# An Axiomatisation of Error Intolerant Estimation

Michael Brand\*

**Abstract.** Point estimation is a fundamental statistical task. Given the wide selection of available point estimators, it is unclear, however, what, if any, would be universally-agreed theoretical reasons to generally prefer one such estimator over another. In this paper, we define a class of estimation scenarios which includes commonly-encountered problem situations such as both "high stakes" estimation and scientific inference, and introduce a new class of estimators, Error Intolerance Candidates (EIC) estimators, which we prove is optimal for it.

EIC estimators are parameterised by an externally-given loss function. We prove, however, that even without such a loss function if one accepts a small number of incontrovertible-seeming assumptions regarding what constitutes a reasonable loss function, the optimal EIC estimator can be characterised uniquely.

The optimal estimator derived in this second case is a previously-studied combination of maximum a posteriori (MAP) estimation and Wallace-Freeman (WF) estimation which has long been advocated among Minimum Message Length (MML) researchers, where it is derived as an approximation to the information-theoretic Strict MML estimator. Our results provide a novel justification for it that is purely Bayesian and requires neither approximations nor coding, placing both MAP and WF as special cases in the larger class of EIC estimators.

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### 1 Introduction

One of the fundamental problems in statistics is point estimation: even when the distribution of the observed data given any potential hypothesis is fully known, one typically still needs to select one hypothesis as one's ultimate "decision" given the data.

The reason for the need to select a single hypothesis is often for simplicity of presentation or for pragmatic reasons such as to streamline downstream decision-making, but it can also have deeper justifications. For example, scientific inference explicitly aims to decide on a single theory that best fits the known set of experimental results (Gauch Jr, 2012).

While much of statistical learning theory deals with optimisation algorithms that select the learnt hypothesis, there is no universal agreement on the question of what these algorithms should optimise for.

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 $<sup>{\</sup>rm *School\ of\ Computing\ Technologies,\ RMIT\ University,\ michael.brand@rmit.edu.au}$ 

R.A. Fisher famously advocated for likelihood maximisation (Aldrich, 1997),

$$\hat{\theta}_{\text{MLE}}(x) \stackrel{\text{def}}{=} \underset{\theta \in \Theta}{\operatorname{argmax}} f(x|\theta),$$

but other optimisation criteria have since been in wide use, such as the wider class of extremum estimators (Hayashi, 2011). Even among users of maximum likelihood estimation (MLE), most, today, will use Penalised MLE (PMLE) (Cole et al., 2014),

$$\hat{\theta}_{\text{PMLE}(g)}(x) \stackrel{\text{def}}{=} \underset{\theta \in \Theta}{\operatorname{argmax}} [f(x|\theta)g(\theta)],$$

in order to provide regularisation. However, different researchers will choose distinct penalty functions, g, leading to algorithms such as LASSO regression (Tibshirani, 1996) when using  $L_1$  regularisation, ridge (Tikhonov) regression (Hoerl and Kennard, 1970) when using  $L_2$  regularisation, or elastic net (Zou and Hastie, 2005) when using both. These are examples of regularisation techniques that have additional parameters that require tuning. Other methods, such as the Akaike information criterion (AIC) (Akaike, 1998; Bozdogan, 1987) and the Bayesian information criterion (BIC) (Schwarz, 1978), do not require further parameters, but do require a level of semantic understanding of the estimation problem in order to determine, e.g., the number of observations and the number of parameters estimated by the model, which are not necessarily straightforward to determine in the general case.

Moreover, the methods cited above are frequentist methods. In a Bayesian setting, one is provided with the additional information of the prior distribution over the hypothesis space, opening the door to a much wider array of possibilities. Maximum a posteriori (MAP) (DeGroot, 2005), for example, maximises the posterior probability (or probability density) of the chosen hypothesis, whereas Bayes estimation (Berger, 2013) minimises an expected loss function.

Other methods are defined in ways that do not bring a goodness-of-fit function to an extremum. Two such examples are posterior expectation (Carlin and Louis, 2010) and minimum message length (MML) (Wallace, 2005). Notably, while neither is defined via a goodness-of-fit extremum, posterior expectation can at least be re-defined equivalently as a Bayes estimator over the quadratic loss function. MML, by contrast, optimises an information criterion that cannot be formulated as an extremum estimator at all, even in a Bayesian sense.

When faced with this wide array of potential methods to choose a single point estimate given exactly the same data, exactly the same hypotheses and exactly the same priors, a natural question to ask is whether there is any disciplined, systematic method that one can use in order to determine a single "best" point estimate.

For one commonly-encountered type of estimation scenarios, which we refer to as *error intolerant estimation*, we show that such a method exists and develop it in this paper. For this, we use the Axiomatic Method (Thomson, 2001), a powerful mathematical technique that has proved successful in determining a "best" solution in many problems that otherwise appear to require qualitative judgement. (For examples, see Von Neumann and Morgenstern (1947); Nash (1950); Shapley (1953).)

The essence of the Axiomatic Method is that one states an explicit set of assumptions that appear to be incontrovertible, in the sense that each appears to be an essential minimal criterion that any solution that purports to be at all "good" must satisfy. One then proves that there exists only a single, unique solution that satisfies all desiderata, making it, therefore, the only possible "best" solution.

Axiomatisation has already proved to be a powerful tool in providing an underlying theory for multiple aspects of Bayesian analysis (Dupré and Tipler, 2009; Majumdar, 2004), and specifically for understanding inference (Myung et al., 2005; Geiger et al., 1991; Wolfowitz, 1962; Marin et al., 2022). Here, we use it to create a unified framework, applicable to both discrete (inference) and continuous (estimation) problems.

#### 1.1 Notation, assumptions and basic definitions

#### **Estimation problems**

In a Bayesian setting, an estimation problem is a pair of random variables  $(\boldsymbol{x}, \boldsymbol{\theta})$  with a known joint distribution that has some  $X \times \Theta$  as its support. I.e., it assigns positive probability,  $\mathbf{P}(x,\theta) = \mathbf{P}(\boldsymbol{x}=x,\boldsymbol{\theta}=\theta)$ , or otherwise positive probability density,  $f(x,\theta) = f^{(\boldsymbol{x},\theta)}(x,\theta)$ , to any  $(x,\theta) \in X \times \Theta$  (and analogously also for other cases, such as where  $\boldsymbol{\theta}$  is continuously distributed but  $\boldsymbol{x}$  only takes values in a discrete set). In addition to sharing the common support X, the conditional data distributions  $P_{\theta} = P_{\boldsymbol{x}|\boldsymbol{\theta}=\theta}$  (sometimes referred to simply as "data distributions") in an estimation problem are assumed to all be distinct and absolutely continuous with respect to each other. We refer to x as the observation,  $X \subseteq \mathbb{R}^N$  as observation space,  $\theta$  as the parameter and  $\Theta \subseteq \mathbb{R}^M$  as parameter space.

Throughout, we use the following notation. Boldface symbols ("x") indicate random variables. Regular symbols ("x") indicate variable instantiations. For continuous x,  $f_{\theta} = f_{\theta}^{(x,\theta)}$  is the probability density function (pdf) of the data distribution  $P_{\theta}$  (i.e.,  $f_{\theta}^{(x,\theta)}(x) \stackrel{\text{def}}{=} f^{(x,\theta)}(x|\theta)$ ).  $\Lambda = \Lambda^{(x,\theta)}$  is the prior. (For discrete  $\theta$ ,  $\Lambda(\theta)$  conveys  $\mathbf{P}(\theta = \theta)$ , for continuous —  $f(\theta)$ .)

Where available, we use pdfs interchangeably with the distributions they represent (e.g., " $\boldsymbol{x} \sim f_{\theta}$ " is equivalent to " $\boldsymbol{x} \sim P_{\theta}$ ").

In this paper, we analyse three classes of estimation problems:

**Discrete:** For every  $x \in X$  there exists a  $\theta \in \Theta$ , such that  $\mathbf{P}(\theta = \theta | \mathbf{x} = x) > 0$ .

**Continuous:** Both x and  $\theta$  are continuous random variables, and  $f(x,\theta)$  is well-defined.

**Semi-continuous:** The random variable  $\theta$  is continuous with a well-defined  $f(\theta|x)$  for every  $x \in X$ , but the random variable x is discrete.

We refer to the latter two, collectively, as  $\theta$ -continuous estimation problems.

A problem's *type* will be a description of

- 1. The problem's class (discrete, continuous or semi-continuous), and
- 2. The problem's dimensions, M and N.

For discrete problems we require no further restriction, but for  $\theta$ -continuous estimation problems our analysis, for reasons of mathematical convenience and simplicity of presentation, will be restricted to only problems satisfying the following criteria:

- 1. For all x,  $f(\theta|x)$  is three-times continuously differentiable in  $\theta$ , and
- 2. For continuous problems,  $f(x,\theta)$  is piecewise continuous in x.

We refer to such problems as *elementary* estimation problems.

We note that by "differentiable", we mean that the derivative is finite and well-defined as a standard derivative in an open neighbourhood of every  $\theta \in \Theta$ .

#### Point estimators

A point estimator is a function  $\hat{\theta}: X \to \mathbb{R}^M$ . For example:

- $\hat{\theta}_{\text{Bayes}}^L(x) \stackrel{\text{def}}{=} \operatorname{argmin}_{\theta \in \Theta} \mathbf{E}[L(\theta, \theta) | \boldsymbol{x} = x]$  is the *Bayes estimator* defined over the loss function  $L = L_{(\boldsymbol{x}, \theta)} : \Theta \times \Theta \to \mathbb{R}^{\geq 0}$ .
- $\hat{\theta}_{\text{c-MAP}}(x) \stackrel{\text{def}}{=} \operatorname{argmax}_{\theta \in \Theta} f^{(\boldsymbol{x}, \boldsymbol{\theta})}(\theta | x)$  and  $\hat{\theta}_{\text{d-MAP}}(x) \stackrel{\text{def}}{=} \operatorname{argmax}_{\theta \in \Theta} \mathbf{P}(\theta | x)$  are the continuous and discrete maximum a posteriori (MAP) estimators, respectively. For our purposes, these are treated as entirely separate estimators.

Note that all standard point estimators are defined, like the ones above, by means of an argmin or an argmax. Such functions intrinsically allow estimation results to be a subset of  $\mathbb{R}^M$ , rather than an element of  $\mathbb{R}^M$ . Thus, they are technically set estimators rather than point estimators.

Though our error intolerant estimators will be explicitly defined as point estimators rather than as set estimators, much of our analysis involves estimators that, in the general case, may sometimes return more than a single  $\theta$  value as their estimates. In those cases, the use of point-estimator notation should be considered solely a notational convenience. Our theorems will be worded most generally, however, to account also for set estimation.

In order to express the practical usability of a set estimator, we define the following. We say that an estimator is a well-defined point estimator for  $(\boldsymbol{x}, \boldsymbol{\theta})$  if it returns a single-element set for every  $x \in X$ , in which case we take this element to be its estimate. A general set estimator is considered well-defined on  $(\boldsymbol{x}, \boldsymbol{\theta})$  if it returns a finite-sized, nonempty set as its estimate for every  $x \in X$ .

#### Loss functions

A loss function is a function  $L = L_{(\boldsymbol{x},\boldsymbol{\theta})} : \Theta \times \Theta \to \mathbb{R}^{\geq 0}$ , where  $L(\theta_1,\theta_2)$  represents the cost of choosing  $\theta_2$  when the true value of  $\boldsymbol{\theta}$  is  $\theta_1$ . We assume for all loss functions

$$L(\theta_1, \theta_2) = 0 \Leftrightarrow \theta_1 = \theta_2 \Leftrightarrow P_{\theta_1} = P_{\theta_2}. \tag{1.1}$$

Bayes estimators may also be defined, equivalently, over a *utility function*, G, where G is the result of a monotone strictly decreasing affine transform on a loss function.<sup>1</sup>

We say that a loss function L is discriminative for an estimation problem  $(\boldsymbol{x}, \boldsymbol{\theta})$  if for every  $\theta \in \Theta$  and every open neighbourhood  $\Phi$  of  $\theta$  such that  $\Theta \setminus \Phi \neq \emptyset$ ,  $\inf_{\theta' \in \Theta \setminus \Phi} L_{(\boldsymbol{x},\boldsymbol{\theta})}(\theta',\theta) > 0$ .

We say a loss function L is *smooth* if for every elementary  $\boldsymbol{\theta}$ -continuous estimation problem,  $(\boldsymbol{x}, \boldsymbol{\theta})$ , the function  $L_{(\boldsymbol{x}, \boldsymbol{\theta})}(\theta_1, \theta_2)$  is three times continuously differentiable in  $\theta_1$ .

We say a loss function L is *sensitive* on a particular type of  $\theta$ -continuous estimation problems if there is at least one elementary estimation problem  $(x, \theta)$  of that type and at least one choice of  $\theta_0$ , i and j such that

$$\left.\frac{\partial^2 L_{(\boldsymbol{x},\boldsymbol{\theta})}(\boldsymbol{\theta},\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}(i)\partial \boldsymbol{\theta}(j)}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} \neq 0.$$

Lastly, a loss function L is said to be *problem-continuous* (or " $\mathcal{M}$ -continuous") if for every sequence of elementary continuous estimation problems  $((\boldsymbol{x}_i, \boldsymbol{\theta}))_{i \in \mathbb{N}}$  and elementary continuous estimation problem  $(\boldsymbol{x}, \boldsymbol{\theta})$  such that at every  $\theta \in \Theta$  the sequence satisfies  $(f_{\theta}^{(\boldsymbol{x}_i, \boldsymbol{\theta})}) \xrightarrow{\mathcal{M}} f_{\theta}^{(\boldsymbol{x}, \boldsymbol{\theta})}$ , it is true that for every  $\theta_1, \theta_2 \in \Theta$ ,

$$\lim_{i\to\infty} L_{(\boldsymbol{x}_i,\boldsymbol{\theta})}(\theta_1,\theta_2) = L_{(\boldsymbol{x},\boldsymbol{\theta})}(\theta_1,\theta_2).$$

The symbol " $\stackrel{\mathcal{M}}{\longrightarrow}$ ", used above, indicates convergence in measure (Halmos, 2013). This is defined as follows. Let  $\mathcal{M}$  be the space of normalisable, non-atomic measures over some  $\mathbb{R}^s$ , let f be a function  $f: \mathbb{R}^s \to \mathbb{R}^{\geq 0}$  and let  $(f_i)_{i \in \mathbb{N}}$  be a sequence of such functions. Then  $(f_i) \stackrel{\mathcal{M}}{\longrightarrow} f$  if

$$\forall \epsilon > 0, \lim_{i \to \infty} \mu(\{x \in \mathbb{R}^s : |f(x) - f_i(x)| \ge \epsilon\}) = 0, \tag{1.2}$$

where  $\mu$  can be, equivalently, any measure in  $\mathcal{M}$  whose support is at least the union of the support of f and all  $f_i$  (recalling that all measures in  $\mathcal{M}$  are normalisable and non-atomic).

<sup>&</sup>lt;sup>1</sup>Note that loss functions and utility functions are always indexed by the estimation problem, even when we omit this subscript as shorthand. A single "loss function" represents a separate function for each estimation problem. We do not require every loss function to necessarily be applicable to every possible estimation problem. Rather, when discussing point estimators defined over a specific loss function, we narrow our discussion to a particular class of estimation problems. The loss function is then expected to be defined over that particular class.

We will usually take f and all  $f_i$  to be pdfs. When this is the case,  $\mu$ 's support only needs to equal the support of f. Furthermore, because  $\mu$  is normalisable, one can always choose a value t such that  $\mu(\{x:f(x)>t\})$  is arbitrarily small, for which reason one can substitute the absolute difference " $\geq \epsilon$ " in (1.2) with a relative difference " $\geq \epsilon f(x)$ ", and reformulate it in the case that f and all  $f_i$  are pdfs as

$$\forall \epsilon > 0, \lim_{i \to \infty} \mathbf{P}_{\boldsymbol{x} \sim f}(|f(\boldsymbol{x}) - f_i(\boldsymbol{x})| \ge \epsilon f(\boldsymbol{x})) = 0.$$
(1.3)

This reformulation makes it clear that convergence in measure over pdfs is a condition independent of the representation of the random variable. I.e., it is invariant to transformations of the kind defining the IRO axiom in Section 3.1.

Not all commonly used loss functions satisfy discriminativity, smoothness, sensitivity and problem continuity, for which reason these properties should not be taken as general "well-behavedness" criteria. Nevertheless, in specific claims we allow ourselves to restrict our choice of loss function somewhat in favour of ensuring the function's good behaviour. These are assumptions made for mathematical convenience.

#### 1.2 The plan, and a detailed result overview

This paper is divided into two parts.

In the first part (Section 2) we define the error intolerant scenario and introduce "Axiom 0" of error intolerant estimation, which we refer to as AIA. This part of the paper combines a definitional approach with an axiomatic approach. The basic notion of error intolerance, which we describe using terminology from econometric theory, is a definition, which we introduce in Section 2.1. It may be objected to, e.g. on the grounds that it is a Bayesian notion, or that error intolerance is defined relative to a loss function. However, once one accepts this basic definition of what we mean by an error intolerant situation, we deem AIA, introduced in Section 2.2, to be an incontrovertible requirement for any reasonable estimator to be applied in such a situation.

In Section 2.3, we introduce a new estimator, which we refer to as the Error Intolerance Candidates (EIC) estimator, that is defined relative to one's choice of loss function. We prove that for both discrete and  $\theta$ -continuous estimation problems, AIA is enough to uniquely characterise EIC. On discrete problems, EIC coincides with discrete MAP, but on  $\theta$ -continuous ones it is a novel estimator and depends on one's choice of loss function.

In Section 2.4, we show that EIC, on its own, is equivalent to PMLE.

By itself, this is quite a weak characterisation, but EIC has one advantage over PMLE, namely that the EIC estimate is completely determined by one's choice of loss function. This is discussed in Section 2.5.

In the second part of the paper (Section 3), which is entirely axiomatic, we explore what would be a reasonable loss function to use in an error intolerant scenario. We begin this in Section 3.1 by introducing 4 axioms that describe incontrovertible-seeming

minimal criteria for such loss functions. Two of these are arguably incontrovertible also in more general estimation scenarios.

Notably, our axioms are not regarding the estimator itself, but rather directly regarding the underlying loss functions. The ability to do this is in itself an assumption of our construction. We refer to it as *the loss principle*, and discuss its own justifications and incontrovertibility in Section 3.2.

In Section 3.3, we prove regarding our first three axioms on loss that together they uniquely characterise a single EIC estimator, no longer parameterised by a loss function, in the continuous case, and that all four axioms on loss together uniquely characterise this estimator also in the semi-continuous case. This EIC estimator is equivalent to the Wallace-Freeman estimator (WF) (Wallace and Freeman, 1987), defined as

$$\hat{\theta}_{\mathrm{WF}}(x) \stackrel{\mathrm{def}}{=} \underset{\theta}{\operatorname{argmax}} \frac{f(\theta|x)}{\sqrt{|\mathcal{I}_{\theta}|}},$$

where  $\mathcal{I}_{\theta}$  is the Fisher information matrix (Lehmann and Casella, 2006), whose (i, j) element is the conditional expectation

$$\mathcal{I}_{\theta}(i,j) \stackrel{\text{def}}{=} \mathbf{E} \left[ \left( \frac{\partial \log f(\boldsymbol{x}|\theta)}{\partial \theta(i)} \right) \left( \frac{\partial \log f(\boldsymbol{x}|\theta)}{\partial \theta(j)} \right) \middle| \boldsymbol{\theta} = \boldsymbol{\theta} \right].$$

We conclude the paper in Section 4, where we follow the conventions of the axiomatic approach, and prove for all our axioms necessity and feasibility.

Necessity is the property that no subset of the axioms suffices for our results of unique characterisation. We show this by providing counterexamples for each potentially-omitted axiom. In particular, we show that the loss principle, despite being merely a guiding principle and not a mathematical axiom, is fundamental to our results. We do this by proving that removing it opens the door even to non-Bayesian estimators, demonstrating by this not only the necessity of the loss principle to our derivation, but its encapsulation of certain aspects of Bayesian thinking.

Feasibility is the property that the set of loss functions meeting our requirements is non-empty. Interestingly, while our characterisation of the optimal error intolerant estimator is unique, we show that the underlying choice of a loss function is not. In fact, many standard loss functions, such as all f-metrics satisfying certain mild criteria, meet our requirements. One interpretation of our results is therefore that while many different loss functions, including standard ones, are reasonable choices in an error intolerant scenario, all ultimately lead to the same estimator: discrete MAP for discrete problems, WF otherwise.

A short conclusion section follows.

Regarding empirical support for our error intolerant estimator, we note that the particular combined usage we describe of discrete MAP and WF has been advocated by MML researchers since at least Comley and Dowe (2005), with recent examples of its empirical success being Bregu et al. (2021b,a); Hlaváčková-Schindler and Plant

(2020); Bourouis et al. (2021); Sumanaweera et al. (2018); Schmidt and Makalic (2016); Saikrishna et al. (2016); Jin and Rumantir (2015); Kasarapu et al. (2014). We do not, in this paper, attempt to add more such examples. See, however, the Supplementary Information (SI), Section B.1 for an analysis of the similarities and the fundamental differences between our error intolerant estimator and that advocated in MML, as well as of the differences in their justifications.

In brief, whereas MML views this estimator as a computationally convenient approximation to the Strict MML point estimator, with no independent justification of its own, and whereas the Strict MML estimator itself is justified through information theoretical means, our derivation is purely Bayesian, requires neither approximations nor coding, and places the combined estimator as a point in the larger continuum of all EIC estimators.

## 2 Rational estimation given error intolerance

#### 2.1 Background

In econometrics (Pratt, 1978), an agent's utility from an outcome can be described either in objective terms (e.g., how many dollars the agent has gained) or subjectively (e.g., how happy they are with their winnings). The agent's risk attitude function is the mapping from objective to subjective utility, and the agent is considered rational if their choices maximise their expected subjective utility. This definition for rationality has been previously axiomatised in many different ways. For examples, see Werner (2005) and references therein. For our purposes, we will take it as given.

Despite the fact that risk attitude functions are not directly observable, they are a critical tool in explaining why rational actors prefer lotteries with a lower win variance over ones with a higher win variance in situations where both lotteries have the same win expectation, and are even willing to trade off win expectation for a decrease in win variance.<sup>2</sup>

A person willing to perform such trade offs is described as *risk averse*, and the rationality of their behaviour is explained by positing that their risk attitude function, unobservable though it is, is concave. Conversely, a person with a convex risk attitude function is described as *risk loving*: they will be willing to pay a premium in order to participate in a lottery with a high win variance.

Switching to the language of loss functions, L, rather than utilities, and taking the risk attitude function, T, from now on to be a mapping converting objective losses to subjective losses, we say that a rational estimation procedure should select for each observed value x the estimate  $\hat{\theta}(x)$  that minimises the expected subjective loss  $\mathbf{E}[T(L(\boldsymbol{\theta}, \hat{\theta}(x)))|x]$ . Formally, we require of such risk attitude functions the following.

**Definition 2.1.** A function,  $T: \mathbb{R}^{\geq 0} \to \mathbb{R}$ , will be called a *risk attitude function* if it is continuously differentiable and monotone weakly increasing.

<sup>&</sup>lt;sup>2</sup>In econometrics, any situation that has a stochastic outcome is referred to as a lottery.

