A Topological View on Integration and Exterior Calculus

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Abstract

A construction of integration, function calculus, and exterior calculus is made, allowing for integration of unital magma valued functions against (compactified) unital magma valued measures over arbitrary topological spaces. The Riemann integral, geometric product integral, and Lebesgue integral are shown as special cases. Notions similar to chain complexes are developed to allow this form of integration to define notions of exterior derivative for differential forms, and of derivatives of functions as well. Resulting conclusions on integration, orientation, dimension, and differentiation are discussed. Applications include calculus on fractals, stochastic analysis, discrete analysis, and other novel forms of calculus.

Keywords— Topology, Analysis, Exterior Calculus, Measure Theory, Fundamentals

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1 Introduction

Analysis is and has been for a long time one of the most prominent fields studied in mathematics. As such, there have been many, many attempts at generalizing it to various settings, both specific and general. Ranging over discrete calculus [5, 30, 11, 10], graph analysis [16], calculus and differential forms on fractals via approaches such as harmonic analysis [12, 6, 26, 17], the fractal-fractional calculus [18, 8] and others [1, 13, 23], as well as other novel forms of analysis such as the p-adic calculus [20, 15], stochastic and rough path analysis [7, 19, 29, 24], generalizations of analysis to more abstract algebraic structures [14, 4, 25, 9, 27], noncommutative geometry [3], and more, inclusing works that attempt to unify various forms of calculus under a single framework [23, 21, 22]. This is yet another such attempt. Inspired by the realization remarked on by David [28] that the usual harmonic analysis approach to fractal calculus is near purely topological, the works of Harrison [23] trying to extend the standard exterior calculus to apply on novel topological spaces embedded in \mathbb{R}^n , and the following line of thought: "Since a topological space is enough information to define measurable sets, shouldn't it be enough information to define integration? And if it is, would that be enough to define notions of differentiation via the generalized Stokes theorem, taken as a defining axiom?"

The answer, as will be shown in this paper, seems to be mostly yes. Section 2 of this paper will define integration over measurable subset of an arbitrary topological space, of unital magma¹ valued functions, against measures valued in compactifications of a unital magma that satisfy certain conditions. A form of integration I call IA integration will be developed, and I will demonstrate the Riemann integral, geometric product integral, and Lebesgue integral on the real number line to be special cases of it. Section 3 will construct notions of orientation and order of operations suited to the possibly noncommutative notions of integration considered in this paper, resulting in a structure I call an "integrable chain complex". Integrable chain complexes will be reminiscent of traditional chain complexes, but suited specifically for the form of calculus this paper defines. Section 4 will develop the resulting basic notions of exterior calculus, including differential forms, the exterior derivative, and an integrable cochain complex dual to the integrable chain complex on a topological space. Section 5 will use a similar construction on integrable chain complexes to define the derivatives of functions defined on certain subsets of the base topological space, as determined by the integrable chain complex considered. This will lead to a type of differential operator on functions with notion of dimension, replacing "functions on a k-dimensional surface" with "functions on the (closure of base sets of) integrable k-chains". Of particular note will be resulting conclusions on the nature of integration, orientation, a notion of integer dimension induced by the integrable chain complex on a topological space, and the behavior of noncommutative integrals. Sadly, due to time constraints, I have not yet been able to explore questions of what are the conditions for existence and uniqueness of many structures defined in this paper, see further work for details. Although some questions of existence and uniqueness were not dealt with, some conclusions were reached on what properties these objects must have when well-defined. This paper doesn't proclaim to be contain a fully developed theory, but rather the skeleton of one - it gives the outline to defining a form of integration, the definition of an integrable chain complex, a resulting basic form of exterior calculus, and the definition of a resulting differential operator. Choosing a specific form of IA integral, constructing an appropriate integrable chain complex, and fully deriving the possible resulting differential operators is left for future work, and likely requires a case-by-case treatment for different forms of analysis defined on different topological spaces. Future works may be on fleshing this skeleton of a theory out more rigorously, constructing further notions of exterior and function calculus in its general setting, applying said the theory developed here as a general guideline to defining specific choices of calculus on specific topological spaces, and more.

2 A Topological Construction of Integration

2.1 Topological unital magmas and measures on the Borel σ -algebra of a topological space

2.1.1 Topological unital magmas

First, I will define the algebraic structure that will be key to defining functions and measures to be integrated in the first place.

Definition 1

A topological unital magma is a unital magma $(M, +_M)$ that has an identity element 0_M , where M is endowed with a Hausdorff topology in which the magma operation $+_M : M \times M \to M$ is a continuous map.

¹A unital magma is a set closed under a binary operation, such that the set includes an identity element of said binary operation.

An **extended** topological unital magma $(\overline{M}, +_M)$ is any compactification of a topological unital magma that is Hausdorff, where the magma operation $+_M$ is extended to be defined on all of $\overline{M} \times M$ and all of $M \times \overline{M}$ such that the identity element 0_M remains an identity element over all of \overline{M} .

Note, in this definition the magma operation $+_M$ doesn't have to be defined on $(\overline{M} \setminus M) \times (\overline{M} \setminus M)$.

Notation 1

The elements of $\overline{M} \setminus M$ are denoted the **infinities** of M.

Notation 2

The topology of a topological unital magma $(M, +_M)$ is marked τ_M accordingly. The topology of the corresponding extended topological unital magma is marked τ_M .

Recall, a net $x_{\bullet}: A \to \overline{M}$ converges to a limit $x^* \in \overline{M}$ if for any open neighborhood U of x^* in \overline{M} , there exists some $\alpha_0 \in A$ such that for all $A \ni \alpha > \alpha_0$, $x_{\alpha} \in U$. Since an extended topological magma is by definition a compact space, any net $x_{\bullet}: A \to \overline{M}$ has a convergent subnet, and since the extended topological magma is required to be Hausdorff, then the limit of any net, if it exists, is unique in \overline{M} . If a net $x_{\bullet}: A \to \overline{M}$ converges to a limit point in M, it's said to **converge to a point** in M. If it converges to a limit point in $\overline{M} \setminus M$, the net is said to **converge to an infinity** of M.

2.1.2 Measures on the Borel σ -algebra of a topological space

Now I will define a sense of "measure", as general as I can make it, that allows for notions of integration to be considered. Given a topological space X, let Σ_X be the Borel σ -algebra of X.

Definition 3

A subset $U \subseteq X$ is called a **measurable subset** of X if it is an element of the Borel σ -algebra Σ_X of X.

Definition 4

An **orientable measure**, or just **measure** for short, μ on a topological space X is a function $\mu: \Sigma_X \to \overline{M}$ from the Borel σ -algebra of X to an extended topological unital magma $(\overline{M}, +_M)$, such that $\mu(\emptyset) = \mathbf{0}_M$.

In this context, \overline{M} is called the **measure codomain**. This notion of measure is in fact, as will be seen later in this paper, a generalization not only of ordinary measures, but of signed measures too. The fact that signed measures are included as a special case of orientable measures will be needed to define integration on oriented sets. Given a measure μ , a measurable subset $A \in \Sigma_X$ is said to be of **zero measure** with respect to μ if $\mu(A) = \mathbf{0}_M$, of **finite measure** with respect to μ if $\mu(A) \in \overline{M} \setminus M$.

Definition 5

The collection of all possible measures over a topological space X with measure codomain \overline{M} is called the **space of measures** on X with target \overline{M} , and denoted as $\Xi(X,\overline{M})$.

Corollary 1

The space of measures $\Xi(X, \overline{M})$ inherits the unital magma operation of M via

$$\forall U \in \Sigma_X, \forall \mu, \nu \in \Xi(X, \overline{M}) : (\mu +_M \nu)(U) := \mu(U) +_M \nu(U) \tag{1}$$

And inherits its identity element via

$$\mathbf{0}_{\Xi}: \Sigma_X \to \mathbf{0}_M \tag{2}$$

So "addition of measures" is well-defined on all pairs of measurable subsets of X that are not both of infinite measure. Instead of denoting $\mathbf{0}_{\Xi}$ and $\mathbf{0}_{M}$ separately, I will denote $\mathbf{0}_{M}$ for both.

2.2 Functions and integration on the Borel σ -algebra of a topological space

2.2.1 Simple integrals

Notation 3

The collection of all functions $f: X \to Y$, with $(Y, +_Y)$ a topological unital magma titled in this context as the **function codomain**, is to be called the **space of functions** on X with target Y, and marked $\mathcal{F}(X,Y)$.

Corollary 2

The space of functions $\mathcal{F}(X,Y)$ inherits the unital magma operation of Y via

$$\forall f, h \in \mathcal{F}(X, Y), \forall x \in X : (f +_Y h)(x) := f(x) +_Y h(x) \tag{3}$$

and the identity element of Y via

$$\mathbf{0}_{\mathcal{F}}: X \to \mathbf{0}_{Y} \tag{4}$$

So "addition of functions" is well-defined on X. Instead of denoting $\mathbf{0}_{\mathcal{F}}$ and $\mathbf{0}_{Y}$ separately, I will denote $\mathbf{0}_{Y}$ for both.

