}|\mathbf{v}|^2\nabla.\mathbf{w}, \\ -\left(\mathbf{v}.\nabla\right)S_{22} + 2S_{22}\dfrac{\overline{\partial v}}{\partial y} = -\dfrac{1}{|\mathbf{v}|^2}\left(\mathbf{v}.\nabla\right)\left(\hat{\nu}|\mathbf{v}|^2\right) = -\left(\mathbf{v}.\nabla\right)\hat{\nu} - \dfrac{\hat{\nu}}{|\mathbf{v}|^2}\left(\mathbf{v}.\nabla\right)|\mathbf{v}|^2, \\ 2S_{12}H_{12} + 2S_{12}\dfrac{\overline{\partial v}}{\partial x} = -\dfrac{2\hat{\mu}}{|\mathbf{v}|}\left(u\dfrac{\partial v}{\partial t} - v\dfrac{\partial u}{\partial t}\right). \end{cases} \tag{29}$$

Following the work of Fattal and Kupferman [<**empty citation**>] (see also Afonso et al. [<**empty citation**>]), the matrix $\mathbf{M}$ can be decomposed as

$$\overline{\nabla}\mathbf{v} = \mathbf{M} = \mathbf{\Omega} + \mathbf{B} + \mathbf{N}\mathbf{S}^{-1} \tag{30}$$

FattalDec

where $\mathbf{\Omega}, \mathbf{N}$ are anti-symmetric, and $\mathbf{B}$ is symmetric and commutes with $\mathbf{S}$.
Substituting Eq. (30) in (21) results

$$\frac{D\mathbf{S}}{Dt} = \left(\mathbf{SH} - \mathbf{HS}\right) + \left(\mathbf{\Omega S} - \mathbf{S\Omega}\right) + 2\mathbf{BS} + \frac{1}{Wi}\left(\mathbf{I} - \mathbf{S}\right) + \frac{\kappa}{Wi}\mathbf{g}\left(\mathbf{S}\right). \tag{31}$$

The eigendecomposition of $\mathbf{S}$ is

$$\mathbf{S} = \mathbf{O\Lambda O}^T \Rightarrow \mathbf{\Lambda} = \mathbf{O}^T\mathbf{SO}, \tag{32}$$

EgeinS

applying the material derivative results

$$\frac{D\mathbf{\Lambda}}{Dt} = \left(\frac{D\mathbf{O}^T}{Dt}\mathbf{O}\right)\mathbf{\Lambda} + \mathbf{\Lambda}\left(\mathbf{O}^T\frac{D\mathbf{O}}{Dt}\right) + \mathbf{O}^T\frac{D\mathbf{S}}{Dt}\mathbf{O} \tag{33}$$

Lambda1

Defining the matrix $\mathbf{V}$ as

$$\mathbf{V} = \mathbf{O}^T \frac{D\mathbf{O}}{Dt}, \tag{34}$$

we note that $\mathbf{V}^T = -\mathbf{V}$, i.e., $\mathbf{V}$ is an anti symmetric matrix. In fact,

$$\mathbf{0} = \frac{D\left(\mathbf{O}\mathbf{O}^T\right)}{Dt} = \frac{D\mathbf{O}}{Dt}\mathbf{O}^T + \mathbf{O}\frac{D\mathbf{O}^T}{Dt} \Rightarrow \frac{D\mathbf{O}}{Dt}\mathbf{O}^T = -\mathbf{O}\frac{D\mathbf{O}^T}{Dt}, \tag{35}$$

which results in

$$\mathbf{V}^T = -\mathbf{V}. \tag{36}$$

Then, equation (33) becomes

$$\frac{D\mathbf{\Lambda}}{Dt} = (\mathbf{\Lambda}\mathbf{V} - \mathbf{V}\mathbf{\Lambda}) + \mathbf{O}^T \frac{D\mathbf{S}}{Dt}\mathbf{O}. \tag{37}$$

EqInter

Substituting equation (**??**) in to equation (37), results

$$\frac{D\mathbf{\Lambda}}{Dt} = (\mathbf{\Lambda}\mathbf{V} - \mathbf{V}\mathbf{\Lambda}) + \left(\mathbf{\Lambda}\widetilde{\mathbf{H}} - \widetilde{\mathbf{H}}\mathbf{\Lambda}\right) + \left(\widetilde{\mathbf{\Omega}}\mathbf{\Lambda} - \mathbf{\Lambda}\widetilde{\mathbf{\Omega}}\right) + 2\tilde{\mathbf{B}}\mathbf{\Lambda} + \frac{1}{Wi}(\mathbf{I} - \mathbf{\Lambda}) + \frac{\kappa}{Wi}\mathbf{g}(\mathbf{\Lambda}), \tag{38}$$