Our aim is to characterise the optimal behaviour in the face of an *error intolerant* scenario. This we define as a situation in which there is only one right answer, and all mistakes are considered equally bad (subjectively). Within the confines of risk-aversion theory, we can approximate such a risk attitude by saying that all losses above some  $t_0$  are equally bad. Formally, we define as follows.

**Definition 2.2.** A risk attitude function T is said to have an *error limit* of  $t_0$  if there exists a  $V_{\text{max}} > T(0)$ , such that for all  $t \geq t_0$ ,  $T(t) = V_{\text{max}}$ .

A risk attitude function's minimal error limit will be called its *error tolerance*.

Informally, we can now say that error *intolerance*, the object of our study, is the scenario characterised by a vanishingly low error tolerance.

To be sure, much of the use of statistical estimation is not in an error intolerant context. A machine-learning algorithm running hundreds of thousands of times to make hundreds of thousands of essentially-interchangeable choices, such as, for example, in bidding for online ads, should optimally choose the strategy that optimises its on-average success, and hence the minimisation of expected loss used within the standard Bayes-estimation framework would be ideal for it.

Such an approach explicitly chooses its estimate,  $\theta$ , so as to be an optimal trade-off between how close it is to other candidate values  $\theta'$  and how likely those other  $\theta'$  values are to be correct. In other words, a  $\theta$  value can be chosen not because  $f(\theta|x)$  is high, but rather because there are sufficient other  $\theta'$  for which  $f(\theta'|x)$  is high, and  $L(\theta',\theta)$  is relatively small. We refer to such estimation methods as trade-off based methods.

Error intolerant estimation, where we deem all errors equally bad, aims explicitly to avoid such trade-offs. We want each  $\theta$  to be judged solely on its own merits, because the aim is to commit to a single hypothesis. Scientific inference is an example of such a scenario. Another example is high-stakes "bet the ranch"-type situations where the costs of any wrong decision are prohibitively high.

For a concrete example, consider a machine learning algorithm whose purpose is to recommend a medication dosage to a hospital patient. A Bayes estimator would recommend a dosage that would average over the recommended dosages for each of the patient's possible diagnoses, with averaging weights determined by diagnosis probability and the severity of alterations over the correct dosage given any such possible diagnosis. By contrast, an error intolerant algorithm would choose its dosage recommendation, and, indeed, its recommendation for the entire treatment regime, based on an underlying choice regarding what specific condition the patient is diagnosed as having, not on a probability cloud of such diagnoses.

Such an algorithm would provide consistent recommendations regarding the patient's care, and, because of its underlying diagnosis, will score better in ease to interrogate (related to explainability), making it likely to be the more trusted choice among medical staff and patients alike.

For more intuition regarding error intolerant estimation and how it explicitly avoids trade-offs, see the discussion in the SI, Section B.2.

#### 2.2 Axiom 0

The purpose of this paper is to define estimators that are suitable for error intolerant scenarios, i.e. for scenarios where one's error tolerance is vanishingly low. Though it is initially unclear how to choose these well, we present minimal criteria that avoid those decisions that are clearly undesirable.

Consider a spectrum of risk attitudes that one might take, each mapped according to its error limit (noting that for the purpose of investigating the error intolerant scenario we can safely ignore any risk attitude that does not have an error limit).

**Definition 2.3.** A set of risk attitude functions  $\mathcal{T} = \{T_{\epsilon}\}_{{\epsilon} \in \mathbb{R}^{>0}}$  where each index value  ${\epsilon}$  is an error limit of its respective  $T_{\epsilon}$  will be called a *risk attitude spectrum*.

For a given loss function L and a given risk attitude spectrum  $\mathcal{T}$ , here are some choices that should clearly be avoided. We avoid them because they present no advantage: they are inferior at *every* low error tolerance.

**Definition 2.4.** An estimate  $\theta_1$  is said to be *error-intolerance inferior* (EI-inferior, as well as "EI-inferior relative to  $\theta_2$ ") given an observation x, a loss function L and a risk attitude spectrum  $\mathcal{T}$ , if there exists a  $\theta_2$  and an  $\epsilon_0 > 0$ , such that for every  $\epsilon \in (0, \epsilon_0]$ ,

$$\mathbf{E}[T_{\epsilon}(L(\boldsymbol{\theta}, \theta_1))|x] > \mathbf{E}[T_{\epsilon}(L(\boldsymbol{\theta}, \theta_2))|x]. \tag{2.1}$$

With this in mind, we formulate our first axiom, which is a minimal admissibility criterion for error intolerant estimators.

#### Axiom 0: Avoidance of Inferior Alternatives (AIA)

An error intolerant estimator over a loss function L should not choose estimates that are always inferior, regardless of one's risk attitude spectrum. Formally: An estimator  $\hat{\theta}$  over a loss function L is said to satisfy AIA if for every  $x \in X$  there exists at least one risk attitude spectrum  $\mathcal{T}$  such that  $\hat{\theta}(x)$  is not EI-inferior at x over L and  $\mathcal{T}$ .

#### 2.3 The EIC estimator

In this section, we show that for a given loss function L, AIA uniquely characterises the following estimator.

**Definition 2.5.** The *Error Intolerance Candidates* (EIC) estimator is defined by

$$\hat{\theta}_{\mathrm{EIC}}^{L}(x) \stackrel{\mathrm{def}}{=} \begin{cases} \operatorname{argmax}_{\theta \in \Theta} \mathbf{P}(\theta|x) & \text{if } \max_{\theta \in \Theta} \mathbf{P}(\theta|x) > 0\\ \operatorname{argmax}_{\theta \in \Theta} \frac{f(\theta|x)}{\sqrt{|H_{H}^{\theta}|}} & \text{otherwise,} \end{cases}$$
(2.2)

where  $H_L^{\theta}$  is the Hessian matrix of L, whose (i,j) element is defined as

$$H_L^{\theta}(i,j) \stackrel{\text{def}}{=} \left. \frac{\partial^2 L(\theta',\theta)}{\partial \theta'(i)\partial \theta'(j)} \right|_{\theta'=\theta}.$$

This estimator equals discrete MAP where discrete MAP is well-defined, irrespective of L, but is a novel estimator, which does depend on the choice of L, for  $\theta$ -continuous problems.

The fact that error intolerance implies discrete MAP for problems where discrete MAP is well-defined is highly intuitive: if the purpose is to maximise the probability of the estimate being "right", and if some possibilities have a positive probability of being right, it makes sense that an error intolerant person would choose the option whose probability of being right is greatest.

We prove this characterisation in Theorem 2.6.

**Theorem 2.6.** Let  $(x, \theta)$  be a discrete estimation problem, and let  $\hat{\theta}_L$  be a point estimator satisfying AIA with respect to a loss function L discriminative for  $(x, \theta)$ . At every  $x \in X$ ,

$$\hat{\theta}_L(x) \in \hat{\theta}_{EIC}^L(x) = \hat{\theta}_{d\text{-}MAP}(x).$$

We remind the reader that even though we use the terminology of point estimation, estimators defined by means of an argmin or an argmax are most generally set estimators. This is why Theorem 2.6, and, indeed, all our claims, are worded so as to admit sets as estimates, despite this requiring a slightly more complex wording to the claims. By contrast,  $\hat{\theta}_L$  is specifically defined to be a point estimator.

Proof. For a choice of a risk attitude spectrum,  $\mathcal{T} = \{T_{\epsilon}\}$ , define  $\mathcal{A} = \{A_{\epsilon} = S_{\epsilon} \circ T_{\epsilon}\}$  (where  $S \circ T$  denotes function composition) such that each  $S_{\epsilon}$  is an affine function mapping  $T_{\epsilon}(0)$  to 1 and  $T_{\epsilon}(\epsilon)$ , the maximum of  $T_{\epsilon}$ , to 0. Minimising the expected loss  $\mathbf{E}[T_{\epsilon}(L(\boldsymbol{\theta}, \boldsymbol{\theta}))|\boldsymbol{x} = x]$  for choosing  $\hat{\theta}_{L}(x) = \boldsymbol{\theta}$  at a given  $\epsilon$  and a given x is equivalent to maximising the expected utility  $\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \boldsymbol{\theta}))|\boldsymbol{x} = x]$ .

Fix x, and let  $V_{\epsilon}(\theta)$  be this utility:

$$V_{\epsilon}(\theta) = \mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \theta))|\boldsymbol{x} = x].$$

Let  $\Phi_{\epsilon}(\theta)$  be the set of  $\theta'$  for which  $A_{\epsilon}(L(\theta',\theta))$  is positive. The value of the expected utility is bounded from both sides by

$$\mathbf{P}(\theta|x) < V_{\epsilon}(\theta) < \mathbf{P}(\boldsymbol{\theta} \in \Phi_{\epsilon}(\theta)|x). \tag{2.3}$$

This is due to  $A_{\epsilon}(0) = 1$  and the range of  $A_{\epsilon}$  being bounded in [0, 1].

Because, by discriminativity of L, for any neighbourhood  $\Phi$  of  $\theta$  there is an  $\epsilon_0$  value such that for every  $\epsilon \in (0, \epsilon_0]$ ,  $\Phi_{\epsilon}(\theta) \subseteq \Phi$ , both bounds converge to  $\mathbf{P}(\theta|x)$ . So, this is the limit for  $V_{\epsilon}(\theta)$ .

To prove the claim, suppose to the contrary that  $\hat{\theta}_L(x) \notin \hat{\theta}_{d\text{-MAP}}(x)$ . Choose  $\theta^* \in \hat{\theta}_{d\text{-MAP}}(x)$ , noting that  $\hat{\theta}_{d\text{-MAP}}(x)$  is never the empty set.

By construction,

$$\lim_{\epsilon \to 0} V_{\epsilon}(\hat{\theta}_L(x)) - V_{\epsilon}(\theta^*) = \lim_{\epsilon \to 0} V_{\epsilon}(\hat{\theta}_L(x)) - \lim_{\epsilon \to 0} V_{\epsilon}(\theta^*) = \mathbf{P}(\hat{\theta}_L(x)|x) - \mathbf{P}(\theta^*|x) < 0,$$

so for a small enough  $\epsilon$ ,  $V_{\epsilon}(\hat{\theta}_L(x)) < V_{\epsilon}(\theta^*)$ , and  $\hat{\theta}_L(x)$  is therefore EI-inferior, contradicting AIA.

The claim for  $\theta$ -continuous problems is similar, with minor additional caveats.

**Theorem 2.7.** Let  $(x, \theta)$  be an elementary  $\theta$ -continuous estimation problem over an open set  $\Theta$ , let L be a smooth loss function discriminative for it, such that  $\hat{\theta}_{EIC}^L$  is a well-defined set estimator on  $(x, \theta)$ , and let  $\hat{\theta}_L$  be a point estimator satisfying AIA with respect to L. For all  $x \in X$ ,

 $\hat{\theta}_L(x) \in \hat{\theta}_{EIC}^L(x).$ 

For reasons of brevity, formal, rigorous proofs to Theorem 2.7 and most other claims in this paper are given in the SI, Section A, rather than in the main paper. We present here, instead, only general proof ideas.

*Proof idea.* For any  $\theta$ , one can use a Taylor approximation in order to find a neighbourhood around  $\theta$  where  $L(\theta', \theta)$  is in the range defined by

$$L(\theta', \theta) = \frac{1}{2} (\theta' - \theta)^T H_L^{\theta}(\theta' - \theta) \pm \frac{m}{6} |\theta' - \theta|^3, \tag{2.4}$$

for some constant error term m.

Choose  $\mathcal{A} = \{A_{\epsilon}\}$  as in the proof of Theorem 2.6.

In computing the expected utility  $\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta},\theta))|\boldsymbol{x}=x]$  for  $\epsilon$  tending to zero, if one chooses a small enough  $\delta$  value such that  $B(\theta,\delta)$ , the ball of radius  $\delta$  around  $\theta$ , is entirely within the neighbourhood where (2.4) holds, and if one further chooses an  $\epsilon$  value small enough that outside this neighbourhood both  $L(\theta',\theta)$  and  $\frac{1}{2}(\theta'-\theta)^T H_L^{\theta}(\theta'-\theta)$  are greater than  $\epsilon$  (which is always possible to do because L is discriminative for the problem), then

$$\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \boldsymbol{\theta}))|x] = \int_{\Theta} f(\boldsymbol{\theta}'|x) A_{\epsilon}(L(\boldsymbol{\theta}', \boldsymbol{\theta})) d\boldsymbol{\theta}' = \int_{B(\boldsymbol{\theta}, \boldsymbol{\delta})} f(\boldsymbol{\theta}'|x) A_{\epsilon}(L(\boldsymbol{\theta}', \boldsymbol{\theta})) d\boldsymbol{\theta}'$$

$$\approx \int_{B(\boldsymbol{\theta}, \boldsymbol{\delta})} f(\boldsymbol{\theta}'|x) A_{\epsilon} \left(\frac{1}{2} (\boldsymbol{\theta}' - \boldsymbol{\theta})^T H_L^{\boldsymbol{\theta}}(\boldsymbol{\theta}' - \boldsymbol{\theta})\right) d\boldsymbol{\theta}'.$$

For a sufficiently small  $\delta$  (and a correspondingly small  $\epsilon$ ) this can then be further approximated as

$$\begin{split} f(\theta|x) \int_{B(\theta,\delta)} A_{\epsilon} \left( \frac{1}{2} (\theta' - \theta)^T H_L^{\theta}(\theta' - \theta) \right) \mathrm{d}\theta' \\ &= f(\theta|x) \int_{\mathbb{R}^M} A_{\epsilon} \left( \frac{1}{2} (\theta' - \theta)^T H_L^{\theta}(\theta' - \theta) \right) \mathrm{d}\theta', \end{split}$$

and this, in turn, can be computed by a Jacobian transformation, ultimately yielding the following approximation:

$$\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \boldsymbol{\theta}))|x] \approx \frac{f(\boldsymbol{\theta}|x)}{\sqrt{|H_L^{\boldsymbol{\theta}}|}} \int_{\mathbb{R}^M} A_{\epsilon} \left(\frac{1}{2}|\omega|^2\right) d\omega. \tag{2.5}$$

The integral in (2.5) is a multiplicative factor independent of  $\theta$ , so gets dropped when computing the ratio

$$\frac{\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \theta_1))|x]}{\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \theta_2))|x]} \approx \frac{f(\theta_1|x)/\sqrt{|H_L^{\theta_1}|}}{f(\theta_2|x)/\sqrt{|H_L^{\theta_2}|}},$$

for which the approximation becomes equality in the limit as  $\epsilon$  goes to zero. If, contrary to the claim,  $\theta_1 = \hat{\theta}_L(x) \notin \hat{\theta}_{\mathrm{EIC}}^L$ , we can choose  $\theta_2 \in \hat{\theta}_{\mathrm{EIC}}^L$ , for which this limit is smaller than 1. This proves that  $\theta_1$  is EI-inferior, and therefore that  $\hat{\theta}_L$  does not satisfy AIA, contradicting our assumption.

#### 2.4 EIC and PMLE

We have shown that AIA implies EIC, but EIC depends on one's choice of a loss function. In this section, we show that if one is free to choose any loss function then, for  $\theta$ -continuous problems, the span of EIC estimators equals the span of PMLE estimators.

**Theorem 2.8.** Let  $(x, \theta)$  be a  $\theta$ -continuous estimation problem.

For every loss function L such that  $\hat{\theta}_{EIC}^L$  is well-defined on  $(\boldsymbol{x}, \boldsymbol{\theta})$ , there is a function  $g: \Theta \to \mathbb{R}^{>0}$ , for which  $\hat{\theta}_{PMLE(q)}$  is well-defined on  $(\boldsymbol{x}, \boldsymbol{\theta})$  and

$$\hat{\theta}_{PMLE(q)} = \hat{\theta}_{EIC}^L. \tag{2.6}$$

Conversely, for every function  $g: \Theta \to \mathbb{R}^{>0}$  such that  $\hat{\theta}_{PMLE(g)}$  is well-defined on  $(\boldsymbol{x}, \boldsymbol{\theta})$  there is a smooth loss function L that is discriminative for  $(\boldsymbol{x}, \boldsymbol{\theta})$  such that  $\hat{\theta}_{EIC}^L$  is well-defined on  $(\boldsymbol{x}, \boldsymbol{\theta})$  and  $\hat{\theta}_{PMLE(g)} = \hat{\theta}_{EIC}^L$ .

*Proof.* From (2.2), we know that given L we can set

$$g(\theta) = \frac{f(\theta)}{\sqrt{\left|H_L^{\theta}\right|}}$$

to satisfy (2.6). By definition of an estimation problem we know that  $f(\theta)$  is always positive, and because EIC is well-defined, we know that  $|H_L^{\theta}|$  is always defined and positive. Altogether, these imply that g is also always positive, as required.

Given g, on the other hand, we can choose

$$L(\theta_1, \theta_2) = \left(\frac{f(\theta_2)}{g(\theta_2)}\right)^{2/M} L_2(\theta_1, \theta_2), \tag{2.7}$$

where  $L_2$  is the quadratic loss function,

$$L_2(\theta_1, \theta_2) \stackrel{\text{def}}{=} |\theta_1 - \theta_2|^2, \tag{2.8}$$

and M is (as always) the dimension of  $\Theta$ .

This works because only the  $L_2$  portion depends on  $\theta_1$ . The rest acts as a constant in the calculation of  $H_L^{\theta}$ . As a result:

$$H_L^{\theta} = \left(\frac{f(\theta)}{g(\theta)}\right)^{2/M} \times H_{L_2}^{\theta} = \left(\frac{f(\theta)}{g(\theta)}\right)^{2/M} \times 2I_M$$

and

$$\left|H_L^\theta\right| = \left(\frac{f(\theta)}{g(\theta)}\right)^2 \times 2^M \times |I_M| = \left(\frac{f(\theta)}{g(\theta)}\right)^2 \times 2^M,$$

where  $I_M$  is the  $M \times M$  identity matrix.

Thus,

$$\underset{\theta \in \Theta}{\operatorname{argmax}} \, \frac{f(\theta|x)}{\sqrt{|H_L^{\theta}|}} = \underset{\theta \in \Theta}{\operatorname{argmax}} \, \frac{f(\theta|x)}{f(\theta)} g(\theta) = \underset{\theta \in \Theta}{\operatorname{argmax}} \, f(x|\theta) g(\theta).$$

This proves that  $\hat{\theta}_{\text{PMLE}(g)} = \hat{\theta}_{\text{EIC}}^L$ , and because we know that  $\hat{\theta}_{\text{PMLE}(g)}$  is well-defined, so is EIC.

Because  $L(\theta_1, \theta_2)$  is for every  $\theta_2$  a constant positive multiple of  $L_2(\theta_1, \theta_2)$ , we also know that it is smooth and discriminative for every problem.

#### 2.5 Discussion

Before concluding the first part of this paper, let us discuss why the results so far, characterising EIC as the class of estimators satisfying AIA and essentially equating between the range of EIC estimators and the range of PMLE estimators, are important conclusions. This is for four reasons.

- 1. Without specifying a loss function, we see that EIC can essentially be any member of the class of PMLE estimators. This class is very broad, making EIC without a specified loss function a very weak characterisation. This fact is important, when using the axiomatic method, in order to demonstrate that our single AIA axiom is not an overreach: it does not overly constrain the list of potential solutions.
- 2. Conversely, it is equally important to note that AIA whittled away exactly those estimators that we wished to discard. If we compare EIC/PMLE against all estimators listed in Section 1, we see that, with the single exception of the general class of extremum estimators, all estimators mentioned are either special cases of PMLE, or are trade-off based methods, which is exactly the class of methods that AIA was specifically designed to carve out because they are unsuitable for the error intolerant scenario.
- 3. EIC is not only defined on  $\theta$ -continuous problems. It is defined equally on all types of estimation problems, and by this constructs a bridge between the different types

of problems. If we choose  $L_2$  as our loss function, for example, the EIC estimator over this loss will be discrete MAP in the discrete case and continuous MAP in the  $\theta$ -continuous case (as can be verified by substituting  $L=L_2$  into (2.2)). Thus, EIC provides a formal justification for the use of continuous MAP as a continuous analogue of discrete MAP. This connection was previously investigated by Bassett and Deride (2018) in the context of the limiting behaviour of Bayes estimators, but our work places this relationship between discrete MAP and continuous MAP inside a much larger context, where the continuous analogue of discrete MAP may be  $\hat{\theta}_{\text{PMLE}}(g)$  for essentially any function g simply by use of an appropriate loss function. The choice of "equating" discrete MAP specifically with continuous MAP is therefore akin to accepting  $L_2$  loss (or an equivalent) as one's loss function of choice.

4. Generalising the previous point: there is one feature that differentiates the class of  $\hat{\theta}_{\text{EIC}}^L$  estimators introduced here from the well-known class of  $\hat{\theta}_{\text{PMLE}}(g)$  estimators: whereas there is no intrinsic reason to choose one g over another in PMLE (and, indeed, we see many researchers using many different g functions), each EIC estimator is indexed by a specific loss function, L, and this enables us to tackle the question of what estimator is most suitable for use in a given scenario (such as in the case of the error intolerant scenario) by reformulating this question as "Which loss functions would be appropriate for use in this situation?". It is this question that will be explored in Section 3.

# 3 Choosing a reasonable loss function

#### 3.1 Axioms on loss

In standard econometrics risk-aversion theory, the objective loss (or utility) is part of the problem definition. In such a case where L is known, Section 2 proves that  $\hat{\theta}_{\mathrm{EIC}}^L$  is the best choice of estimator for the error intolerant scenario, if one accepts the AIA axiom.

In estimation, however, L is typically chosen by the statistician, and is not simply given. In our case, this allows us to choose judiciously which of the many EIC estimators to select, by transforming this choice into one about deciding which loss function would be most appropriate to use in our scenario of interest. We now present the axioms that we claim to be incontrovertible for any "reasonable" loss function to be used in the error intolerant problem scenario.

In all axioms, our requirement is that the loss function L (parameterised by its estimation problem as  $L_{(\boldsymbol{x},\boldsymbol{\theta})}$ ) satisfies the specified conditions for every elementary estimation problem  $(\boldsymbol{x},\boldsymbol{\theta})$  and every pair of parameters  $\theta_1$  and  $\theta_2$  in parameter space.

As always, we take the parameter space to be  $\Theta \subseteq \mathbb{R}^M$  and the observation space to be  $X \subseteq \mathbb{R}^N$ .

We present the axioms here in their formal formulation. Readers looking for a deeper analysis of the rationale and the claim of incontrovertibility of each axiom can find this discussion in the SI, Section B.3. We claim that two of the axioms (IRP and IIA) are incontrovertible in error intolerant scenarios, while the remaining axioms (IRO and ISI) are incontrovertible even in the general estimation scenario, beyond error intolerant estimation.

