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Jaynes' "Caliber" is Proportional to "Exertion" in Quantitative Geometrical Thermodynamics

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Abstract. It is demonstrated analytically in an exact treatment that the varying part of the "caliber" C, defined originally by Edwin Jaynes in the context of a treatment of entropy production, is simply proportional to what Parker & Jeynes have previously defined as the (entropic) "exertion" X in the context of their general treatment of Quantitative Geometrical Thermodynamics. Exertion is the entropic isomorph of the energetic Action, where action is quantised by Planck's constant and entropy is quantised by Boltzmann's constant k_B as usual. The coefficient of proportionality is the entropic mass m_S, defined as $m_S \equiv i\kappa k_B$, where κ can be considered as a scale factor for the entity being described (and $i^2 = -1$ as usual): $C = -X/m_S$.

INTRODUCTION

Quantitative Geometrical Thermodynamics (QGT) has recently emerged as an important new entropic organising principle able to describe and provide explanation for many natural phenomena [1, 2]. In particular, QGT provides a quantitative and analytical framework for the role of entropy in dynamically shaping many natural structures and processes, from spiral galaxies at cosmic scales [1] down to sub-atomic nuclei at the picoscopic scale [3]. We have previously shown that certain holomorphic geometries (the double-logarithmic spiral and the double helix [1] and the sphere [4]) can be represented by *entropic* Lagrangian/Hamiltonian formulations that correctly obey the Cauchy-Riemann relations in hyperbolic Minkowski spacetime. This formulation underpins the variational and entropic Principle of Least Exertion (PLE) conforming to the Euler-Lagrange equations, as does the (complementary) kinematic Principle of Least Action (PLA). The PLE is isomorphic to the PLA: "exertion" is the path integral of the entropic Lagrangian quantised by the Boltzmann constant k_B; and "action" is the path integral of the kinematic Lagrangian quantised by the Planck constant $\hbar$.

Briefly, the double logarithmic spiral is a fundamental eigenvector of the entropic Hamiltonian, as is also the double helix (a special case of the double logarithmic spiral) [1]. It turns out that the sphere may be represented as the (linear) superposition of two double helices [4].

With its variational calculus properties, including for example the principle of maximum entropy (MaxEnt), QGT must inevitably have a connection with Jaynes' concept of Maximum Caliber (MaxCal) [5, 6] which is also based on variational principles. Unfortunately, Jaynes was only able to provide an outline sketch for his theory, since he considered the Maximum Caliber principle to be: ". . . *probably beyond our mathematical ability to do the indicated calculations explicitly for any really nontrivial problem; that is perhaps a task for the computers of the next* [*21st*] *Century*." However, we will show that exploitation of the double-helical eigenvector geometry of QGT allows a straightforward analytic demonstration both of the validity of MaxCal and also of the simple expression of MaxCal in QGT terms.

Recent years has seen a resurgence of interest in Jaynes' MaxCal ideas, with a recent overview by Pressé *et al.* [7] and application of the theory to systems far from equilibrium [8, 9]. These more recent papers explore the MaxCal principle in a quantitative and analytical fashion; in particular, by exploiting the clear analogy with Feynman's path integral formalism in quantum mechanics. However, in exploring the variational properties of the MaxCal integral, they define a Lagrangian based upon the conventional kinematic (energetic) definitions (particularly, employing differentials

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with respect to time), such that their path probability in effect becomes a conventional action integral based upon the Planck constant $\hbar$. They correctly recognise that their path entropy integral is real-valued, in contrast to the conventional Feynman complex-valued quantum mechanical path integral formalism, with interesting implications for this present paper. While being elegant, one key difference compared to our current paper is that their MaxCal path integral formalism is based on the Planck constant $\hbar$ and the conventional kinematical action integral, rather than the Boltzmann constant k_B of thermodynamics as here.

The same authors have also used the MaxCal principle to derive the well-known Jarzynski equality [10], which represents an important nonequilibrium identity relating the free energy difference between two equilibrium thermodynamic states with the work performed when going between those states via an average over the path ensemble. The implicit use in [8–10] of the Planck-based maximum path entropy approach (with the entropy defined via temperature in energy terms) is actually somewhat independent of the underlying action, so that the final (important) conclusions of these papers are not necessarily affected by the choice of quantisation.