Inter

where

$$\mathbf{g}(\mathbf{\Lambda}) = \begin{cases} \mathbf{0}, & \text{Oldroyd-B}, \\ tr\,(\mathbf{\Lambda} - \mathbf{I})(\mathbf{I} - \mathbf{\Lambda}), & \text{PTT}, \\ (\mathbf{\Lambda} - \mathbf{I})(\mathbf{I} - \mathbf{\Lambda}), & \text{Giesekus}, \end{cases} \tag{39}$$

Considering in Eq. (38) $\mathbf{W} = \mathbf{V} + \widetilde{\mathbf{H}}$ results in

$$\frac{D\mathbf{\Lambda}}{Dt} = (\mathbf{\Lambda}\mathbf{W} - \mathbf{W}\mathbf{\Lambda}) + \left(\widetilde{\mathbf{\Omega}}\mathbf{\Lambda} - \mathbf{\Lambda}\widetilde{\mathbf{\Omega}}\right) + 2\tilde{\mathbf{B}}\mathbf{\Lambda} + \frac{1}{Wi}(\mathbf{I} - \mathbf{\Lambda}) + \frac{\kappa}{Wi}\mathbf{g}(\mathbf{\Lambda}), \tag{40}$$

which we get

$$\begin{cases} W_{12} = V_{12} + \widetilde{H}_{12} = \widetilde{\Omega}_{12} \Rightarrow V_{12} = \widetilde{\Omega}_{12} - \widetilde{H}_{12}, \\ \dfrac{D\Lambda_{ii}}{Dt} = 2\widetilde{B}_{ii}\Lambda_{ii} + \dfrac{1}{Wi}(1 - \Lambda_{ii}) + \dfrac{\kappa}{Wi} g(\mathbf{\Lambda})_{ii}, \quad i = 1, 2. \end{cases} \tag{41}$$

Note that,

$$\frac{D\mathbb{K}(\mathbf{\Lambda})}{Dt} = \frac{D\mathbf{\Lambda}}{Dt}\frac{\partial\mathbb{K}(\mathbf{\Lambda})}{\partial\mathbf{\Lambda}} = \frac{D\mathbf{\Lambda}}{Dt}\mathbf{J} \tag{42}$$

where

$$\mathbf{J} = \frac{\partial\mathbb{K}(\mathbf{\Lambda})}{\partial\mathbf{\Lambda}} = \begin{pmatrix} \frac{\partial\mathbb{K}(\Lambda_{11})}{\partial\Lambda_{11}} & 0 \\ 0 & \frac{\partial\mathbb{K}(\Lambda_{22})}{\partial\Lambda_{22}} \end{pmatrix}. \tag{43}$$

Then,

$$\frac{D\mathbb{K}(\mathbf{\Lambda})}{Dt} = \left[2\tilde{\mathbf{B}}\mathbf{\Lambda} + \frac{1}{Wi}(\mathbf{I} - \mathbf{\Lambda}) + \frac{\kappa}{Wi}\mathbf{g}(\mathbf{\Lambda})\right]\mathbf{J}. \tag{44}$$

Applying a kernel transformation in Eq. (32), i.e.,

$$\begin{aligned} &\mathbb{K}(\mathbf{S}) = \mathbf{O}\mathbb{K}(\boldsymbol{\Lambda})\mathbf{O}^T \Rightarrow \\ &\Rightarrow \frac{D\mathbb{K}(\mathbf{S})}{Dt} = \mathbf{V}\mathbb{K}(\mathbf{S}) - \mathbb{K}(\mathbf{S})\mathbf{V} + \mathbf{O}\frac{D\mathbb{K}(\boldsymbol{\Lambda})}{Dt}\mathbf{O}^T. \end{aligned} \tag{45}$$

