非線性控制第六章作業

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1. Consider the following nonlinear system

$$\dot{x}_1 = -x_1 + x_2 - x_3,\ \dot{x}_2 = -x_1 x_3 - x_2 + u,\ \dot{x}_3 = -x_1 + u$$

(1)

Express the system as $\dot{x} = f(x) + g(x)u$, we can obtain that

$$f(x) = \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 x_3 - x_2 \\ -x_1 \end{bmatrix},\ g(x) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Step1.

Check whether the controllability matrix satisfies

$$Rank\begin{bmatrix} g & ad_f g & ad_f^2 g \end{bmatrix} = 3$$

$$ad_f g = [f, g] = \nabla g \cdot f - \nabla f \cdot g = 0 - \begin{bmatrix} -1 & 1 & -1 \\ -x_3 & -1 & -x_1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 + x_1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} ad_f^2 g &= \begin{bmatrix} f & ad_f g \end{bmatrix} = \nabla ad_f g \cdot f - \nabla f \cdot ad_f g \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 x_3 - x_2 \\ -x_1 \end{bmatrix} - \begin{bmatrix} -1 & 1 & -1 \\ -x_3 & -1 & -x_1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 + x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 - x_1 \\ 1 + x_2 - x_3 \\ 0 \end{bmatrix} \end{aligned}$$

$$\therefore Rank\begin{bmatrix} 0 & 0 & -1 - x_1 \\ 1 & 1 + x_1 & 1 + x_2 - x_3 \\ 1 & 0 & 0 \end{bmatrix} = 3 \ \text{ (full rank)}$$

Therefore, we know that this system is controllable, which means that $x = \Phi^{-1}(z)$ exists.

Step2.

Due to the fact that involutive condition will be automatically met as long as we successfully find $\phi_1(x)$ which satisfies $L_g L_f^{i-1} \phi_1 = 0,\ i = 1,2$, we're going to go ahead and do so.

We can then derive $\phi_1(x)$ by applying Eq.(6.5.26) from the text book.

(a) Condition 1: $\nabla\phi_1 \cdot g(x) = 0$

$$\nabla\phi_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \frac{\partial\phi_1}{\partial x_3} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{\partial\phi_1}{\partial x_2} + \frac{\partial\phi_1}{\partial x_3} = 0$$

(b) Condition 4: $\phi_2(x) = \nabla\phi_1 \cdot f(x)$

$$\begin{aligned} \phi_2(x) &= \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \frac{\partial\phi_1}{\partial x_3} \end{bmatrix} \cdot \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 x_3 - x_2 \\ -x_1 \end{bmatrix} \\ &= \frac{\partial\phi_1}{\partial x_1}(-x_1 + x_2 - x_3) + \frac{\partial\phi_1}{\partial x_2}(-x_1 x_3 - x_2) + \frac{\partial\phi_1}{\partial x_3}(-x_1) \end{aligned}$$

(c) Condition 2: $\nabla\phi_2 \cdot g(x) = 0$

$$\begin{bmatrix} \frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \frac{\partial\phi_2}{\partial x_3} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{\partial\phi_2}{\partial x_2} + \frac{\partial\phi_2}{\partial x_3} = \left(\frac{\partial\phi_1}{\partial x_1} - \frac{\partial\phi_1}{\partial x_2} \right) + \left(-\frac{\partial\phi_1}{\partial x_1} - x_1 \frac{\partial\phi_1}{\partial x_2} \right) = 0$$

From (c), we know that $-(1+x_1)\frac{\partial\phi_1}{\partial x_2} = 0 \Rightarrow \frac{\partial\phi_1}{\partial x_2} = 0$ or $x_1 = -1$

Note that it's possible that $x_1 = -1$, which will fulfill condition (c) without having $\frac{\partial\phi_1}{\partial x_2} = 0$. However, this is proved not adoptable when designing control signal u and will be further discussed later.

Plugging $\frac{\partial\phi_1}{\partial x_2} = 0$ back into (a) gives $\frac{\partial\phi_1}{\partial x_3} = 0$

Therefore, we know that both x_2 and x_3 have nothing to do with ϕ_1.

(d) Condition 5: $\phi_3(x) = \nabla\phi_2 \cdot f(x)$

$$\phi_3(x) = \begin{bmatrix} \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_2}{\partial x_3} \end{bmatrix} \cdot \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1 x_3 - x_2 \\ -x_1 \end{bmatrix}$$

$$= \frac{\partial \phi_2}{\partial x_1}(-x_1 + x_2 - x_3) + \frac{\partial \phi_2}{\partial x_2}(-x_1 x_3 - x_2) + \frac{\partial \phi_2}{\partial x_3}(-x_1)$$