#### Axiom 1: Invariance to Representation of Parameter Space (IRP)

Loss (and the subsequent estimate) should not depend on how estimates are presented (e.g., in metric or imperial measurements). Formally:

A loss function L is said to satisfy IRP if for every diffeomorphism  $F: \mathbb{R}^M \to \mathbb{R}^M$  for which  $(\mathbf{x}, F(\boldsymbol{\theta}))$  is an elementary estimation problem,

$$L_{(\boldsymbol{x},\boldsymbol{\theta})}(\theta_1,\theta_2) = L_{(\boldsymbol{x},F(\boldsymbol{\theta}))}(F(\theta_1),F(\theta_2)).$$

#### Axiom 2: Invariance to Representation of Observation Space (IRO)

Loss (and the subsequent estimate) should not depend on how measurements are presented (e.g., in polar or Cartesian coordinates). Formally:

A loss function L is said to satisfy IRO if for every invertible, piecewise-diffeomorphic function  $G: \mathbb{R}^N \to \mathbb{R}^N$  for which  $(G(\boldsymbol{x}), \boldsymbol{\theta})$  is an elementary estimation problem,

$$L_{(\boldsymbol{x},\boldsymbol{\theta})}(\theta_1,\theta_2) = L_{(G(\boldsymbol{x}),\boldsymbol{\theta})}(\theta_1,\theta_2).$$

#### Axiom 3: Invariance to Irrelevant Alternatives (IIA)

Loss from  $\theta_1$  to  $\theta_2$  should not depend on details of the problem at alternatives other than  $\theta_1$  and  $\theta_2$  (and consequently, the estimate should not base its preference between these two on such irrelevancies). Formally:

A loss function L is said to satisfy IIA if for every elementary estimation problem  $(\boldsymbol{x}, \boldsymbol{\theta'})$ , where  $\boldsymbol{\theta'}$  is distributed over  $\Theta' \subseteq \mathbb{R}^M$ , and every  $\theta_1, \theta_2 \in \Theta \cap \Theta'$ , if at  $\theta \in \{\theta_1, \theta_2\}$   $\Lambda^{(\boldsymbol{x}, \boldsymbol{\theta})}(\theta) = \Lambda^{(\boldsymbol{x}, \boldsymbol{\theta'})}(\theta)$  and  $P_{\theta}^{(\boldsymbol{x}, \boldsymbol{\theta})} = P_{\theta}^{(\boldsymbol{x}, \boldsymbol{\theta'})}$ , then

$$L_{(\boldsymbol{x},\boldsymbol{\theta})}(\theta_1,\theta_2) = L_{(\boldsymbol{x},\boldsymbol{\theta'})}(\theta_1,\theta_2).$$

#### Axiom 4: Invariance to Superfluous Information (ISI)

The addition of independent noise should not influence loss (nor the subsequent estimate). Formally:

A loss function L is said to satisfy ISI if for any random variable y such that y is independent of  $\theta$  given x,

$$L_{((\boldsymbol{x},\boldsymbol{y}),\boldsymbol{\theta})}(\theta_1,\theta_2) = L_{(\boldsymbol{x},\boldsymbol{\theta})}(\theta_1,\theta_2),$$

assuming  $((x, y), \theta)$  is an elementary estimation problem.

Table 1 provides a graphical representation of the four axioms.

We remark that the choice of these particular four axioms, both those universally applicable and those specific to the error intolerant scenario, is not arbitrary. It is heavily

	Invariance to Representation	Invariance to Irrelevancies
In Parameter Space	IRP	IIA
In Observation Space	IRO	ISI

Table 1: The axioms, represented graphically. The table demonstrates the symmetry of the construction, with two axioms requiring invariance to representation (one in parameter space, one in observation space) and two requiring invariance to irrelevancies (one in parameter space, and one in observation space).

implied (and, technically, constrained) by the structure of the EIC estimator (which, in turn, is characterised entirely by AIA, and therefore by our understanding of error intolerance).

To explain: on discrete problems, EIC coincides with discrete MAP, which is an estimator that is well known to exhibit exactly these types of invariance as we are now requiring of the loss functions: it is agnostic both to representation and to irrelevancies. EIC is an estimator that extends MAP into the continuous domain, and ideally we would have liked it to exhibit MAP's good properties also in its extensions. All four properties listed above are ones that, if they are exhibited by EIC's underlying loss function, L, then the corresponding estimator-level invariances will also be manifested by  $\hat{\theta}_{\text{EIC}}^L$ :

- Regarding IRO and ISI,  $\hat{\theta}_{\text{EIC}}^L(x)$  can only depend on the representation of x or superfluous information therein if this dependency comes from L.
- Regarding IIA, if L satisfies its definition above, then  $\hat{\theta}_{\mathrm{EIC}}^L(x)$  satisfies IIA in the sense originally defined for this term by Nash (1950): if, for an estimation problem  $(\boldsymbol{x}, \boldsymbol{\theta})$  over some domain  $X \times \Theta$  we have  $\hat{\theta}_{\mathrm{EIC}}^L(x) = \theta$ , IIA in the Nash sense stipulates that for any  $\Theta' \subseteq \Theta$  such that  $\theta \in \Theta'$ , if  $\boldsymbol{\theta}'$  is  $\boldsymbol{\theta}$  restricted to  $\Theta'$  then for the estimation problem  $(\boldsymbol{x}, \boldsymbol{\theta}')$  the equality  $\hat{\theta'}_{\mathrm{EIC}}^L(x) = \theta$  should also be satisfied.
- Lastly, regarding IRP, EIC is invariant to parameter representation in the sense that if for some estimation problem  $(\boldsymbol{x}, \boldsymbol{\theta})$  we define a new estimation problem  $(\boldsymbol{x}, \boldsymbol{\phi})$  by choosing  $\boldsymbol{\phi} = F(\boldsymbol{\theta})$  for some diffeomorphism  $F : \mathbb{R}^M \to \mathbb{R}^M$  such that for all  $\theta_1$  and  $\theta_2$ ,  $L_{(\boldsymbol{x},\boldsymbol{\theta})}(\theta_1, \theta_2) = L_{(\boldsymbol{x},\boldsymbol{\phi})}(F(\theta_1), F(\theta_2))$ , then EIC will satisfy for

<sup>&</sup>lt;sup>3</sup>This actually requires a slightly stronger IIA condition than what we have assumed in Axiom 3, but the stronger axiom is not required for our derivation. (The estimator property described is, however, nevertheless implied by the combination of our IIA and IRP axioms.)

all 
$$x$$
,  $\hat{\phi}_{\mathrm{EIC}}^{L}(x) = F\left(\hat{\theta}_{\mathrm{EIC}}^{L}(x)\right)$ . This is because 
$$\hat{\phi}_{\mathrm{EIC}}^{L}(x) = \underset{\phi}{\operatorname{argmax}} \frac{f^{(x,\phi)}(\phi|x)}{\sqrt{\left|H_{L_{(x,\phi)}}^{\phi}\right|}} = F\left(\underset{\theta}{\operatorname{argmax}} \frac{f^{(x,\phi)}(F(\theta)|x)}{\sqrt{\left|H_{L_{(x,\phi)}}^{F(\theta)}\right|}}\right)$$
$$= F\left(\underset{\theta}{\operatorname{argmax}} \frac{f^{(x,\phi)}(\phi|x)/|J_{F}(\theta)|}{\sqrt{\left|H_{L_{(x,\phi)}}^{\theta}\right|}/|J_{F}(\theta)|}\right) = F\left(\hat{\theta}_{\mathrm{EIC}}^{L}(x)\right),$$

where  $J_F$  is the Jacobian of F.

Thus, the choice of axioms reflects good properties already present at the core structure of EIC, which we wish the final estimator,  $\hat{\theta}_{\text{EIC}}^L$  to also manifest.

#### 3.2 The loss principle

So far, we have seen multiple justifications for all five of our axioms (AIA, IRP, IRO, IIA and ISI). Another assumption that needs justification, however, albeit remaining implicit so far, is that it is legitimate for us to word our axioms as axioms about the loss functions, rather than as axioms about the estimators themselves, axioms that do not assume any underlying structure to the estimators. This is a departure from the standard approach of the axiomatic method, but is necessary, as we demonstrate in Section 4.3.

In terms of this assumption's incontrovertibility, we submit to the reader than if an estimator is reasonable, it should be formulatable as an estimator over a reasonable loss function. We refer to this assertion as the loss principle. This primacy of the loss function over the estimator is used here in the context of error intolerant estimators, but it is an underlying principle in much of Bayesian thought: Bayesian researchers often justify their choice of estimator by the reasonableness of the underlying loss function (Williamson, 2023) or, indeed, consider the loss functions themselves as objects of study in their own right (Hasan et al., 2013).

We deem the loss principle incontrovertible because in Bayesian thought it is almost definitional: not only does the loss function completely determine the estimator, it is the loss function that defines the estimator to begin with. Challenging the loss principle is challenging this Bayesian approach, and, indeed, what is demonstrated in Section 4.3 is that omitting it, with all else remaining equal, opens the door even to decidedly non-Bayesian solutions.

#### 3.3 EIC estimation over reasonable loss

Our main theorem for continuous problems is as follows. It requires only 3 of our 4 axioms.

**Theorem 3.1.** If  $(x, \theta)$  is an elementary continuous estimation problem for which  $\hat{\theta}_{WF}$  is a well-defined set estimator, and if L is a smooth and problem continuous loss function, sensitive on  $(x, \theta)$ 's type, that satisfies all of IIA, IRP and IRO, then

$$\hat{\theta}_{EIC}^{L} = \hat{\theta}_{WF}.$$

We prove the theorem through a progression of lemmas. As before, M and N will always represent the dimensions of the parameter space and the observation space, respectively.

**Lemma 3.2.** For  $\theta$ -continuous estimation problems  $(x, \theta)$ , if L satisfies both IIA and IRP then  $L_{(x,\theta)}(\theta_1, \theta_2)$  is a function only of the data distributions  $P_{\theta_1}$  and  $P_{\theta_2}$  and of M, the dimension of the parameter space.

*Proof idea.* The IIA axiom is tantamount to stating that  $L_{(\boldsymbol{x},\boldsymbol{\theta})}(\theta_1,\theta_2)$  is dependent only on the following:

- 1. The function's inputs  $\theta_1$  and  $\theta_2$ ,
- 2. The data distributions  $P_{\theta_1}$  and  $P_{\theta_2}$ ,
- 3. The priors  $f(\theta_1)$  and  $f(\theta_2)$ , and
- 4. The problem's parameter space dimension M.

We can assume without loss of generality that  $\theta_1 \neq \theta_2$ , or else the value of  $L(\theta_1, \theta_2)$  can be determined to be zero by (1.1).

IRP excludes dependence on the priors as well as on the values of  $\theta_1$  and  $\theta_2$  themselves, because the parameter space can be distorted diffeomorphically to change these to any desired values, without impacting the loss.

In light of Lemma 3.2, we will henceforth use the notation  $L(P_{\theta_1}, P_{\theta_2})$  (or, when  $\boldsymbol{x}$  is known to be continuous,  $L(f_{\theta_1}, f_{\theta_2})$ ) instead of  $L_{(\boldsymbol{x},\boldsymbol{\theta})}(\theta_1, \theta_2)$ . A loss function that can be expressed in this way is referred to as a *conditional-distribution-based* loss function.

We now introduce the function c, which will form the backbone for much of the rest of our construction.

For distributions P and Q with a common support  $X \subseteq \mathbb{R}^N$ , absolutely continuous with respect to each other, let  $r_{P,Q}(x)$  be the Radon-Nikodym derivative  $\frac{\mathrm{d}P}{\mathrm{d}Q}(x)$  (Royden and Fitzpatrick, 1988). For a value of x that has positive probability in both P and Q, this is simply  $\mathbf{P}_{x\sim P}(x=x)/\mathbf{P}_{x\sim Q}(x=x)$ , whereas for pdfs f and g it is f(x)/g(x) within the common support.

The function  $c[P,Q]:[0,1]\to\mathbb{R}^{\geq 0}$  is defined by

$$c[P,Q](t) \stackrel{\text{def}}{=} \inf \left( \left\{ r \in \mathbb{R}^{\geq 0} : t \leq \mathbf{P}_{\boldsymbol{x} \sim Q}(r_{P,Q}(\boldsymbol{x}) \leq r) \right\} \right).$$

Up to how it resolves certain edge cases, this function is simply the inverse function to the cumulative distribution function of the random variable  $r_{P,Q}(x)$ , where x is distributed according to Q. The next two lemmas demonstrate the usefulness of c to our analysis.

**Lemma 3.3.** The function c is  $\mathcal{M}$ -continuous for continuous distributions, in the sense that if both  $(f_i)_{i\in\mathbb{N}} \xrightarrow{\mathcal{M}} f$  and  $(g_i)_{i\in\mathbb{N}} \xrightarrow{\mathcal{M}} g$ , where f and g are pdfs and  $(f_i)_{i\in\mathbb{N}}$  and  $(g_i)_{i\in\mathbb{N}}$  are pdf sequences, then  $(c[f_i,g_i])_{i\in\mathbb{N}} \xrightarrow{\mathcal{M}} c[f,g]$ .

Proof idea. By definition of convergence in measure.

**Lemma 3.4.** If L is a problem continuous conditional-distribution-based loss function that satisfies IRO, and p and q are piecewise-continuous probability density functions that are data distributions in an estimation problem  $(\mathbf{x}, \boldsymbol{\theta})$ , then  $L_{(\mathbf{x}, \boldsymbol{\theta})}(p, q)$  depends only on c[p, q] and on the type of  $(\mathbf{x}, \boldsymbol{\theta})$ .

*Proof idea*. Fix the problem type.

Because L is conditional-distribution-based, it is solely a function of p and q (in addition to the now fixed problem type).

Consider the case N=1, which is the case where demonstrating the idea of the proof is easiest. We will show that given any p and q, we can alter the observation space using a transformation that preserves the loss so as to map the data distributions p and q into a canonical form, dependent only on c[p,q]. This being the case, starting from any p' and q' (also data distributions in an estimation problem of the same type) for which c[p',q']=c[p,q] will reach the same canonical form, and must therefore have L(p',q')=L(p,q), proving the claim.

To do this, let us first transform the observation x into  $\mathrm{cdf}(x) \stackrel{\mathrm{def}}{=} \mathbf{P}_{\boldsymbol{x} \sim q}(\boldsymbol{x} \leq x)$ . By IRO, if an estimation problem has  $p = f_{\theta_1}^{(\boldsymbol{x}, \boldsymbol{\theta})}$  and  $q = f_{\theta_2}^{(\boldsymbol{x}, \boldsymbol{\theta})}$ , then the estimation problem  $(\mathrm{cdf}(\boldsymbol{x}), \boldsymbol{\theta})$  will satisfy  $L_{(\boldsymbol{x}, \boldsymbol{\theta})}(\theta_1, \theta_2) = L_{(\mathrm{cdf}(\boldsymbol{x}), \boldsymbol{\theta})}(\theta_1, \theta_2)$ .

However, this transform maps  $f_{\theta_2}^{(\text{cdf}(\boldsymbol{x}),\boldsymbol{\theta})}$  to the uniform distribution on [0,1] and maps  $f_{\theta_1}^{(\text{cdf}(\boldsymbol{x}),\boldsymbol{\theta})}(\text{cdf}(\boldsymbol{x}))$  to  $r_{p,q}(\boldsymbol{x})$ .

We next take the new problem,  $(\operatorname{cdf}(\boldsymbol{x}), \boldsymbol{\theta})$ , and transform it again. This time, we partition the range [0,1] into segments of length  $\delta$ , and then rearrange the parts so that they will be ordered by increasing  $f_{\theta_1}^{(\operatorname{cdf}(\boldsymbol{x}),\boldsymbol{\theta})}$ . Again, by IRO, the loss cannot change.

We can do this for arbitrarily small  $\delta$ . In the limit, the resulting  $f_{\theta_1}^{(\boldsymbol{x'},\boldsymbol{\theta})}$  is simply c[p,q]. The loss in the limit must also remain the same, namely due to Lemma 3.3 and the problem-continuity assumption.

This brings us, as desired, to a canonical form with the same loss, which is determined only by c[p,q] and must therefore be shared also by any other p' and q' for which c[p,q] = c[p',q'].

Similar can also be done for a general N, in which case in the canonical form of the problem the domain of x is the N-dimensional unit cube, and in all dimensions except the first all marginal distributions are uniform.

**Lemma 3.5.** Let L be a smooth conditional-distribution-based loss function satisfying that L(P,Q) is a function only of c[P,Q] and of the problem type, and let  $(\boldsymbol{x},\boldsymbol{\theta})$  be an elementary  $\boldsymbol{\theta}$ -continuous estimation problem of a type on which L is sensitive.

If one of the following conditions holds true:

- 1. The problem  $(x, \theta)$  is a continuous estimation problem, or,
- 2. The problem  $(x, \theta)$  is a semi-continuous estimation problem and L satisfies ISI,

then there exists a nonzero constant  $\gamma$ , dependent only on the choice of L and the problem type, such that for every  $\theta \in \Theta$  the Hessian matrix  $H_L^{\theta}$  equals  $\gamma$  times the Fisher information matrix  $\mathcal{I}_{\theta}$ .

*Proof idea.* Fix the problem type.

By assumption, L(P,Q) can be rewritten as  $\tilde{L}(r_{P,Q})$ . To determine the derivatives of L(P,Q), we can use this compositional form. This assumption also poses multiple constraints on the form of these derivatives (which we derive in the SI), from which we ultimately conclude the existence of functions  $\Delta_{\rm x}$ ,  $\Delta_{\rm xy}$  and  $\Delta_{\rm xx}$ , dependent only on the distribution of  $r_{f_{\theta_1},f_{\theta_2}}(\boldsymbol{x})$  with  $\boldsymbol{x} \sim f_{\theta_2}$ , such that the second derivative of the loss function can be written as

$$\frac{\partial^{2}L(\theta_{1},\theta_{2})}{\partial\theta_{1}(i)\partial\theta_{1}(j)} = \int_{X} \Delta_{\mathbf{x}}(r_{f_{\theta_{1}},f_{\theta_{2}}}(x)) \frac{\partial^{2}r_{f_{\theta_{1}},f_{\theta_{2}}}(x)}{\partial\theta_{1}(i)\partial\theta_{1}(j)} f_{\theta_{2}}(x) dx 
+ \int_{X} \int_{X} \Delta_{\mathbf{x}\mathbf{y}}(r_{f_{\theta_{1}},f_{\theta_{2}}}(x_{1}), r_{f_{\theta_{1}},f_{\theta_{2}}}(x_{2})) 
- \frac{\partial r_{f_{\theta_{1}},f_{\theta_{2}}}(x_{1})}{\partial\theta_{1}(i)} \frac{\partial r_{f_{\theta_{1}},f_{\theta_{2}}}(x_{2})}{\partial\theta_{1}(j)} f_{\theta_{2}}(x_{2}) dx_{2} f_{\theta_{2}}(x_{1}) dx_{1} 
+ \int_{X} \Delta_{\mathbf{x}\mathbf{x}}(r_{f_{\theta_{1}},f_{\theta_{2}}}(x)) \frac{\partial r_{f_{\theta_{1}},f_{\theta_{2}}}(x)}{\partial\theta_{1}(i)} \frac{\partial r_{f_{\theta_{1}},f_{\theta_{2}}}(x)}{\partial\theta_{1}(j)} f_{\theta_{2}}(x) dx.$$
(3.1)

The Hessian matrix element  $H_L^{\theta}(i,j)$  equals  $\partial^2 L(\theta_1,\theta_2)/\partial \theta_1(i)\partial \theta_1(j)$  at  $\theta_1=\theta_2=\theta$ . When this is the case,  $r_{f_{\theta_1},f_{\theta_2}}$  equals 1 everywhere in X.

The value of (3.1) in this case becomes

$$\Delta_{\mathbf{x}}(1) \int_{X} \frac{\partial^{2} r_{f_{\theta_{1}}, f_{\theta}}(x)}{\partial \theta_{1}(i) \partial \theta_{1}(j)} \bigg|_{\theta_{1} = \theta} f_{\theta}(x) dx 
+ \Delta_{\mathbf{x}\mathbf{y}}(1, 1) \left( \int_{X} \frac{\partial r_{f_{\theta_{1}}, f_{\theta}}(x)}{\partial \theta_{1}(i)} \bigg|_{\theta_{1} = \theta} f_{\theta}(x) dx \right) \left( \int_{X} \frac{\partial r_{f_{\theta_{1}}, f_{\theta}}(x)}{\partial \theta_{1}(j)} \bigg|_{\theta_{1} = \theta} f_{\theta}(x) dx \right) (3.2) 
+ \Delta_{\mathbf{x}\mathbf{x}}(1) \int_{X} \left( \frac{\partial r_{f_{\theta_{1}}, f_{\theta}}(x)}{\partial \theta_{1}(i)} \bigg|_{\theta_{1} = \theta} \right) \left( \frac{\partial r_{f_{\theta_{1}}, f_{\theta}}(x)}{\partial \theta_{1}(j)} \bigg|_{\theta_{1} = \theta} \right) f_{\theta}(x) dx.$$

Note, however, that because  $(x, \theta)$  is an estimation problem, i.e. all its data distributions are probability measures, not general measures, it is the case that

$$\int_X r_{f_{\theta_1}, f_{\theta}}(x) f_{\theta}(x) dx = \int_X f_{\theta_1}(x) dx = 1,$$

and is therefore a constant independent of either  $\theta_1$  or  $\theta$ . Its various derivatives in  $\theta_1$  are accordingly all zero. This makes the first two summands in (3.2) zero.

Moreover, because r is, again, simply 1 throughout all of X, and does not depend on the estimation problem, the remaining function  $\Delta_{xx}$  also does not depend on the estimation problem, and in particular  $\Delta_{xx}(1)$  is simply a constant. We therefore define  $\gamma = \Delta_{xx}(1)$ , and rewrite the equation as

$$H_{L}^{\theta}(i,j) = \gamma \int_{X} \left( \frac{\partial r_{f_{\theta_{1}},f_{\theta}}(x)}{\partial \theta_{1}(i)} \Big|_{\theta_{1}=\theta} \right) \left( \frac{\partial r_{f_{\theta_{1}},f_{\theta}}(x)}{\partial \theta_{1}(j)} \Big|_{\theta_{1}=\theta} \right) f_{\theta}(x) dx$$

$$= \gamma \int_{X} \left( \frac{\partial f_{\theta_{1}}(x)/f_{\theta}(x)}{\partial \theta_{1}(i)} \Big|_{\theta_{1}=\theta} \right) \left( \frac{\partial f_{\theta_{1}}(x)/f_{\theta}(x)}{\partial \theta_{1}(j)} \Big|_{\theta_{1}=\theta} \right) f_{\theta}(x) dx$$

$$= \gamma \int_{X} \left( \frac{\partial \log f_{\theta}(x)}{\partial \theta(i)} \right) \left( \frac{\partial \log f_{\theta}(x)}{\partial \theta(j)} \right) f_{\theta}(x) dx$$

$$= \gamma \mathbf{E}_{\boldsymbol{x} \sim f_{\theta}} \left[ \left( \frac{\partial \log f_{\theta}(\boldsymbol{x})}{\partial \theta(i)} \right) \left( \frac{\partial \log f_{\theta}(\boldsymbol{x})}{\partial \theta(j)} \right) \right]$$

$$= \gamma \mathcal{I}_{\theta}(i,j).$$

Hence,  $H_L^{\theta} = \gamma \mathcal{I}_{\theta}$ .

The only difference in this derivation for the case where  $\boldsymbol{x}$  is discrete, is that in this case the final result would have used probabilities rather than probability densities. This is consistent, however, with the way Fisher information is defined in this more general case. In fact, some sources (e.g., Bobkov et al., 2014) define the Fisher information directly from Radon-Nikodym derivatives.

As a final point in the proof, we remark that  $\gamma$  must be nonzero, because had it been zero,  $H_L^{\theta}$  would have been zero for every  $\theta$  in every elementary  $\theta$ -continuous estimation problem of the same type, contrary to our sensitivity assumption on L.

We now turn to prove our main claim.