Finally, the Kernel-NSF equation is given by

$$\frac{D\mathbb{K}(\mathbf{S})}{Dt} = \mathbf{V}\mathbb{K}(\mathbf{S}) - \mathbb{K}(\mathbf{S})\mathbf{V} + \left[2\mathbf{B}\mathbf{S} + \frac{1}{Wi}(\mathbf{I} - \mathbf{S}) + \frac{\kappa}{Wi}\mathbf{g}(\mathbf{S})\right]\widetilde{\mathbf{J}}, \tag{46}$$ KernelEq

where

$$\widetilde{\mathbf{J}} = \mathbf{O}\mathbf{J}\mathbf{O}^T. \tag{47}$$

$$\begin{aligned} &\begin{pmatrix} \frac{D\mathbb{K}(\mathbf{S})_{11}}{Dt} & \frac{D\mathbb{K}(\mathbf{S})_{12}}{Dt} \\ \frac{D\mathbb{K}(\mathbf{S})_{12}}{Dt} & \frac{D\mathbb{K}(\mathbf{S})_{22}}{Dt} \end{pmatrix} = \begin{pmatrix} 2V_{12}\mathbb{K}(\mathbf{S})_{12} & V_{12}\left(\mathbb{K}(\mathbf{S})_{22} - \mathbb{K}(\mathbf{S})_{11}\right) \\ V_{12}\left(\mathbb{K}(\mathbf{S})_{22} - \mathbb{K}(\mathbf{S})_{11}\right) & -2V_{12}\mathbb{K}(\mathbf{S})_{12} \end{pmatrix} + \\ &\begin{pmatrix} 2(B_{11}S_{11} + B_{12}S_{12}) + \frac{1}{Wi}(1 - S_{11}) + \frac{\kappa}{Wi}g(\mathbf{S})_{11} & 2(B_{11}S_{12} + B_{12}S_{22}) + \frac{1}{Wi}(-S_{12}) + \frac{\kappa}{Wi}g(\mathbf{S})_{12} \\ 2(B_{12}S_{11} + B_{22}S_{12}) + \frac{1}{Wi}(-S_{12}) + \frac{\kappa}{Wi}g(\mathbf{S})_{12} & 2(B_{12}S_{12} + B_{22}S_{22}) + \frac{1}{Wi}(1 - S_{22}) + \frac{\kappa}{Wi}g(\mathbf{S})_{22} \end{pmatrix} \begin{pmatrix} \tilde{J}_{11} & \tilde{J}_{12} \\ \tilde{J}_{12} & \tilde{J}_{22} \end{pmatrix} \end{aligned} \tag{48}$$

where

$$\widetilde{\mathbf{J}} = \begin{pmatrix} O_{11}^2 J_{11} + O_{12}^2 J_{12} & O_{11}O_{21}J_{11} + O_{12}O_{22}J_{22} \\ O_{11}O_{21}J_{11} + O_{12}O_{22}J_{22} & O_{22}^2 J_{22} + O_{21}^2 J_{11} \end{pmatrix} \tag{49}$$

4. Numerical method

5. Numerical results

The meshes used in the numerical simulations are described in Table 1.

Mesh	Space-step	Number of cells
M_1	$\delta x_{min} = \delta y_{min} = 0.008$	400×380
M_2	$\delta x_{min} = \delta y_{min} = 0.004$	

Meshes

Table 1: Details of the meshes.

5.1. Verification of the asymptotic results

5.2. Numerical verification: Corner vortex, vortex intensity and Couette correction

In Table **??** we present the Corner vortex, vortex intesity and Couette correction for several kernel functions with $Wi = 1$.

5.3. Study of the positivity of the conformation tensor

5.4. Numerical analysis of the kernel functions

6. Conclusions

Method	X_R	Ψ_R	C
$\mathbf{T}^p$	1.496845	1.165636	0.335247
$\sqrt{\mathbf{A}}$	–	–	–
$\ln \mathbf{A}$	–	–	–
$\mathbf{N}$	1.496845	1.391620	0.335991
$\sqrt{\mathbf{N}}$	–	–	–
$\ln \mathbf{N}$	–	–	–

XrWi01

Table 2: Vortex size X_R, vortex intensity Ψ_R and Couette correction C using $Wi = 0.1$.

Method	X_R	Ψ_R	C
$\mathbf{T}^p$	1.496845	1.151294	0.291796
$\sqrt{\mathbf{A}}$	–	–	–
$\ln \mathbf{A}$	–	–	–
$\mathbf{N}$	1.496845	1.372660	0.290968
$\sqrt{\mathbf{N}}$	–	–	–
$\ln \mathbf{N}$	–	–	–

XrWi02

Table 3: Vortex size X_R, vortex intensity Ψ_R and Couette correction C using $Wi = 0.2$.