(e) Condition 3: $\nabla \phi_3 \cdot g(x) \neq 0$

$$\begin{bmatrix} \frac{\partial \phi_3}{\partial x_1} & \frac{\partial \phi_3}{\partial x_2} & \frac{\partial \phi_3}{\partial x_3} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{\partial \phi_3}{\partial x_2} + \frac{\partial \phi_3}{\partial x_3} = \left(\frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_2}{\partial x_2} \right) + \left(-\frac{\partial \phi_2}{\partial x_1} - x_1 \frac{\partial \phi_2}{\partial x_2} \right) \neq 0$$

From (e), we obtain that $-(1 + x_1)\frac{\partial \phi_2}{\partial x_2} \neq 0 \Rightarrow \frac{\partial \phi_2}{\partial x_2} = \left(\frac{\partial \phi_1}{\partial x_1} - \frac{\partial \phi_1}{\partial x_2} \right) \neq 0$

Due to the fact that $\frac{\partial \phi_1}{\partial x_2} = 0$, we derive that $\frac{\partial \phi_1}{\partial x_1} \neq 0$.

We can then conclude that $\phi_1(x) = \phi_1(x_1)$.

Therefore, we're going to assume that $\phi_1(x) = x_1$ as this is very straightforward and it satisfies what we have derived so far, as well as the Condition 6 which is $\phi_1(0) = 0$.

<u>Step 3.</u>

Set up the state coordinate transformation z:

$$z_1 = \phi_1(x) = x_1$$
$$z_2 = \phi_2(x) = 1 \cdot (-x_1 + x_2 - x_3) = -x_1 + x_2 - x_3$$
$$z_3 = \phi_3(x) = -1 \cdot (-x_1 + x_2 - x_3) + 1 \cdot (-x_1 x_3 - x_2) + (-1) \cdot (-x_1) = 2x_1 - 2x_2 + x_3 - x_1 x_3$$

Also, set up the control signal transformation $u = \alpha(x) + \beta(x)v$

$$\alpha(x) = -\frac{\nabla\phi_3 \cdot f(x)}{\nabla\phi_3 \cdot g(x)} = -\frac{\begin{bmatrix} 2-x_3 & -2 & 1-x_1 \end{bmatrix} \cdot \begin{bmatrix} -x_1+x_2-x_3 \\ -x_1x_3-x_2 \\ -x_1 \end{bmatrix}}{\begin{bmatrix} 2-x_3 & -2 & 1-x_1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

$$= -\frac{(2-x_3)(-x_1+x_2-x_3)-2(-x_1x_3-x_2)+(1-x_1)(-x_1)}{-1-x_1}$$

$$\beta(x) = \frac{1}{\nabla\phi_3 \cdot g(x)} = \frac{1}{-1-x_1}$$

Step 4.

Design the new control law as $v = -Kz$

First, we can obtain the linear equation:

$$\dot{z} = A_c z + B_c v \Rightarrow \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v \Rightarrow \dot{z}_1 = z_2,\ \dot{z}_2 = z_3,\ \dot{z}_3 = v$$

$$\dot{z} = (A_c - B_c K) z \Rightarrow \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

where $k_{1,2,3}$ is dependent on closed-loop eigenvalues $\lambda_{1,2,3}$

In this case, $\lambda_{1,2,3} = -2$

$$\Rightarrow \det(\lambda I - (A_c - B_c K)) = (\lambda+2)^3 = \lambda^3 + 6\lambda^2 + 12\lambda + 8$$

Since we already know that $v = -(k_1 z_1 + k_2 z_2 + k_3 z_3)$, we can derive

$$k_1 = 8,\ k_2 = 12,\ k_3 = 6$$

Step 6.

Finally, we can derive the control law u by plugging $\alpha(x),\ \beta(x),\ v$

$$u=\alpha(x)+\beta(x)v$$

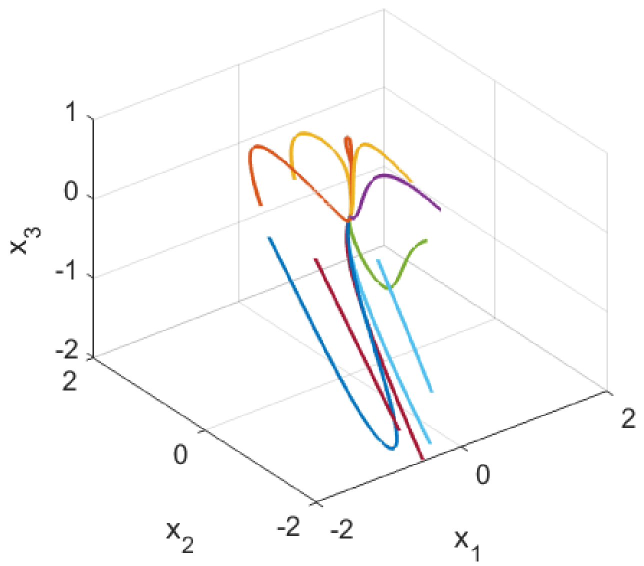
$$\alpha(x)=-\frac{(2-x_3)(-x_1+x_2-x_3)-2(-x_1x_3-x_2)+(1-x_1)(-x_1)}{-1-x_1}$$

where $\beta(x)=\dfrac{1}{-1-x_1}$

$$v=-8(x_1)-12(-x_1+x_2-x_3)-6(2x_1-2x_2+x_3-x_1x_2)$$

(2)