Proof of Theorem 3.1. By Lemma 3.2 we know L to be a conditional-distribution-based loss function, and by Lemma 3.4 we know its value at L(P,Q) depends only on the value of c[P,Q] and the problem type. With these prerequisites, we can use Lemma 3.5 to conclude that up to a nonzero constant multiple  $\gamma$  its  $H_L^{\theta}$  is the Fisher information matrix  $\mathcal{I}_{\theta}$ . This, in turn, we've assumed to be positive definite by requiring  $\hat{\theta}_{\mathrm{WF}}$  to be well-defined.

Furthermore,  $\gamma$  cannot be negative, as in combination with a positive definite Fisher information matrix this would indicate that  $H_L^{\theta}$  is not positive semidefinite, causing L to attain negative values in the neighbourhood of  $\theta$ .

It is therefore the case that  $H_L^{\theta}$  must be positive definite, and therefore  $\hat{\theta}_{\text{EIC}}^L$  is well-defined and equal to  $\hat{\theta}_{\text{WF}}$ .

The final case remaining is that of semi-continuous estimation problems. Unlike in the continuous case, here we use all 4 of our axioms on loss.

**Theorem 3.6.** If  $(x, \theta)$  is an elementary, semi-continuous estimation problem for which  $\hat{\theta}_{WF}$  is a well-defined set estimator, and if L is a smooth loss function, sensitive on  $(x, \theta)$ 's type, that satisfies all of IIA, IRP, IRO and ISI, then

$$\hat{\theta}_{EIC}^L = \hat{\theta}_{WF}.$$

*Proof idea.* The proof is essentially identical to that of Theorem 3.1. The only change is that we can no longer apply Lemma 3.4. However, because of ISI we can nevertheless show that L(P,Q) still only depends on c[P,Q] and M, for which reason we can still apply Lemma 3.5, as before, to complete the proof.

# 4 Feasibility and necessity

Customarily, when using the axiomatic method, one proves two additional claims:

**Feasibility:** It is possible to construct a loss function and an estimator that meet all criteria, and

**Necessity:** No criterion is superfluous for our main uniqueness results.

We prove these here. Note that, as above, some technical proofs have been moved to the SI, Section A, and only their basic ideas are provided here.

#### 4.1 AIA

**Theorem 4.1.** The AIA axiom is always feasible, in the sense that both for Theorem 2.6 and for Theorem 2.7, if  $(\mathbf{x}, \boldsymbol{\theta})$  and L satisfy the conditions of the theorem, then there exists a point estimator  $\hat{\theta}_L$  that satisfies AIA on  $(\mathbf{x}, \boldsymbol{\theta})$  with respect to L.

*Proof.* The AIA axiom requires an estimator not to choose estimates that are always EI-inferior regardless of the choice of one's risk attitude spectrum,  $\mathcal{T}$ . We will prove that even given an arbitrary choice of  $\mathcal{T}$ , at every x one can always find at least one  $\theta \in \Theta$  that is not EI-inferior.

For an estimation problem  $(x, \theta)$  that matches the conditions of Theorem 2.6, the theorem proves that all other  $\theta \in \Theta$  are EI-inferior to those in  $\hat{\theta}_{\text{d-MAP}}(x)$ . For an estimation problem that matches the conditions of Theorem 2.7, the theorem proves that all other  $\theta \in \Theta$  are EI-inferior to those in  $\hat{\theta}_{\text{EIC}}^L(x)$ .

In both cases, the theorem describes a set,  $\Theta'$ , of  $\theta$  values not EI-inferior to any  $\theta' \in \Theta \setminus \Theta'$ , so these  $\theta$  values, if they are EI-inferior, can only be EI-inferior to each other. Furthermore, in both cases  $\Theta'$  is nonempty and finite.

In the case of Theorem 2.6, it is nonempty and finite by the assumption that there exists a  $\theta \in \Theta$  with a positive posterior probability p given x, from which we can conclude that at least one and no more than  $\lfloor 1/p \rfloor$   $\theta$  values exist with a posterior probability of p or more. In the case of Theorem 2.7,  $\Theta'$  is nonempty and finite because  $\hat{\theta}_{\text{EIC}}^L$  was assumed to be a well-defined set estimator.

Given any choice of  $\mathcal{T}$  and x, EI-inferiority induces a partial ordering of  $\Theta'$ . However, any partial ordering among a finite, nonempty set of elements admits at least one maximal element. Hence, one can always define an estimator satisfying AIA by choosing a maximal element for each x.

**Theorem 4.2.** All axioms used in Theorem 2.6 and Theorem 2.7 are necessary.

*Proof.* This is trivial, because only one axiom, AIA, was used in these theorems. Omitting this axiom means that the choice of estimator is not constrained at all.  $\Box$ 

#### 4.2 Axioms on loss

**Theorem 4.3.** Our system of axioms is feasible, in the sense that there exist smooth and problem continuous loss functions, L, sensitive on all  $\theta$ -continuous problem types, that satisfy all of IRP, IRO, IIA and ISI.

*Proof idea.* We prove that any f-divergence (Ali and Silvey, 1966) meets the requirements, if its F-function satisfies the following:

- 1. F has 3 continuous derivatives,
- 2. F''(1) > 0, and
- 3.  $\lim_{x\to 0} F(x) < \infty$  and  $\lim_{x\to \infty} F'(x) < \infty$ ,

where F' and F'' are the first two derivatives of F, which we assumed exist.

One example of such a loss function is squared Hellinger distance (Pollard, 2002).  $\Box$ 

The last of the above criteria is only needed to ensure problem continuity, which is a requirement in Theorem 3.1 but not in Theorem 3.6. An example of an f-divergence that meets all requirements except this last, and is therefore suitable for semi-continuous problems but not necessarily for continuous ones, is Kullback-Leibler divergence (Kullback, 1997).

Notably, there are many other loss functions, not f-divergences, that also meet all requirements. One example is Bhattacharyya distance (Bhattacharyya, 1946).

$$D_{\rm B}(P,Q) \stackrel{\text{def}}{=} -\ln(1 - H^2(P,Q)),$$

where  $H^2$  is the squared Hellinger distance.

It can easily be shown to satisfy all 4 axioms, as well as problem continuity, because it is a function of the squared Hellinger distance. Also, the ratio of the Bhattacharyya distance to the squared Hellinger distance tends to 1 for small Hellinger distances, so it also satisfies smoothness and sensitivity. We leave it for follow-up research to provide a more complete explicit characterisation of the possible loss functions satisfying our desiderata.

**Theorem 4.4.** All axioms used in Theorem 3.1 are necessary.

**Theorem 4.5.** All axioms used in Theorem 3.6 are necessary.

*Proof idea for Theorem 4.4 and Theorem 4.5.* We demonstrate both claims by constructing counterexamples for each case.

For example, IRP can be shown to be necessary by considering quadratic loss, which leads to continuous MAP rather than to  $\hat{\theta}_{WF}$ . It satisfies all of IRO, IIA and ISI, but not IRP.

## 4.3 The loss principle

Now that we have shown that all of our formal axioms are necessary, we return to the underlying loss principle, which allowed us, from the start, to describe the axioms in terms of L rather than in term of the estimator itself.

To demonstrate that this is also necessary, we need to be able to apply IRP, IRO, IIA and ISI to estimators rather than to loss functions. For this, we could have used the descriptions of IRP, IRO, IIA and ISI at the estimator level as given at the end of Section 3.1. However, in order to define estimator level properties in a way that clearly does not alter the meaning of the original axioms, we define the following.

**Definition 4.6.** Let P be a property,  $P \in \{IRP, IRO, IIA, ISI\}$ . We say that estimator  $\hat{\theta}$  satisfies property P if it equals  $\hat{\theta}_{EIC}^L$  for a loss function L that satisfies P.

With this definition, we can now demonstrate that the loss principle is not only necessary, but also that it encapsulates, in fact, important elements of Bayesianism. We do this by providing an example of another estimator, which is neither discrete MAP nor WF, and is not even a Bayesian estimator at all, that also meets all four of our axioms on loss.

**Theorem 4.7.** MLE satisfies IRP, IRO, IIA and ISI.

*Proof idea.* We demonstrate that MLE can be described as a  $\hat{\theta}_{EIC}^L$  using two separate loss functions, one satisfying IRP and the other satisfying IRO, IIA and ISI.

We remark that MLE also satisfies all of IRP, IRO, IIA and ISI if taking the definitions of these at the estimator level as per their descriptions in Section 3.1.

#### 5 Conclusions and future work

In this paper, we have rigorously defined "error intolerant estimation", a common and important scenario for point estimation that merits its own specialised solutions, and defined a new framework for creating Bayesian estimators, akin to the framework of Bayes estimation, specifically tailored for it.

We defined an axiom, AIA, that describes a minimal criterion for any good error intolerant estimator, and this sufficed to characterise a class of estimators which we call Error Intolerance Candidates (EIC) estimators, parameterised by their underlying loss function.

We defined four axioms that appear incontrovertible within the context of choosing a reasonable loss function for error intolerant estimation (of which two seem highly compelling beyond error intolerant estimation as well) and have proved that the combination of discrete MAP (for estimation problems where at least one hypothesis has positive probability) and WF (otherwise) is the only error intolerant estimator to satisfy our axioms, and that the axioms are all necessary for the uniqueness of this characterisation, as well as being jointly feasible.

By this, we have provided a first theoretical justification for the use of WF estimation that does not rely on approximations, and the first justification at all for the use of the combination of WF together with discrete MAP, even though the empirical evidence for the power of this combination has been accumulating within the context of the MML literature for the past three decades.

Our justification for WF is fully Bayesian, does not incorporate information theory, and is unrelated to MML. Furthermore, the class of EIC estimators now includes both discrete MAP and WF as special cases.

The paper opens up any number of potential avenues for future research, but perhaps the most intriguing follow-up question is whether such an axiomatic framework can be built without the loss principle, i.e. whether similarly incontrovertible axioms can be introduced for error intolerant estimation which will refer directly to the estimators investigated, rather than to their underlying loss functions. We have shown that in our approach, the loss principle is a necessary assumption.

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# Supplementary Material

Supplementary Information for "An Axiomatisation of Error Intolerant Estimation". A. Supplementary proofs: The full text of the proofs that were too lengthy and too technical to be included in the main paper. B. Supplementary discussion: Supplementary discussion on the connection between error intolerance and trade-off base methods, on the rationale and putative incontrovertibility of the axioms presented for error intolerant estimation, and also on the similarities and the fundamental differences between the error intolerant estimators developed here and estimators advocated for in the MML literature, as well as on how analysing the empirical successes of MML estimators through the lens of error intolerant estimation differs from their MML analysis.

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# Supplementary Information for "An Axiomatisation of Error Intolerant Estimation"

Michael Brand\*

# Appendix A: Supplementary proofs

We provide in this section complete proofs for the claims of the main paper.

#### A.1 The EIC estimator

**Theorem A.1** (Theorem 2.7 of the main text). Let  $(x, \theta)$  be an elementary  $\theta$ -continuous estimation problem over an open set  $\Theta$ , let L be a smooth loss function discriminative for it, such that  $\hat{\theta}_{EIC}^L$  is a well-defined set estimator on  $(x, \theta)$ , and let  $\hat{\theta}_L$  be a point estimator satisfying AIA with respect to L. For all  $x \in X$ ,

$$\hat{\theta}_L(x) \in \hat{\theta}_{EIC}^L(x)$$
.

Proof. For a choice of a risk attitude spectrum,  $\mathcal{T} = \{T_{\epsilon}\}$ , define  $\mathcal{A} = \{A_{\epsilon} = S_{\epsilon} \circ T_{\epsilon}\}$  such that each  $S_{\epsilon}$  is an affine function mapping  $T_{\epsilon}(0)$  to 1 and the maximum of  $T_{\epsilon}$  to 0. Minimising the expected loss  $\mathbf{E}[T_{\epsilon}(L(\boldsymbol{\theta}, \boldsymbol{\theta}))|\boldsymbol{x} = x]$  for choosing  $\hat{\theta}_{L}(x) = \boldsymbol{\theta}$  at a given  $\epsilon$  and a given x is equivalent to maximising the expected utility  $\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \boldsymbol{\theta}))|\boldsymbol{x} = x]$ .

When computing this expected utility,

$$\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \boldsymbol{\theta}))|x] = \int_{\Theta} f(\boldsymbol{\theta}'|x) A_{\epsilon}(L(\boldsymbol{\theta}', \boldsymbol{\theta})) d\boldsymbol{\theta}', \tag{A.1}$$

for  $\epsilon$  values tending to zero, one only needs to consider the integral over  $B(\theta, \delta)$ , the (closed set) ball of radius  $\delta$  around  $\theta$ , for any  $\delta > 0$ , as for a sufficiently small  $\epsilon$ , the rest of the integral values will be zero by the discriminativity assumption. We will, in particular, assume that any chosen  $\delta$  is small enough so that  $B(\theta, \delta) \subseteq \Theta$ .

Because L is smooth, the second derivative (i.e., the Hessian)  $H_L^{\theta}$  is defined and finite everywhere, and its own derivative (i.e., the third derivative) is continuous. Thus, this third derivative is bounded inside the closed set  $B(\theta, \delta)$ . Let m be its maximum absolute value in any direction.

Using a second-order Taylor approximation, we can therefore write, for  $\theta' \in B(\theta, \delta)$ ,

$$L(\theta', \theta) = \frac{1}{2} (\theta' - \theta)^T H_L^{\theta}(\theta' - \theta) \pm \frac{m}{6} |\theta' - \theta|^3.$$
 (A.2)

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<sup>\*</sup>School of Computing Technologies, RMIT University, michael.brand@rmit.edu.au

Let  $H_{\min}^{\theta} = H_L^{\theta} - \frac{m\delta}{3}I_M$  and  $H_{\max}^{\theta} = H_L^{\theta} + \frac{m\delta}{3}I_M$ , where  $I_M$  is the  $M \times M$  identity matrix. For all  $\theta'$  values in  $B(\theta, \delta)$ ,

$$\frac{1}{2}(\theta' - \theta)^T H_{\min}^{\theta}(\theta' - \theta) \le L(\theta', \theta) \le \frac{1}{2}(\theta' - \theta)^T H_{\max}^{\theta}(\theta' - \theta). \tag{A.3}$$

Because we also assumed that the posterior distribution  $f(\theta|x)$  is continuous, inside  $B(\theta, \delta)$  it is also bounded from above and below. Let its bounds be  $f_{\text{max}}$  and  $f_{\text{min}}$ .

From (A.3), we know that the value of (A.1) can be bounded by

$$f_{\min} \int_{B(\theta,\delta)} A_{\epsilon} \left( \frac{1}{2} (\theta' - \theta)^{T} H_{\max}^{\theta} (\theta' - \theta) \right) d\theta' \leq \int_{\Theta} f(\theta'|x) A_{\epsilon} (L(\theta',\theta)) d\theta'$$

$$\leq f_{\max} \int_{B(\theta,\delta)} A_{\epsilon} \left( \frac{1}{2} (\theta' - \theta)^{T} H_{\min}^{\theta} (\theta' - \theta) \right) d\theta'.$$
(A.4)

Because we assumed that  $\hat{\theta}_{\mathrm{EIC}}^L$  is well-defined, we know that  $H_L^{\theta}$  is positive definite, so for a small enough  $\delta$   $H_{\mathrm{min}}^{\theta}$  and  $H_{\mathrm{max}}^{\theta}$  will be, too. Thus, for H being either  $H_{\mathrm{min}}^{\theta}$  or  $H_{\mathrm{max}}^{\theta}$  there is a positive infimum to the value of  $\frac{1}{2}(\theta'-\theta)^T H(\theta'-\theta)$  outside  $B(\theta,\delta)$ .

When  $\epsilon$  is smaller than this infimum, the integration bounds in (A.4) can be switched from  $B(\theta, \delta)$  to  $\mathbb{R}^M$ , without this impacting the truth of the inequality, because all  $\theta'$  values outside  $B(\theta, \delta)$  will contribute zero to the integral.

Values of the form

$$f \int_{\mathbb{R}^M} A_{\epsilon} \left( \frac{1}{2} (\theta' - \theta)^T H(\theta' - \theta) \right) d\theta'$$

can be computed via a Jacobian transformation as

$$\frac{f}{\sqrt{|H|}} \int_{\mathbb{R}^M} A_{\epsilon} \left(\frac{1}{2} |\omega|^2\right) d\omega, \tag{A.5}$$

where the integral is a multiplicative factor independent of  $\theta$ .

As  $\epsilon$ , and therefore also  $\delta$ , tends to zero, both  $|H_{\min}^{\theta}|$  and  $|H_{\max}^{\theta}|$  tend to  $|H_{L}^{\theta}|$ , and both  $f_{\min}$  and  $f_{\max}$  converge to  $f(\theta|x)$ .

Noting that  $f(\theta|x)$  is positive by our assumption on estimation problems, and that  $|H_L^{\theta}|$  is positive by our assumption that  $\hat{\theta}_{\text{EIC}}^L$  is well-defined, we reach

$$\lim_{\epsilon \to 0} \frac{\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \boldsymbol{\theta}))|x]}{\int_{\mathbb{R}^M} A_{\epsilon}\left(\frac{1}{2}|\omega|^2\right) d\omega} = \frac{f(\boldsymbol{\theta}|x)}{\sqrt{|H_L^{\boldsymbol{\theta}}|}} > 0.$$

Suppose, contrary to the claim, that  $\hat{\theta}_L(x) \notin \hat{\theta}_{\mathrm{EIC}}^L(x)$ . Because  $\hat{\theta}_{\mathrm{EIC}}^L$  was assumed to be a well-defined set estimator, we know that  $\hat{\theta}_{\mathrm{EIC}}^L(x)$  is not empty. Choose  $\theta^* \in \hat{\theta}_{\mathrm{EIC}}^L(x)$ .

By its definition, EIC maximises the metric  $f(\theta|x)/\sqrt{|H_L^{\theta}|}$ , so

$$\lim_{\epsilon \to 0} \frac{\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \hat{\theta}_L(x)))|x]}{\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \theta^*))|x]} = \frac{f(\hat{\theta}_L(x)|x)/\sqrt{\left|H_L^{\hat{\theta}_L(x)}\right|}}{f(\theta^*|x)/\sqrt{\left|H_L^{\theta^*}\right|}} < 1.$$

This proves that for every low enough  $\epsilon$ ,

$$\mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \hat{\theta}_L(x)))|x] < \mathbf{E}[A_{\epsilon}(L(\boldsymbol{\theta}, \boldsymbol{\theta}^*))|x],$$

contradicting our assumption that  $\hat{\theta}_L$  satisfies AIA.

#### A.2 EIC estimation over reasonable loss

Our main theorem for continuous problems is as follows.

**Theorem A.2** (Theorem 3.1 of the main text). If  $(\mathbf{x}, \boldsymbol{\theta})$  is an elementary continuous estimation problem for which  $\hat{\theta}_{WF}$  is a well-defined set estimator, and if L is a smooth and problem continuous loss function, sensitive on  $(\mathbf{x}, \boldsymbol{\theta})$ 's type, that satisfies all of IIA, IRP and IRO, then

$$\hat{\theta}_{EIC}^{L} = \hat{\theta}_{WF}$$
.

In the main text, we prove the theorem through a progression of lemmas. While the proof of the main theorem exists in the main text, we supplement it here with the complete proofs of its underlying lemmas.

Recall that, throughout, M and N always represent the dimensions of the parameter space and the observation space, respectively.

**Lemma A.3** (Lemma 3.2 of the main text). For  $\theta$ -continuous estimation problems  $(x, \theta)$ , if L satisfies both IIA and IRP then  $L_{(x,\theta)}(\theta_1, \theta_2)$  is a function only of the data distributions  $P_{\theta_1}$  and  $P_{\theta_2}$  and of M, the dimension of the parameter space.

*Proof.* The IIA axiom is tantamount to stating that  $L_{(\boldsymbol{x},\boldsymbol{\theta})}(\theta_1,\theta_2)$  is dependent only on the following:

- 1. The function's inputs  $\theta_1$  and  $\theta_2$ ,
- 2. The data distributions  $P_{\theta_1}$  and  $P_{\theta_2}$ ,
- 3. The priors  $f(\theta_1)$  and  $f(\theta_2)$ , and
- 4. The problem's parameter space dimension M (noting that all other elements of the problem type are already fixed by specifying the data distributions).

We can assume without loss of generality that  $\theta_1 \neq \theta_2$ , or else the value of  $L(\theta_1, \theta_2)$  can be determined to be zero by (1.1).

Our first claim is that, due to IRP, L can also not depend on the problem's prior probability densities  $f(\theta_1)$  and  $f(\theta_2)$ . To show this, construct a diffeomorphism  $F: \mathbb{R}^M \to \mathbb{R}^M$  in the following way.

Let  $(\vec{v}_1, \dots, \vec{v}_M)$  be an orthogonal basis for  $\mathbb{R}^M$  wherein  $\vec{v}_1 = \theta_2 - \theta_1$ . We design F as

$$F\left(\theta_{1} + \sum_{i=1}^{M} b_{i} \vec{v}_{i}\right) = \theta_{1} + F_{1}(b_{1}) \vec{v}_{1} + \sum_{i=2}^{M} b_{i} \vec{v}_{i},$$

where  $F_1 : \mathbb{R} \to \mathbb{R}$  is a continuous, differentiable function onto  $\mathbb{R}$ , with a derivative that is positive everywhere, satisfying

- 1.  $F_1(0) = 0$  and  $F_1(1) = 1$ , and
- 2.  $F'_1(0) = d_0$  and  $F'_1(1) = d_1$ , for some arbitrary positive values  $d_0$  and  $d_1$  to be chosen later.

Such a function is straightforward to construct for any values of  $d_0$  and  $d_1$ , and by an appropriate choice of these values, it is possible to map  $(\boldsymbol{x}, \boldsymbol{\theta})$  into  $(\boldsymbol{x}, F(\boldsymbol{\theta}))$  in a way that does not change  $P_{\theta_1}$  or  $P_{\theta_2}$ , but adjusts  $f(\theta_1)$  and  $f(\theta_2)$  to any desired positive values.

Lastly, we show that  $L(\theta_1, \theta_2)$  can also not depend on the values of  $\theta_1$  and  $\theta_2$  other than through  $P_{\theta_1}$  and  $P_{\theta_2}$ . For this we once again invoke IRP: by applying a similarity transform on  $\Theta$ , we can map any  $\theta_1$  and  $\theta_2$  values into arbitrary new values, again without this affecting their respective conditional data distributions.

**Lemma A.4** (Lemma 3.3 of the main text). The function c is  $\mathcal{M}$ -continuous for continuous distributions, in the sense that if both  $(f_i)_{i\in\mathbb{N}} \xrightarrow{\mathcal{M}} f$  and  $(g_i)_{i\in\mathbb{N}} \xrightarrow{\mathcal{M}} g$ , where f and g are pdfs and  $(f_i)_{i\in\mathbb{N}}$  and  $(g_i)_{i\in\mathbb{N}}$  are pdf sequences, then  $(c[f_i,g_i])_{i\in\mathbb{N}} \xrightarrow{\mathcal{M}} c[f,g]$ .

*Proof.* For any  $t_0 \in [0,1]$ , let  $r_0 = c[f,g](t_0)$ , and let  $t_{\min}$  and  $t_{\max}$  be the infimum t and the supremum t, respectively, for which  $c[f,g](t) = r_0$ .