Definition 6

Given a topological space X, a measure codomain $(\overline{M}, +_M)$, and a function codomain $(Y, +_Y)$, an **integration element** is a mapping $g: Y \times \overline{M} \to \overline{G}$, such that g is bi-distributive over the magma operations $+_Y$ and $+_M$:

1. $\forall a, b \in Y, m \in \overline{M} : g(a +_Y b, m) = g(a, m) +_G g(b, m)$

2.
$$\forall a \in Y, m \in M, n \in \overline{M} : \begin{cases} g(a, m +_M n) = g(a, m) +_G g(a, n) \\ g(a, n +_M m) = g(a, n) +_G g(a, m) \end{cases}$$

And such that

$$\forall m \in \overline{M} : g(\mathbf{0}_Y, m) = \mathbf{0}_G$$

Where $(\overline{G}, +_G)$ is an extended topological unital magma titled the **integration codomain**.

The integration codomain $(\overline{G}, +_G)$ is an extended topological unital magma that partially inherits its magma operation $+_G$ from $+_Y$ and $+_M$, and fully inherits its identity element via $\forall a \in Y, m \in \overline{M} : g(\mathbf{0}_Y, m) = \mathbf{0}_G$.

Definition 7

Given a mapping $f: A \times B \to C$ from two unital magmas $(A, +_A), (B, +_B)$ to a unital magma $(C, +_C)$ that is bi-distributive in the sense shown in definition 6, the **arithmetic structure** of f is the expression of $f(a +_A b, c +_B d)$ in terms of f(a, c), f(b, c), f(a, d), f(b, d) and $+_C$ for all $a, b \in A, c, d \in B$. Note, if the unital magma operations $+_A, +_B, +_C$ are commutative and associative, there is only one possible arithmetic structure of f. If $+_A, +_B, +_C$ aren't commutative and associative, there is more than one possible arithmetic structure, but the number of possibilities is finite.

Note that integration elements $g: Y \times \overline{M} \to \overline{G}$ are such bi-distributive maps over $(Y, +_Y), (M, +_M)$ and $(G, +_G)$, so any integration element must have an arithmetic structure accordingly. Now that the integration element is defined, a notion of integration can be defined accordingly. Let X be a topological space with Borel σ -algebra Σ_X , let $(Y, +_Y)$ be a function codomain, let $\Xi(X, \overline{M})$ be a space of measures on X with measure codomain $(\overline{M}, +_M)$, and let $g: Y \times \overline{M} \to \overline{G}$ be the element of integration with integration codomain $(\overline{G}, +_G)$.

Definition 8

Given the above setting, an **indicator function** of a measurable subset $S \in \Sigma_X$ is any function $I_S \in \mathcal{F}(X,Y)$ defined as follows:

$$\forall x \in X : I_S(c:x) = \begin{cases} c & , x \in S \\ 0_Y & , x \notin S \end{cases}$$
 (5)

Where $0_Y \neq c \in Y$ is some non-identity constant in a unital magma $(Y, +_Y)$.

Given the above setting, a **simple integral** $\int_{U} g(f, d\mu)$ of a function $f: X \to Y$ against a measure $\mu: \Sigma_X \to \overline{M}$ over a measurable subset $U \in \Sigma_X$ is any map $\int_{U} : \mathcal{F}(X, Y) \times \Xi(X, \overline{M}) \to \overline{G}$ that satisfies the following properties:

1. Arithmetic structure:

The simple integral operation $\int_U : \mathcal{F}(X,Y) \times \Xi(X,\overline{M}) \to \overline{G}$ must be a bi-distributive mapping sharing the same arithmetic structure that the integration element $g: Y \times \overline{M} \to \overline{G}$ has.

2. Simple integration of indicator functions:

Given an indicator function $I_S(c:x) = \begin{cases} c & , x \in S \\ 0_Y & , x \notin S \end{cases}$ with $U, S \in \Sigma_X$ measurable subsets of X, and given a measure $\mu \in \Xi(X, \overline{M})$, the simple integral of $I_S(c:x)$ is

$$\int_{U} g(I_{S}(c:x), d\mu) := g\left(c, \mu\left(U \cap S\right)\right)$$
(6)

Note, the simple integral doesn't have to be defined on all of $\mathcal{F}(X,Y) \times \Xi(X,\overline{M})$, only on indicator functions and on *integrable simple functions*, see definitions 10, 11 and theorem 1.

Example of the arithmetic structure demand of definition 9:

If
$$\forall a, b \in Y, m, n \in M : g(a +_Y b, m +_M n) = g(a, m) +_G g(b, m) +_G g(a, n) +_G g(b, n)$$
 then $\forall f, h \in \mathcal{F}(X, Y), \mu, \nu \in \Xi(X, \overline{M}) : \int_U g(f +_{\mathcal{F}} h, d\mu +_\Xi d\nu) = \int_U g(f, d\mu) +_G \int_U g(h, d\mu) +_G \int_U g(f, d\nu) +_G \int_U g(h, d\nu) +_G \int$

$$\begin{array}{l} \textbf{If} \ \forall a,b \in Y, m,n \in M: g(a+_{Y}b,m+_{M}n) = g(a,m) +_{G}g(a,n) +_{G}g(b,m) +_{G}g(b,n) \ \textbf{then} \\ \forall f,h \in \mathcal{F}(X,Y), \mu,\nu \in \Xi(X,\overline{M}): \int\limits_{U}g\left(f+_{\mathcal{F}}h \,,\, d\mu +_{\Xi}d\nu\right) = \int\limits_{U}g\left(f,d\mu\right) \, +_{G}\int\limits_{U}g\left(f,d\nu\right) \, +_{G}\int\limits_{U}g\left(h,d\mu\right) \, +_{G}\int\limits_{U}g\left(h,d\mu\right) +_{G}\int\limits_{U}g\left(h,d\mu\right) \, +_{$$

With the relations holding whenever the RHS of 4 equations above is well-defined. In short, going from integration element to simple integral is a structure-preserving process, similar to the notion of homomorphism.

Corollary 3

A simple integral $\int_U : \mathcal{F}(X,Y) \times \Xi(X,\overline{M}) \to \overline{G}$ is by definition bi-distributive over functions in $\mathcal{F}(X,Y)$ and measures in $\Xi(X,\overline{M})$ the same way its integration element $g:Y\times\overline{M}\to\overline{G}$ is bi-distributive over elements of Y and \overline{M} :

$$\forall f, h \in \mathcal{F}(X, Y), \mu \in \Xi(X, \overline{M}) : \int_{U} g(f +_{Y} h, \mu) = \int_{U} g(f, \mu) +_{G} \int_{U} g(h, \mu)$$
 (7)

$$\forall f \in \mathcal{F}(X,Y), \mu, \nu \in \Xi(X,\overline{M}) : \int_{U} g(f,\mu +_{M} \nu) = \int_{U} g(f,\mu) +_{G} \int_{U} g(f,\nu)$$
(8)

2.2.2 Simple functions and IA integration

Notation 4

Given a unital magma $(A, +_A)$ and a sequence of elements $a_1, ..., a_n \in A$, the following notation is used:

$$\sum_{i=1}^{n} a_i := a_1 +_A \dots +_A a_n \tag{9}$$

Where the order of operations is chosen to be from left to right. An analogous theory exists for an order of operations chosen to be from right to left, and for any other choice of ordering - the choice need only be consistent.

A **simple function** on the topological space X is a finite magma sum of indicator functions of measurable subsets of X:

$$\forall x \in X : f_n(x) := \sum_{k=1}^n I_{S_k}(c_k : x)$$
 (10)

Where $\forall k = 1, ..., n : S_k \in \Sigma_X$.

Definition 11

Given a measure $\mu \in \Xi(X, \overline{M})$ on X, a simple function $f_n(x) = \sum_{k=1}^n {}_Y I_{S_k}(c_k : x)$ is said to be integrable with respect to μ on a measurable subset $U \in \Sigma_X$ if

$$\forall k = 1, ..., n : \mu\left(S_K \bigcap U\right) \in M \tag{11}$$

Theorem 1

Simple integration of simple functions: It trivially follows from definition 9 that given a simple function $f_n(x) = \sum_{k=1}^n {}_Y I_{S_k}(c_k : x)$ that is integrable with respect to a measure $\mu \in \Xi(X, \overline{M})$ on a measurable subset $U \in \Sigma_X$, the simple integral of f_n against μ on U must be

$$\int_{U} g(f_n, d\mu) = \sum_{k=1}^{n} g\left(c_k, \mu\left(S_k \cap U\right)\right)$$
(12)

And definition 11 guarantees that the unital magma sum on the RHS of eq. 12 is a well-defined element of the integration codomain \overline{G} .

Now that I have a notion for integration of simple functions, I can use it to define integration of non-simple functions as well. As per notation 2, let τ_Y be the topology of the topological unital magma $(Y, +_Y)$.

Definition 12

Given a function $f \in \mathcal{F}(X,Y)$ and a measure $\mu \in \Xi(X,\mathcal{M})$, an **integrable approximation** (IA) of f with respect to μ on U, denoted $\{s_k|U\}_{k=1}^{\infty} \xrightarrow{\mu} f$, on a measurable subset $U \in \Sigma_X$ is a series of simple functions $\{s_k\}_{k=1}^{\infty}$ such that:

- 1. For all $k \in \mathbb{N}$, s_k is integrable on U with respect to μ
- 2. The sequence $\{s_k\}_{k=1}^{\infty}$ converges pointwise to f for all $x \in U$: For all $x \in U$, and for any open neighborhood $V \in \tau_Y$ such that $f(x) \in V$, there exists some $K \in \mathbb{N}$, such that for all k > K, $s_k(x) \in V$.