Finally, Dixit *et al*., 2018 [11] (involving the same group as Pressé *et al*., 2013 [7]) also provide an overview of the principles underlying MaxCal, giving a deeper historical context to the concept. They show how MaxCal can be used to generate additional important results of nonequilibrium statistics, such as the Green-Kubo relations for systems near equilibrium, Prigogine's principle of minimum entropy production, Fick's law of diffusion (including the "few-molecule" limit), and Markovian dynamical processes. Together, this shows the remarkable power and generality (both for near-to and far-from equilibrium processes) of the MaxCal approach.

Interestingly, Pressé *et al*. [7] also suggest that their MaxCal quantity should be preserved under coordinate transformation (that is, it should be relativistically invariant). Clearly, such a requirement is not satisfied with the conventional definition of entropy, but does accord with our recent relativistically-invariant identification of entropy production $\dot{S} \equiv \frac{\partial S}{\partial t}$ with the product of the speed of light c and the entropic Hamiltonian H_S, where H_S is also equivalent to p_0, the (time-like) entropic momentum [12]: $\dot{S} \equiv cH_S$. Note that entropic quantities are denoted with a subscript S to indicate their thermodynamic (as opposed to conventional kinematic) quality.

We now demonstrate here (in Section §3, after a summary of the QGT formalism in Section §2) that the varying part of the caliber C is proportional to what we have previously defined as the entropic exertion X, with the coefficient of proportionality being a quantity we call the "entropic mass", m_S, such that $C = -X/m_S$. Since the exertion is defined as the path integral of the entropic Lagrangian, $X \equiv \int L_S dx_3$, it is clear that Jaynes was justified when he wrote [6]: "*The caliber of a space-time process thus appears as the fundamental quantity that "presides over" the theory of irreversible processes in much the same way that the Lagrangian presides over mechanics*."

THE ENTROPIC LAGRANGIAN L_S of QGT

The Quantitative Geometrical Thermodynamics (QGT) formalism [1] constructs a consistent entropic Lagrangian-Hamiltonian system in a complex hyperbolic Minkowski 4-space, but immediately does a contour integration to eliminate the time variable (see Appendix A of [1]) thus giving a geometrical (rather than a dynamical) representation which takes account of *information* (which is the Hodge dual of entropy in the formalism) and also automatically building the Second Law into the geometry. Then, the holomorphic geometries considered (the double helix or the double logarithmic spiral) have a preferred (axial) direction x_3, and represent the Maximum Entropy condition which is a stationary point in the variational calculus. It is worth underlining that a MaxEnt geometry expresses the Second Law even if it has zero entropy production.

Therefore, the entropic Lagrangian is constructed from the entropic momentum $\boldsymbol{p}$ and the hyperbolic position $\boldsymbol{q}$, together with the (Euclidean) axis measure x_3 which behaves as a "geometric time" coordinate. In the energy (kinematical) representations of action the variation is with respect to (w.r.t.) time t, but in the entropic representation the variation is w.r.t. geometric time x_3. Hyperbolic space is needed since the expression for the information entropy is logarithmic. Parker & Jeynes [1] have shown in detail that info-entropy (the combination of information and entropy) is holomorphic just as is the electromagnetic field in free space (the combination of the electric and magnetic fields).

The fundamental eigenvector of QGT is the double helix, which represents a holomorphic pair of spatial trajectories in Minkowski spacetime, and which represents a structure with two degrees of freedom: its radius R, and its wavelength λ (and wavenumber $\kappa \equiv \frac{2\pi}{\lambda}$). In conventional Euclidean space, the double spiral structure can be represented in a 3D co-ordinate framework with transverse co-ordinates x_1 and x_2, and the longitudinal (direction of propagation) co-ordinate given by x_3. Similar to the case of DNA and the loci of the electric and magnetic fields of circularly-polarised light, the two trajectories are $\frac{\pi}{2}$ out of phase with respect to each other, which allows the x_1 and x_2 co-ordinates of each trajectory to be described by a single holomorphic function, where $i^2 = -1$ as conventional:

$$x_1 = Re^{i\kappa x_3} \tag{1a}$$

$$x_2 = -iRe^{i\kappa x_3} \tag{1b}$$

Taking both co-ordinates together, the double helix spiral can be considered as a single holomorphic trajectory Σ that is the spatial equivalent of a pair of mutually-orthogonal plane waves propagating along the $\hat{x}_3$-axis:

