DS284: Numerical Linear Algebra

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Assignment 3

Q2 You are one of the scientists working at NASA's Goddard Space Flight Center in Greenbelt, Maryland and have been researching Wide-Field Slitless Spectroscopy to capture galaxy spectra of the distant universe. With the help of NASA's James Webb Space Telescope, you have successfully captured the deepest and sharpest infrared image of the distant universe to date. It is an image of the galaxy cluster SMACS 0723 and has been named Webb's First Deep Field. Unfortunately, due to some technical difficulties, the space telescope has not been able to transmit full-resolution images to Earth. However, an onboard computer can be programmed remotely from Earth to transmit the image in a compressed format until the difficulties are resolved. The control station on Earth has decided to use SVD to compress the image. As a scientist tasked with programming the onboard computer, think about the following

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(a) How many singular values are required to approximate the image i.e., make it look in distinguishable from the original image?

We will first load the image in python. We can see that the image is of shape ***(l,w,3) = (2000, 1968, 3)***. Let's assume that we are using *uint*8 to store each of the value in the array and it takes 1 unit space. Therefore total space taken is given by:

$$Originalentries = l \cdot w \cdot 3 = 2000 \cdot 1968 \cdot 3 = 11808000 \; units.$$

We can use SVD on each individual channel to get the number of singular values for each channel that gives us 99 percent of the energy. Energy is defined as

$$E = \sum_{i=1}^{n} (\sigma_i)^2$$

For the first k values,

$$E_k = \sum_{i=1}^{k} (\sigma_i)^2$$

For 99 percent energy to be conserved, we want smallest k such that

$$E_k >= 0.99E$$

Number of singular values to capture 99% energy for ***(kR,kG,kB): (276, 176, 292)***

We will perform an **SVD** for each channel and extract the following:

$$U_{2000\times2000}, S_{1968\times1968}, V^T_{1968\times1968}$$

We will reduce it to only use k singular by deleting rows and columns from the matrices to reduce it to:

$$U_{2000\times k}, S_{k\times k}, V^T_{k\times1968}$$

We will multiply the above 3 matrices to reconstruct the image:

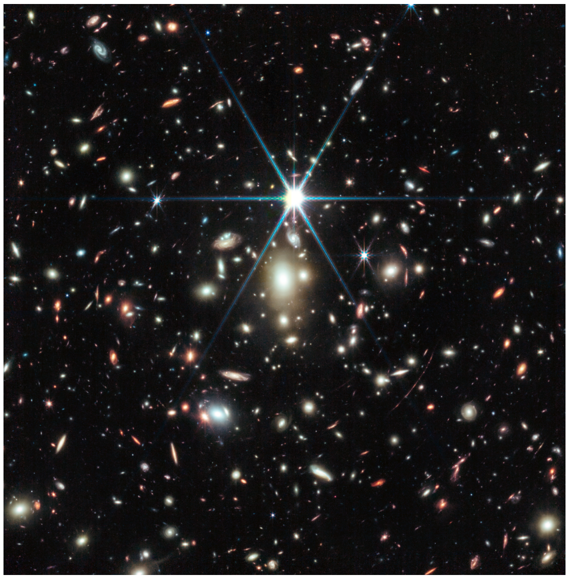

Figure 1: Original Image

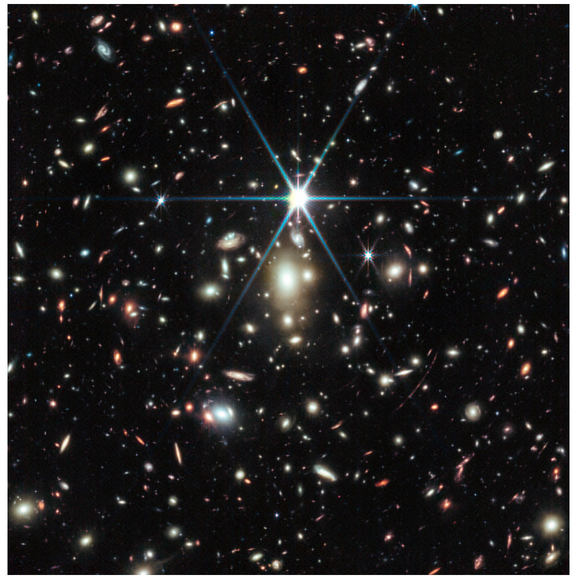

Figure 2: Compressed Image

We can see from the Figure 1 and 2 that the compressed image looks indistinguishable from the original image by using only a fraction of the singular values for each channel.

$$(kR, kG, kB) : (276, 176, 292) \ out \ of \ 1968 \ for \ each \ channel.$$

(b) Based on your observation in (a), how many entries need to be transmitted to earth to reconstruct the approximate image as opposed to sending the original image?

We will multiply $U.S$ to reduce the number of values even more. If we are able to transmit these matrices, someone on Earth can recompute the images from these matrices.

$$US_{2000\times k}, V^T_{k\times 1968}$$

Therefore, for each channel we need matrix US and V^T. Hence the total number of entries to be sent is given by :

$$\sum_{for \ each \ channel \ D} (l+w)\cdot KD$$

$$= (l+w)(KR+KG+KB)$$

$$reducedentries = (1968+2000)\cdot(276+176+292) = 2952192 \ entries$$

One can reconstruct the channels by using *reducedentries = 2952192*.

Ratio of units of space $= \frac{reducedentries}{Originalentries} = \frac{2952192}{11808000} = 0.250016$

Therefore we can compress the data by a factor of 4 and still maintain an indistinguishable image!

(c) What is the 2-Norm and Frobenius-Norm error between the matrix representation of the original image and the approximate image obtained for different number of singular values. Check if the following theorems hold for these errors.

From the below computed table, we know that both the theorems $\|A - A_k\|_2 = \sigma_{k+1}$ and $\|A - A_k\|_F = \sqrt{\sum_{i=k+1}^{r}(\sigma_i^2)}$ hold true.

Channels	$\|A - A_k\|_2$	σ_{k+1}	$\|A - A_k\|_F$	$\sqrt{\sum_{i=k+1}^{r}(\sigma_i^2)}$
Red	592.4452782705443	592.4452782705449	9265.015175504825	9265.015175504825
Green	852.9572802975003	852.9572802975013	8786.227763249091	8786.227763249091
Blue	533.9876575800014	533.9876575800021	8530.339367679042	8530.339367679042

Table 1: $\|A - A_k\|_2$, σ_{k+1}, $\|A - A_k\|_F$, and $\sqrt{\sum_{i=k+1}^{r}(\sigma_i^2)}$ for Red, Green, and Blue channels.