For any IA $\{s_k|U\}_{k=1}^{\infty} \xrightarrow{\mu} f$ of a function $f \in \mathcal{F}(X,Y)$ on U with respect to a measure $\mu \in \Xi(X,\mathcal{M})$, there is an associated sequence of simple integrals $\left\{\int_U g(s_k,d\mu)\right\}_{k=1}^{\infty}$ that by theorem 1 are all well-defined. Therefore, I can now define:

Given a function $f \in \mathcal{F}(X,Y)$ and a collection of infinite sequences of simple functions $L[f](X,Y) \subseteq (\mathcal{F}(X,Y))^{\mathbb{N}}$, the L[f](X,Y)-IA integral of f on U with respect to a measure $\mu \in \Xi(X,\mathcal{M})$, is, if it exists, the unique value $I \in \overline{G}$ of the integration codomain \overline{G} such that

$$I = \lim_{k \to \infty} \int_{I} g(s_k, d\mu)$$

for all possible IAs $\{s_k|U\}_{k=1}^{\infty} \stackrel{\mu}{\to} f$ such that $\{s_k\}_{n=1}^{\infty} \in L[f](X,Y)$. The L[f](X,Y)-IA integral of a function f on a region U against measure μ is simply denoted as $\int_U g(f,d\mu)$, the same as in definition 9.

In the context of definition 13, an IA $\{s_k|U\}_{k=1}^{\infty} \stackrel{\mu}{\to} f$ such that $\{s_k\}_{n=1}^{\infty} \in L[f](X,Y)$ is called an L[f](X,Y)-IA of f.

Definition 14

If a function $f \in \mathcal{F}(X,Y)$ has any two L[f](X,Y)-IAs, $\{s_{1,k}|U\}_{k=1}^{\infty} \stackrel{\mu}{\to} f$ and $\{s_{2,k}|U\}_{k=1}^{\infty} \stackrel{\mu}{\to} f$ such that $\lim_{k\to\infty} \int\limits_{U} g(s_{1,k}d\mu) \neq \lim_{k\to\infty} \int\limits_{U} g(s_{2,k}d\mu)$, then the L[f](X,Y)-IA integral of f on U with respect to μ is said to be **undefined**.

For functions $f \in \mathcal{F}(X,Y)$ whose L[f](X,Y)-IA integral is undefined on a region $U \in \Sigma_X$ with respect to a measure $\mu \in \Xi(X,\overline{M})$, a construction analogous to Cauchy's principal value may be achieved by defining a systematic way to choose a specific reduced collection $\mathcal{L}[f](X,Y) \subset L[f](X,Y)$ on U, such that the $\mathcal{L}[f](X,Y)$ -IA integral of f is (hopefully) defined. The specific choice of $\mathcal{L}[f](X,Y)$ would then fulfill the same role as the classic Cauchy principal value (or any of its variations) does in ordinary analysis. However, there may be some cases where even that can't be done:

Definition 15

Given a function $f \in \mathcal{F}(X,Y)$, if there doesn't exist any L[f](X,Y)-IA of f on U with respect to μ that has a convergent integral series, then the L[f](X,Y)-IA integral of f on U with respect to μ is said to be **fundamentally undefined**.

Note, that while by theorem 1 the simple integral of an integrable simple function s is always defined, the question of whether or not its L[s](X,Y)-IA integral is defined depends on the choice of L[s](X,Y).

Example: IA-Riemann integration

Let the topological space in question be $X = \mathbb{R}$, let the function codomain be $(Y, +_Y) = (\mathbb{R}, +)$, let the measure codomain be $(\overline{M}, +_M) = ([0, \infty], +)$, and let the element of integration be

$$\forall a \in Y, m \in \overline{M} : g(a, m) = a \cdot m$$

With integration codomain $(\overline{G}, +_G) = ([-\infty, \infty], +)$, with $+, \cdot$ being ordinary real number addition and multiplication. For an interval $[a, b] \subset \mathbb{R}$, a **tagged partition** of [a, b] is a structure of the form $P = \{t_i, [x_i, x_{i+1}]\}_{i=1}^n$ with $n \in \mathbb{N}$, such that $t_i \in [x_i, x_{i+1}]$ and $x_{i+1} > x_i$ for all i = 1, ..., n, and $[a, b] = \bigcup_{i=1}^n [x_i, x_{i+1}]$. Given a function $f \in \mathcal{F}(X, Y)$, let the P-approximation of f on [a, b] be the simple function

on
$$[a, b]$$
 defined as $\forall x \in [a, b] : f_P(x) = \sum_{i=1}^n I_{[x_i, x_{i+1}]}(f(t_i); x)$.

The IA-Riemann integral, of a function f on [a,b] against a measure $\mu \in \Xi([a,b],[0,\infty])$ is the $L[f]([a,b],\mathbb{R})$ -IA integral of f on [a,b] against μ , with $L[f]([a,b],\mathbb{R})$ chosen such that only IAs made of P-approximations of f on [a,b], for all partitions P of [a,b], are allowed. The IA-Riemann integral clearly coincides with the ordinary Riemann integral, as partition refinements define an IA for all Riemann-integrable functions on [a,b], and it's easy to show that a function is Riemann-integrable in the ordinary sense on [a,b] if and only if it's IA-Riemann integrable on [a,b]. Definition 13 naturally gives that the IA-Riemann integral is to be denoted as $\int_{[a,b]} f d\mu$, accordingly. The domain of this integral can be generalized from an [a,b]

interval [a, b] to arbitrary unions of such intervals.

Example: The IA geometric product integral

Let the topological space in question be $X = \mathbb{R}$, let the function codomain be $(Y, +_Y) = (\mathbb{R}^+, \cdot)$, let the measure codomain be $(\overline{M}, +_M) = ([0, \infty], +)$, and let the element of integration be

$$\forall a \in Y, m \in \overline{M} : g(a, m) = a^m$$

With integration codomain $(\overline{G}, +_G) = ([0, \infty], \cdot)$, with $+, \cdot$ being ordinary real number addition and multiplication, and a^m being a to the power of m. Defining partitions of the interval [a, b] the same way as in the previous example, given a function $f \in \mathcal{F}(X,Y)$, let the **geometric** P-approximation of f on [a,b] be the simple function on [a,b] defined as $\forall x \in [a,b]: f_P(x) = \prod_{i=1}^n I_{[x_i,x_{i+1}]}(f(t_i);x)$. Note, since the function codomain's magma operation $+_Y$ is in this case real number multiplication, the identity element used in the definition of indicator functions is $\mathbf{0}_Y = 1$. The **IA geometric product integral** of a function f on [a,b] against a measure $\mu \in \Xi([a,b],[0,\infty])$ is the $L[f]([a,b],\mathbb{R})$ -IA integral of f on [a,b] against μ , with $L[f]([a,b],\mathbb{R})$ chosen such that only IAs made of P-approximations of f on [a,b], for all partitions P of [a,b], are allowed. Definition 13 naturally gives that the IA geometric product integral is to be denoted as $\int_{[a,b]}^{f} f^{d\mu}$, accordingly. The domain of this integral can be generalized from an interval [a,b] to arbitrary unions of such intervals

[a,b] to arbitrary unions of such intervals.

Example: The IA Lebesgue integral

Let the topological space in question be $X = \mathbb{R}$, let the function codomain be $(Y, +_Y) = (\mathbb{R}, +)$, let the measure codomain be $(\overline{M}, +_M) = ([0, \infty], +)$, and let the element of integration be

$$\forall a \in Y, m \in \overline{M} : q(a, m) = a \cdot m$$

With integration codomain $(\overline{G}, +_G) = ([-\infty, \infty], +)$, with $+, \cdot$ being ordinary real number addition and multiplication. The **IA Lebesgue integral** of a function $f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ on \mathbb{R} against a measure $\mu \in \Xi(\mathbb{R}, [0, \infty])$ is the $L[f](\mathbb{R}, \mathbb{R})$ -IA integral of f, with $L[f](\mathbb{R}, \mathbb{R})$ chosen to only allow IAs $\{s_k|U\}_{k=1}^{\infty} \stackrel{\mu}{\to} f$ that satisfy the following conditions:

- 1. $\forall k \in \mathbb{N}, x \in \mathbb{R} : |s_k(x)| \le |f(x)|$
- 2. $\forall k \in \mathbb{N}, x \in \mathbb{R} : |s_k(x)| \le |s_{k+1}(x)|$

IAs $\{s_k|U\}_{k=1}^{\infty} \xrightarrow{\mu} f$ that satisfy these conditions are to be known as **Lebesgue IAs**. Definition 13 naturally gives that the IA Lebesgue integral above is to be denoted as $\int f d\mu$, accordingly.