Method	X_R	Ψ_R	C
$\mathbf{T}^p$	1.496845	1.136462	0.245351
$\sqrt{\mathbf{A}}$	–	–	–
$\ln \mathbf{A}$	–	–	–
$\mathbf{N}$	1.496845	1.351782	0.243935
$\sqrt{\mathbf{N}}$	–	–	–
$\ln \mathbf{N}$	–	–	–

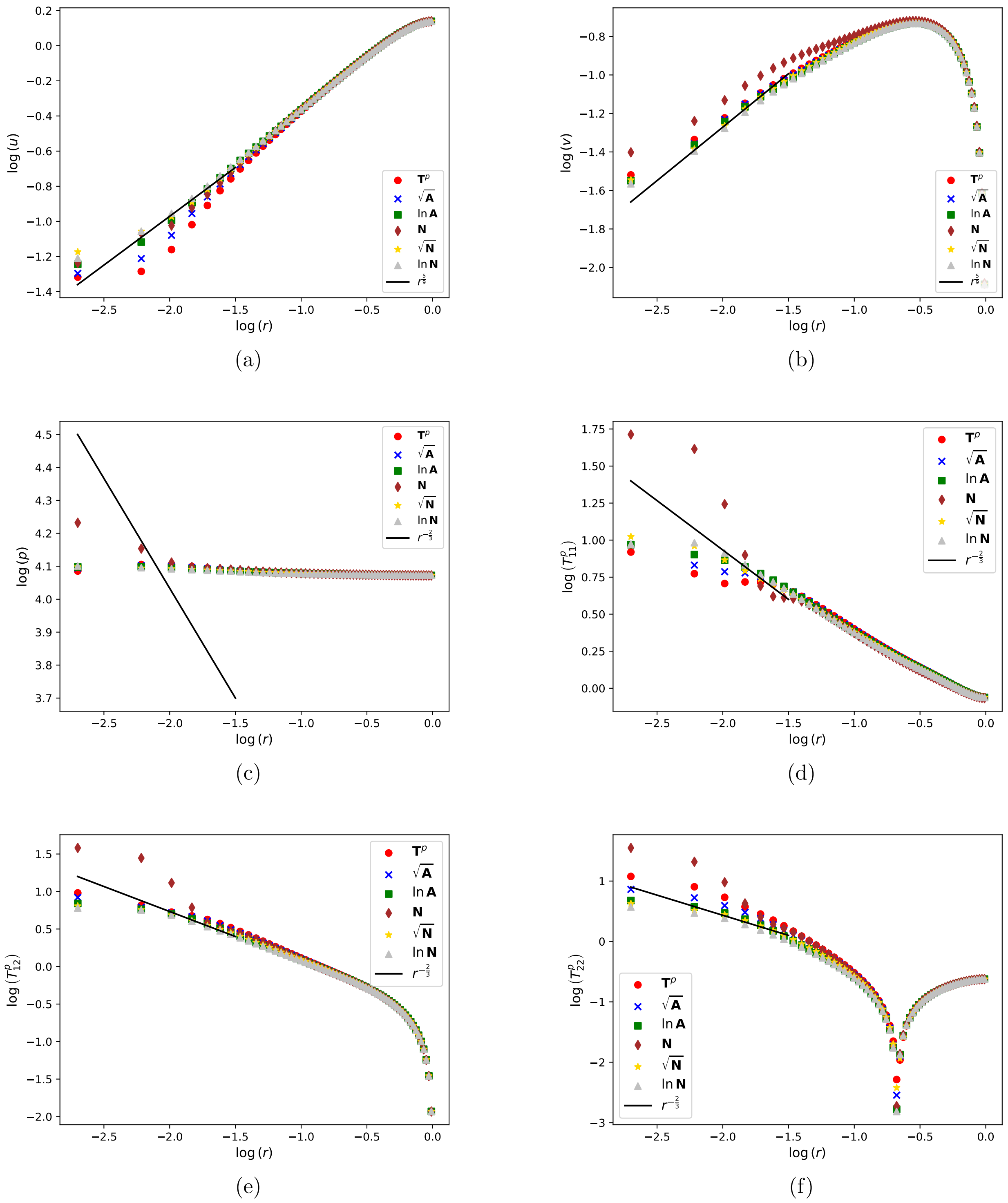
XrWi03

Table 4: Vortex size X_R, vortex intensity Ψ_R and Couette correction C using $Wi = 0.3$.