To see whether the origin is asymptotically stable, we put in 10 different initial $(x_1(0), x_2(0), x_3(0))$ and draw the corresponding path in the phase plane:

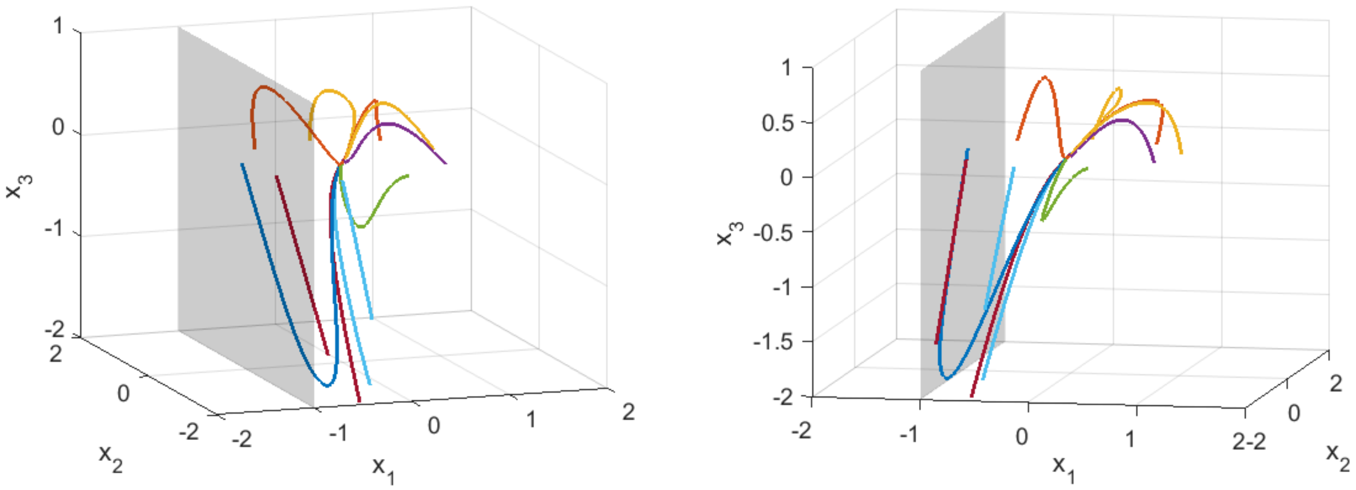


From the figure, we know that the origin is indeed asymptotically stable because there are several points ended up converging to the origin. This is because after we've done the transformation $z = \Phi(x)$ in (1), $\Phi(0) = 0$ is still satisfied so that the equilibrium point remains $(0,0)$. Since $f(0) = 0$, we just have to ensure that $\phi_1(0) = 0$ to check whether the origin is the equilibrium point after the transformation process.

After the transformation, the new state z will be exponentially stable as well as globally stable, since it became a linear system. During the transformation, each z will have a corresponding x and that's how we transform the whole thing into a linear system. Therefore, the original state x will also be exponentially stable, but we can't promise that it will be globally stable. This is because when we design the control law u, it's possible that there will be more constraints to avoid this control law being singular.

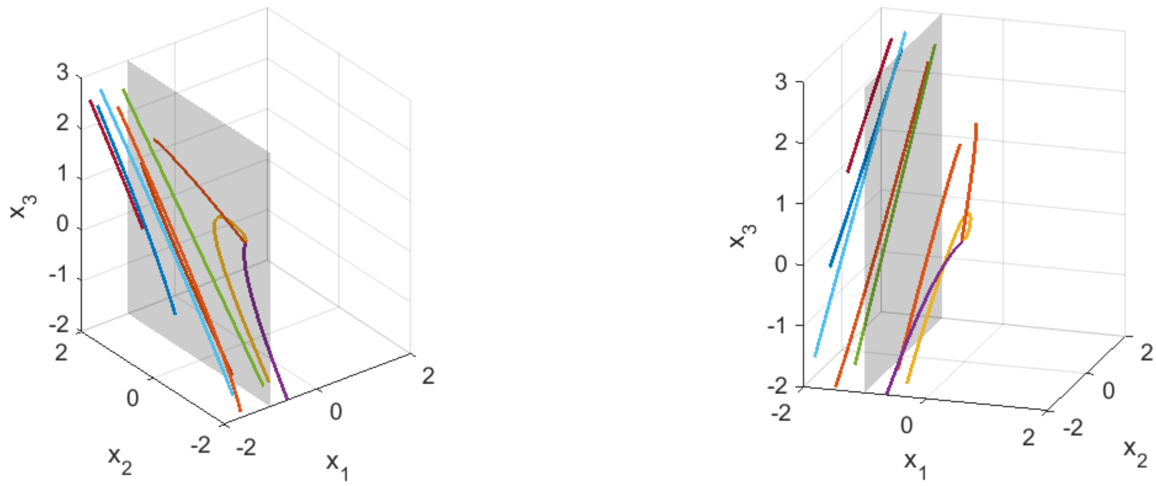
In the figure above, we notice that some of the points diverged. This is because during the transformation of control law, the denominator of $\alpha(x), \beta(x)$ is $-1 - x_1$, which means that whenever the path goes through the surface of $x_1 = -1$, they will end up diverging. We then conclude that this surface is "singular".