 $\textbf{Claim:} \ \ \text{The IA Lebesgue integral is, in fact, the ordinary Lebesgue integral.}$

Proof: Let $f(x) = f^+(x) - f^-(x)$, where $f^+(x) := \begin{cases} f(x) & f(x) \geq 0 \\ 0 & f(x) \leq 0 \end{cases}$ and $f^-(x) := \begin{cases} -f(x) & f(x) \leq 0 \\ 0 & f(x) \geq 0 \end{cases}$ It's clear by corollary 3 and the definition of IA integration in terms of simple integrals that for any measure $\mu \in \Xi(\mathbb{R}, [0, \infty])$, $\int\limits_{\mathbb{R}} f d\mu = \int\limits_{\mathbb{R}} f^+ d\mu - \int\limits_{\mathbb{R}} f^- d\mu$. First looking at $\int\limits_{\mathbb{R}} f^+ d\mu$, note that the simple integrals of simple functions coincide in this setting with the ordinary Lebesgue integration of simple functions. Thus, the monotone convergence theorem for non-negative functions can be applied, and it's clear that any Lebesgue IA $\left\{s_k^+|U\right\}_{k=1}^\infty \stackrel{\mu}{\to} f^+$ converges to a limit in $[0,\infty]$. Furthermore, the monotone convergence theorem implies that $\lim_{k\to\infty} \int\limits_{\mathbb{R}} s_k^+ d\mu = \int\limits_{\mathbb{R}} \lim_{k\to\infty} s_k^+ d\mu$, so therefore for any two Lebesgue IAs $\left\{s_{1,k}^+|U\right\}_{k=1}^\infty \stackrel{\mu}{\to} f^+$, it results that $\lim_{k\to\infty} \int\limits_{\mathbb{R}} s_{1,k}^+ d\mu - \lim_{k\to\infty} \int\limits_{\mathbb{R}} s_{2,k}^+ d\mu = \int\limits_{\mathbb{R}} \lim_{k\to\infty} \left(s_{1,k}^+ - s_{2,k}^+\right) d\mu = \int\limits_{\mathbb{R}} 0 \cdot d\mu = 0$, so all Lebesgue IAs $\left\{s_k^+|U\right\}_{k=1}^\infty \stackrel{\mu}{\to} f^+$ converge to the same limit, implying that they all converge to the supremum $\sup \int\limits_{\mathbb{R}} s d\mu$ over all simple functions s satisfying $\forall x \in \mathbb{R} : 0 \leq s(x) \leq f^+(x)$, meaning that the

IA Lebesgue integral of the non-negative function f^+ equals its standard Lebesgue integral. Repeating the same process with f^- , and noting that for all Lebesgue IAs $\left\{s_k^+|U\right\}_{k=1}^\infty \stackrel{\mu}{\to} f^+$, $\left\{s_k^-|U\right\}_{k=1}^\infty \stackrel{\mu}{\to} f^-$, the

sequence $\{s_k\}_{k=1}^{\infty}$ defined by $s_k(x) := \begin{cases} s_k^+ & s_k^+ \ge 0 \\ -s_k^- & s_k^- \ge 0 \end{cases}$ is a Lebesgue IA of f(x), and in fact, and Lebesgue

IA of f can be written that way. Therefore, the IA Lebesgue integral of f is exactly equal to the subtraction of standard Lebesgue integrals $\int\limits_{\mathbb{R}} f^+ d\mu - \int\limits_{\mathbb{R}} f^- d\mu$, thus making the IA Lebesgue integral of f simply its ordinary Lebesgue integral.

Further examples such Riemann integration on PCF fractals a-la Strichartz[12], stochastic Itô integration, and more, can similarly be defined as special cases of IA integration. Note that while both of the

above examples used measure codomains corresponding to unsigned measures for simplicity's sake, as will be needed in the next section, measures in this paper are generally meant as a generalization of signed measures.

From this point on in the paper, only IA integrals will be considered, as their construction naturally avoids problems with integration on sets of infinite measure.

3 A Topological Construction of Integrable Chain Complexes

3.1 Measures, functions, orientation, and IA integration on integrable chain spaces of a topological space

3.1.1 Measures and IA integration on subspace topologies

Given a topological space X, sometimes it makes sense to define a non-trivial "size", aka measure, on subsets of X that are either not members of the Borel σ -algebra Σ_X , or may have zero measure for all non-singular measures on Σ_X . For example, the Lebesgue measure of a smooth 1D curve c embedded in \mathbb{R}^n is zero under the standard \mathbb{R}^n topology, but one may still want a non-trivial "length" measure of the curve. For this purpose, I will now define:

Definition 16

Given a topological space X and a subset $S \subseteq X$, the **subspace Borel** σ -algebra of S is the Borel σ -algebra Σ_S generated by the subspace topology τ_S^X induced on S by X.

From there, all previous definitions and resulting properties of measures can be readily applied, simply replacing Σ_X with Σ_S . As such, I will simply denote:

Notation 5

For a subset $S \subseteq X$ of a topological space X, the space of measures on the subspace Borel σ -algebra of S is denoted as $\Xi(\Sigma_S, \overline{M}_S)$.

For example, for a 1D curve c embedded in \mathbb{R}^n , the Lebesgue measure under the standard \mathbb{R}^n topology will give $\mu_{\mathbb{R}^n}(c) = 0$, but a suitable Lebesgue measure under the subspace topology of c in \mathbb{R}^n will give the arclength of c.

Remark 1

If a subset $U \subseteq X$ is an open set of τ_X , then $\Sigma_U = \{U \cap S \mid S \in \Sigma_X\}$. In such a case, the subspace Borel σ -algebra of U is simply the restriction of Σ_X to U. Thus, subsets $S \subseteq X$ with non-empty topological interior in X can have non-zero continuous measure under Σ_X as is, and using Σ_S should generally only be necessary for subsets $S \subseteq X$ whose interior in X is empty.

Going on from defining measures on subspace topologies, integration of a function $f: S \to Y$ on a subset $S \subseteq X$ can be defined relative to the subspace Borel σ -algebra Σ_S instead of Σ_X , and all the above definitions of general integration maps and IA integrations then readily apply.

3.1.2 Integrable chain spaces

For the domains of integration on a topological space, one may be tempted to just use subsets of the the space in question, but such a construction lacks much of the necessary information. First, besides the sets to integrate on, a notion of the orientation of the sets to integrate over is needed - e.g. when integrating around a curve, is one integrating clockwise, or counterclockwise? For this reason, chain complexes are often used to define integration, but they may not always be sufficiently general. What if the "addition" operation underlying the chosen notions of integration is noncommutative, or even nonassociative? What if not just orientation matters, but also the *order of operations*? Hence, a construction similar to but not the same as chain complexes is made to accommodate such cases, to encode the information not just of orientation, but also of the ordering.

Integrable chains

First, I need to consider the collection of subsets integration is carried over:

Definition 17

Given a topological space X, an **integrable collection** on X is a collection $B \subseteq \mathcal{P}(X)$ of subsets of X, where $\mathcal{P}(X)$ is the power set of X, such that:

- 1. Closure under Borel σ -algebra: If S is a member of B, then all measurable subsets $V \in \Sigma_S$ are also members of B, where Σ_S is as defined in definition 16.
- 2. Closure under finite unions: If U and V are members of B, then $U \bigcup V$ is a member of B.

Notation 6

Note, integrable collections on a topological space x can be easily generated by taking any collection of A subsets of X, taking all of their finite unions, and then taking the subspace Borel σ -algebra on each of resulting subsets of X relative to X. In this setting, A is to be called a **generating collection** of the integrable collection B.

For example, the collection of all subsets of a topological space X that are locally homeomorphic to the unit interval (0,1), corresponding to the set of all smooth 1D paths through X, is a generating collection of an integrable collection on X, allowing to define line integrals on X.

Definition 18

Given a topological space X, an integrable collection B on X, and a non-empty set \mathcal{O} , the corresponding **integrable basis** is the set $\mathcal{B} = \mathcal{O} \times B$, endowed with a function codomain $(Y, +_Y)$, a measure codomain $(\overline{M}, +_M)$, an integration codomain $(\overline{G}, +_G)$, and a corresponding notion of IA integration $\int_S : \mathcal{F}(S, Y) \times \Xi(S, \overline{M}) \to \overline{G}$ with integration element $g: Y \times \overline{M} \to \overline{G}$ on all basic sets $S \in B$.

- 1. The elements of B are called the **basic sets** of \mathcal{B} .
- 2. The elements of \mathcal{O} are called the **orientations** of \mathcal{B} .
- 3. The elements of the \mathcal{B} are called the **oriented sets** of \mathcal{B} and denoted $\iota_k S$, where $\iota_k \in \mathcal{O}$ and $S \in \mathcal{B}$.

Each orientation $\iota_k \in \mathcal{O}$ must correspond to a function $\iota_k : M \to M$ that is continuous in the unital magma topology of M, and there must be a unique orientation symbol $e \in \mathcal{O}$ that corresponds to the identity map on M via $\forall m \in M : e(m) := \mathbf{0}_M +_M m = m +_M \mathbf{0}_M = m$.

Notation 7

In definition 18, the specific choice of orientations \mathcal{O} , function codomain $(Y, +_Y)$, measure codomain $(\overline{M}, +_M)$, integration codomain $(\overline{G}, +_G)$, and IA integration $\int_S : \mathcal{F}(S, Y) \times \Xi(S, \overline{M}) \to \overline{G}$ with integration element $g : Y \times \overline{M} \to \overline{G}$ is called the **choice** of calculus forming \mathcal{B} , and denoted by $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$. The integrable basis \mathcal{B} is said to be formed on a topological space X by the integrable collection B and the choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$.