$$\Sigma = Re^{i\kappa x_3}\hat{x}_1 - iRe^{i\kappa x_3}\hat{x}_2 \tag{1c}$$

where $\hat{x}_1$ and $\hat{x}_2$ are the transverse axes. The two spiral trajectories are coupled to each other to form a pair of differential equations, where the wavenumber of the double-helix $\kappa = \frac{2\pi}{\lambda}$ is also equivalent to the coupling coefficient:

$$\frac{\partial x_1}{\partial x_3} = x_1' = i\kappa Re^{i\kappa x_3} = -\kappa x_2 \tag{2a}$$

$$\frac{\partial x_2}{\partial x_3} = x_2' = \kappa Re^{i\kappa x_3} = \kappa x_1 \tag{2b}$$

and the prime symbol indicates differentiating with respect to not time but "geometric time" (the longitudinal x_3 coordinate). Although defined in *local* Euclidean 3-space $\boldsymbol{x}$, the QGT entropic Lagrangian $L_S(\boldsymbol{p},\boldsymbol{q},x_3)$ requires its position variable $\boldsymbol{q}$ to be defined in a *global* hyperbolic 3-space, where the hyperbolic scale is given by the double- helical radius R:

$$q = R\ ln\ ln\ \frac{x}{R} \tag{3}$$

Eq.(3) allows the hyperbolic 3-space positions $\boldsymbol{q}$ to be mapped to each of the Euclidean 3-space $\boldsymbol{x}$ co-ordinates, respectively. For more details see the basic and general treatment in [1] which we summarise here for convenience. The entropic Lagrangian also requires a momentum term $\boldsymbol{p}$ (defined in 3-space) which we define as the product of the entropic mass m_S and the hyperbolic (*phase*) velocity q'_φ in strict isomorphism to the conventional kinematic momentum which is the product of inertial mass and velocity. The entropic mass is a geometric quantity, based on the Boltzmann constant k_B, and for a given double helix structure is determined by its wavelength:

$$m_S \equiv i\kappa k_B = i\frac{2\pi}{\lambda}k_B \tag{4}$$

The hyperbolic *group* velocity $\boldsymbol{q}$2 is given as the differential of hyperbolic space $\boldsymbol{q}$ w.r.t. geometric time (for more details see the extended treatment in [2]):

$$q' \equiv \frac{\partial q}{\partial x_3} \tag{5}$$

Note, the hyperbolic group velocity $\boldsymbol{q}$2 is distinct from the hyperbolic phase velocity q'_φ but they are reciprocally related (again, in complete isomorphism to the kinematic case):

$$q'_\varphi = \frac{1}{q'} \tag{6}$$

where the maximum possible hyperbolic group velocity is unity, such that $q' \leq 1$, in isomorphism to the absolute kinematic speed of light c. The entropic momentum $\boldsymbol{p}$ is now simply given by the product of the entropic mass Eq.(4) and the hyperbolic phase velocity Eq.(6):

$$p = m_S q'_\varphi = \frac{m_S}{q'} \tag{7}$$

We have defined the QGT conjugate variables $\{\boldsymbol{q},\boldsymbol{p}\}$ and the differentiating variable x_3 (a geometric time isomorphic to the conventional time variable t in kinematics), and we can use them to define the entropic equations of state. Again, in complete isomorphism to the conventional kinematic concepts, we define the kinetic entropy T_S (isomorphic to the kinetic energy) as follows:

$$T_S \equiv \int \quad p dq' = \int \quad \frac{m_S}{q'} dq' = m_S \, ln \, ln \, q' \; = \frac{1}{2} m_S \, ln \, ln \, q'^2 \tag{8}$$