Because c[f,g] is a monotone increasing function,  $t_{\min}$  is also the supremum t for which  $c[f,g](t) < r_0$  (unless no such t exists, in which case  $t_{\min} = 0$ ), so by definition  $t_{\min}$  is the supremum of  $\mathbf{P}_{\boldsymbol{x} \sim g}(r_{f,g}(\boldsymbol{x}) \leq r)$ , for all  $r < r_0$ , from which we conclude that  $t_{\min} = \mathbf{P}_{\boldsymbol{x} \sim g}(r_{f,g}(\boldsymbol{x}) < r_0)$ .

Because  $(f_i) \xrightarrow{\mathcal{M}} f$  and  $(g_i) \xrightarrow{\mathcal{M}} g$ , we can use (1.3) to determine that using a large enough i both  $f_i(x)/f(x)$  and  $g_i(x)/g(x)$  are arbitrarily close to 1 in all but a diminishing measure of X. Hence,

$$\lim_{i \to \infty} \mathbf{P}_{\boldsymbol{x} \sim g_i}(r_{f_i,g_i}(\boldsymbol{x}) < r_0) = \lim_{i \to \infty} \mathbf{P}_{\boldsymbol{x} \sim g}(r_{f_i,g_i}(\boldsymbol{x}) < r_0) = \mathbf{P}_{\boldsymbol{x} \sim g}(r_{f,g}(\boldsymbol{x}) < r_0) = t_{\min}.$$

We conclude that for any  $t^+ > t_{\min}$  a large enough i will satisfy  $\mathbf{P}_{\boldsymbol{x} \sim g_i}(r_{f_i,g_i}(\boldsymbol{x}) < r_0) < t^+$ , and hence  $c[f_i,g_i](t^+) \geq r_0$ . For all such  $t^+$ , and in particular for all  $t^+ > t_0$ ,

$$\lim_{i \to \infty} \inf c[f_i, g_i](t^+) \ge r_0 = c[f, g](t_0). \tag{A.6}$$

A symmetrical analysis on  $t_{\text{max}}$  yields that for all  $t^- < t_0$ ,

$$\limsup_{i \to \infty} c[f_i, g_i](t^-) \le c[f, g](t_0). \tag{A.7}$$

Consider, now, the functions

$$c_{\sup}(t) \stackrel{\text{def}}{=} \limsup_{i \to \infty} c[f_i, g_i](t)$$

and

$$c_{\inf}(t) \stackrel{\text{def}}{=} \liminf_{i \to \infty} c[f_i, g_i](t).$$

Because each  $c[f_i, g_i]$  is monotone increasing, so are  $c_{\text{sup}}$  and  $c_{\text{inf}}$ . Monotone functions can only have countably many discontinuity points (for a total of measure zero). For any  $t_0$  that is not a discontinuity point of either function, we have from (A.6) and (A.7) that  $\lim_{i\to\infty} c[f_i, g_i](t_0)$  exists and equals  $c[f, g](t_0)$ , so the conditions of convergence in measure hold.

**Lemma A.5** (Lemma 3.4 of the main text). If L is a problem continuous conditional-distribution-based loss function that satisfies IRO, and p and q are piecewise-continuous probability density functions that are data distributions in an estimation problem  $(\mathbf{x}, \boldsymbol{\theta})$ , then  $L_{(\mathbf{x}, \boldsymbol{\theta})}(p, q)$  depends only on c[p, q] and on the type of  $(\mathbf{x}, \boldsymbol{\theta})$ .

*Proof.* Fix the problem type.

Let us first note that the following conditions are equivalent.

- 1. c[p,q] equals the indicator function on (0,1] in all but a measure zero of values,
- 2. p equals q in all but a measure zero of X, the joint support of p and q,
- 3. p and q are  $\mathcal{M}$ -equivalent, in the sense that a sequence of elements all equal to p nevertheless satisfies the condition of  $\mathcal{M}$ -convergence to q, and
- 4. L(p,q) = 0.

The equivalence of the first and second conditions stems from the definition of c[p,q], the equivalence of the second and third conditions stems from the definition of  $\mathcal{M}$ -convergence, and the equivalence of the last two conditions follows from problem continuity, together with (1.1).

Hence, if c[p,q] equals the indicator function on (0,1], this uniquely determines the value of L(p,q) to be 0, in accordance with the lemma.

It remains to be proved that such a functional relationship from c[p,q] to L(p,q) holds for all other c[p,q] as well, but for this, in accordance with the second condition stated, we can safely assume that p and q differ in a positive measure of X. Because both p and q integrate to 1, we can consequently also assume that the two are not linearly dependent. The remainder of this proof follows under this assumption.

Note first that because L is known to be conditional-distribution-based, the value of L(p,q) is not dependent on the full details of the estimation problem: it will be the same in any estimation problem of the same type that contains the conditional data distributions p and q. Let us therefore design an estimation problem that is easy to analyse but contains these two conditional data distributions.

Let  $(\boldsymbol{x}, \boldsymbol{\theta})$  be an estimation problem with  $\Theta = [0, 1]^M$  and a uniform prior on  $\boldsymbol{\theta}$ . Its conditional data distribution at  $\theta_0 = (0, \dots, 0)$  will be p, at  $\theta_1 = (1, 0, \dots, 0)$  will be q, and we will choose piecewise continuous conditional data distributions,  $f_{\theta}$ , over the rest of the  $\theta \in \{0, 1\}^M$  so all  $2^M$  are linearly independent, share the same support, and differ from each other over a positive measure of X, and so that their respective  $r_{f_{\theta},q}$  values are all monotone weakly increasing with  $r_{p,q}$  and with each other.

If M>1, we further choose  $f_{\theta_2}$  at  $\theta_2=(0,1,0,\ldots,0)$  to satisfy that  $c[f_{\theta_2},q]$  is monotone strictly increasing. If M=1, this is not necessary and we, instead, choose  $\theta_2=\theta_0$ .

We then extend this description of  $f_{\theta}^{(x,\theta)}$  at  $\{0,1\}^M$  into a full characterisation of all the problem's conditional data distributions by setting these to be multilinear functions of the coordinates of  $\theta$ .

We now create a sequence of estimation problems,  $(\boldsymbol{x}_i, \boldsymbol{\theta})$  to satisfy the conditions of L's problem-continuity assumption. We do this by constructing a sequence  $(S_i)_{i \in \mathbb{N}}$  of subsets of  $\mathbb{R}^N$  such that for all  $\theta \in \{0,1\}^M$ ,  $\mathbf{P}(\boldsymbol{x} \in S_i | \boldsymbol{\theta} = \theta)$  tends to 1, and for every  $x \in S_i$ ,

$$\left| f_{\theta}^{(\boldsymbol{x},\boldsymbol{\theta})}(x) - f_{\theta}^{(\boldsymbol{x}_i,\boldsymbol{\theta})}(x) \right| < \epsilon_i, \tag{A.8}$$

for an arbitrarily-chosen sequence  $(\epsilon_i)_{i\in\mathbb{N}}$  tending to zero. By setting the remaining conditional data distribution values as multilinear functions of the coordinates of  $\theta$ , as above, the sequence  $(\boldsymbol{x}_i, \boldsymbol{\theta})$  will satisfy the problem-continuity condition and will guarantee

$$\lim_{i \to \infty} L_{(\boldsymbol{x}_i, \boldsymbol{\theta})}(\theta_0, \theta_1) = L_{(\boldsymbol{x}, \boldsymbol{\theta})}(\theta_0, \theta_1) = L(p, q).$$

We now construct the sequence  $(S_i)_{i\in\mathbb{N}}$  explicitly.

Each  $S_i$  will be describable by the positive parameters (a,b,d,r) as follows. Let  $C_d^N = \{x \in \mathbb{R}^N : |x|_\infty \le d/2\}$ , i.e. the axis-parallel, origin-centred, N-dimensional cube of side length d.  $S_i$  will be chosen to contain all  $x \in C_d^N$  such that for all  $\theta \in \{0,1\}^M$ ,  $a \le f_\theta^{(\boldsymbol{x},\boldsymbol{\theta})}(x) \le b$  and x is at least a distance of r away from the nearest discontinuity point of  $f_\theta^{(\boldsymbol{x},\boldsymbol{\theta})}$ , as well as from the origin. By choosing small enough a and r and large enough b and d, it is always possible to make  $\mathbf{P}(\boldsymbol{x} \in S_i | \boldsymbol{\theta} = \theta)$  arbitrarily close to 1, so the sequence can be made to satisfy its requirements.

We will choose d to be a natural.

We now describe how to construct each  $f_{\theta}^{(\boldsymbol{x}_i,\boldsymbol{\theta})}$  from its respective  $f_{\theta}^{(\boldsymbol{x},\boldsymbol{\theta})}$ . We first describe for each  $\theta \in \{0,1\}^M$  a new function  $g_{\theta}^i: \mathbb{R}^N \to \mathbb{R}^{\geq 0}$  as follows. Begin by setting  $g_{\theta}^i(x) = f_{\theta}^{(\boldsymbol{x},\boldsymbol{\theta})}(x)$  for all  $x \in S_i$ . If  $x \notin C_d^N$ , set  $g_{\theta}^i(x)$  to zero. Otherwise, complete the  $g_{\theta}^i$ 

functions so that all are linearly independent and so that each is positive and continuous inside  $C_d^N$ , and integrates to 1. Note that because a neighbourhood around the origin is known to not be in  $S_i$ , it is never the case that  $S_i = C_d^N$ . This allows enough degrees of freedom in completing the functions g in order to meet all their requirements.

As all  $g_{\theta}^{i}$  are continuous functions over the compact domain  $C_{d}^{N}$ , by the Heine-Cantor Theorem (Rudin, 1964) they are uniformly continuous. There must therefore exist a natural n, such that we can tile  $C_{d}^{N}$  into sub-cubes of side length 1/n such that by setting each  $f_{\theta}^{(x_{i},\theta)}$  value in each sub-cube to a constant for the sub-cube equal to the mean over the entire sub-cube tile of  $g_{\theta}^{i}$ , the result will satisfy for all  $x \in C_{d}^{N}$  and all  $\theta \in \{0,1\}^{M}$ ,  $\left|f_{\theta}^{(x_{i},\theta)}(x) - g_{\theta}^{i}(x)\right| < \epsilon_{i}$ . Because  $f^{(x_{i},\theta)}$  is by design multi-linear in  $\theta$ , this implies that for all  $\theta \in [0,1]^{M}$  and all  $x \in S_{i}$ , condition (A.8) is attained. Furthermore, by choosing a large enough n, we can always ensure, because the g functions are continuous and linearly independent, that also the  $f_{\theta}^{(x_{i},\theta)}$  functions, for  $\theta \in \{0,1\}^{M}$  are linearly independent and differ in more than a measure zero of  $\mathbb{R}^{N}$ . Together, these properties ensure that the new problems constructed are well-defined and elementary.

We have therefore constructed  $(x_i, \theta)$  as a sequence of elementary estimation problems that  $\mathcal{M}$ -approximate  $(x, \theta)$  arbitrarily well, while being entirely composed of  $f_{\theta}^{(x_i, \theta)}$  functions whose support is  $C_{d_i}^N$  for some natural  $d_i$  and whose values within their support are piecewise-constant inside cubic tiles of side-length  $1/n_i$ , for some natural  $n_i$ .

We now use IRO to reshape the observation space of the estimation problems in the constructed sequence by a piecewise-continuous transform.

Namely, we take each constant-valued cube of side length  $1/n_i$  and transform it using a scaling transformation in each coordinate, as follows. Consider a single cubic tile, and let the value of  $f_{\theta_1}^{(x_i,\theta)}(x)$  at points x that are within it be  $G_i$ . We scale the first coordinate of the tile to be of length  $G_i/n_i^N$ , and all other coordinates to be of length 1. Notably, this transformation increases the volume of the cube by a factor of  $G_i$ , so the probability density inside the cube, for each  $f_{\theta}$ , will drop by a corresponding factor of  $G_i$ .

We now place the transformed cubes by stacking them along the first coordinate, sorted by increasing value of  $f_{\theta_2}^{(\boldsymbol{x}_i,\boldsymbol{\theta})}(x)/f_{\theta_1}^{(\boldsymbol{x}_i,\boldsymbol{\theta})}(x)$ .

Notably, because the probability density  $f_{\theta_1}$  in all transformed cubes is  $G_i/G_i = 1$ , it is possible to arrange all transformed cubes in this way so that, together, they fill exactly the unit cube in  $\mathbb{R}^N$ . Let the new estimation problems created in this way be  $(\boldsymbol{x}_i', \boldsymbol{\theta})$ , let  $t_i : \mathbb{R}^N \to \mathbb{R}^N$  be the transformation,  $t_i(\boldsymbol{x}_i) = \boldsymbol{x}_i'$ , applied on the observation space and let  $t_i^1(x)$  be the first coordinate value of  $t_i(x)$ .

By IRO, 
$$L(f_{\theta_0}^{(\boldsymbol{x}_i',\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x}_i',\boldsymbol{\theta})})=L(f_{\theta_0}^{(\boldsymbol{x}_i,\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x}_i,\boldsymbol{\theta})})$$
, which we know tends to  $L(p,q)$ .

Consider the probability density of each  $f_{\theta}^{(x'_i,\theta)}$  over its support  $[0,1]^N$ . This is a probability density that is uniform along all axes except the first, but has some marginal,

 $s=s_{\theta}^{i}$ , along the first axis. We denote such a distribution by  $D_{N}(s)$ . Specifically, for  $\theta=\theta_{2}$ , because of our choice of sorting order, we have  $s_{\theta_{2}}^{i}=c[f_{\theta_{2}}^{(\boldsymbol{x}_{i},\boldsymbol{\theta})},f_{\theta_{1}}^{(\boldsymbol{x}_{i},\boldsymbol{\theta})}]$ , so by Lemma A.4, this is known to  $\mathcal{M}$ -converge to  $c[f_{\theta_{2}}^{(\boldsymbol{x},\boldsymbol{\theta})},f_{\theta_{1}}^{(\boldsymbol{x},\boldsymbol{\theta})}]$ .

If M = 1, the above is enough to show that the  $\mathcal{M}$ -limit problem of  $(\boldsymbol{x}_i', \boldsymbol{\theta})$  exists. If M > 1, consider the following.

Let  $t: X \to [0,1]$  be the transformation mapping each  $x \in X$  to the supremum of all  $\tilde{t}$  for which  $c[f_{\theta_2}^{(\boldsymbol{x},\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}](\tilde{t}) \leq f_{\theta_2}^{(\boldsymbol{x},\boldsymbol{\theta})}(x)/f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}(x)$ . By design,  $c[f_{\theta_2}^{(\boldsymbol{x},\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}]$  is left continuous, so  $c[f_{\theta_2}^{(\boldsymbol{x},\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}](t(x)) = f_{\theta_2}^{(\boldsymbol{x},\boldsymbol{\theta})}(x)/f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}(x)$ .

Summing up the entire construction, in all but a diminishing measure of x we have that  $f_{\theta_2}^{(\boldsymbol{x}_i,\boldsymbol{\theta})}(x)/f_{\theta_1}^{(\boldsymbol{x}_i,\boldsymbol{\theta})}(x)$  approaches  $f_{\theta_2}^{(\boldsymbol{x},\boldsymbol{\theta})}(x)/f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}(x)$ , which in turn equals the value  $c[f_{\theta_2}^{(\boldsymbol{x},\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}](t(x))$ . On the other hand, we have that  $s_{\theta_2}^i=c[f_{\theta_2}^{(\boldsymbol{x}_i,\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x}_i,\boldsymbol{\theta})}]$  also  $\mathcal{M}$ -approaches  $c[f_{\theta_2}^{(\boldsymbol{x},\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}]$  by Lemma A.4, and satisfies  $f_{\theta_2}^{(\boldsymbol{x}_i,\boldsymbol{\theta})}(x)/f_{\theta_1}^{(\boldsymbol{x}_i,\boldsymbol{\theta})}(x)=s_{\theta_2}^i(t_i^i(x))$ .

Together, this indicates 
$$\left(c[f_{\theta_2}^{(\boldsymbol{x},\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}](t_i^1(x))\right) \xrightarrow{\mathcal{M}} c[f_{\theta_2}^{(\boldsymbol{x},\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}](t(x)).$$

Because  $c[f_{\theta_2}^{(\boldsymbol{x},\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}]$  is monotone strictly increasing, it follows that  $t_i^1(x)$   $\mathcal{M}$ -converges to t(x). For all other  $\theta \in [0,1]^M$  this then implies that  $s_{\theta}^i$   $\mathcal{M}$ -converges to  $c[f_{\theta}^{(\boldsymbol{x},\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}]$ , because by construction all the problem's  $r_{f_{\theta},q}$  are monotone increasing with each other.

For all  $\theta$ , the  $f_{\theta}^{(\boldsymbol{x}_i',\boldsymbol{\theta})}$  sequence therefore has a limit, that limit being the distribution  $D_N(c[f_{\theta}^{(\boldsymbol{x},\boldsymbol{\theta})},f_{\theta_1}^{(\boldsymbol{x},\boldsymbol{\theta})}])$ .

In particular, the limit at  $\theta = \theta_0$  is  $D_N(c[p,q])$  and the limit at  $\theta = \theta_1$  is  $U([0,1]^N)$ , the uniform distribution over the unit cube.

By problem-continuity of L,  $L(D_N(c[p,q]), U([0,1]^N)) = L(p,q)$ . Hence, L(p,q) is a function of only c[p,q].

**Lemma A.6** (Lemma 3.5 of the main text). Let L be a smooth conditional-distribution-based loss function satisfying that L(P,Q) is a function only of c[P,Q] and of the problem type, and let  $(\mathbf{x}, \boldsymbol{\theta})$  be an elementary  $\boldsymbol{\theta}$ -continuous estimation problem of a type on which L is sensitive.

If one of the following conditions holds true:

- 1. The problem  $(x, \theta)$  is a continuous estimation problem, or,
- 2. The problem  $(x, \theta)$  is a semi-continuous estimation problem and L satisfies ISI,

then there exists a nonzero constant  $\gamma$ , dependent only on the choice of L and the problem type, such that for every  $\theta \in \Theta$  the Hessian matrix  $H_L^{\theta}$  equals  $\gamma$  times the Fisher information matrix  $\mathcal{I}_{\theta}$ .

*Proof.* Fix the problem type.

We wish to calculate the derivatives of  $L(P_{\theta_1}, P_{\theta_2})$  according to  $\theta_1$ . For convenience, let us define a new function,  $\tilde{L}$ , which describes L in a one-parameter form, by

$$\tilde{L}(r_{P,Q}) = L(P,Q).$$

This is possible by our assumption that L(P,Q) is only a function of c[P,Q].

We differentiate  $\tilde{L}(r_{P,Q})$  as we would any composition of functions. The derivatives of  $r_{P_{\theta_1},P_{\theta_2}}$  in  $\theta_1$  are straightforward to compute, so we concentrate on the question of how minute perturbations of r affect  $\tilde{L}(r)$ .

For this, we first extend the domain of  $\tilde{L}(r)$ . Natively,  $\tilde{L}(r)$  is only defined when  $\mathbf{E}_{\boldsymbol{x}\sim Q}[r(\boldsymbol{x})]=1$ . However, to be able to perturb r more freely, we define, for finite expectation  $r(\boldsymbol{x})$ ,  $\tilde{L}(r)=\tilde{L}(r/\mathbf{E}_{\boldsymbol{x}\sim Q}[r(\boldsymbol{x})])$ .

Let  $Y \subseteq X$  be a set with  $\mathbf{P}_{x \sim Q}(x \in Y) = \epsilon > 0$  such that for all  $x \in Y$ , r(x) is a constant,  $r_0$ . The derivative of  $\tilde{L}(r)$  in Y is defined as

$$\left[\nabla_Y(\tilde{L})\right](r) = \lim_{\Delta \to 0} \frac{\tilde{L}(r + \Delta \cdot \chi_Y) - \tilde{L}(r)}{\Delta},$$

where for any  $S \subseteq \mathbb{R}^s$ , we denote by  $\chi_S$  the function over  $\mathbb{R}^s$  that yields 1 when the input is in S and 0 otherwise.

By our smoothness assumption on L, this derivative is known to exist, because it is straightforward to construct an elementary, continuous estimation problem for which these would be (up to normalisation) the  $r_{P_{\theta_0},P_{\theta_0+\Delta e_1}}$  values for some  $\theta_0$  and basis vector  $e_1$ .

Consider, now, what this derivative's value can be. By assumption that L(P,Q) only depends on c[P,Q], we know that the derivative's value can depend on  $\epsilon$ ,  $r_0$  and on the distribution of  $r(\boldsymbol{x})$  for  $\boldsymbol{x} \sim Q$ , a distribution we will name  $D_Q(r)$ , but it cannot depend on any other properties of Y or r, because any changes to Y and r that do not affect  $\epsilon$ ,  $r_0$  and  $D_Q(r)$  will yield the same c[P,Q] values throughout the calculation of  $[\nabla_Y(\tilde{L})](r)$ . Hence, we can describe the derivative as  $\Delta_{\mathbf{x}}(\epsilon, r_0|D_Q(r))$ .

As a next step, consider what would happen if we were to partition Y into 2 sets, each of measure  $\epsilon/2$  in Q. (Such a partition is straightforward to produce in the continuous case. If X is discrete, one can create it by utilising ISI: translate  $\boldsymbol{x}$  to  $(\boldsymbol{x}, \boldsymbol{y})$ , where  $\boldsymbol{y}$  is a Bernoulli random variable with p=1/2.)

The marginal impact of each set on the value of L is  $\Delta_{\mathbf{x}}(\epsilon/2, r_0|D_Q(r))$ , but their total impact is  $\Delta_{\mathbf{x}}(\epsilon, r_0|D_Q(r))$ . More generally, either using the problem's continuity or using ISI and L's smoothness, we can describe  $\Delta_{\mathbf{x}}(\epsilon, r_0|D_Q(r))$  as  $\epsilon \cdot \Delta_{\mathbf{x}}(r_0|D_Q(r))$ .

Utilising  $L(\theta_1, \theta_2)$ 's representation as  $\tilde{L}(r)$ , where  $Q = P_{\theta_2}$  and  $r = r_{P_{\theta_1}, P_{\theta_2}}$ , and noting that  $\epsilon$  was, in the calculation above, the measure of Y in  $Q = P_{\theta_2}$ , we can now write the first derivative of L in some direction i of  $\theta_1$  explicitly as an integral in this measure, i.e. in " $dP_{\theta_2}$ ".

For clarity of presentation, we will write this as an integral in " $f_{\theta_2}(x) dx$ ", using here and throughout the remainder of the proof pdf notation, as would be appropriate when x is known to be continuous. This change is meant merely to simplify the notation, and in no way restricts the proof. Readers are welcome to verify that all steps are equally valid for any  $P_{\theta_1}$  and  $P_{\theta_2}$  distributions.