Now I can form domains of integration from the integrable basis, in a similar fashion to how chains are formed from simplexes in singular homology:

Given a topological space X and an integrable basis \mathcal{B} on X, a \mathcal{B} -integrable chain is a formal magma sum of a finite number of oriented sets:

$$c := \sum_{k=1}^{n} \iota_k S_k \tag{13}$$

Where each $\iota_k S_k \in \mathcal{B}$ is an oriented set, with each ι_k an orientation and each S_k a basic set. The set $S_c := \bigcup_{k=1}^n S_k$ is called the **base set** of c. The magma operation $+_{\mathcal{B}}$ is defined as a purely formal operation, similar to formal group sums but not satisfying any properties other than being a binary operation on oriented sets.

Note, definition 17 guarantees that the base set S_c of a \mathcal{B} -integrable chain c is always a member of the integrable collection B.

Notation 8

Given an integrable basis \mathcal{B} formed on a topological space X by an integrable collection B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$, two \mathcal{B} -integrable chains c_1, c_2 are said to be **over the same** domain and denoted $c_1 \stackrel{S}{\sim} c_2$ if their base sets are the same:

$$c_1 \stackrel{S}{\sim} c_2 \Leftrightarrow S_{c_1} = S_{c_2}$$

Measure, orientation and IA integration on integrable chains

Definition 20

Given an integrable basis \mathcal{B} formed on a topological space X by an integrable collection B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$, the **space of measures** on \mathcal{B} is the bundle defined as follows:

$$\Xi(\mathcal{B}) := \left\{ \left(S, \Xi(S, \overline{M}) \right) \middle| S \in B \right\} \tag{14}$$

Measures on a specific set $S \in B$ will be denoted either as $\mu \in \Xi(S, \overline{M})$ or $(S, \mu) \in (S, \Xi(S, \overline{M}))$, interchangeably.

Definition 21

Given an integrable basis \mathcal{B} formed on a topological space X by an integrable collection B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$, an oriented set $\iota_k S \in \mathcal{B}$, and a measure $\mu \in \Xi(S, \overline{M})$, the corresponding **oriented measure** on $\iota_k S$ is the measure $\iota_k \mu \in \Xi(S, \overline{M})$ defined by

$$\forall A \in \Sigma_S : \iota_k \mu(A) := (\iota_k \circ \mu) (A) \tag{15}$$

Note that $\iota_k\mu(A)$ may be undefined if A is a set of infinite measure with respect to μ , but that doesn't matter in the context of IA-integration against μ , because the integral is computed as a limit of integrals of integrable simple functions, which by definition ensures only sets of finite measure with respect to μ , and thus also finite with respect to $\iota_k\mu$, are considered in the first place.

Remark 2 (A conclusion on the nature of orientation)

If the measure codomain $(\overline{M}, +_M)$ is such that $(M, +_M)$ is a topological group, it makes most sense to have 2 orientation symbols: $\mathcal{O} = \{e, -_M\}$, corresponding to the identity map on M and the group inverse map on M. This can be seen as the reason why there are only two possible orientations of domains of integration in ordinary exterior calculus. Instead of being an underlying property of the topological space X, the possible orientations of subsets of X can rather be seen as a property of the choice of calculus. Specifically, of the part (signed) measures play in defining the choice of calculus over an integrable collection of X.

The above definitions now allow me to define integration on an integrable chain, of a function on its base set against a measure on its base set:

Definition 22

Given a \mathcal{B} -integrable chain $c = \sum_{k=1}^{n} \mathcal{B}^{l} \iota_{k} S_{k}$ of an integrable basis \mathcal{B} formed by an integrable collection B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$ on a topological space X, the **integral** on c of a function $f \in \mathcal{F}(S_{c}, Y)$ against a measure $\mu \in \Xi(S_{c}, \overline{M})$ is defined as follows:

$$\int_{C} g(f, d\mu) := \sum_{k=1}^{n} \int_{S_k} g(f, \iota_k d\mu)$$

$$\tag{16}$$

Where each integral $\int_{S_k} g(f, \iota_k d\mu)$ is carried out with respect to function f restricted to S_k , and the measure $\iota_k \mu$ (see definition 21) restricted to Σ_{S_k} .

Integrable chain spaces

Definition 23

Given an integrable basis \mathcal{B} formed by an integrable collection B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$ on a topological space X, two \mathcal{B} -integrable chains c_1, c_2 are said to be **integration** equivalent if:

- 1. The chains c_1, c_2 are over the same domain a-la notation 8: $c_1 \stackrel{S}{\sim} c_2$
- 2. Denoting $S = S_{c_1} = S_{c_2}$, for all functions $f \in \mathcal{F}(S, Y)$, all measures $\mu \in \Xi(\Sigma_S, \overline{M})$, and any choice of IA-integration:

$$\int_{G} g(f, d\mu) = \int_{G} g(f, d\mu)$$

Meaning that if one integral is undefined, both are undefined, and if one is defined, both are defined and equal each other.

Notation 9

As integration equivalence is an equivalence relation, I denote the integration equivalence class of a chain c as [c]. For reasons that'll be clarified by definition 24 and theorem 2, the integration equivalence classes are to be called C-integrable chains.

Since integration equivalence is an equivalence relation, the topological quotient space can be defined:

Definition 24

Given a topological space X and an integrable basis \mathcal{B} formed on X by an integrable collection B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$, let C be the set of all possible chains of \mathcal{B} . The corresponding **integrable chain space** \mathcal{C} is the topological quotient space of C by the integration-equivalence relation.

Notation 10

Just like the integrable basis \mathcal{B} , the integrable chain space \mathcal{C} is said to be formed on a topological space X by an integrable collection B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$.

Theorem 2

An integrable chain space C formed on a topological space X by an integrable collection B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$ is a unital magma with respect to a magma operation $+_{\mathcal{C}}$ defined via $[c_1] +_{\mathcal{C}} [c_2] := [c_1 +_{\mathcal{B}} c_2]$, for all \mathcal{B} -integrable chains c_1, c_2 . The identity element of $(C, +_{\mathcal{C}})$ is the empty set \emptyset .

Proof: Let X be a topological space, and let \mathcal{C} be an integrable chain space formed on X by an integrable collection B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$. The claims to be proven are:

- 1. For all $[a], [b] \in \mathcal{C}$, and all $a_1, a_2 \in [a], b_1, b_2 \in [b]$: $[a_1 +_{\mathcal{B}} b_1] = [a_2 +_{\mathcal{B}} b_2]$.
- 2. For all $[a] \in \mathcal{C}$: $[a +_{\mathcal{B}} \emptyset] = [\emptyset +_{\mathcal{B}} a] = [a]$.

To prove claim 1, first note that by definition 23, $S_{a_1} = S_{a_2}$ and $S_{b_1} = S_{b_2}$. As such, it must also be that $S_{a_1+\mathcal{B}b_1} = S_{a_2+\mathcal{B}b_2}$. Now denoting $S_a = S_{a_1}$, $S_b = S_{b_1}$, and $S_{a+\mathcal{B}b} = S_{a_2+\mathcal{B}b_1}$, note that again by definition 23 it must be the case that for all functions $f \in \mathcal{F}(S_a, Y)$, $h \in \mathcal{F}(S_b, Y)$ and all measures $\mu \in \Xi(S_a, \overline{M}), \nu \in \Xi(S_b, \overline{M})$, it must be that

$$\int\limits_{a_1}g(f,d\mu)=\int\limits_{a_2}g(f,d\mu)$$

$$\int\limits_{b_1}g(h,d\nu)=\int\limits_{b_2}g(h,d\nu)$$

Thus, for all functions $f \in \mathcal{F}(S_{a+Bb}, Y)$ and all measures $\mu \in \Xi(S_{a+Bb}, \overline{M})$, it must be the case that

$$\int\limits_{a_1} g(f,d\mu) +_G \int\limits_{b_1} g(f,d\mu) = \int\limits_{a_2} g(f,d\mu) +_G \int\limits_{b_2} g(f,d\mu)$$

Taking all of this together, it results that by definitions 22 and 23: $[a_1 +_{\mathcal{B}} b_1] = [a_2 +_{\mathcal{B}} b_2]$, meaning that claim 1 is now proven. To prove claim 2, simply note that by definition 4, the measure of the empty set \emptyset is always $\mathbf{0}_M$. As such, due to definitions 9 and 13, as well as theorem 1, it trivially follows that $\int_{\emptyset} g(f, d\mu) = \mathbf{0}_G$ for any function $f \in \mathcal{F}(\emptyset, Y)$ and any measure $\mu \in \Xi(\emptyset, \overline{M})$. As such, it follows

from definitions 22 and 23, and the fact that \overline{G} is an extended topological unital magma satisfyinh $\forall K \in \overline{G} : K +_G \mathbf{0}_G = \mathbf{0}_G +_G K = K$, that for all $[a] \in \mathcal{C}$: $[a +_{\mathcal{B}} \emptyset] = [\emptyset +_{\mathcal{B}} a] = [a]$, proving claim 2.