The similarities to the kinetic energy ½mv^2 are clear, but the presence of the logarithm function is the clear signature of an entropic quantity (as well as the associated need for a hyperbolic space representation). Note that $\boldsymbol{q'}$ as a hyperbolic velocity is dimensionless since it is a spatial gradient in hyperbolic space, and so can directly appear in the argument of the logarithm; in contrast to a kinematic velocity $v \equiv \dot{x} \equiv \frac{\partial x}{\partial t}$ which does have dimension. The potential entropy V_S is also isomorphic to the kinematic potential energy, although the hyperbolic acceleration $\boldsymbol{\Gamma}$ is given as usual by $\Gamma = \frac{q''}{q'^2}$ so that:

$$V_S = m_S \Gamma q = m_S \frac{q''}{q'^2} q \tag{9}$$

where $\boldsymbol{q}$ represents the displacement in hyperbolic space. (Note that for a double helix, isomorphic to a freely-propagating particle represented by a plane wave, there is no entropic field and the associated potential entropy is zero, V_S=0.) The entropic Hamiltonian is then simply the sum of the kinetic entropy T_S and potential entropy V_S, again in complete isomorphism with conventional kinematics:

$$H_S = T_S + V_S \tag{10}$$

The entropic Lagrangian can now be straightforwardly expressed via the Legendre transformation:

$$L_S = q'_n p^n - H_S \qquad n = \{1,2,3\} \tag{11}$$

with the Einstein summation convention over the n=\{1,2,3\} spatial dimensions as usual. Equation (7) also tells us that $q'_n p^n = m_S$, so that the entropic Lagrangian and entropic Hamiltonian are simply mutual inverses. The path integral of the entropic Lagrangian L_S over the geometric time variable x_3 is the exertion X (isomorphic to action: the path integral of the energetic Lagrangian L over the time variable t):

$$X \equiv \int \quad L_S dx_3 \tag{12}$$

For a stable system, the Principle of Least Exertion (PLE) (isomorphic to the Principle of Least Action, PLA) holds, since the entropic Lagrangian can be deployed in an entropic version of the Euler-Lagrange equation (true for each n=\{1,2,3\} spatial dimension) given by:

$$\frac{d}{dx_3}\frac{\partial L_S}{\partial q'_n} - \frac{\partial L_S}{\partial q_n} = 0 \tag{13}$$

In which case the PLE ensures that the variation of the exertion is zero:

$$\delta X = \delta \int \quad L_S dx_3 = 0 \tag{14}$$

In QGT the integral of the entropic Hamiltonian over the geometric time (propagation axis x_3) yields the entropy S of the system under consideration:

$$S = \int H_S dx_3 \tag{15}$$

and since variation of the exertion is zero according to Eqs.(13),(14) for a stable system, the variation of the entropy S must also be zero, in accordance with the Principle of Maximum Entropy:

$$\delta S = \delta \int H_S dx_3 = 0 \tag{16}$$

Thus a double helix geometry, as a stable, maximum entropy holomorphic trajectory Σ in 3-space, obeys the following canonical equations of state:

$$p'_n = \frac{\partial L_S}{\partial q_n} \tag{17a}$$

$$q'_n = -\frac{\partial L_S}{\partial p_n} \tag{17b}$$

$$\frac{\partial L_S}{\partial x_3} = -\frac{\partial H_S}{\partial x_3} \tag{17c}$$

all of which (in contrast to the conventional kinematic equations of state based on the Planck constant $\hbar$) are fundamentally based upon the dimension of the Boltzmann constant k_B and are independent of the system temperature.

Having defined the entropic Lagrangian for the double-helical trajectory in QGT, and indicated how the associated entropic equations of state are comprehensive and coherent, yet intrinsically thermodynamic in character, we now show how they relate to Jaynes' caliber C.

MAXIMUM CALIBER THEORY

We use the notation of González *et al*. [8] to mathematically describe the MaxCal concept. The caliber C is defined as a functional integral over all space trajectories, denoted D$\boldsymbol{x}$ ( ) , with ρ [$\boldsymbol{x}$ ( )] being a probability functional, such that the functional similarity of the caliber C to the Shannon entropy is obvious, it being given by:

$$C = -\int \rho[x(\ \ )] ln\, ln\, \rho[x(\ \ )]\, Dx(\ \ ) \tag{18}$$

Such that as an extremal function (i.e. Maximum Caliber) its variation is equal to zero:

$$\delta C = -\delta \int \rho[x(\ \ )] lnln\, \rho[x(\ \ )]\, Dx(\ \ ) = 0 \tag{19}$$