Let  $R_{\theta_1,\theta_2}$  be  $D_{f_{\theta_2}}(r_{f_{\theta_1},f_{\theta_2}})$ . If  $c[f_{\theta_1},f_{\theta_2}]$  is a piecewise-constant function, the derivative of L can be written as follows.

$$\frac{\partial L(\theta_1,\theta_2)}{\partial \theta_1(i)} = \int_X \Delta_{\mathbf{x}}(r_{f_{\theta_1},f_{\theta_2}}(x)|R_{\theta_1,\theta_2}) \frac{\partial r_{f_{\theta_1},f_{\theta_2}}(x)}{\partial \theta_1(i)} f_{\theta_2}(x) \mathrm{d}x.$$

The same reasoning can be used to describe the second derivative of L (this time in the directions i and j of  $\theta_1$ ). The second derivative of  $\tilde{L}(r)$  when perturbing r relative to two subsets  $Y_1$  and  $Y_2$  is defined as

$$\lim_{\Delta \to 0} \frac{\left[\nabla_{Y_1}(\tilde{L})\right] \left(r + \Delta \cdot \chi_{Y_2}\right) - \left[\nabla_{Y_1}(\tilde{L})\right] \left(r\right)}{\Delta},$$

and once again using the assumption that L(P,Q) only depends on c[P,Q], we can see that if  $Y_1$  and  $Y_2$  are disjoint, if the measures of  $Y_1$  and  $Y_2$  in Q are, respectively,  $\epsilon_1$  and  $\epsilon_2$ , both positive, and if the value of  $r_{P,Q}(x)$  for x values in each subset is a constant, respectively  $r_1$  and  $r_2$ , then any such  $Y_1$  and  $Y_2$  will perturb  $\tilde{L}(r)$  in exactly the same amount over any r with the same  $D_Q(r)$ . We name the second derivative coefficient in this case  $\Delta_{xy}(r_1, r_2|D_Q(r))$ .

The caveat that  $Y_1$  and  $Y_2$  must be disjoint is important, because if  $Y = Y_1 = Y_2$  the symmetry no longer holds. This is a second case, and for it we must define a different coefficient  $\Delta_{xx}(r_0|D_Q(r))$ , where  $r_0 = r_1 = r_2$ .

In the case where  $c[f_{\theta_1}, f_{\theta_2}]$  is a piecewise-constant function, the second derivative can therefore be written as

$$\frac{\partial^{2}L(\theta_{1},\theta_{2})}{\partial\theta_{1}(i)\partial\theta_{1}(j)} = \int_{X} \Delta_{\mathbf{x}}(r_{f_{\theta_{1}},f_{\theta_{2}}}(x)|R_{\theta_{1},\theta_{2}}) \frac{\partial^{2}r_{f_{\theta_{1}},f_{\theta_{2}}}(x)}{\partial\theta_{1}(i)\partial\theta_{1}(j)} f_{\theta_{2}}(x) dx 
+ \int_{X} \int_{X} \Delta_{\mathbf{x}\mathbf{y}}(r_{f_{\theta_{1}},f_{\theta_{2}}}(x_{1}), r_{f_{\theta_{1}},f_{\theta_{2}}}(x_{2})|R_{\theta_{1},\theta_{2}}) 
- \frac{\partial r_{f_{\theta_{1}},f_{\theta_{2}}}(x_{1})}{\partial\theta_{1}(i)} \frac{\partial r_{f_{\theta_{1}},f_{\theta_{2}}}(x_{2})}{\partial\theta_{1}(j)} f_{\theta_{2}}(x_{2}) dx_{2} f_{\theta_{2}}(x_{1}) dx_{1} 
+ \int_{X} \Delta_{\mathbf{x}\mathbf{x}}(r_{f_{\theta_{1}},f_{\theta_{2}}}(x)|R_{\theta_{1},\theta_{2}}) \frac{\partial r_{f_{\theta_{1}},f_{\theta_{2}}}(x)}{\partial\theta_{1}(i)} \frac{\partial r_{f_{\theta_{1}},f_{\theta_{2}}}(x)}{\partial\theta_{1}(j)} f_{\theta_{2}}(x) dx.$$
(A.9)

In order to compute the Hessian matrix  $H_L^{\theta}$ , we consider each matrix element  $H_L^{\theta}(i,j)$  in turn.

This equals  $\partial^2 L(\theta_1, \theta_2)/\partial \theta_1(i)\partial \theta_1(j)$  where  $\theta_1 = \theta_2 = \theta$ . In particular,  $c[f_{\theta_1}, f_{\theta_2}]$  is  $\chi_{(0,1]}$  and  $R_{\theta_1,\theta_2}$  is the distribution 1, which yields the constant value 1.

The value of (A.9) in this case becomes

$$\Delta_{\mathbf{x}}(1|\mathbf{1}) \int_{X} \frac{\partial^{2} r_{f_{\theta_{1}},f_{\theta}}(x)}{\partial \theta_{1}(i) \partial \theta_{1}(j)} \bigg|_{\theta_{1}=\theta} f_{\theta}(x) dx 
+ \Delta_{\mathbf{x}\mathbf{y}}(1,1|\mathbf{1}) \left( \int_{X} \frac{\partial r_{f_{\theta_{1}},f_{\theta}}(x)}{\partial \theta_{1}(i)} \bigg|_{\theta_{1}=\theta} f_{\theta}(x) dx \right) \left( \int_{X} \frac{\partial r_{f_{\theta_{1}},f_{\theta}}(x)}{\partial \theta_{1}(j)} \bigg|_{\theta_{1}=\theta} f_{\theta}(x) dx \right) 
+ \Delta_{\mathbf{x}\mathbf{x}}(1|\mathbf{1}) \int_{X} \left( \frac{\partial r_{f_{\theta_{1}},f_{\theta}}(x)}{\partial \theta_{1}(i)} \bigg|_{\theta_{1}=\theta} \right) \left( \frac{\partial r_{f_{\theta_{1}},f_{\theta}}(x)}{\partial \theta_{1}(j)} \bigg|_{\theta_{1}=\theta} \right) f_{\theta}(x) dx.$$
(A.10)

Note, however, that because  $(x, \theta)$  is an estimation problem, i.e. all its conditional data distributions are probability measures, not general measures, it is the case that

$$\int_X r_{f_{\theta_1}, f_{\theta}}(x) f_{\theta}(x) dx = \int_X f_{\theta_1}(x) dx = 1,$$

and is therefore a constant independent of either  $\theta_1$  or  $\theta$ . Its various derivatives in  $\theta_1$  are accordingly all zero. This makes the first two summands in (A.10) zero. What is left, when setting  $\gamma = \Delta_{xx}(1|1)$ , is

$$H_L^{\theta}(i,j) = \gamma \int_X \left( \frac{\partial r_{f_{\theta_1},f_{\theta}}(x)}{\partial \theta_1(i)} \Big|_{\theta_1 = \theta} \right) \left( \frac{\partial r_{f_{\theta_1},f_{\theta}}(x)}{\partial \theta_1(j)} \Big|_{\theta_1 = \theta} \right) f_{\theta}(x) dx$$

$$= \gamma \int_X \left( \frac{\partial f_{\theta_1}(x)/f_{\theta}(x)}{\partial \theta_1(i)} \Big|_{\theta_1 = \theta} \right) \left( \frac{\partial f_{\theta_1}(x)/f_{\theta}(x)}{\partial \theta_1(j)} \Big|_{\theta_1 = \theta} \right) f_{\theta}(x) dx$$

$$= \gamma \int_X \left( \frac{\partial \log f_{\theta}(x)}{\partial \theta(i)} \right) \left( \frac{\partial \log f_{\theta}(x)}{\partial \theta(j)} \right) f_{\theta}(x) dx$$

$$= \gamma \mathbf{E}_{\mathbf{x} \sim f_{\theta}} \left[ \left( \frac{\partial \log f_{\theta}(\mathbf{x})}{\partial \theta(i)} \right) \left( \frac{\partial \log f_{\theta}(\mathbf{x})}{\partial \theta(j)} \right) \right]$$

$$= \gamma \mathcal{I}_{\theta}(i, j).$$

Hence,  $H_L^{\theta} = \gamma \mathcal{I}_{\theta}$ .

The only difference in this derivation for the case where x is discrete, is that in this case the final result would have used probabilities rather than probability densities. This is consistent, however, with the way Fisher information is defined in this more general case. In fact, some sources (e.g., Bobkov et al., 2014) define the Fisher information directly from Radon-Nikodym derivatives.

As a final point in the proof, we remark that  $\gamma$  must be nonzero, because had it been zero,  $H_L^{\theta}$  would have been zero for every  $\theta$  in every elementary  $\theta$ -continuous estimation problem of the same type, contrary to our sensitivity assumption on the loss function.

**Theorem A.7** (Theorem 3.6 of the main text). If  $(x, \theta)$  is an elementary, semi-continuous estimation problem for which  $\hat{\theta}_{WF}$  is a well-defined set estimator, and if L is a smooth loss function, sensitive on  $(x, \theta)$ 's type, that satisfies all of IIA, IRP, IRO and ISI, then

$$\hat{\theta}_{EIC}^L = \hat{\theta}_{WF}.$$

*Proof.* The proof is essentially identical to that of Theorem A.2. The only change is that we can no longer apply Lemma A.5. However, we claim that L(P,Q) only depends on c[P,Q] and M despite this, for which reason we can still apply Lemma A.6, as before, to complete the proof.

Fix M, the dimension of  $\Theta$ .

To prove that L(P,Q) only depends on c[P,Q], let  $(\boldsymbol{x},\boldsymbol{\theta})$  be an elementary, semi-continuous estimation problem, and let L be any smooth loss function, sensitive on its type, that satisfies all of IIA, IRP, IRO and ISI. Furthermore, fix  $\theta_1,\theta_2\in\Theta$ , and let P be the distribution  $P_{\theta_1}$  and Q be the distribution  $P_{\theta_2}$ . We will show that L(P,Q) can only depend on c[P,Q].

We note that we can freely assume  $P \neq Q$ , because

$$c[P,Q] = \chi_{(0,1]} \Leftrightarrow P = Q \Leftrightarrow L(P,Q) = 0,$$

where the first equivalence follows from the definition of c[P,Q] (for discrete distributions P and Q) and the second equivalence follows from (1.1).

The assumption  $P \neq Q$  also implies that the two distributions are not linearly dependent.

We begin by describing a new estimation problem  $((\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}), \boldsymbol{\theta})$ , where  $\boldsymbol{y}$  is a random variable independent of  $\boldsymbol{x}$  and of  $\boldsymbol{\theta}$ , which is uniformly distributed in  $\{1, \dots, 2^M\}$ , and where  $\boldsymbol{z} = r_{P,Q}(\boldsymbol{x})$ .

We know that  $L_{(\boldsymbol{x},\boldsymbol{\theta})}(\theta_1,\theta_2) = L_{((\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}),\boldsymbol{\theta})}(\theta_1,\theta_2) = L_{((\boldsymbol{z},\boldsymbol{y},\boldsymbol{x}),\boldsymbol{\theta})}(\theta_1,\theta_2)$ , because the first equality stems from ISI and the second from IRO.

In principle, we now wish to apply ISI again, in order to show that

$$L_{((\boldsymbol{z},\boldsymbol{y},\boldsymbol{x}),\boldsymbol{\theta})}(\theta_1,\theta_2) = L_{((\boldsymbol{z},\boldsymbol{y}),\boldsymbol{\theta})}(\theta_1,\theta_2), \tag{A.11}$$

because if we can establish (A.11) then we are done. The reason for this is that from Lemma A.3 we know that the value of  $L_{((z,y),\theta)}(\theta_1,\theta_2)$  can only depend on the conditional data distributions at  $\theta_1$  and  $\theta_2$ , and by construction these conditional data distributions depend only on the distribution of  $z = r_{P,Q}(x)$  at  $x \sim P$  and at  $x \sim Q$ , in both of which the distribution of z is fully constructible from knowledge of c[P,Q].

Unfortunately, it is not possible for us to apply ISI directly to prove (A.11). The reason for this is that even though (as we will demonstrate) x is independent of the choice of  $\theta_1$  versus  $\theta_2$  given (z, y), this independence may not extend to all other parameter choices in  $\Theta$ .

We therefore first apply Lemma A.3 in order to show that  $L_{((\boldsymbol{z},\boldsymbol{y},\boldsymbol{x}),\boldsymbol{\theta})}(\theta_1,\theta_2) = L_{((\boldsymbol{z},\boldsymbol{y},\boldsymbol{x}),\boldsymbol{\theta}')}(\theta_1',\theta_2')$ , for any elementary estimation problem  $((\boldsymbol{z},\boldsymbol{y},\boldsymbol{x}),\boldsymbol{\theta}')$  over any parameter domain  $\Theta'$  of the same dimension, as long as the distribution of  $(\boldsymbol{z},\boldsymbol{y},\boldsymbol{x})$  given  $\boldsymbol{\theta}' = \theta_1'$  is the same as its distribution given  $\boldsymbol{\theta} = \theta_1$ , and its distribution given  $\boldsymbol{\theta}' = \theta_2'$  is the same as given  $\boldsymbol{\theta} = \theta_2$ .

We will construct such an estimation problem where x is independent of  $\theta'$  given (z, y) across the entire range of  $\theta'$ .

To do this, let  $\Theta'$  be the unit cube of dimension M, and let  $\theta'_1 = (1, 0, ..., 0)$  and  $\theta'_2 = (0, ..., 0)$ . The conditional data distributions at  $\theta'_1$  and  $\theta'_2$  will be the same as in  $\theta_1$  and  $\theta_2$  in the original problem, so as to enable us to apply Lemma A.3.

We will retain the definition of z as  $r_{P,Q}(x)$ . The variable y will remain independent of x, but in the new problem it will depend on  $\theta'$ . Specifically, it will remain uniformly distributed over  $\{1, \ldots, 2^M\}$  only for  $\theta'$  values of the form  $(\alpha, 0, \ldots, 0)$ . In the remaining  $2^M - 2$  corners of the unit cube, y will have other distributions, arbitrarily chosen from all distributions over the same support, such that all  $2^M - 1$  such distributions of y are linearly independent (which is clearly possible to do).

The distribution of x will be P at all corners of the unit cube other than in the origin.

Elsewhere in the unit cube, away from its corners, the distribution of (z, y, x) will be the multilinear continuation of their distributions at the corners. Because, by the construction of y, we know the corner distributions to all be linearly independent other than at  $\theta'_1$  and  $\theta'_2$ , the constructed problem is well-defined (in that it satisfies (1.1)), and is elementary.

We will now show that  $\boldsymbol{x}$  is independent of  $\boldsymbol{\theta'}$  given  $(\boldsymbol{z}, \boldsymbol{y})$ . Consider any specific value x of  $\boldsymbol{x}$  and y of  $\boldsymbol{y}$ , and let  $z = r_{P,Q}(x)$  be the  $\boldsymbol{z}$  value corresponding to  $\boldsymbol{x} = x$ . Furthermore, let  $X_z$  be the set of all values of  $\boldsymbol{x}$  that share the same  $\boldsymbol{z}$  value. The variable  $\boldsymbol{x}$  is independent of  $\boldsymbol{\theta'}$  given  $(\boldsymbol{z}, \boldsymbol{y})$  if for every such x, the conditional probability  $\mathbf{P}(\boldsymbol{x} = x | (\boldsymbol{z}, \boldsymbol{y}) = (z, y), \boldsymbol{\theta'} = \boldsymbol{\theta'})$  is not a function of  $\boldsymbol{\theta'}$ . This conditional probability can be computed as

$$\mathbf{P}(\boldsymbol{x} = \boldsymbol{x} | (\boldsymbol{z}, \boldsymbol{y}) = (\boldsymbol{z}, \boldsymbol{y}), \boldsymbol{\theta'} = \boldsymbol{\theta'}) = \frac{\mathbf{P}(\boldsymbol{x} = \boldsymbol{x}, \boldsymbol{y} = \boldsymbol{y} | \boldsymbol{\theta'} = \boldsymbol{\theta'})}{\sum_{x' \in X_z} \mathbf{P}(\boldsymbol{x} = \boldsymbol{x'}, \boldsymbol{y} = \boldsymbol{y} | \boldsymbol{\theta'} = \boldsymbol{\theta'})}$$

$$= \frac{\mathbf{P}(\boldsymbol{x} = \boldsymbol{x} | \boldsymbol{\theta'} = \boldsymbol{\theta'}) \mathbf{P}(\boldsymbol{y} = \boldsymbol{y} | \boldsymbol{\theta'} = \boldsymbol{\theta'})}{\sum_{x' \in X_z} \mathbf{P}(\boldsymbol{x} = \boldsymbol{x'} | \boldsymbol{\theta'} = \boldsymbol{\theta'}) \mathbf{P}(\boldsymbol{y} = \boldsymbol{y} | \boldsymbol{\theta'} = \boldsymbol{\theta'})}$$

$$= \frac{\mathbf{P}(\boldsymbol{x} = \boldsymbol{x} | \boldsymbol{\theta'} = \boldsymbol{\theta'})}{\sum_{x' \in X_z} \mathbf{P}(\boldsymbol{x} = \boldsymbol{x'} | \boldsymbol{\theta'} = \boldsymbol{\theta'})}.$$
(A.12)

Consider the distribution of x at  $\theta' = \theta'$ . By construction, it is  $\gamma P + (1 - \gamma)Q$ , for some  $0 \le \gamma \le 1$ . Substituting this into (A.12), we get

$$\mathbf{P}(\boldsymbol{x} = \boldsymbol{x} | (\boldsymbol{z}, \boldsymbol{y}) = (\boldsymbol{z}, \boldsymbol{y}), \boldsymbol{\theta'} = \boldsymbol{\theta'}) = \frac{\gamma \mathbf{P}_{\boldsymbol{x} \sim P}(\boldsymbol{x}) + (1 - \gamma) \mathbf{P}_{\boldsymbol{x} \sim Q}(\boldsymbol{x})}{\sum_{\boldsymbol{x'} \in X_*} \gamma \mathbf{P}_{\boldsymbol{x} \sim P}(\boldsymbol{x'}) + (1 - \gamma) \mathbf{P}_{\boldsymbol{x} \sim Q}(\boldsymbol{x'})}$$

$$\begin{split} &= \frac{\gamma r_{P,Q}(x)\mathbf{P}_{\boldsymbol{x}\sim Q}(x) + (1-\gamma)\mathbf{P}_{\boldsymbol{x}\sim Q}(x)}{\sum_{x'\in X_z}\gamma r_{P,Q}(x')\mathbf{P}_{\boldsymbol{x}\sim Q}(x') + (1-\gamma)\mathbf{P}_{\boldsymbol{x}\sim Q}(x')} \\ &= \frac{\gamma z\mathbf{P}_{\boldsymbol{x}\sim Q}(x) + (1-\gamma)\mathbf{P}_{\boldsymbol{x}\sim Q}(x)}{\sum_{x'\in X_z}\gamma z\mathbf{P}_{\boldsymbol{x}\sim Q}(x') + (1-\gamma)\mathbf{P}_{\boldsymbol{x}\sim Q}(x')} \\ &= \frac{(\gamma z + (1-\gamma))\mathbf{P}_{\boldsymbol{x}\sim Q}(x)}{\sum_{x'\in X_z}(\gamma z + (1-\gamma))\mathbf{P}_{\boldsymbol{x}\sim Q}(x')} \\ &= \frac{\mathbf{P}_{\boldsymbol{x}\sim Q}(x)}{\sum_{x'\in X_z}\mathbf{P}_{\boldsymbol{x}\sim Q}(x')}. \end{split}$$

This is clearly not a function of  $\theta'$ , and hence the independence of x is proved, and ISI can be employed to show

$$L_{((\boldsymbol{z},\boldsymbol{y},\boldsymbol{x}),\boldsymbol{\theta'})}(\theta'_1,\theta'_2) = L_{((\boldsymbol{z},\boldsymbol{y}),\boldsymbol{\theta'})}(\theta'_1,\theta'_2) = L_{((\boldsymbol{z},\boldsymbol{y}),\boldsymbol{\theta})}(\theta_1,\theta_2),$$

where the final equality is given, once again, by Lemma A.3.

Thus, L(P,Q) depends only on c[P,Q] and M, and we can use Lemma A.6, as in Theorem A.2, to complete the proof.

# A.3 Feasibility and necessity

**Theorem A.8** (Theorem 4.3 of the main text). Our system of axioms is feasible, in the sense that there exist smooth and problem continuous loss functions, L, sensitive on all  $\theta$ -continuous problem types, that satisfy all of IRP, IRO, IIA and ISI.

While the existence of such a loss function L may not seem trivial, there are, in fact, many loss functions that satisfy the necessary requirements. We will show how to construct an infinite family of such L.

Our family of loss functions will be a subset of the f-divergences (Ali and Silvey, 1966).

An f-divergence is a loss function L that can be computed, most generally, as

$$L(P,Q) = \mathbf{E}_Q [F(r_{P,Q})].$$

This is usually expressed as

$$L(P,Q) = \int_{X} F(f_{P}(x)/f_{Q}(x))f_{Q}(x)dx,$$

which represents the case where P and Q are continuous distributions.

We call  $F: \mathbb{R}^{>0} \to \mathbb{R}$  the *F*-function of the *f*-divergence (refraining from using the more common term "*f*-function" so as to avoid unnecessary confusion with our probability density functions). It should be convex and satisfy F(1) = 0.

**Lemma A.9.** All f-divergences L satisfy all of IRP, IRO, IIA and ISI.

*Proof.* As we have already established in Lemma A.3, satisfying IRP and IIA, our two parameter-space axioms, is tantamount to requiring that L is a conditional-distribution-based loss function. The two observation-space axioms, IRO and ISI, add that the loss must depend only on a sufficient statistic for the observation. The former is proved directly from the definition of f-divergences. The latter is among the most basic properties of f-divergences (See, e.g., Amari and Nagaoka (2000)).

While all f-divergences satisfy IRP, IRO, IIA and ISI, not all of them satisfy smoothness, sensitivity and problem continuity.

**Lemma A.10.** Any f-divergence whose F-function satisfies the following conditions

- 1. F has 3 continuous derivatives, and
- 2. F''(1) > 0

is smooth and is sensitive on all heta-continuous problem types. If it also satisfies

- 1.  $\lim_{x\to 0} F(x) < \infty$ , and
- 2.  $\lim_{x\to\infty} F'(x) < \infty$ ,

it is also problem continuous.

Here, F' and F'' are the first two derivatives of F, which we assumed exist.

The main difficulty in Lemma A.10 is proving problem continuity. We will build up to this using the following two lemmas.

**Lemma A.11.** If  $F: \mathbb{R}^{>0} \to \mathbb{R}$  is continuously differentiable and convex, and the limit  $\lim_{x\to 0} F(x)$  is bounded, then there is a value  $x_0 > 0$ , such that for every x in  $(0, x_0]$ , F'(x) > -1/x, where F' is the derivative of F.

*Proof.* Because we know that F is continuously differentiable, we know that F' exists and is continuous in this range. Furthermore, because F is convex, F' is monotone increasing.

Let us assume, to the contrary, that no  $x_0$  as in the claim exists. Then there exists an infinite decreasing sequence  $x_1, \ldots$ , converging to zero, for which  $F'(x_i) \leq -1/x_i$ .

Because we know that F' is monotone increasing, an upper bound for it in the range  $(x_{i+1}, x_i]$  is  $F'(x_i)$  and therefore also  $-1/x_i$ .

Integrating this, we see that a lower bound for  $F(x_{i+1})$  is

$$F(x_{i+1}) \ge F(x_1) + \sum_{j=1}^{i} (x_j - x_{j+1})(1/x_j).$$

We will show, however, that as i goes to infinity, this lower bound also diverges to infinity, contradicting our assumption that  $\lim_{x\to 0} F(x)$  is bounded.

To see this, consider the series summand. This equals  $1 - x_{j+1}/x_j$ . For the series to converge, this summand must converge to 0, so for all j at least as large as some  $j_0$  we have that there is some r for which  $r \leq x_{j+1}/x_j \leq 1$ . In this range, we know that

$$\log(x_{j+1}/x_j) \ge (x_{j+1}/x_j - 1) \frac{\log(r)}{r - 1},$$

by the concavity of the log function.