Remark 3 (Region-additivity of integrals, commutativity, and noncommutative integrals)

Note that in the definitions 4, 9 and 13, measure and integration (both simple and IA) have not been required to be additive over unions of disjoint domains in the topological space X being measured/integrated over. For integration, this means situations of

$$\int_{A \sqcup LB} g(f, d\mu) \neq \int_{A} g(f, d\mu) +_{G} \int_{B} g(f, d\mu)$$

with A, B disjoint may be allowed. This was because (a) it turned out that property was unnecessary to define integration, and (b) the union operation of sets is commutative, but the magma operations $+_M$ and $+_G$ are allowed in this paper to be noncommutative - the operation of set union doesn't carry the necessary information for order of operations needed in the case of $+_G$ noncommutativity. Further note, that the requirement of integration being additive over union of integration domains can be framed in terms of integration equivalence: Given an integrable basis \mathcal{B} formed on a topological space X by an integrable collection \mathcal{B} and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$, let us say that the integral $\int g$ is **region additive** over the basic sets of B if for all $\iota_* \in \mathcal{O}$, and all \mathcal{B} -integrable chains $c = \sum_{k=1}^n {}_{\mathcal{B}} \iota_* S_k$ such that all S_k are pairwise disjoint, $[c] = [\iota_* S_c]$. It follows trivially that a necessary condition for region additivity is commutativity of the integration "addition" operation $+_G$. This doesn't mean that notions of integration based on noncommutative $+_G$ magma operations are ill-defined - it just means that they won't be region additive the way more familiar notions of integration usually are.

3.2 Integrable chain complexes and disintegration of measures

3.2.1 Integrable chain complexes

Now that integrable chain spaces have been defined, I can define integrable chain complexes in analogy to the traditional definition of chain complexes.

Definition 25

Given a topological space X, an **integrable chain complex** $\{C_n, \partial_n\}$ on X is a sequence of integrable chain spaces $\{\emptyset\}$, $C_0, C_1, ...$, with each C_n formed by an integrable collection B_n and a choice of calculus $(\mathcal{O}_n, Y_n, \overline{M}_n, \overline{G}, \int g_n)$, together with a family of **boundary operators** $\partial_n : C_n \to C_{n-1}$ that satisfy:

1. The boundary operators are unital magma homomorphisms:

$$\forall n \in \mathbb{N}, \forall [c_1], [c_2] \in \mathcal{C}_n : \partial_n \left([c_1] +_{\mathcal{C}_n} [c_2] \right) = \partial_n \left([c_1] \right) +_{\mathcal{C}_{n-1}} \partial_n \left([c_2] \right)$$

2. Given two C_n -integrable chains defined over the same domain as each other, their boundaries must also defined over the same domain as each other:

$$\forall n \in \mathbb{N}, \forall [c_1], [c_2] \in \mathcal{C}_n : [c_1] \stackrel{S}{\sim} [c_2] \Rightarrow \partial_n[c_1] \stackrel{S}{\sim} \partial_n[c_2]$$

3. For all C_n -integrable chains $[c] \in C_n$, the base set $S_{\partial_n[c]}$ is contained in the closure of the base set $S_{[c]}$ under the topology of X:

$$\forall n \in \mathbb{N}, \forall [c] \in \mathcal{C}_n : S_{\partial_n[c]} \subseteq \operatorname{cl}_X S_{[c]}$$

4. The boundary of a boundary is empty:

$$\forall n \in \mathbb{N}, \forall [c] \in \mathcal{C}_{n+1} : (\partial_n \circ \partial_{n+1}) [c] = \emptyset$$

5. The boundary of 0-chains is empty:

$$\forall [c] \in \mathcal{C}_0 : \partial_0[c] = \emptyset$$

Important things to note:

- 1. Each choice of calculus $(\mathcal{O}_n, Y_n, \overline{M}_n, \overline{G}, \int g_n)$ in definition 25 is allowed to have different $(\mathcal{O}_n, Y_n, \overline{M}_n, \int g_n)$ for different values of n, but all must share the same integration codomain \overline{G} .
- 2. Given a sequence of integrable chain spaces $\{\emptyset\}$, \mathcal{C}_0 , \mathcal{C}_1 , ... as in definition 25, it's not in general guaranteed that a corresponding integrable chain complex $\{\mathcal{C}_n, \partial_n\}$ can be defined the question of whether or not suitable non-trivial boundary operators exist is crucial. If they don't, it'll be impossible to define differentiation of functions and differential forms in the manner done in this paper.

Notation 11

Given an integrable chain complex $\{C_n, \partial_n\}$ on a topological space X, for any $k \in \mathbb{N}$, the C_k -integrable chains belonging to C_k are called the k-chains of $\{C_n, \partial_n\}$.

Notation 12

Following the notation used in discussion of traditional chain complexes, all integrable chain complexes are said to be **bounded below**, since they don't extend to non-empty integrable chain spaces $C_{-1}, C_{-2}, ...$. Following from this, an integrable chain complex is said to be **bounded above**, and thus fully **bounded**, if there exists some $N \in \mathbb{N}$ such that for all n > N, $C_n = \{\emptyset\}$.

Notation 13

Given a topological space X, an integrable chain complex $\{C_n, \partial_n\}$ is said to be **reduced** if the integrable collection forming C_0 is $B_0 \subseteq X$. This corresponds to the intuition that 0-chains should represent points of the topological space X. Note, in this context the union of points in X is undefined, so B_0 is vacuously closed under unions. Similarly, the subspace Borel σ -algebra of a point $x \in X$ is simply $\{\emptyset, x\}$.

Remark 4 (Integrable chain complexes define an integer dimension on subsets of a topological space)

Note that definition 25 and notation 11 suggest that integrable chain complexes on a topological space X impose a sort of **integer dimension** on certain subsets of X - specifically, on the base sets of integrable chains in the complex. Given the topological space x, this notion of integer dimension may have different geometric interpretations depending on the choice of integrable chain complex on X.

3.2.2 Measures and disintegrations on integrable chain complexes

Now that I have integrable chain complexes on a topological space, I can define one last crucial ingredient needed to define differentiation of functions on integrable chain complexes - the concept of how measures on a k-chain relate to measures on its boundary.

Definition 26

Given an integrable chain complex $\{C_n, \partial_n\}$ formed on a topological space X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y_n, \overline{M}_n, \overline{G}, \int g_n)$, let

$$\mathcal{U}(\mathcal{B}_n) = \left\{ \left(S, \mathcal{U}(S, \overline{M}_n) \right) \middle| S \in B_n, \mathcal{U}(S, \overline{M}_n) \subseteq \Xi(S, \overline{M}_n) \right\}$$

Be a subset of the space of measures on the integrable bases \mathcal{B}_n , called the **decomposable measures** on \mathcal{B}_n . A sequence of **measure disintegrations** on $\{\mathcal{C}_n, \partial_n\}$ is a sequence of mappings $\pi_n : \mathcal{U}(\mathcal{B}_n) \to \mathcal{U}(\mathcal{B}_{n-1})$, relating the disintegrable measures on \mathcal{C} -integrable chains to corresponding measures on their boundaries:

$$\forall [c] \in \mathcal{C}_n, (S_c, \mu) \in \mathcal{U}(\mathcal{B}_n) : \pi_n \left(S_{[c]}, \mu \right) \in \left(S_{\partial_n[c]}, \mathcal{U}(S_{\partial_n[c]}, \overline{M}_{n-1}) \right)$$

Where $S_{[c]}$ is the base set of [c], and $S_{\partial_n[c]}$ is the base set of $\partial_n[c]$. Note, condition 2 in definition 25 guarantees that $S_{\partial_n[c]}$ is well defined. If the C-integrable chain [c] is known from context, then I will also simply denote $\pi_n \mu := \pi_n(\mu) \in \mathcal{U}(S_{\partial_n[c]}, \overline{M}_{n-1})$, accordingly.

Corollary 4 (The measure disintegration of a measure disintegration is the zero measure)

Note, due to the fact that by definition 25 the boundary of a boundary is empty a-la

$$\partial_n \circ \partial_{n+1} = \emptyset$$

It results from definitions 4 and 26 that the measure disintegration of a measure disintegration is the zero measure (unital magma identity element) of the corresponding measure codomain:

$$\forall [c] \in \mathcal{C}_{n+1}, (S_c, \mu) \in \mathcal{U}(\mathcal{B}_{n+1}) : (\pi_n \circ \pi_{n+1}) \left(S_{[c]}, \mu \right) = (\emptyset, \mathbf{0}_{M-1})$$

Notation 14

An integrable chain complex (C_n, ∂_n) formed on a topological space X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y_n, \overline{M}_n, \overline{G}, \int g_n)$, the integrable chain complex is said to be **measured**, and denoted as (C_n, ∂_n, π_n) , if a sequence of disintegrations $\pi_n : \mathcal{U}(\mathcal{B}_n) \to \mathcal{U}(\mathcal{B}_{n-1})$ exists on it as per definition 26.

Notation 15

Given a base set $S_{[c]}$ of a \mathcal{C}_n -integrable chain [c] in a measured integrable chain complex $(\mathcal{C}_n, \partial_n)$, the notation " $\mathcal{U}\left(S_{[c]}, \overline{M}_{n+1}\right)$ " denotes specifically the decomposable measures that are defined on $\Sigma_{S_{[c]}}$.

4 The Exterior Calculus on Integrable Chain Complexes

4.1 Differential forms on an integrable chain space

Given an integrable chain space \mathcal{C} formed on a topological space X by an integrable collections B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$, consider the set of all mappings $\alpha: \mathcal{C} \to \overline{G}$, taking in \mathcal{C} -integrable chains and returning values in \overline{G} . That set may be considered the set of all differential forms on \mathcal{C} , and integration of a differential form α on a \mathcal{C} -integrable chain [c] may be defined via $\int_{\mathbb{R}^d} \alpha := \alpha([c])$.

However, that set is a bit too large to deal with, so in this section I will now form a more tractable set of differential forms, whose properties are easier to digest.