The following can be understood as conventional in a partition function analysis, such that the caliber C is maximised under the constraint of normalisation:

$$\int \rho[x(\ \ )] Dx(\ \ ) = \langle 1 \rangle = 1 \tag{20}$$

We also assume an additional constraint due to a known physical parameter of the system. This is often taken to be the total energy, but in the context of this entropic analysis is more appropriately taken to be the timelike entropic momentum p_0 (another conserved quantity), equivalent to the entropic Hamiltonian H_S , as well as being proportional to the total entropy production $\dot{S}_0$ (also, equally conserved.). This constraint is the expected value of a functional F [$\boldsymbol{x}$()], that is $\langle F[x(\ \ )]\rangle = F_0$, such that in the general case we write:

$$\int \rho[x(\ )]F[x(\ )]Dx(\ ) = \langle F[x(\ )]\rangle = F_0 \tag{21}$$

Having established the formalism behind the MaxCal quantity of Eq.(18), we now show how it agrees with the most-likely (MaxEnt) QGT holomorphic double helix trajectory descriptions of the first section of this paper. Note that this treatment also generalises to the logarithmic double spiral.

We first note that the probability functional $\rho\,[\boldsymbol{x}\,(\ )]$ describes a spatial distribution of possible trajectories; we have identified specific (most likely) double-helical trajectory solutions, which are not probabilistic distributions. Rather, the double helix represents a definite trajectory in space. As such, the associated "probability distribution" conforms to a sharply-defined distribution analogous to that of a Dirac delta function, i.e. $\rho\,[\boldsymbol{x}\,(\ )] = \delta_{Dirac}\,(\boldsymbol{x} - \boldsymbol{x}_{spiral})$, where $\boldsymbol{x}_{spiral}$ represents the description of the double spiral Σ trajectory in Euclidean 3-space. It is clear for such a sharply-defined trajectory that we require $\rho = 1$ anywhere on the spiral trajectory, and $\rho = 0$ elsewhere, according to the Dirac delta functional relationship for the trajectory functional probability distribution. Therefore, we define the probability distribution by inspection as follows:

$$\rho[x(\ )] = \frac{x_1(x_3)}{ix_2(x_3)} \tag{22}$$

where the functionals x_1 and x_2 are defined in Eqs.(1), such that $\rho = 1$ anywhere on the spiral trajectory Σ. Particularly, since ρ is integrated along the path of the trajectory in Eq.(18), where $\rho = 1$, the path integral continues to add up in a "constructive" fashion along the double-helix. However, for locations in space away from the double-helix, the unity ratio of x_2 and x_1 no longer holds (as in Eqs.1,22, that force the ratio ρ to unity along the spiral); but off the double helical trajectory, both the amplitude and the *phase* of ρ vary. It can be shown that integrating ρ along the path integral of Eq.(18) or Eq.(21) for such an off-track trajectory, with its varying phase values of ρ, causes the summed amplitudes to "destructively" cancel, in the same way that alternative path histories cancel in Feynman's path integral formulation. Just as for the PLA, the PLE leaves a non-zero integral only on the double-helix.

For the specific double-helical geometry we are interested in, the functional derivative quantity is simply given by $D\boldsymbol{x}(\ )\ \alpha\ dx_3$. In which case, substituting Eq.(22) into the expression for the MaxCal expression of Eq.(18) we find that the caliber is given by:

$$C = -\int \left(\frac{x_1(x_3)}{ix_2(x_3)}\right) ln\, ln\, \left(\frac{x_1(x_3)}{ix_2(x_3)}\right)\, dx_3 \tag{23}$$

The first quotient quantity within the integral of Eq.(23) is just unity along the double-helix. The second, logarithmic quantity, can be re-expressed using the rules of logarithms as follows (where, adopting the variational version of Eq.(23) also allows us to ignore the constant term associated with ln i):

$$\delta C = -\delta \int\ ln\, ln\, \left(\frac{x_1(x_3)}{x_2(x_3)}\right)\, dx_3 \tag{24}$$

Using the pair of cross-coupled equations, Eqs.(2), for the double helix: $x_1' = -\kappa x_2$ and $x_2' = \kappa x_1$, we re-write Eq.(24) as follows, where we also remove any unvarying constants in the variation:

$$\delta C = -\delta \int\ ln\, ln\, \left(\frac{x_1}{x_2}\right)\, dx_3 = -\delta \int\ ln\, ln\, \left(\frac{\kappa x_1}{x_1'}\right)\, dx_3 \tag{25a}$$