Method	X_R	Ψ_R	C
$\mathbf{T}^p$	1.496845	1.122511	0.197289
$\sqrt{\mathbf{A}}$	–	–	–
$\ln \mathbf{A}$	–	–	–
$\mathbf{N}$	1.496845	1.331674	0.195152
$\sqrt{\mathbf{N}}$	–	–	–
$\ln \mathbf{N}$	–	–	–

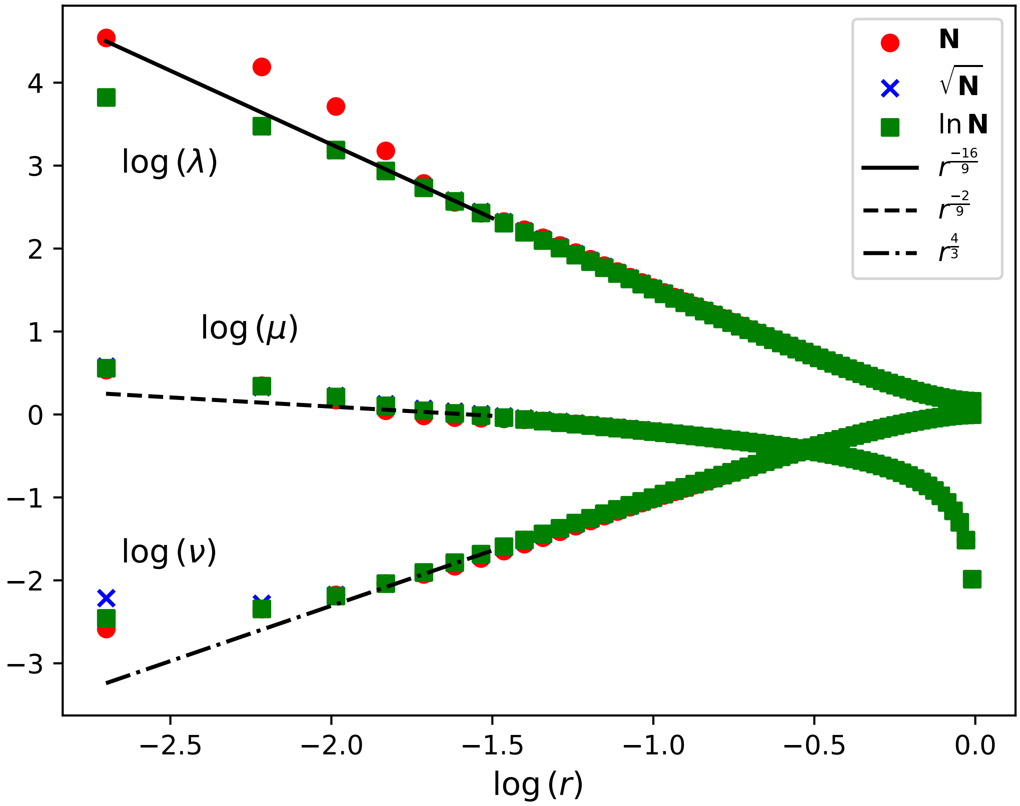
XrWi04

Table 5: Vortex size X_R, vortex intensity Ψ_R and Couette correction C using $Wi = 0.4$.



NumVer2

Figure 1: Numerical verification of the asymptotic behavior of the velocity components u, v, pressure p and the components of the extra-stress tensor $\mathbf{T}^p$ with $Wi = 1$ for the 4:1 contraction flow.