To see this clearer, I drew the surface of $x_1 = -1$ with the phase plane.



In the figure above, I put all 10 initials on the right side of $x_1 = -1$ and we can see as before that some points converged to the origin and some diverged.

I also tried putting some of the points on the left side of $x_1 = -1$:

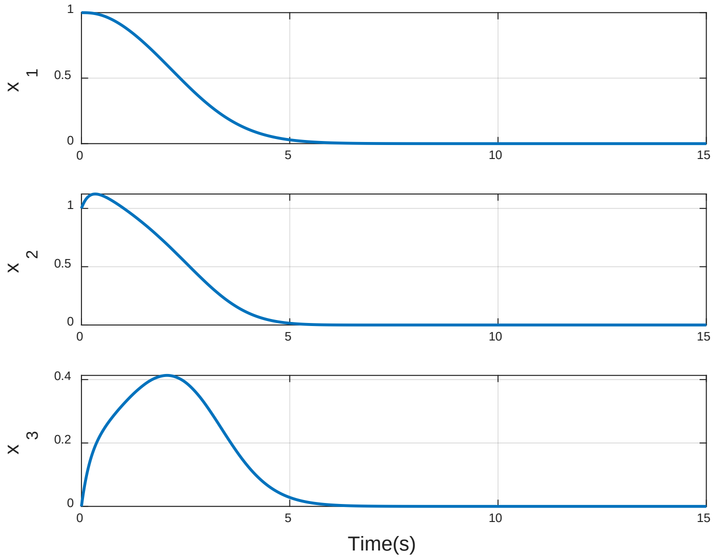


In the figure above, we can see that all the points started from the left side of $x_1 = -1$ end up diverging and being on the right side of $x_1 = -1$ doesn't promise a convergent result either.

From my point of view, this kind of situation often happens when we transform the nonlinear system to linear system. This is the shortcoming of this type of method and we have to be conscious about it. To avoid this, we should carefully choose the initials and try our best to find the stable region.

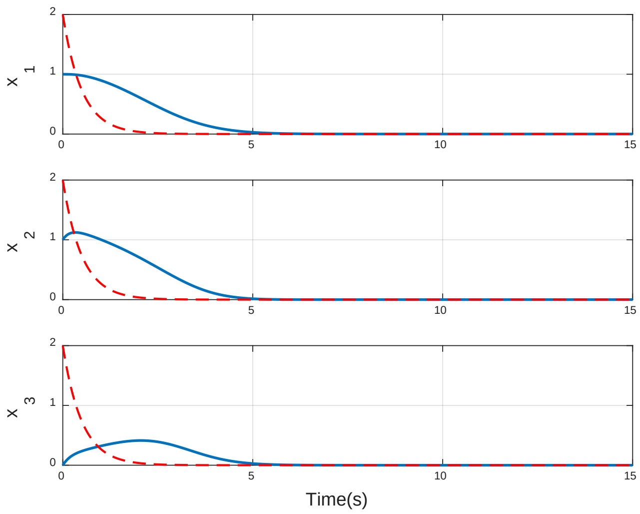
(3)

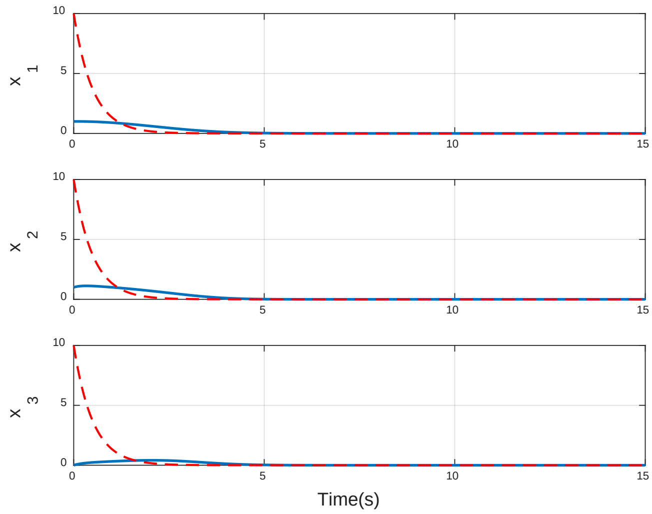
The time response of x_1, x_2, x_3 can be drawn as follows.