Definition 27

Given an integrable chain space \mathcal{C} formed on a topological space X by an integrable collections B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$, a **basic differential form** is a mapping $\alpha : \mathcal{C} \to \overline{G}$ that satisfies the following conditions:

- 1. $\forall [c_1], [c_2] \in \mathcal{C} : \alpha([c_1] +_{\mathcal{C}} [c_2]) = \alpha([c_1]) +_{\mathcal{G}} \alpha([c_2])$
- 2. For all $[c] \in \mathcal{C}$, there exists a function $f \in \mathcal{F}(S_{[c]}, Y)$ and a measure $\mu \in \Xi(S_{[c]}, \overline{M})$ such that $\alpha([c]) = \int_{[c]} g(f, d\mu)$.

The function f and measure μ taken together are called a **representation** of α on [c].

3. For all $[a], [b] \in \mathcal{C}$ such that $S_{[a]} \subseteq S_{[b]}$, the representation of α on [a] equals the restriction to $S_{[c]}$ of the representation of α on [b]. Meaning, if α is represented on [b] via a function $f \in \mathcal{F}(S_{[b]}, Y)$ and a measure $\mu \in \Xi(S_{[b]}, \overline{M})$, then α is represented on [a] via the restriction of f to $S_{[a]}$, and the restriction of μ to $\Sigma_{S_{[a]}}$.

Definition 28

Given an integrable chain space \mathcal{C} formed on a topological space X by an integrable collections B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$, and two basic differential forms α_1, α_2 on \mathcal{C} , the **sum** of α_1, α_2 is defined as follows:

$$\forall [c] \in \mathcal{C} : (\alpha_1 +_{\Omega} \alpha_2)([c]) := \alpha_1([c]) +_G \alpha_2([c])$$

Definition 29

Given an integrable chain space \mathcal{C} formed on a topological space X by an integrable collections B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$, the **differential form space** on \mathcal{C} is the set $\Omega(\mathcal{C})$ of all mappings $\omega : \mathcal{C} \to \overline{G}$ that can be formed as a finite sum of basic differential forms:

$$\exists n \in \mathbb{N}, \forall [c] \in \mathcal{C} : \omega([c]) = \left(\sum_{k=1}^{n} \alpha_k\right)([c]) = \sum_{k=1}^{n} \alpha_k([c])$$

Where each α_k is a basic differential form on \mathcal{C} . The elements of $\Omega(\mathcal{C})$ are called the **differential** forms on \mathcal{C} .

Theorem 3

Given an integrable chain space \mathcal{C} formed on a topological space X by an integrable collections B and a choice of calculus $(\mathcal{O}, Y, \overline{M}, \overline{G}, \int g)$, the differential form space $\Omega(\mathcal{C})$ defined in definition 29 is a unital magma under the sum operation defined in 28, with an identity element $\mathbf{0}_{\Omega}$ defined via $\forall [c] \in \mathcal{C} : \mathbf{0}_{\Omega}([c]) = \mathbf{0}_{G}$

The proof is trivial, as the sum of any two differential forms is itself a finite sum of basic differential forms and is thus a differential form as well, and $\mathbf{0}_{\Omega}$ is a basic differential form easily seen to be the identity element via $\forall [c] \in \mathcal{C}, \omega \in \Omega(\mathcal{C}) : (\omega +_{\Omega} \mathbf{0}_{\Omega}) ([c]) = \omega([c]) +_{G} \mathbf{0}_{\Omega}([c]) = \omega([c]) +_{G} \mathbf{0}_{G} = \omega([c]),$ and much the same with $(\mathbf{0}_{\Omega} +_{\Omega} \omega) ([c]) - \omega([c])$.

4.2 The exterior derivative and integrable cochain complexes

Now that I have differential form spaces, I can easily define the exterior derivative by taking Stokes' theorem as an axiom, and form the resulting integrable cochain complex.

Definition 30

Given an integrable chain complex $\{C_n, \partial_n\}$ formed on a topological space X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y_n, \overline{M}_n, \overline{G}, \int g_n)$, and given the differential form spaces Ω_n on $\{C_n, \partial_n\}$, the **explicit exterior derivative operators** are a sequence of operators $d_n: B_{n+1} \times \Omega_n \to \Omega_{n+1}$ defined as follows for all $n \in \mathbb{N}$:

For all $S \in B_{n+1}$ and all $\omega \in \Omega_n$:

$$\forall [c] \in \mathcal{C}_{n+1} : S_{[c]} \subseteq S \Rightarrow d_n [S, \omega] ([c]) := \omega \left(\partial_{n+1} [c] \right)$$

$$\tag{17}$$

Where if $S_{[c]} \nsubseteq S$, then $d_n[S,\omega]([c])$ can be anything.

Note, equation 17 is in fact the generalized Stokes theorem, taken here as axiom to define the exterior derivative operators.

Corollary 5

In the context of definition 30, the requirement needed for $d_n[S,\omega]([c])$ to be defined on a chain $[c] \in \mathcal{C}_{n+1}$ such that $S_c \subseteq S$ is that $\omega(\partial_{n+1}[c])$ is a well defined value in \overline{G} .

Notation 16

In the context of definition 30 and corollary 5, given a basic set $S \in B_{n+1}$, it is denoted that a differential form $\omega \in \Omega_n$ is **exterior differentiable** on S if for all $[c] \in \mathcal{C}_{n+1}$ such that $S_c \subseteq S$, $\omega (\partial_{n+1}[c])$ is defined in \overline{G} . If a differential form $\omega \in \Omega_n$ is differentiable on all basic sets $S \in B_{n+1}$, then it's **differentiable everywhere**.

Corollary 6

For all $S, T \in B_{n+1}$ such that $S \subseteq T$, all differential forms $\omega \in \Omega_n$ that are exterior-differentiable on T, and all [c] such that $S_{[c]} \subseteq S$:

$$d_n\left[S,\omega\right]\left([c]\right) = \omega\left(\partial_{n+1}[c]\right) = d_n\left[T,\omega\right]\left([c]\right)$$

Definition 31

Given an integrable chain complex $\{C_n, \partial_n\}$ formed on a topological space X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y_n, \overline{M}_n, \overline{G}, \int g_n)$, given the differential form spaces Ω_n on $\{C_n, \partial_n\}$, and given a sequence of explicit exterior derivative operators $d_n : B_{n+1} \times \Omega_n \to \Omega_{n+1}$, the **exterior derivative operators** are a sequence of operators $d_n : \Omega_n \to \Omega_{n+1}$ defined as follows:

$$\forall \omega \in \Omega_n, [c] \in \mathcal{C}_{n+1} : d_n \omega([c]) := d_n \left[S_{[c]}, \omega \right] ([c])$$

Corollary 7

(The exterior derivative of an exterior derivative is trivial) Given an integrable chain complex $\{C_n, \partial_n\}$ formed on a topological space X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y_n, \overline{M}_n, \overline{G}, \int g_n)$, given the differential form spaces Ω_n on $\{C_n, \partial_n\}$, and given a sequence of exterior derivative operators $d_n : \Omega_n \to \Omega_{n+1}$, the exterior derivative of an exterior derivative is trivial:

$$\forall [c] \in \mathcal{C}_{n+2}, \omega \in \Omega_n : (d_{n+1} \circ d_n) \, \omega([c]) = d_n \omega \, (\partial_{n+2}[c])$$

$$= \omega \, ((\partial_{n+1} \circ \partial_{n+2}) \, [c])$$

$$\stackrel{*}{=} \omega(\emptyset)$$

$$\stackrel{**}{=} \mathbf{0}_G$$

Where in (*) I used definition 25, and in (**) I used definitions 4,27, and 29.

Definition 32

Given an integrable chain complex $\{C_n, \partial_n\}$ formed on a topological space X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y_n, \overline{M}_n, \overline{G}, \int g_n)$, the collection (Ω_n, d_n) of differential form spaces taken together with the exterior derivative operators is said to be the **integrable cochain complex** formed on X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y_n, \overline{M}_n, \overline{G}, \int g_n)$.

5 The Function Calculus on Integrable Chain Complexes

To consider differentiation of functions defined on integrable chain complexes, I can now use the exterior derivatives of basic differential forms, together with the concept of a measured integrable chain complex. While usually differentiation of functions is first defined, and used to define disintegration of measures as is the purview of disintegration theorems a-la Pachl[2], my approach will be going the other way around; Using measure disintegrations to define the differentiation of functions. First, however, some notation I'll need to move forward:

Notation 17

An integrable chain complex $\{C_n, \partial_n\}$ formed on a topological space X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y_n, \overline{M}_n, \overline{G}, \int g_n)$ is said to be **f-monovalued** if $Y_n = Y$ are the same function codomain for all $n \in \mathbb{N}$.