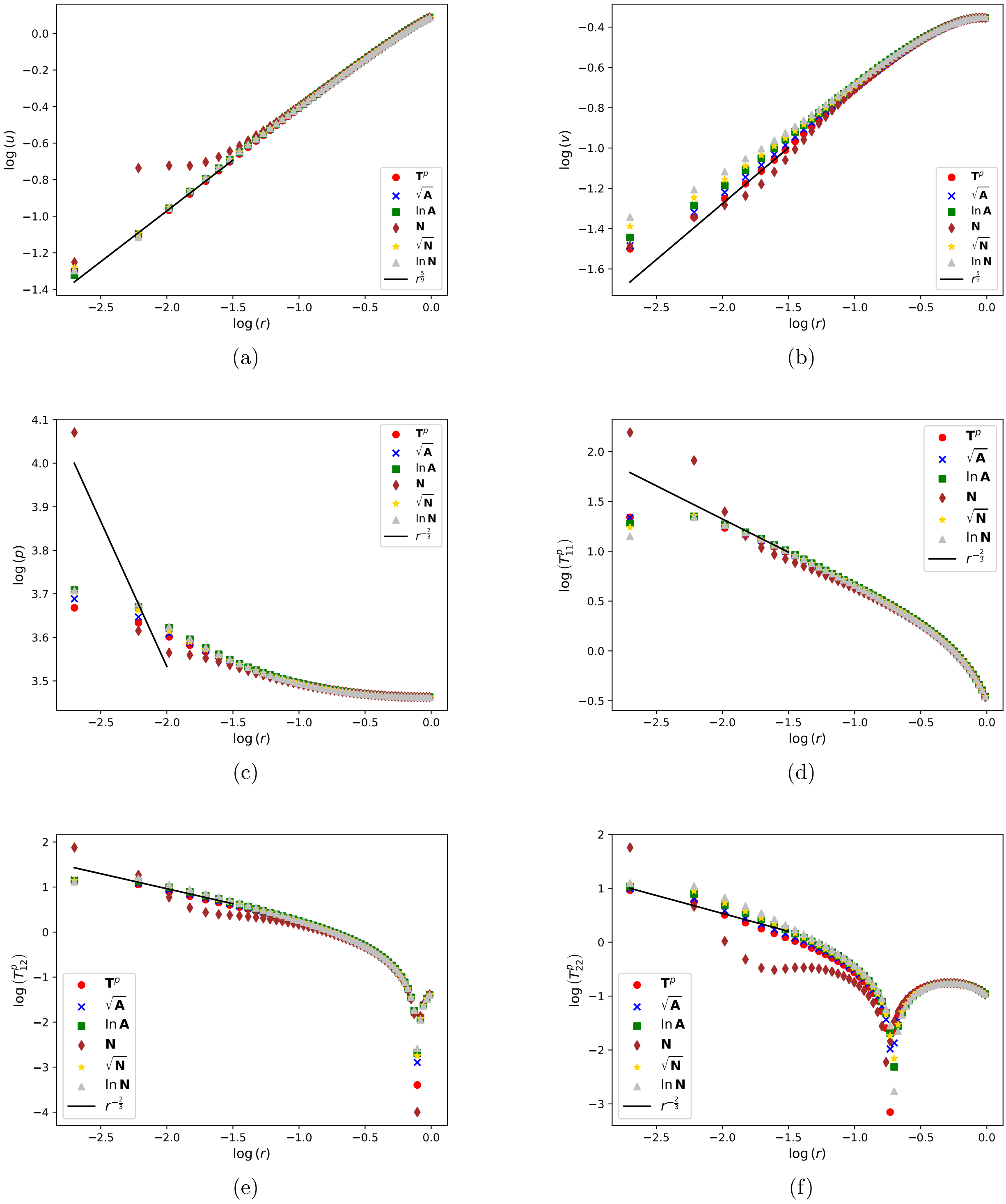
NumVer

Figure 2: Numerical verification of the asymptotic behavior of the natural stress components λ, μ and ν with $Wi = 1$ for the 4:1 contraction flow.

Method	X_R	Ψ_R	C
$\mathbf{T}^p$	1.451163	1.108696	0.147367
$\sqrt{\mathbf{A}}$	–	–	–
$\ln \mathbf{A}$	–	–	–
$\mathbf{N}$	–	–	–
$\sqrt{\mathbf{N}}$	1.496845	1.311177	0.145068
$\ln \mathbf{N}$	–	–	–