To check the speed of convergent when $\lambda = -2$, we're going to draw Ae^{-2t} where A stands for amplitude.

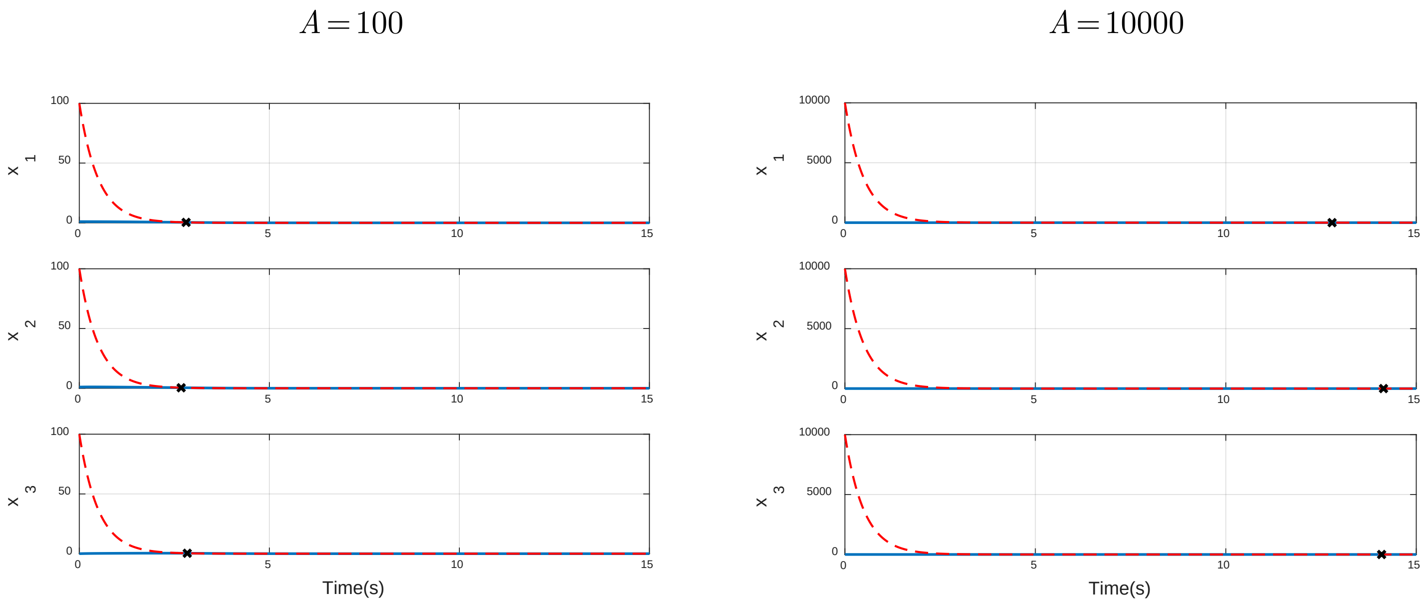
$A = 2$ $A = 10$





In the figure, we can see that at some point Ae^{-2t} actually intercepts with $x_{1,2,3}$. This means Ae^{-2t} is convergent "faster" than $x_{1,2,3}$.

To ensure that our inference is correct, I also tried putting bigger value of A.



From the figures above, we can see that even though the time of interception happening is larger, the speed of convergent for $x_{1,2,3}$ is still slower than Ae^{-2t}. To say that this system is asymptotically stable in a specific speed, we have to find $Ae^{-\lambda t}$ of which its curve will "cover" the time response curve at all times and there exists no interception.

(4)

When we look back at (1), we know that the solution for $u(x)$ is certainly not unique.

Because it's dependent on $\alpha(x)$, $\beta(x)$, v of which the value will change if we select a different $\phi_1(x)$. In other words, we will obtain different $u(x)$ when choosing a $\phi_1(x)$ different from what we had ($\phi_1(x)=x_1$) but still satisfies our conditions.

To verify our derivation, we're going to choose $\phi_1(x)=2x_1$ here instead and repeat the same procedure starting from Step 3. in (1).