However, we realise that the reciprocal of ρ in Eq.(22) could equally have been deployed as our probability distribution along the double-helix trajectory, such that we can also assume:

$$\delta C = -\delta \int\ ln\, ln\, \left(\frac{x_2}{x_1}\right)\, dx_3 = -\delta \int\ ln\, ln\, \left(\frac{\kappa x_2}{x_2'}\right)\, dx_3 \tag{25b}$$

We superpose these two independent (i.e. dependent, respectively, either on the x_1 or x_2 axes) variational expressions for the caliber, as follows:

$$\delta C = -\delta \int \left(\frac{\kappa x_2}{x_2'}\right) dx_3 \tag{26}$$

Recall that the entropic Hamiltonian for a double helix is given from Eq.(8) by $H_S = ½ m_S \ln \ln q_n' q'^n$ for $n = \{1,2,3\}$ (Einstein summation convention applies), noting $q_3' = 1$ for a double helix, such that $H_S = m_S(\ln \ln q_1' + \ln \ln q_2')$ with the Legendre transformation therefore offering us the entropic Langrangian as $L_S = q_n' p^n - H_S = m_S - H_S$. From the definition of the hyperbolic position $\boldsymbol{q}$ of Eq.(3), we can write $q_n' = \frac{R x_n'}{x_n}$, where R is the radius of the double helix. Thus it becomes clear that we can rewrite Eq.(26) as:

$$\delta C = -\delta \int \left(\frac{\kappa R}{q_2'}\right) dx_3 \tag{27}$$

Noting that the parameters κ and R are constant (invariant) for a double helix allows us to simplify:

$$\delta C = \delta \int (q_2') dx_3 \tag{28}$$

and we can therefore express the variation of the caliber C directly in terms of the entropic Lagrangian L_S:

$$\delta C = -\delta \int \frac{L_S}{m_S} dx_3 = -\frac{1}{m_S} \delta \int L_S dx_3 = 0 \tag{29}$$

Thus we see that the varying part of the caliber C is simply proportional to what we have previously defined as the exertion X, Eq.(12), with the proportional factor simply being the entropic mass m_S:

$$C = -\frac{X}{m_S} \tag{30}$$

DISCUSSION

In this analysis we have therefore proven, for the first time, that the entropic Lagrangian $L_S(\boldsymbol{p},\boldsymbol{q},x_3)$ of QGT is intimately connected to the well-known caliber C of Jayne's Principle of Maximum Caliber. In so doing, we have also created an important connection between the emerging QGT theory of geometrical entropy and the well-established studies of MaxEnt and MaxCal in the context of statistical mechanics.

This has much wider implications. QGT is not a statistical theory, although an entropic Partition Function is readily derived from the entropic Liouville Theorem [2] (as may also have been inferred from the idea of Caliber, invoking as it does a phase space). The derivations of QGT are independent of any sort of ensemble, indeed treatments of cosmic entropy [12] and sub-atomic entropy [3] are analytical (not statistical) with black holes having the same (analytic) description as alpha particles [3].

This means that Thermodynamics possesses the same fundamental status as the *Principle of Least Action* (PLA), and is not dependent on (or "emergent" from) statistical or kinematical theories. QGT makes this explicit as it entails a *Principle of Least Exertion* [1] **isomorphic** to the PLA (and hence just as fundamental).

CONCLUSIONS

We have shown how Jaynes' MaxCal theory can be understood within the context of the simple geometry of a double helical trajectory; and moreover how Jayne's MaxCal is straightforwardly satisfied by a double helical geometry. Indeed, as suggested by Jaynes [6], we can see from Equations (29) & (30) that the caliber C is simply determined by the entropic Lagrangian L_S of the system. Thus we have proven the intrinsic correctness of Jaynes' MaxCal theory, without the need to resort to complex numerical calculations.

This treatment is quite general: the same result can be obtained for the double logarithmic spiral (of which the double-helix is a special case). Future work will be directed towards applying the explicit entropic Lagrangian formalism of QGT to specific, known MaxCal processes (e.g. for systems far from equilibrium) and also exploiting the covariant transformation properties of QGT to understand the entropy production and time-variation characteristics of the MaxCal phenomena from a geometrical entropy perspective.

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