Now, to define the derivatives of functions:

Definition 33

Given a measured f-monovalued integrable chain complex $\{C_n, \partial_n, \pi_n\}$ formed on a topological space X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y, \overline{M}_n, \overline{G}, \int g_n)$, and given the closure $\overline{S} := \operatorname{cl}_X S$ of a basic set $S \in B_{n+1}$, a function $D_n f \in \mathcal{F}(\overline{S}, Y)$ is said to be a D_n -derivative of a function $f \in \mathcal{F}(\overline{S}, Y)$ if the following relation holds:

$$\forall [c] \in C_{n+1}\left(\overline{S}\right), \mu \in \mathcal{U}\left(S_{[c]}, \overline{M}_{n+1}\right) : \int_{[c]} g_{n+1}\left(D_n f, \mu\right) = \int_{\partial_{n+1}[c]} g_n\left(f, d\pi_{n+1} \mu\right)$$
(18)

Where $C_{n+1}\left(\overline{S}\right)$ is the set of all C_{n+1} -integrable chains that satisfy $S_{[c]}\subseteq S$, $\mathcal{U}\left(S_{[c]},\overline{M}_{n+1}\right)$ is the set of decomposable measures on $S_{[c]}$ as denoted in 15, the integral $\int\limits_{[c]}g_{n+1}\left(D_{n}f,\mu\right)$ is done with $D_{n}f$ restricted to $S_{[c]}$,

and the integral $\int_{\partial_{n+1}[c]} g_n(f, \pi_{n+1}\mu)$ is done with f restricted to $S_{\partial_{n+1}[c]}$.

Note, just like eq. 17 in definition 30, eq. 18 in definition 33 is once again the generalized Stokes theorem taken as axiom, this time to define the derivative of a function.

Given a measured f-monovalued integrable chain complex $\{C_n, \partial_n, \pi_n\}$ formed on a topological space X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y, \overline{M}_n, \overline{G}, \int g_n)$,

let $S_n := \{ \operatorname{cl}_X S | S \in B_n \}$ be the set of X-closures of the basic sets of B_n .

A family of derivative operators

$$D_n: \left(\bigcup_{S \in \mathcal{S}_n} \mathcal{F}\left(S, Y\right)\right) \to \left(\bigcup_{S \in \mathcal{S}_{n+1}} \mathcal{F}\left(S, Y\right)\right)$$

is defined on $\{C_n, \partial_n, \pi_n\}$ as follows:

For all $S \in \mathcal{S}_n$, and all $f \in \mathcal{F}(\overline{S}, Y)$, if $D_n f$ is defined, it is a D_n -derivative of f on S as per definition 33.

Notation 18 (The ply and order of derivatives - differentiability classes)

In this context, a function $f \in \mathcal{F}(\overline{S}, Y)$ for which $D_n f$ is defined is to be called a

 D_n -differentiable function on \overline{S} . For each D_n , the integer subscript n is to be called the **ply** of the derivative. This is to distinguish the ply from the unrelated notion of the **order** of a derivative, which is to refer to the amount of times D_n is applied to a function $f \in \mathcal{F}(\overline{S}, Y)$.

For example, $D_n^2 f := (D_n \circ D_n) f$ is the **second order** D_n -**derivative** of f. D_n -**Differentiability** classes can then be defined, as sets of functions that have kth order D_n -derivatives.

Corollary 8 (Connection to exterior calculus)

Given a measured f-monovalued integrable chain complex $\{C_n, \partial_n, \pi_n\}$ formed on a topological space X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y, \overline{M}_n, \overline{G}, \int g_n)$, and given a function $f \in \mathcal{F}(\overline{S}, Y)$ with $\overline{S} \in \mathcal{S}_{n+1}$, consider any differential form $\alpha \in \Omega_n$ that satisfies

$$\forall [c] \in \mathcal{C}_{n+1} : S_{[c]} \subseteq S \Rightarrow \alpha \left(\partial_{n+1}[c] \right) = \int_{\partial_{n+1}[c]} g_n \left(f, d\pi_{n+1} \mu \right)$$

For some $\mu \in \mathcal{U}\left(S_{[c]}, \overline{M}_{n+1}\right)$ where the integral $\int_{\partial_{n+1}[c]} g_n\left(f, d\pi_{n+1}\mu\right)$

is done with f restricted to $\partial_{n+1}[c]$.

Now consider the basic differential form $\alpha' \in \Omega_{n+1}$ that satisfies

$$\forall [c] \in \mathcal{C}_{n+1} : S_{[c]} \subseteq S \Rightarrow \alpha'([c]) = \int_{[c]} g_{n+1}(D_n f, \mu)$$

Where the integral $\int_{[c]} g_{n+1}(D_n f, \mu)$ is done with f restricted to $S_{[c]}$.

Clearly, $\alpha' = d_n \left[\overline{S}, \alpha \right]$ is the exterior derivative of α on \overline{S} .

From here on, given a measured f-monovalued integrable chain complex $\{C_n, \partial_n, \pi_n\}$ formed on a topological space X by integrable collections B_n and choices of calculus $(\mathcal{O}_n, Y, \overline{M}_n, \overline{G}, \int g_n)$, further notions of function calculus can be easily defined for the derivative operators D_n of different ply n, each defined on functions $f \in \mathcal{F}(\overline{S}_n, Y)$, where $\overline{S}_n \in \mathcal{S}_{n+1}$.

Remark 5 (D_n -differentiation as a generalization of n-variables differentiation)

Note, in the standard exterior calculus, 1-chains correspond to 1D curves, making the D_0 operator equivalent to single-variable differentiation. 2-chains correspond to 2D surfaces, making D_1 a 2-variables differential operator. Going on with this logic, it results that:

The above function calculus defines a differential operator D_n , where the ply n serves as a generalization to (n+1)-dimensional domains of functions. This conclusion can be seen as a direct result of remark 4.

6 Conclusions

This paper has done quite a lot, so allow me to summarize:

Conclusion 1

Section 2 defined a highly abstract notion of integration, of functions taking value in general topological unital magmas against measures taking value in general extended (compactified) topological unital magmas, on the Borel σ -algebra of a general topological space. The Riemann integral, geometric product integral, and Lebesgue integral have all been shown as special cases of the IA integration defined in this paper.

Conclusion 2

Section 3 defined notions reminiscent of the traditional definition of chains in a chain complex, specifically suited to integration that may be noncommutative and nonassociative. Integration on these "integrable chains" was defined, and it's been shown that the collection of such integrable chains, aka the integrable chain space, is endowed with a unital magma structure by the choice of calculus on a topological space. Furthermore, **integrable chain complexes** and further constructions on them were also defined, to accommodate the exterior calculus and function calculus defined in sections 4 and 5 after.

Conclusion 3

Also of note in section 3 are remark 2 on a conclusion as to the nature of orientations of sets in a chain, remark 3 on a certain property of noncommutative integrals, and remark 4 as to how integrable chain complexes define an integer dimension on subsets of a topological space.

Conclusion 4

Section 4 defined notions of differential forms, exterior derivatives, and integrable cochain complexes for this general setting of integration based in unital magma operations, generalizing standard theory on the topic.

Conclusion 5

Section 5 has done the same for function calculus, defining the derivatives of functions defined on certain closed subsets of a topological space. In particular, a differential operator has been found reminiscent to the notion of derivatives of functions with n variables, see remark 5.

In doing all of the above, this paper is hoping to generalize standard theories of integration, function calculus, and exterior calculus, in a fairly general manner that as far as the author is aware, has yet to be explored in literature. It is not a complete theory, as questions of existence, uniqueness, and well-definiteness have not been explored due to time constraints. It is, however, the skeleton of one, and can still be used to derive some fairly general conclusions, as well as a helpful start to defining specific novel forms of calculus in practical applications.

7 Further work

There are several options for further work on this topic:

- 1. Fleshing out the theory outlined in this paper:
 - What are the conditions needed to for an integrable chain complex to be well defined on a topological space, given the integrable collections and choices of calculus on it? What are the conditions for boundary operators defined in section 3 to exist, in particular?
 - Given a measured f-monovalued integrable chain complex, what are the conditions needed for existence and uniqueness of the derivatives of functions defined in section 5?
- 2. Developing the general theory further:
 - Can more structures of exterior calculus be derived in this general setting, or are additional assumptions needed for it?

- What does the chain rule look like in this setting? Can structures such as the codifferential, Laplacian, Hodge star, etc, be defined? Can notions of orthogonality and other key structures be described in this general setting, perhaps through some generalization of integral inner products?
- 3. Since differentiable functions and differentiability classes have been successfully defined in this paper, can diffeomorphisms be? If yes, can this be used to generalize notions of differentiable manifolds to more novel notions of local calculus?
- 4. Duality between functions and measures if the same basic differential form can be represented by different function-measure pairs, can that be used to find useful dualities between functions and measures? Can the choices of calculus be varied, keeping only the integration codomain fixed, to transform between different kinds of calculus? For example, can a duality between a calculus of rough functions on smooth domains and a calculus of smooth functions on rough domains be reached?
- 5. Attempting to adapt J. Harrison[23]'s ideas of taking limits of chains to the setting of this theory Harrison successfully used limits of chains² to derive in a unified setting the exterior calculi of many topological spaces embedded in \mathbb{R}^n , including highly singular and nonsmooth spaces including discrete calculus and calculus on fractals. Can a wider notion be reached via limits of integrable chains, of induced exterior calculi on topological spaces embedded in a topological space some form of calculus was defined on via the theory in this paper?
- 6. Specific applications, defining integration and exterior calculus on novel topological spaces, defining novel forms of differential and exterior calculus in general, and more. As the references of this paper imply, novel forms of calculus are aplenty, and it is my belief that the theory developed here may be applicable in some capacity to many of them, as well as to generalizing and gaining new perspectives on quite a few existing theories built on top of differential calculus.

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²Recall, both my definition of integrable chains and the standard definition of chains in a chain complex require them to be made of a finite sum of oriented sets - J. Harrison[23] found that further structures of exterior calculus can be derived, under certain conditions, by taking the limit of these sums from finite to infinite.

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