Let $\phi_1(x)=2x_1$

<u>Step 3.</u>

Set up the state coordinate transformation z:

$$
\begin{aligned}
z_1 &= \phi_1(x) = 2x_1 \\
z_2 &= \phi_2(x) = 2\cdot(-x_1+x_2-x_3) = -2x_1+2x_2-2x_3 \\
z_3 &= \phi_3(x) = -2\cdot(-x_1+x_2-x_3)+2\cdot(-x_1x_3-x_2)+(-2)\cdot(-x_1) = 4x_1-4x_2+2x_3-2x_1x_3
\end{aligned}
$$

Also, set up the control signal transformation $u=\alpha(x)+\beta(x)v$

$$
\alpha(x) = -\frac{\nabla\phi_3\cdot f(x)}{\nabla\phi_3\cdot g(x)} = -\frac{\begin{bmatrix}4-2x_3 & -4 & 2-2x_1\end{bmatrix}\cdot\begin{bmatrix}-x_1+x_2-x_3\\ -x_1x_3-x_2\\ -x_1\end{bmatrix}}{\begin{bmatrix}4-2x_3 & -4 & 2-2x_1\end{bmatrix}\cdot\begin{bmatrix}0\\1\\1\end{bmatrix}}
$$

$$
= -\frac{(4-2x_3)(-x_1+x_2-x_3)-4(-x_1x_3-x_2)+(2-2x_1)(-x_1)}{-2-2x_1}
$$

$$
\beta(x) = \frac{1}{\nabla\phi_3\cdot g(x)} = \frac{1}{-2-2x_1}
$$

<u>Step 4.</u>

Design the new control law as $v=-Kz$

In this case, $\lambda_{1,2,3} = -2$

$$\Rightarrow \det\left(\lambda I - \left(A_c - B_c K\right)\right) = \left(\lambda + 2\right)^3 = \lambda^3 + 6\lambda^2 + 12\lambda + 8$$

Since we already know that $v = -\left(k_1 z_1 + k_2 z_2 + k_3 z_3\right)$, we can derive

$$k_1 = 8,\ k_2 = 12,\ k_3 = 6$$

<u>Step 6.</u>

Finally, we can derive the control law u by plugging $\alpha(x),\ \beta(x),\ v$

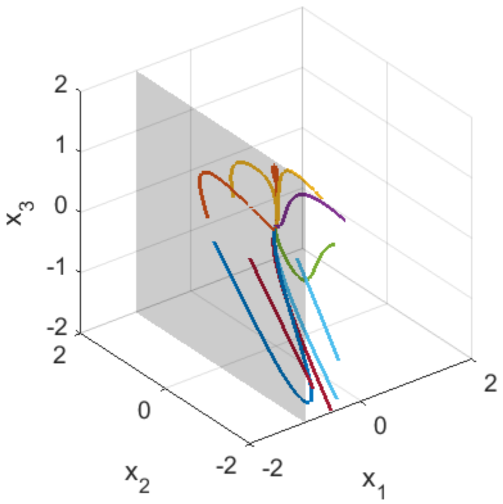
$$u = \alpha(x) + \beta(x) v$$

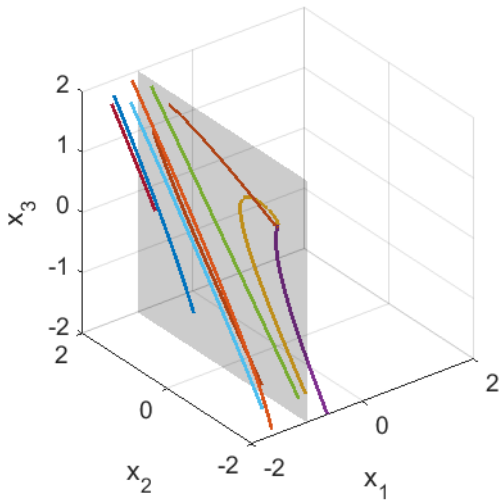
$$\alpha(x) = -\frac{(4-2x_3)(-x_1+x_2-x_3) - 4(-x_1x_3 - x_2) + (2-2x_1)(-x_1)}{-2-2x_1}$$

where $\beta(x) = \dfrac{1}{-2-2x_1}$

$$v = -8(2x_1) - 12(-2x_1 + 2x_2 - 2x_3) - 6(4x_1 - 4x_2 + 2x_3 - 2x_1x_3)$$

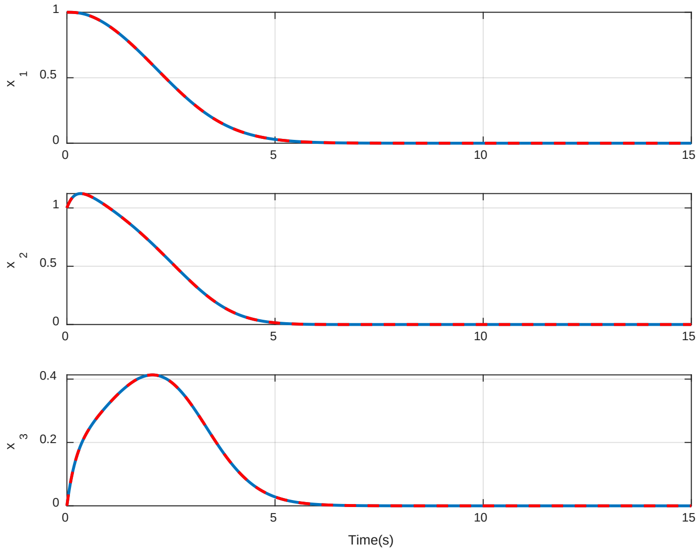
Now that we've gotten a different control law u, we can do the same as we did in (2).