We can therefore conclude the following regarding the series partial sum.

$$\sum_{j=j_0}^{\infty} 1 - x_{j+1}/x_j \ge -\frac{r-1}{\log(r)} \sum_{j=j_0}^{\infty} \log(x_{j+1}/x_j) = -\frac{r-1}{\log(r)} \lim_{j \to \infty} \log(x_j/x_{j_0}) = \infty,$$

proving the claim.

**Lemma A.12.** If  $F: \mathbb{R}^{>0} \to \mathbb{R}$  is the F-function of an f-divergence, F is continuously differentiable,  $\lim_{x\to 0} F(x) < \infty$  and  $\lim_{x\to \infty} F'(x) < \infty$ , where F' is the derivative of F, then the f-divergence is a problem-continuous loss function.

*Proof.* Let P and Q be probability distributions over a ground set X, and let  $\{P_i\}$  and  $\{Q_i\}$  be sequences of distributions over ground sets  $X_i$ , converging in measure to P and Q, respectively.

The condition of convergence in measure is that for any  $\epsilon > 0$  and any  $\delta > 0$  there is an  $i_0$ , such that for every  $i \geq i_0$ ,

$$\mathbf{P}_{\boldsymbol{x} \sim P} \left( 1 - \epsilon \le \frac{\mathrm{d}P_i}{\mathrm{d}P}(\boldsymbol{x}) \le 1 + \epsilon \right) \ge 1 - \delta,$$

and

$$\mathbf{P}_{\boldsymbol{x} \sim Q} \left( 1 - \epsilon \le \frac{\mathrm{d}Q_i}{\mathrm{d}Q}(\boldsymbol{x}) \le 1 + \epsilon \right) \ge 1 - \delta.$$

We wish to prove that if these conditions hold, then the f-divergences between  $P_i$  and  $Q_i$  converge to the f-divergence between P and Q, where the f-divergence between P and Q is calculated as

$$D(P,Q) = \mathbf{E}_{\boldsymbol{x} \sim P} \left[ F\left(\frac{\mathrm{d}Q}{\mathrm{d}P}(\boldsymbol{x})\right) \right].$$

While these formulations, using the Radon-Nikodym derivative dQ/dP, are the most general, for convenience of presentation, we will consider all distributions as continuous, describable by their probability density functions (pdfs). We will use  $f_P$  to describe the pdf of P. Our proof will remain entirely general, but the use of pdfs to describe distributions will make their presentations easiest.

Using pdfs, the claims can be reworded as follows. We know that for all  $\epsilon > 0$  and  $\delta > 0$  a sufficiently large i will satisfy

$$\mathbf{P}_{\boldsymbol{x} \sim P} \left( 1 - \epsilon \le \frac{f_{P_i}(\boldsymbol{x})}{f_{P}(\boldsymbol{x})} \le 1 + \epsilon \right) \ge 1 - \delta$$

and

$$\mathbf{P}_{\boldsymbol{x} \sim Q} \left( 1 - \epsilon \le \frac{f_{Q_i}(\boldsymbol{x})}{f_Q(\boldsymbol{x})} \le 1 + \epsilon \right) \ge 1 - \delta,$$

and wish to prove that

$$\lim_{i \to \infty} \int_{X_i} F\left(\frac{f_{Q_i}(x)}{f_{P_i}(x)}\right) f_{P_i}(x) dx = \int_X F\left(\frac{f_Q(x)}{f_P(x)}\right) f_P(x) dx. \tag{A.13}$$

For a given  $\epsilon$  and a given i, let  $X_i^{\epsilon}$  be the subset of X wherein

$$1 - \epsilon \le \frac{f_{P_i}(x)}{f_P(x)} \le 1 + \epsilon \tag{A.14}$$

and

$$1 - \epsilon \le \frac{f_{Q_i}(x)}{f_{O}(x)} \le 1 + \epsilon \tag{A.15}$$

are both satisfied.

To prove (A.13), we will separate the difference between the left-hand side and the right-hand side to four elements, then show that all four must equal zero. The four sub-differences we will look at are:

$$\lim_{i \to \infty} \int_{X \setminus X_i^{\epsilon}} F\left(\frac{f_Q(x)}{f_P(x)}\right) f_P(x) \mathrm{d}x,\tag{A.16}$$

$$\lim_{i \to \infty} \int_{X_i \setminus X_i^{\epsilon}} F\left(\frac{f_{Q_i}(x)}{f_{P_i}(x)}\right) f_{P_i}(x) \mathrm{d}x,\tag{A.17}$$

$$\lim_{i \to \infty} \int_{X_i^{\epsilon}} \left| F\left(\frac{f_{Q_i}(x)}{f_{P_i}(x)}\right) \right| |f_{P_i}(x) - f_P(x)| \mathrm{d}x \tag{A.18}$$

and

$$\lim_{i \to \infty} \int_{X_i^{\epsilon}} \left| F\left(\frac{f_{Q_i}(x)}{f_{P_i}(x)}\right) - F\left(\frac{f_Q(x)}{f_P(x)}\right) \right| f_P(x) dx. \tag{A.19}$$

Regarding  $X \setminus X_i^{\epsilon}$ , we know that it has some measure  $\mu_P = \mu_P(\epsilon, i)$  in P and some measure  $\mu_Q = \mu_Q(\epsilon, i)$  in Q, both tending to 0 as i goes to infinity, for any  $\epsilon$ . Correspondingly,  $X_i^{\epsilon}$  has measure  $1 - \mu_P$  in P and  $1 - \mu_Q$  in Q.

Because within  $X_i^{\epsilon}$  (A.14) and (A.15) hold, the measure of  $X_i^{\epsilon}$  in  $P_i$  is at least  $(1 - \mu_P)(1 - \epsilon)$  and its measure in  $Q_i$  is at least  $(1 - \mu_Q)(1 - \epsilon)$ . Correspondingly, the

measure of  $X_i \setminus X_i^{\epsilon}$  in  $P_i$  is at most  $\mu_P + \epsilon - \mu_P \epsilon$ , and its measure in  $Q_i$  is at most  $\mu_Q + \epsilon - \mu_Q \epsilon$ .

We have assumed that  $F: \mathbb{R}^{>0} \to \mathbb{R}$  is continuously differentiable and that it is convex (because it is an F-function for an f-divergence). Under these preconditions, the conditions assumed,  $\lim_{x\to 0} F(x) < \infty$  and  $\lim_{x\to \infty} F'(x) < \infty$ , can easily be shown to be equivalent to the condition that there exists  $A \geq 0$  and  $B \geq 0$  such that for all x,  $|F(x)| \leq Ax + B$ .

Using A and B, we can now bound (A.16) as follows.

$$\lim_{i \to \infty} \int_{X \setminus X_i^{\epsilon}} F\left(\frac{f_Q(x)}{f_P(x)}\right) f_P(x) \mathrm{d}x \le \lim_{i \to \infty} \int_{X \setminus X_i^{\epsilon}} \left(A \frac{f_Q(x)}{f_P(x)} + B\right) f_P(x) \mathrm{d}x$$

$$= \lim_{i \to \infty} \int_{X \setminus X_i^{\epsilon}} A f_Q(x) \mathrm{d}x + \int_{X \setminus X_i^{\epsilon}} B f_P(x) \mathrm{d}x$$

$$= \lim_{i \to \infty} A \mu_Q + B \mu_P$$

$$= 0.$$

We bound (A.17) similarly.

$$\begin{split} \lim_{i \to \infty} \int_{X_i \setminus X_i^{\epsilon}} F\left(\frac{f_{Q_i}(x)}{f_{P_i}(x)}\right) f_{P_i}(x) \mathrm{d}x &\leq \lim_{i \to \infty} \int_{X_i \setminus X_i^{\epsilon}} \left(A\frac{f_{Q_i}(x)}{f_{P_i}(x)} + B\right) f_{P_i}(x) \mathrm{d}x \\ &= \lim_{i \to \infty} \int_{X_i \setminus X_i^{\epsilon}} A f_{Q_i}(x) \mathrm{d}x + \int_{X_i \setminus X_i^{\epsilon}} B f_{P_i}(x) \mathrm{d}x \\ &\leq \lim_{i \to \infty} A(\mu_Q + \epsilon - \mu_Q \epsilon) + B(\mu_P + \epsilon - \mu_Q \epsilon) \\ &= (A + B) \epsilon. \end{split}$$

Regarding (A.18), note that within  $X_i^{\epsilon}$  we know that  $|f_{P_i}(x) - f_P(x)| \leq f_{P_i}(x) \frac{\epsilon}{1-\epsilon}$ . For this reason, we can bound (A.18) as follows.

$$\lim_{i \to \infty} \int_{X_i^{\epsilon}} \left| F\left(\frac{f_{Q_i}(x)}{f_{P_i}(x)}\right) \right| |f_{P_i}(x) - f_P(x)| dx$$

$$\leq \lim_{i \to \infty} \int_{X_i^{\epsilon}} \left( A \frac{f_{Q_i}(x)}{f_{P_i}(x)} + B \right) \left( \frac{\epsilon}{1 - \epsilon} \right) f_{P_i}(x) dx$$

$$\leq \lim_{i \to \infty} \left( \frac{\epsilon}{1 - \epsilon} \right) \left( \int_{X_i} A f_{Q_i}(x) dx + \int_{X_i} B f_{P_i}(x) dx \right)$$

$$= \left( \frac{\epsilon}{1 - \epsilon} \right) (A + B).$$

It now only remains to bound (A.19). This is once again an integral over  $X_i^{\epsilon}$ , which is a domain where we know that

$$\frac{1-\epsilon}{1+\epsilon} \left( \frac{f_Q(x)}{f_P(x)} \right) \le \frac{f_{Q_i}(x)}{f_{P_i}(x)} \le \frac{1+\epsilon}{1-\epsilon} \left( \frac{f_Q(x)}{f_P(x)} \right).$$

Let us therefore define  $\alpha(x)$  as the value in the range

$$\frac{1-\epsilon}{1+\epsilon} \le 1 + \alpha(x) \le \frac{1+\epsilon}{1-\epsilon}$$

that maximises

$$\left| F\left( (1+\alpha(x)) \frac{f_Q(x)}{f_P(x)} \right) - F\left( \frac{f_Q(x)}{f_P(x)} \right) \right|.$$

By the Mean Value Theorem, for each x there is a  $\gamma(x)$ ,

$$\frac{1-\epsilon}{1+\epsilon} \le 1 + \gamma(x) \le \frac{1+\epsilon}{1-\epsilon},$$

such that

$$F\left((1+\alpha(x))\frac{f_Q(x)}{f_P(x)}\right) - F\left(\frac{f_Q(x)}{f_P(x)}\right) = F'\left((1+\gamma(x))\frac{f_Q(x)}{f_P(x)}\right)\alpha(x)\frac{f_Q(x)}{f_P(x)}.$$

The value of (A.19) can therefore be bounded from above by

$$\begin{split} &\lim_{i \to \infty} \int_{X_i^\epsilon} \left| F\left(\frac{f_{Q_i}(x)}{f_{P_i}(x)}\right) - F\left(\frac{f_Q(x)}{f_P(x)}\right) \right| f_P(x) \mathrm{d}x \\ &\leq \lim_{i \to \infty} \int_{X_i^\epsilon} \left| F'\left((1 + \gamma(x)) \frac{f_Q(x)}{f_P(x)}\right) \right| |\alpha(x)| \frac{f_Q(x)}{f_P(x)} f_P(x) \mathrm{d}x. \end{split}$$

As these equations should be valid for all  $\epsilon$ , let us assume that  $\epsilon \leq 1/4$ , in which case  $\alpha(x)$  and  $\gamma(x)$  are both bounded in the range [-2/5, 2/3].

Because of our assumption that  $\lim_{y\to 0} F(y) < \infty$ , we can now use Lemma A.11 to know that there is a  $y_0$  such that if  $y \le y_0$ , the value of F'(y) must be greater than -1/y.

Let us now partition  $X_i^{\epsilon}$  to those x values for which  $(1 + \gamma(x)) \frac{f_Q(x)}{f_P(x)} \leq y_0$  and  $F'\left((1 + \gamma(x)) \frac{f_Q(x)}{f_P(x)}\right) < 0$ , which we will refer to as  $L_i^{\epsilon}$ , and the rest, which we will refer to as  $H_i^{\epsilon}$ .

Within the domain  $L_i^{\epsilon}$ , we have

$$\lim_{i \to \infty} \int_{L_i^{\epsilon}} \left| F'\left( (1 + \gamma(x)) \frac{f_Q(x)}{f_P(x)} \right) \right| |\alpha(x)| \frac{f_Q(x)}{f_P(x)} f_P(x) dx$$

$$\leq \lim_{i \to \infty} \int_{L_i^{\epsilon}} \frac{|\alpha(x)|}{1 + \gamma(x)} \left( \frac{f_P(x)}{f_Q(x)} \right) \frac{f_Q(x)}{f_P(x)} f_P(x) dx$$

$$\leq \lim_{i \to \infty} 2\epsilon \frac{1 + \epsilon}{(1 - \epsilon)^2} \int_X f_P(x) dx$$

$$\leq \frac{40}{9}\epsilon$$

where the last equality plugs in our condition  $\epsilon \leq 1/4$ .

Finally, within the domain  $H_i^{\epsilon}$ , we know that |F'(x)| is bounded. This is because of the following conditions. First, because F is convex, F' is monotone increasing, so its supremum is  $\lim_{x\to\infty} F'(x)$ , which we assumed to be bounded. On the other hand, its values must either satisfy  $F'\left((1+\gamma(x))\frac{f_Q(x)}{f_P(x)}\right)\geq 0$ , in which case they are bounded from below by 0, or  $(1+\gamma(x))\frac{f_Q(x)}{f_P(x)}>y_0$ , in which case they are bounded from below by  $F'(y_0)$ .

Let  $A' \ge 0$  be an upper bound for |F'(x)|.

We now have

$$\lim_{i \to \infty} \int_{H_i^{\epsilon}} \left| F'\left( (1 + \gamma(x)) \frac{f_Q(x)}{f_P(x)} \right) \right| |\alpha(x)| \frac{f_Q(x)}{f_P(x)} f_P(x) dx \le \lim_{i \to \infty} A' \frac{2\epsilon}{1 - \epsilon} \int_X f_Q(x) dx$$

$$\le \epsilon \frac{8A'}{3},$$

where the last equality, as before, stems from our assumption  $\epsilon \leq 1/4$ .

In total, we've established that

$$\left| \lim_{i \to \infty} \int_{X_i} F\left(\frac{f_{Q_i}(x)}{f_{P_i}(x)}\right) f_{P_i}(x) dx - \int_X F\left(\frac{f_{Q}(x)}{f_{P}(x)}\right) f_{P}(x) dx \right|$$

$$\leq \left(\frac{7}{3}(A+B) + \frac{40}{9} + \frac{8A'}{3}\right) \epsilon,$$
(A.20)

under our assumption  $\epsilon \leq 1/4$ .

Because A, B and A' are nonnegative constants, independent of  $\epsilon$ , and because this equality must hold for any  $\epsilon$  in (0, 1/4], we conclude that the true difference on the left-hand side of (A.20) (which is independent of  $\epsilon$ ) must be zero, so (A.13) holds and the lemma is proved.

Proof of Lemma A.10. Any f-divergence whose F-function has 3 continuous derivatives satisfies the smoothness condition. This can be verified simply by explicitly computing the required derivatives from the f-divergence's formula. In particular, this computation shows that the Hessian equals F''(1) times the Fisher information matrix. Therefore, if F''(1) > 0 the f-divergence also satisfies sensitivity for all problem types. Problem continuity, if relevant, is then proved by Lemma A.12.

Proof of Theorem A.8. From Lemma A.9 and Lemma A.10, we know that it is enough to show an example of a loss function L that is an f-divergence satisfying the extra conditions of Lemma A.10.

A concrete example of a commonly-used L function satisfying all criteria is squared Hellinger distance (Pollard, 2002),

$$H^{2}(p,q) = \frac{1}{2} \int_{X} \left( \sqrt{p(x)} - \sqrt{q(x)} \right)^{2} dx,$$

which is the f-divergence whose F-function is  $F(r) = 1 - \sqrt{r}$ .

**Theorem A.13** (Theorem 4.4 of the main paper). All axioms used in Theorem 3.1 are necessary.

*Proof.* Much as the proof of feasibility was simply a proof by example, proving necessity will be done by counterexample: we will show alternate loss functions, L, leading to alternate estimations, that satisfy all axioms but one. For each such L, we will show that its  $|H_L^{\theta}|$  is not proportional to  $|\mathcal{I}_{\theta}|$ , and hence produces an estimator different to WF.

A well-known loss function that satisfies IRO and IIA but not IRP is quadratic loss,

$$L(\theta_1, \theta_2) = |\theta_1 - \theta_2|^2. \tag{A.22}$$

As was demonstrated in the main paper, an error intolerant estimator with this loss function yields the continuous MAP estimate, different to the WF estimate.

A loss function satisfying IRP and IIA but not IRO is

$$L(P,Q) = \int_{Y} f_{Q}(x)(f_{P}(x) - f_{Q}(x))^{2} dx,$$
 (A.23)

which is the expected square difference between the probability densities at  $x \sim Q$ . Calculating  $H_L^{\theta}$  we get

$$H_L^{\theta}(i,j) = \mathbf{E}_{\boldsymbol{x} \sim f_{\theta}} \left[ 2 \left( \frac{\partial f_{\theta}(\boldsymbol{x})}{\partial \theta(i)} \right) \left( \frac{\partial f_{\theta}(\boldsymbol{x})}{\partial \theta(j)} \right) \right],$$

which is different to the Fisher information matrix, and defines an estimator that is not WF.

To prove necessity of our third axiom, IIA, we construct a loss function L that satisfies IRP and IRO but not IIA as follows. Let  $L_1$  and  $L_2$  be two smooth and problem continuous loss functions, sensitive on the relevant problem type, satisfying all axioms (such as, for example, two f-divergences matching the criteria of Lemma A.10) and let t be a threshold value.

Consider the function

$$P(\theta) = \mathbf{P}(L_1(\theta, \theta) \le t). \tag{A.24}$$

By construction, this function is independent of representation.

Define

$$L(\theta_1, \theta_2) = P(\theta_2)L_2(\theta_1, \theta_2). \tag{A.25}$$

The resulting  $|H_L^{\theta}|$  equals  $P(\theta)^M |H_{L_2}^{\theta}|$ , where the equality stems from the fact that  $P(\theta_2)$  is independent of  $\theta_1$  and therefore acts as a constant multiplier in the calculation of the Hessian.

This new estimator is different to the Wallace-Freeman estimator in the fact that it adds a weighing factor  $P(\theta)^{M/2}$ .

**Theorem A.14** (Theorem 4.5 of the main paper). All axioms used in Theorem 3.6 are necessary.

*Proof.* This proof is a direct continuation of the proof of Theorem A.13, and re-uses the same techniques and some of the same examples.

To begin with, quadratic loss, given in (A.22) as an example of a loss function that satisfies IIA and IRO but not IRP also demonstrates that IRP is necessary in the semi-continuous case because (by not depending on the estimation problem at all) it also satisfies ISI.

Similarly, the example of (A.25) can be re-used here to demonstrate the necessity of IIA also in the semi-continuous case. We have already shown that this loss function satisfies IRP and IRO. To show the remaining condition, namely that it also satisfies ISI, one merely needs to choose two loss functions  $L_1$  and  $L_2$  that satisfy the conditions of Theorem A.8. By assumption, these satisfy ISI, and so by construction (A.24) is independent of superfluous information and (A.25) must be, too, proving the claim.

To demonstrate that ISI is necessary, consider

$$L_1(P,Q) = \sum_{x \in X} \mathbf{P}_{\mathbf{x} \sim Q}(\mathbf{x} = x) \left( \mathbf{P}_{\mathbf{x} \sim P}(\mathbf{x} = x) - \mathbf{P}_{\mathbf{x} \sim Q}(\mathbf{x} = x) \right)^2, \tag{A.26}$$

which is the discrete analogue of (A.23).

This loss function clearly satisfies IRP and IIA, just as (A.23) does. Neither satisfies ISI, however: if we, for example, replace  $\mathbf{x}$  by  $(\mathbf{x}, \mathbf{y})$ , where  $\mathbf{y}$  is a Bernoulli random variable with p = 1/2, losses will not be preserved. (All losses will, in fact, be uniformly scaled down by a factor of 4.)

In the proof of Theorem A.13 we use (A.23) as an example of a loss function that does not satisfy IRO. In the semi-continuous domain, however, (A.26) does satisfy IRO. This is because the impact of applying piecewise-diffeomorphic deformations on discrete domains is quite different to applying them on continuous domains. In the continuous domain, such functions can stretch and contract the probability space, causing probability densities, such as those used in (A.23), to change. In the discrete domain, no such effects are possible: the probabilities used in (A.26) do not change as a result of deformations of the observation space, and so IRO is met.

However, as before, we can use the Hessian, which in this case is

$$H_L^{\theta}(i,j) = \mathbf{E}_{\boldsymbol{x} \sim P_{\theta}} \left[ 2 \left( \frac{\partial P_{\theta}(\boldsymbol{x})}{\partial \theta(i)} \right) \left( \frac{\partial P_{\theta}(\boldsymbol{x})}{\partial \theta(j)} \right) \right],$$

to show that the resulting estimator is different to WF, proving the necessity of ISI.

Lastly, we want to prove that IRO is necessary. For this, let x(1:k) be the value of x's first k dimensions, let x(k) be the value of its k'th dimension alone, and for a distribution P, if  $x \sim P$ , let  $P_k^y$  be the distribution of x(k) given that x(1:k-1) = y. Consider, now, a function  $L_N(P,Q)$  calculated in the following way.

$$L_N(P,Q) = \sum_{k=1}^{N} \mathbf{E}_{x \sim Q} \left[ L_1 \left( P_k^{x(1:k-1)}, Q_k^{x(1:k-1)} \right) \right],$$

where we reuse the function  $L_1$  from (A.26), but use it only to compare between distributions over a one-dimensional observation space.

By construction,  $L_N$  satisfies ISI for any  $L_1$ , because any dimension that does not add information also does not add to the value of  $L_N(P,Q)$ . Therefore, it satisfies all of IRP, IIA and ISI.

However, as was already demonstrated regarding  $L_1$  in the one-dimensional case, it does not lead to WF, proving the necessity of IRO.

**Theorem A.15** (Theorem 4.7 of the main text). MLE satisfies IRP, IRO, IIA and ISI.

*Proof.* Consider loss functions L of the form

$$L(\theta_1, \theta_2) = \gamma(\theta_2) L'(\theta_1, \theta_2),$$

where L' is a smooth and problem-continuous loss function, sensitive on the relevant problem type.

In the proof to Theorem 2.8 of the main text, we showed that any PMLE estimator can be described in this way, with L' being quadratic loss. Substituting the identity function for g in (2.7), we get that MLE can be described by

$$\gamma(\theta_2) = f(\theta_2)^{2/M}.$$

This L satisfies IRO, IIA and ISI, but not IRP. We conclude from it that MLE satisfies IRO, IIA and ISI.

To prove that MLE also satisfies IRP, let L' be squared Hellinger distance and let

$$\gamma(\theta_2) = \left(\frac{f(\theta_2)}{\sqrt{|\mathcal{I}_{\theta_2}|}}\right)^{2/M}.$$

This L, too, leads to MLE, because  $H_{L'}^{\theta_2}$  equals  $\mathcal{I}_{\theta_2}$ .