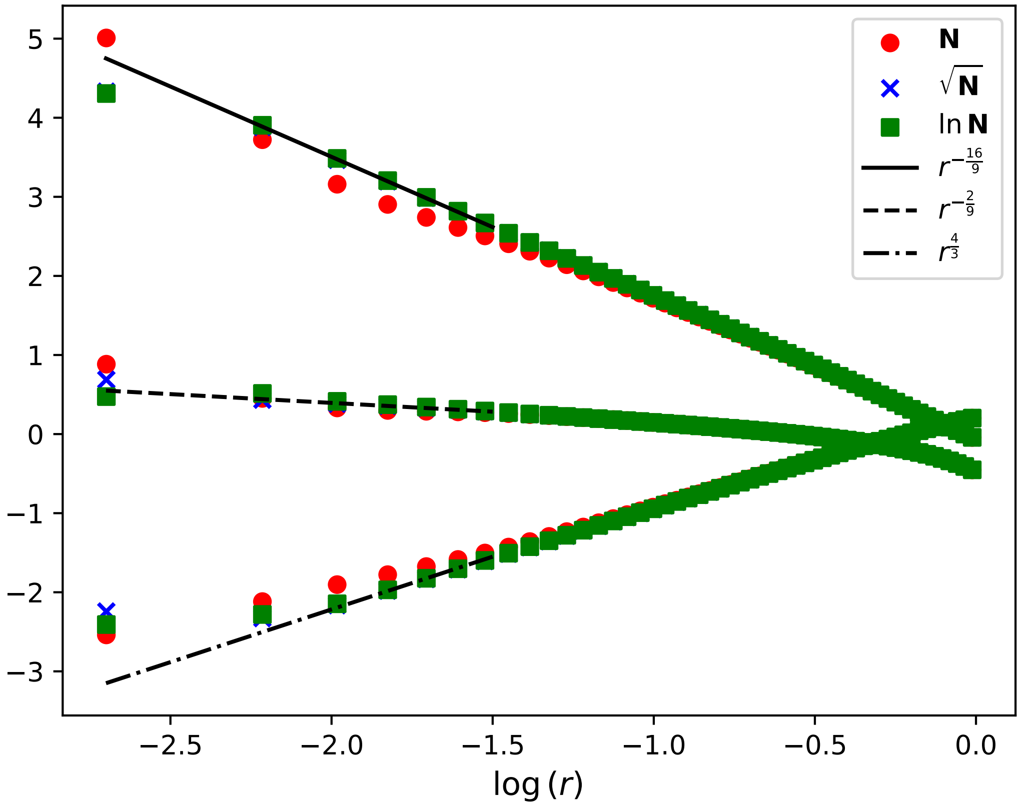
XrWi05

Table 6: Vortex size X_R, vortex intensity Ψ_R and Couette correction C using $Wi = 0.5$.



NumVer3

Figure 3: Numerical verification of the asymptotic behavior of the velocity components u, v, pressure p and the components of the extra-stress tensor $\mathbf{T}^p$ with $Wi = 1$ for the L-shaped flow.



NumVer4 Figure 4: Numerical verification of the asymptotic behavior of the natural stress components λ, μ and ν with $Wi = 1$ for the L-shaped flow.

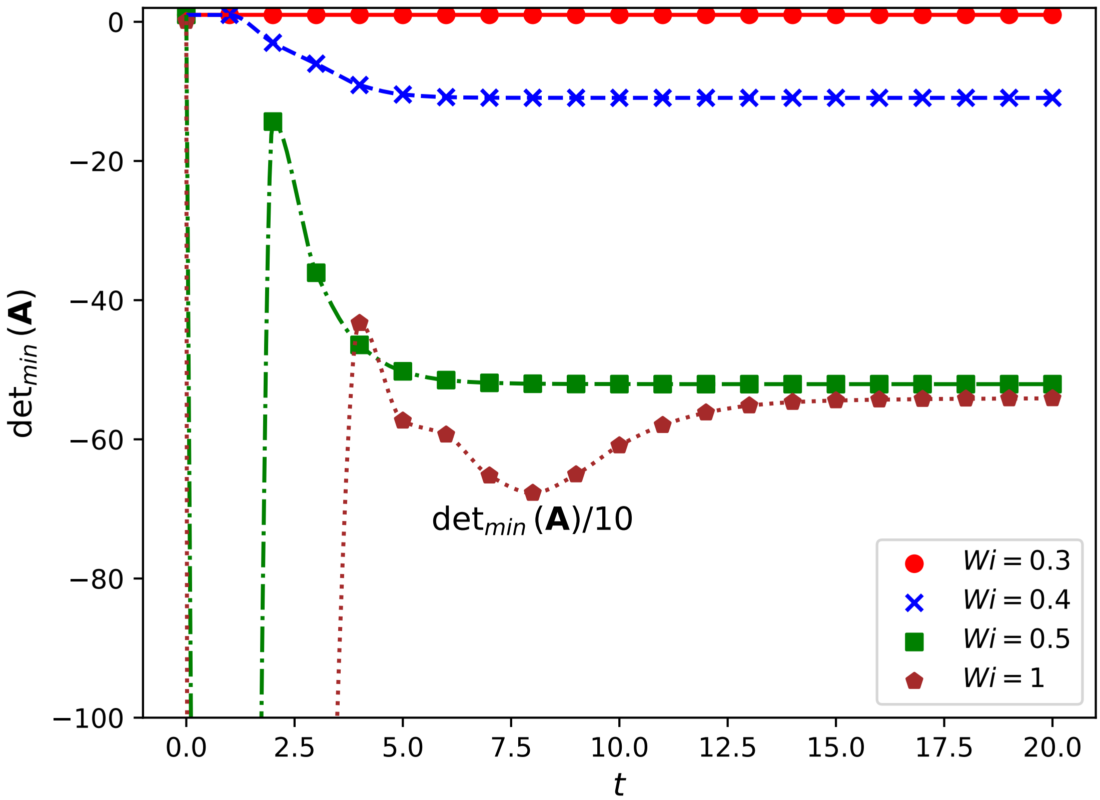
Method	X_R	Ψ_R	C
$\mathbf{T}^p$	1.451163	1.043963	−0.110588
$\sqrt{\mathbf{A}}$	1.451163	1.044549	−0.113106
$\ln \mathbf{A}$	1.451163	1.045088	−0.115949
$\mathbf{N}$	1.451163	1.223956	−0.116135
$\sqrt{\mathbf{N}}$	1.451163	1.043478	−0.115549
$\ln \mathbf{N}$	1.451163	1.046345	−0.116123