Again, we put in 10 different initials to see that what happened is pretty similar to what we had before in (2).

To draw the time response:



where the red dash line is the obtained based on the new u that we designed.

We can observe that the response curves are identical. However, this doesn't represent that all different set of u will make the system have the same time response.

In my opinion, this is because when we decided to choose $\phi_1(x)=2x_1$, the corresponding $\alpha(x)$, $\beta(x)$, v will end up canceling out with one another and therefore cause the system to have the same time response.

To validate this saying, I'm going to try using another $\phi_1(x)$ here and repeat the process from Step 3. in (1) once again.

Let $\phi_1(x)=x_1^2$

<u>Step 3.</u>

Set up the state coordinate transformation z:

$$
\begin{aligned}
z_1 &= \phi_1(x) = x_1^2 \\
z_2 &= \phi_2(x) = 2x_1 \cdot (-x_1 + x_2 - x_3) = -2x_1^2 + 2x_1x_2 - 2x_1x_3 \\
z_3 &= \phi_3(x) = (-4x_1 + 2x_2 - 2x_3)\cdot(-x_1 + x_2 - x_3) + (2x_1)\cdot(-x_1x_3 - x_2) + (-2x_1)\cdot(-x_1) \\
&= 6x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1^2x_3 - 8x_1x_2 - 4x_2x_3 + 6x_1x_3
\end{aligned}
$$

Also, set up the control signal transformation $u = \alpha(x) + \beta(x)v$

$$\alpha(x) = -\frac{\nabla\phi_3 \cdot f(x)}{\nabla\phi_3 \cdot g(x)} = -\frac{\begin{bmatrix} 6x_1 - 4x_1x_3 - 8x_2 + 6x_3 & 4x_2 - 8x_1 - 4x_3 & 4x_3 - 2x_1^2 - 4x_2 + 6x_1 \end{bmatrix} \cdot \begin{bmatrix} -x_1 + x_2 - x_3 \\ -x_1x_3 - x_2 \\ -x_1 \end{bmatrix}}{\begin{bmatrix} 6x_1 - 4x_1x_3 - 8x_2 + 6x_3 & 4x_2 - 8x_1 - 4x_3 & 4x_3 - 2x_1^2 - 4x_2 + 6x_1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$$

$$= -\frac{(6x_1 - 4x_1x_3 - 8x_2 + 6x_3)(-x_1 + x_2 - x_3) + (4x_2 - 8x_1 - 4x_3)(-x_1x_3 - x_2) + \left(4x_3 - 2x_1^2 - 4x_2 + 6x_1\right)(-x_1)}{-2x_1^2 - 2x_1}$$

$$\beta(x) = \frac{1}{\nabla\phi_3 \cdot g(x)} = \frac{1}{-2x_1^2 - 2x_1}$$

<u>Step 4.</u>

Design the new control law as $v = -Kz$

In this case, $\lambda_{1,2,3} = -2$

$$\Rightarrow \det\left(\lambda I - (A_c - B_c K)\right) = (\lambda + 2)^3 = \lambda^3 + 6\lambda^2 + 12\lambda + 8$$