The new L satisfies IRP because both L' and  $\gamma$  are invariant to the representation of  $\theta$ .

We note that even though the theorem is solely about MLE, our construction is equally applicable to any PMLE estimator whose penalty function g satisfies the four expected invariance properties.

# **Appendix B: Supplementary discussion**

# **B.1** Comparison with MML

In the main text we discuss axioms that give rise to the Wallace-Freeman point estimator (WF). This estimator was originally developed in the Minimum Message Length (MML) literature, where it is most commonly referred to as the Wallace-Freeman approximation.

In this section we discuss in greater detail the relationship between Wallace-Freeman estimation as it appears in the MML literature, and how it appears here, in the context of Error Intolerant Estimation.

#### Introduction to MML

MML is a general name for any member of the family of Bayesian statistical inference methods based on the minimum message length principle. They are closely related to the family of minimum description length (MDL) estimators (Grünwald, 2007; Rissanen, 1987, 1999), but predate them.

The minimum message length principle was first introduced in Wallace and Boulton (1968), and the estimator that follows the principle directly, which was first described in Wallace and Boulton (1975), is known as Strict MML (SMML).

SMML can be defined as follows. Given an estimation problem  $(\boldsymbol{x}, \boldsymbol{\theta})$  and given an observation, x, we wish to choose  $\hat{\theta}(x)$  so that it optimally trades off two ideals. First, it must be a good approximation to  $\theta$ , the true value of  $\boldsymbol{\theta}$ , in that the distribution of  $\boldsymbol{x}$  given  $\boldsymbol{\theta} = \hat{\theta}(x)$  is a good approximation to its distribution given  $\boldsymbol{\theta} = \theta$ . Second, the choice of  $\hat{\theta}$  must be "simple", in the sense that it upholds the ideal of Occam's razor. Both these ideals can be formulated in information-theoretic terms.

Namely, suppose we wish to communicate both our estimate  $\hat{\theta}(x)$  and the observation x, and suppose, further, that we choose for this the following protocol known as the MML two-part message. First, we communicate  $\hat{\theta}(x)$  via the optimal communication protocol. (The choice of this optimal protocol is different, depending on what the function  $\hat{\theta}$  is.) Second, we communicate x via a protocol that is optimal only under the assumption that  $\theta = \hat{\theta}(x)$ .

The expected length of the first part of the message is the Shannon entropy (Shannon, 1948) of  $\hat{\theta}(\boldsymbol{x})$ . Thus, a "simpler" function  $\hat{\theta}: X \to \mathbb{R}^M$  will result in a shorter first part message in expectation. On the other hand, the second part of the message introduces inefficiencies due to its approximation that  $\boldsymbol{\theta} = \hat{\theta}(x)$ . Thus, the better the approximation the shorter the expected length of the second part of the message becomes

Strict MML chooses the function  $\hat{\theta}$  so that the overall length of the two-part message is minimised in expectation, creating, according to MML theory, an optimal trade-off between simplicity and accuracy.

Notably, this construction only works if X is countable (because otherwise the second

part of the message becomes infinite) and always results in a  $\hat{\theta}$  function whose range is only a countable subset of  $\Theta$ , even if  $\Theta$  itself is continuous (because otherwise the first part of the message becomes infinite).

SMML is, however, used also for continuous X. This is done by extending the construction above in one of several equivalent ways. For example, instead of optimising the message length, it is possible to optimise the excess message length, which is the difference between the expected message length and the information-theoretical optimum to deliver the information without restrictions on the message format. This optimum is the Shannon entropy of x and is independent of the choice of  $\hat{\theta}$ . By subtracting this entropy, the excess message length can be represented as a Kullback-Leibler divergence. Because this divergence is computable also on problems with a continuous X, it provides a natural way to extend the definition of SMML to the continuous domain.

Nevertheless, even in those cases, SMML will advocate a  $\hat{\theta}$  whose range is countable.

#### Wallace-Freeman Estimation

Strict MML, as a representation-invariant Bayesian point-estimation method, has much theoretical appeal, but it cannot be used in practice because it is computationally and analytically intractable in all but a select few single-parameter cases (Dowty, 2015). It was proved to be NP-Hard to compute even in estimating the parameters of a trinomial distribution (Farr and Wallace, 2002).

Even beyond the computational problems, SMML is a difficult estimate to work with. For example, it provides piecewise-constant estimates and can be asymmetrical even when working on an estimation problem exhibiting symmetry.

Wallace and Freeman (1987b) addressed these problems by developing the Wallace-Freeman estimator (WF) as a computationally convenient approximation to SMML. The approximation itself makes many assumptions regarding the underlying estimation problem, and, furthermore, makes no guarantees regarding the inaccuracy incurred by using it. The main guarantees given are that WF, like SMML, is representation-invariant, and that WF is in general not identical to SMML, because WF's estimates are a continuous function of x for problems with a smooth  $(x, \theta)$ , whereas SMML is piecewise constant.

## The evolution of MML justifications

Interestingly, though the MML criterion, as described above, is at heart an information-theoretic criterion, when Wallace and Boulton (1975) describe their motivation, it is not an information-theoretic one. The paper sets out with a motivation remarkably similar to ours here: it aims to extend discrete MAP into the continuous domain, while preserving certain good properties of discrete MAP that are not met by continuous MAP. Wallace and Boulton reference, specifically, the idea of representation invariance. They do not, however, provide any theoretical rationale regarding why one should start with the discrete MAP estimate, other than the following:

We do not here advance any argument in favour of [discrete MAP], save to note that it in some sense yields the most plausible, or least improbable account of what has been observed.

Wallace and Boulton derive Strict MML (SMML) as a representation invariant extension of discrete MAP by several approximations, chief of which is the discretisation of the observation space. This particular approximation is a choice that makes a significant difference to the meaning of representation invariance, and the approximate nature of the resulting solution can be observed directly, e.g. when applying it on a discrete problem, where discrete MAP can also be used: the SMML estimate and the discrete MAP estimate are not the same.

At the conclusion of Wallace and Boulton (1975), however, the paper points out that the SMML estimate can be given a different interpretation, this time from an information-theoretic perspective rather than a Bayesian perspective, and under this alternate description SMML becomes an exact solution.

According to Dowe (2008), this dual justification is not a fluke, but a reflection of the different authors' perspectives:

Chris [Wallace] was already a Bayesian from his mid-20s in the 1950s while David Boulton was clearly talking here in the spirit of (en)coding. Story has it that they had their separate approaches, went away and did their mathematics separately, re-convened about 6 weeks later and found out that they were doing essentially the same thing.

The paper then goes on to quote Wallace himself, from a talk given 2003, describing the birth of SMML as a synthesis of these two ideas:

[W]e came together again and I looked at his maths and he looked at my maths and we discovered that we had ended up at the same place. There really wasn't any difference between doing Bayesian analyses the way I thought one ought to do Bayesian analyses and doing data compression the way he thought you ought to do data compression. Great light dawned...

By the time of the writing of the other seminal paper of MML, Wallace and Freeman (1987a,b), where the Wallace-Freeman estimator was first introduced, the information-theoretic view of MML was already more deeply entrenched, but the authors nevertheless use justifications that align remarkably well with the justifications for error intolerant estimation.

They note, for example, as a weakness of the Bayes estimator approach that it can yield an estimate outside of the given parameter space, and stress the usefulness of their method in settling on a single hypothesis rather than on a weighted mixture of possibilities, even bringing up, as we do here, the need for this in scientific inference. They write:

[W] ould we be happy with a scientist who proposed a Bayesian mixture of a countably infinite set of incompatible models for electromagnetic fields?

Thus, the early motivations for MML align remarkably well with our motivations in deriving error intolerant estimation. However, they rely on a cascade of approximations, first from Discrete MAP to SMML, and then from SMML to WF. Regarding the second of these, Wallace and Freeman write:

Even if the approximations used in this paper cannot be justified, the general concepts can still be applied.

In other words, the authors acknowledge the lack of theoretical justification for their approximations, leave room for the possibility that better approximations can be found, and stress that the importance is in the underlying concepts that these results embody.

Throughout the years, other good properties were alleged for SMML and WF, which would have provided additional justification for them. For example, a general form of consistency for MML results was argued by Dowe and Wallace (1997). However, these good properties have since been refuted for both WF and Strict MML itself (Brand, 2019).

By 2005, when Wallace (2005), the canonical text of MML literature, was published, the information-theoretic view of MML was the orthodox view. Thus, Wallace (2005) describes Strict MML as an *exact* solution, not as a derivative of Discrete MAP, and WF as an approximation thereof.

In the same year, Comley and Dowe (2005) advocated a simple recipe for the MML practitioner to follow: if one's estimation problem has continuous parameters, one should use WF; if it is discrete—use Discrete MAP. Thus, while originally (working with a Bayesian motivation) Discrete MAP was the ideal starting point, and SMML—a derivative thereof, now, from an information-theoretic lens, SMML was seen as the ideal, and both Discrete MAP and WF were viewed as its approximations.

Figure 1 summarises this discussion, providing a graphical depiction of the relationships between Error Intolerant Estimation and MML, in their view of the Wallace-Freeman estimator: while MML sees the connections between Discrete MAP and WF as approximative and going through SMML as an intermediary, whether in the original Bayesian view in which these results were first presented or in the information-theoretic view that was later adopted, Error Intolerant Estimation derives exactly both Discrete MAP and the Wallace-Freeman estimator (which is not an "approximation" in this context), and does so using purely Bayesian reasoning and the presented axioms.

We believe this places the methodology of working with Discrete MAP in the discrete setting and with Wallace-Freeman estimation in a continuous setting on much stronger theoretical footing.

It not only explains the many successes of this recipe over the past 30 years, of which recent examples include Bregu et al. (2021b,a); Hlaváčková-Schindler and Plant (2020); Bourouis et al. (2021); Sumanaweera et al. (2018); Schmidt and Makalic (2016); Saikrishna et al. (2016); Jin and Rumantir (2015); Kasarapu et al. (2014), it also explains the recipe's failings, because it provides robust reasoning regarding why the method works, and therefore when we should expect it to underperform.

For example, in Webb and Petitjean (2016) the WF results are less competitive, but this is clear when analysed from an Error Intolerant Estimation perspective: the paper utilises WF in a context where inferences have to be made piecewise, serially. The success criterion in such a study is "best-fit for later inferences", not an error intolerant

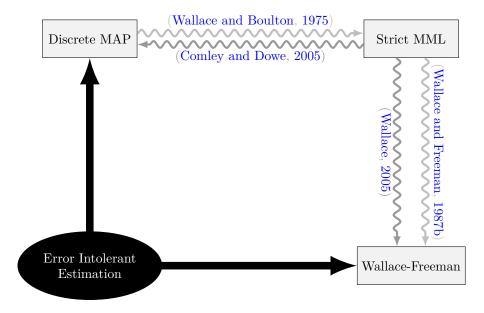


Figure 1: A graphical comparison of Error Intolerant Estimation's derivation of the Wallace-Freeman estimator with MML's. Squiggly lines represent approximations and straight lines represent exact derivations. In light grey: When Strict MML was first introduced in Wallace and Boulton (1975), Discrete MAP was taken as the ideal estimator, and SMML was derived as an approximation. Similarly, when WF was first introduced in Wallace and Freeman (1987b), it was a further, secondary approximation, but still in keeping with the view that these estimators formalise the ideal of finding a "most plausible" estimate. In dark grey: More modern MML texts (Comley and Dowe, 2005; Wallace, 2005) consider Strict MML to be the ideal estimator, and consider both Discrete MAP and WF its approximations. In black: Error Intolerant Estimation, on the other hand, derives both Discrete MAP and WF directly and independently from its axiomatic principles, and requires for this neither approximations nor intermediaries.

choice of a most-plausible "truth". Because of this, the error intolerance criterion is ill-fitting for the task, and the WF results, being overly conservative, were outperformed by other methods. MML reasoning alone does not predict this.

### B.2 Error intolerance and trade-off based methods

In the main paper, we describe error intolerant estimation as a "bet-the-ranch" type situation, where every mistake is equally and prohibitively bad, and mention that trade-off based methods are an ill-fit for such a scenario.

In this supplementary section, we expand on this description in greater detail, and provide intuition regarding why setting all mistakes as equally bad avoids such trade-

offs.

#### A trade-off example

Consider a physicist measuring extremely low electrical charges as part of an experiment. The physicist is trying to determine whether only one electron was discharged as part of the experiment ( $\theta = 1$ ), or whether it was two electrons ( $\theta = 2$ ). For simplicity, let us assume these are the only choices, and that there is no reason to consider either option more likely a priori.

If the physicist's equipment measures the electric charge discharged in the experiment as equivalent to 1.5 electrons (which we'll take to mean that the probability of either event is the same) and if the costs of either possible type of mistake is the same, most estimation methods will have difficulty deciding which way to go.

Lamport (2012) named this problem *Buridan's principle*, in reference to the dilemma of Buridan's ass (named after the fourteenth century French philosopher Jean Buridan) where an ass placed equidistant between two bales of hay must starve to death because it has no reason to choose one bale over the other.

When using Bayes estimators, however, the problem becomes much worse. Consider any value, x, reported by the measuring equipment. Consider any non-trivial joint probability distribution on  $(\boldsymbol{x}, \boldsymbol{\theta})$ . For example, suppose that the posterior distribution given the data is  $\mathbf{P}(\boldsymbol{\theta} = 1|x) = 0.9$  and  $\mathbf{P}(\boldsymbol{\theta} = 2|x) = 0.1$ . Consider, further, any choice of a loss function (as long as loss increases super-linearly with error magnitude). For example, let us assume quadratic loss:

$$L(\theta_1, \theta_2) = |\theta_1 - \theta_2|^2.$$

Recall that a Bayes estimator is defined as a minimiser of expected loss:

$$\hat{\theta}_{\mathrm{Bayes}}^L(x) \stackrel{\mathrm{def}}{=} \operatorname*{argmin}_{\theta \in \Theta} \mathbf{E}[L(\boldsymbol{\theta}, \theta) | \boldsymbol{x} = x].$$

So, for example, the expected loss for the choice  $\hat{\theta}(x) = 1$  is

$$0.9 \times L(1,1) + 0.1 \times L(2,1) = 0.9 \times 0 + 0.1 \times 1 = 0.1,$$

which is clearly superior to the expected loss for the choice  $\hat{\theta}(x) = 2$ ,

$$0.9 \times L(1,2) + 0.1 \times L(2,2) = 0.9 \times 1 + 0.1 \times 0 = 0.9.$$

Given that our chosen L is defined not only over  $\Theta \times \Theta$  but rather over  $\mathbb{R}^M \times \mathbb{R}^M$ , let us, however, consider what would happen if we extend the definition of our Bayes estimator to

$$\operatorname*{argmin}_{\boldsymbol{\theta} \in \mathbb{R}^M} \mathbf{E}[L(\boldsymbol{\theta}, \boldsymbol{\theta}) | \boldsymbol{x} = x]$$

(noting that in general Bayes inference, the space of actions, which is here the possible estimation values, is unrelated to the hypothesis space,  $\Theta$ .)

When all  $\mathbb{R}^M$  is considered, the optimal choice is  $\hat{\theta}(x) = 1.1$ , for which the expected loss is only

$$0.9 \times L(1, 1.1) + 0.1 \times L(2, 1.1) = 0.9 \times 0.01 + 0.1 \times 0.81 = 0.09.$$

So, in this case, such extended Bayes estimation would have advocated a value in the open interval (1,2), a value that clearly cannot be the true value of  $\theta$  because it lies outside  $\Theta$ .

One can repeat this same example with any choice of observations, any choice of positive posterior probabilities, and any choice of loss function that is super-linear with distance, and the estimate will never be within  $\Theta$ .

This is a general problem with trade-off-based estimation methods, and while it is particularly jarring in cases of discrete estimation or of continuous estimation over a non-convex  $\Theta$ , where the estimation can fall outside of  $\Theta$  altogether, the issue exists in all estimation problems, and does not require any extension to the definition of Bayes estimation. Consider exactly the same situation as before, except this time let us define  $\Theta = [1, 2]$ , and choose our prior so that the probability that  $\theta$  is in the open interval (1, 2) is negligible. The Bayes estimate (this time, without any extension to its definition) will be the same as in the extended Bayes one: 1.1, in our example.

Thus, a Bayes estimator is able to choose a highly unlikely  $\theta$ , simply because it is a convenient trade-off between other  $\theta$  values, each of which is far more likely than the one actually chosen. In error intolerant estimation we specifically aim to avoid such trade-off based decisions.

#### How error intolerance limits trade-offs

In the proofs of both Theorem 2.6 and Theorem A.1, we transform the risk attitude functions,  $T_{\epsilon}$ , into new functions,  $A_{\epsilon}$ , that map from objective loss to subjective utility.

One way to interpret this rescaling is that we use a loss function, L, to determine how similar or different  $\theta$  values are to each other, and then use an attenuation function,  $A_{\epsilon}$ , to translate this divergence into a *similarity measure*: a 1 indicates an exact match and a 0 that the two  $\theta$  are not materially similar.

This rescaling gives a better intuitive understanding of why lowering one's error tolerance leads to decisions more appropriate to the error intolerance scenario: for each  $\theta_1$ , the neighbourhood in which  $A_{\epsilon}(L(\theta_1, \theta_2))$  is positive is the neighbourhood for which mistaking  $\theta_1$  for  $\theta_2$  is at all an acceptable error. Beyond that, the utility of choosing a given  $\theta_2$  is zero. Lowering the error tolerance contracts this neighbourhood of partial similarity; at the limit, anything that is not "essentially identical" to  $\theta_1$  according to the loss function contributes zero utility, the minimal possible value. The optimal decision rule in such a scenario is therefore one that strives, as much as possible, to avoid

<sup>&</sup>lt;sup>1</sup>This could model, e.g., a situation in which the scientist, being a scientist, allows some tiny probability to the possibility that everything we think we know about the discrete nature of electric charges will ultimately be proven false.

this possibility, and therefore maximises the probability that its decision is, essentially, "exactly right".

There is, in particular, no reason for such an estimator to ever produce an estimate  $\hat{\theta}(x)$  outside  $\Theta$  even when such values are considered, because the neighbourhood of  $\theta$  values for which  $A_{\epsilon}(L(\theta, \hat{\theta}(x)))$  is positive will ultimately shrink to exclude all  $\theta \notin \Theta$ , thus yielding for such a decision a utility of zero.

It is interesting to note that while the loss function, L, retains in error intolerant estimation its basic meaning from Bayes estimation, it changes how it impacts the estimator. Unlike in standard Bayesian risk minimisation, in our scenario the only parts of the loss function that are of interest are those within an  $\epsilon$ -neighbourhood of the diagonal. One can think of this as using the loss function to convey the local topology of the hypothesis space. This is because error intolerant estimation does not concern itself with errors larger than any  $\epsilon$ : these are all considered equally "wrong".

# B.3 Rationale of the axioms on loss

In the main paper we introduce four axioms on the choice of a loss function to be used in error intolerant estimation. In this section, we provide a deeper analysis of the rationale and incontrovertibility of each.

#### **IRP**

Our first axiom is Invariance to Representation of Parameter Space (IRP). Informally, it states that our loss evaluation (and therefore our choice of estimate) should not depend on how we name our hypotheses.

While the IRP axiom may seem highly intuitive, it is actually the one axiom among our four for which it is least obvious why it is "incontrovertible". After all, common loss functions in use, such as quadratic loss, do not satisfy it.

Not only that, but the very fact that common loss functions do not satisfy IRP is often put to good use. When estimating, say, the size of a cohort it makes a big difference whether one is interested in getting the answer right in terms of the minimum error in number of people or in terms of minimum error in percents. One favours the use of a linear representation, the other—a logarithmic one.

As another example, consider estimation of a probability. When estimating the probability that a person is likely to respond to a marketing message, the error needs to be measured on a linear scale, because the underlying analysis is about maximising the expected size of the total cohort that will respond to the message. On the other hand, when estimating the probability that a dangerous experiment has crossed its safety bounds, the difference between a probability of 0.001% and 0.0001% is far more substantial than between 4% and 5%. Again, a logarithmic scale becomes more appropriate, in order to accentuate the difference between near-zero figures. In yet other domains, both near-zero and near-one probability ranges will need to be magnified, and a log-odds metric may become appropriate.

In light of this, what is the justification for IRP?

The important element to note is that in all these examples the choice of estimation method was led by a desire to manage errors in particular ways. This is not the case in error intolerant estimation. Here, we are in a bet-the-house situation. Everything rests on the question of whether our estimate is *correct*, not how well it approximates the correct answer.

In such a situation, a person faced with, for example, the need to estimate the side length of a cube and estimating it to be, say, 2 metres in length will undoubtedly answer 8 cubic metres if asked to estimate the volume of said cube. Both answers must be informed by that person's view of what the most likely "truth" is regarding the dimensions of the cube, which is why this truth must always be the same, regardless of how the person is asked to express it.

We see, therefore, that while IRP is not universally incontrovertible, and is, in fact, frequently violated, that is not the case within the world of error intolerant estimation. Here, there is no reason to doubt its veracity.

#### **IRO**

Our second axiom, Invariance to Representation of Observation Space (IRO), is also regarding invariance to representation, but this time regarding the representation of observation space. Informally, IRO guarantees that the estimate should remain the same regardless of how the observations are presented to us.

Is IRO incontrovertible?

Statisticians analysing real-world data spend much of their time wrangling this data. This wrangling, bringing the raw input data into a usable format, is a process that only works under the assumption that the raw formatting of the input does not matter: regardless of how the input is initially presented, one spends the necessary effort to bring the information into the format most appropriate for the analysis of the problem.

This is precisely the statement of IRO: the choice of the raw input format should not influence the insights derived from the data.

### IIA

The third axiom, Invariance to Irrelevant Alternatives (IIA), takes its name from Nash (1950), a seminal work in game theory, where it was introduced in the following formulation: if, when faced with a choice among alternatives in a set  $\Theta$ , a certain rational player would choose  $\theta \in \Theta$ , then, with all other conditions being equal, the same rational player would make the same choice if the alternatives were narrowed down to some  $\Theta' \subset \Theta$ , as long as  $\theta \in \Theta'$ . For example, if, given any evidence x, we estimate that the best place to dig for gold in Australia is, say, in Melbourne, then IIA states that we should reach the same conclusion, given the same evidence, also if the question asked is where to dig for gold in only mainland Australia.

Our IIA axiom merely reformulates this principle so that it addresses the underlying loss function rather than the estimate itself.

Informally, our formulation of IIA states that the loss experienced when choosing hypothesis  $\theta_2$  in cases where the true value is  $\theta_1$  should only relate to the nature of these two hypotheses, not any other property of the estimation problem. This follows directly from the semantics of what loss function values are meant to represent.

The IIA axiom has been a hotly debated axiom ever since its introduction in 1950. In our context, however, it is a direct statement regarding the nature of the error intolerant scenario: we aim for an estimate that is "best" according to its own merits, not based on its relationship with other potential estimates (as is the case with trade-off-based methods). This was already stipulated by AIA in the context of the estimator itself, as parameterised by a loss function, and IIA merely extends this requirement also to the underlying loss function, and thus to the estimator as a whole.

### ISI

What ISI states is that merely adding data bits to the observation that do not convey any information about  $\theta$  should not change our estimate of it.

Like our other axioms, ISI, too, seems nearly tautological, as one would be hard put to justify why such extra, information-free data should influence a point estimate in any way.

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