XrWi1 Table 7: Vortex size X_R, vortex intensity Ψ_R and Couette correction C using $Wi = 1$.

Formulation	Loses positivity	Simulation breakdown
$\mathbf{T}^p$	$Wi \geqslant 0.4$	$Wi \geqslant 5$
$\sqrt{\mathbf{A}}$	-	$Wi \geqslant 50$
$\ln \mathbf{A}$	-	$Wi \geqslant 100$
$\mathbf{N}$	$Wi \geqslant \frac{1}{2}$	-
$\sqrt{\mathbf{N}}$	-	-
$\ln \mathbf{N}$	-	-

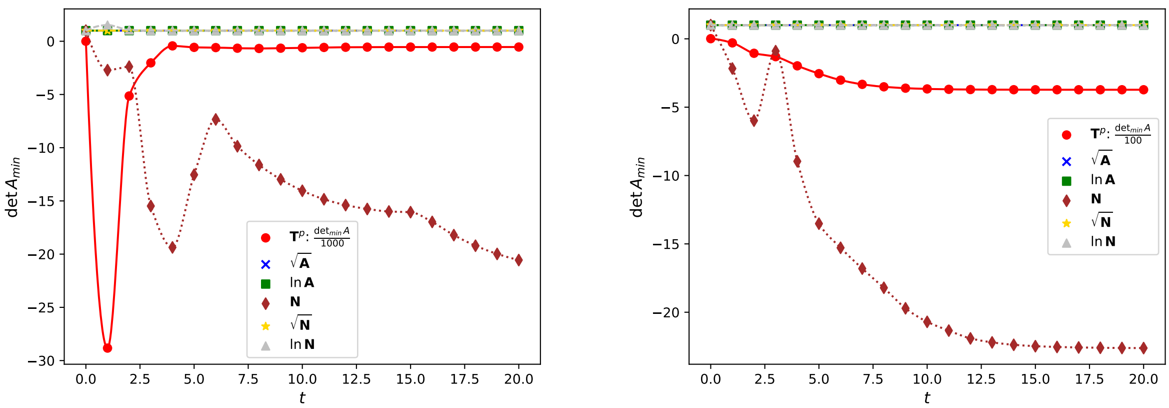
WiCrit

Table 8: Critical Weissenberg number for different methods in the 4:1 contraction flow.



detAmin

Figure 5: The minimum determinant of $\mathbf{A}$, $\det_{min} \mathbf{A}$, along the time t for the CSF in the 4:1 contraction flow.



(a) 4:1 Contraction (b) L-shaped

detA

Figure 6: The minimum determinant of $\mathbf{A}$, $\det \mathbf{A}_{min}$, along the time t with $Wi = 1$.