Since we already know that $v = -(k_1z_1 + k_2z_2 + k_3z_3)$, we can derive

$$k_1 = 8,\ k_2 = 12,\ k_3 = 6$$

<u>Step 6.</u>

Finally, we can derive the control law u by plugging $\alpha(x),\ \beta(x),\ v$

$$u = \alpha(x) + \beta(x)v$$

where

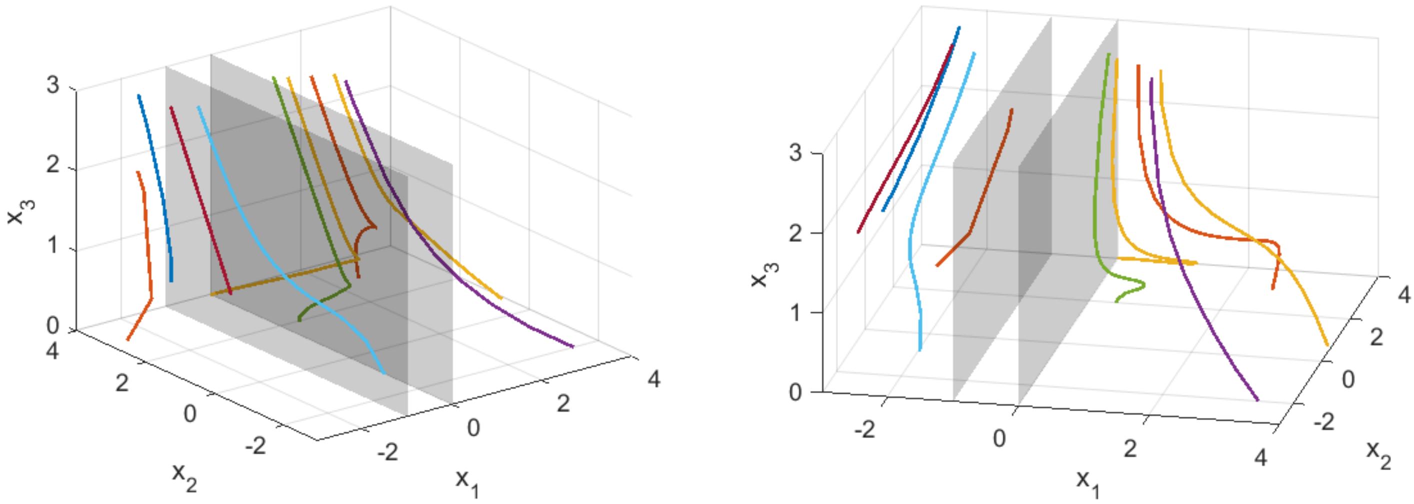
$$\alpha(x) = -\frac{(6x_1 - 4x_1x_3 - 8x_2 + 6x_3)(-x_1 + x_2 - x_3) + (4x_2 - 8x_1 - 4x_3)(-x_1x_3 - x_2) + \left(4x_3 - 2x_1^2 - 4x_2 + 6x_1\right)(-x_1)}{-2x_1^2 - 2x_1}$$

$$\beta(x) = \frac{1}{-2x_1^2 - 2x_1}$$

$$v = -8\left(x_1^2\right) - 12\left(-2x_1^2 + 2x_1x_2 - 2x_1x_3\right) - 6\left(6x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1^2x_3 - 8x_1x_2 - 4x_2x_3 + 6x_1x_3\right)$$

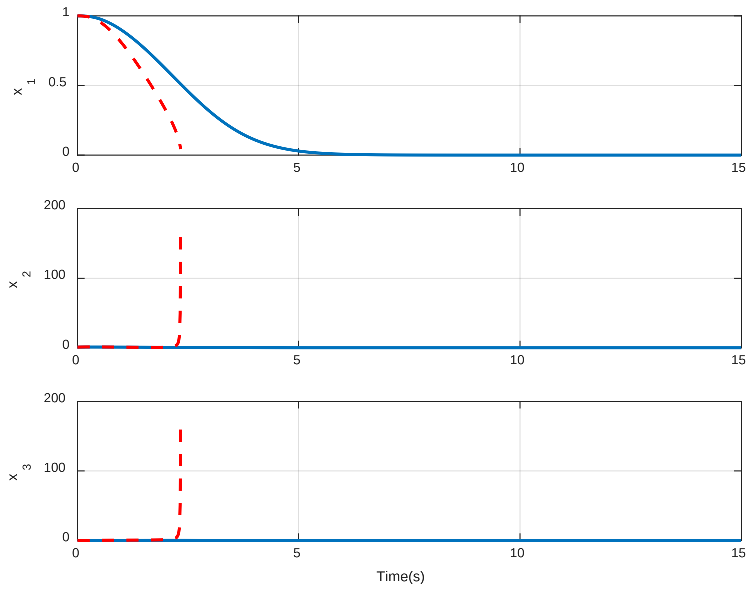
Now that we've gotten a different control law u, we can do the same as we did in (2).

Also, now that we know the denominator part of $\alpha(x)$, $\beta(x)$ $: -2x_1(x_1+1)$ will cause issues when passing $x_1 = 0, \ -1$, we're going to try to avoid it first and draw the phase plane as follows.



From this figure, we can see that by having this different set of u, as it is more complex and there are more "rules" we have to follow, it's even harder for them to converge to the origin. As a matter of fact, it's not even possible for them to converge to $(0,0)$.

To draw the time response:



where the red line is the time response we obtained based on this new control law u. We can now see that clearly; the time response is not the same when using a different control law u even though they have the same $\lambda_{1,2,3} = -2$. The new time response will "diverge" because at that point it probably hit the "singular" surface.

Moreover, we can conclude that this u is actually “worse” than what we have before because looking back at it, its equilibrium isn’t even on the origin due to the fact that the when being near $x_1 = 0$, the system will diverge.

To sum it up, if we want to find the convergent range for the nonlinear system by using this particular method, we should try to look for control law u that doesn’t have the constraints that we had earlier. If we can design a control law of which the denominator part is a constant or always positive, it will give us a better picture of the whole system. Perhaps we can find a couple of them and then combine the convergent range together to determine the range for the nonlinear system. The more we have, the more accurate the range we obtained is.