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Gacs quantum algorithmic entropy in infinite dimensional Hilbert spaces

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We extend the notion of Gacs quantum algorithmic entropy, originally formulated for finitely many qubits, to infinite dimensional quantum spin chains and investigate the relation of this extension with two quantum dynamical entropies that have been proposed in recent years. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4892516]

I. INTRODUCTION

Algorithmic complexity, or descriptive complexity, or, in the following, Kolmogorov complexity, developed by Solomonoff,[1] Kolmogorov[2] and Chaitin,[3] has been an important tool in many different fields[4–8] where it shed light on subtle concepts such as information content, randomness, inductive inference, and also had applications in thermodynamics. In a nutshell, the complexity of a target object is measured by the difficulty to describe it; in the case of targets describable by binary strings, they are algorithmically complex when their shortest binary descriptions are essentially of the same length in terms of necessary bits, the descriptions being binary programs such that any universal Turing machine that runs them outputs the target string.

The recent developments in quantum mechanics that, together with the birth of the so-called quantum computation theory, have also led to the development of a broad quantum information theory, have spurred the attempt to extend the concept of algorithmic complexity to the quantum realm. There exist however different proposals of quantum algorithmic complexity that, while agreeing on quantum states as description targets, differ on what their description should be achieved by. The so-called quantum qubit complexity[9] stresses the fact that programs as well as targets should be quantum, namely, quantum states run by universal quantum Turing machines, while the bit quantum complexities in Ref. 10, respectively,[11] are based on descriptions of quantum states by classical programs, respectively, by classical descriptions of the quantum circuits that operatively provide them at the output of suitable sequences of quantum gate operations. A third option[12] avoids the reference to either quantum or classical Turing machines, and rather extends to the non-commutative setting the notion of universal semi-measure by introducing the notion of universal semi-density matrix.

In all cases a useful guide to sort out the various quantum extensions of algorithmic complexity is provided by the relations between the classical algorithmic complexity and the Shannon entropy. Even when not pretending to exactly reproducing them in a non-commutative context, it is nevertheless important to clarify the connections, if any, between the quantum algorithmic complexities and the von Neumann entropy or related concepts. In particular, in the classical setting a theorem of Brudno[13] states that almost every trajectory of an ergodic classical system has an algorithmic

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complexity rate which equals the entropy rate, the latter being also known as Kolmogorov-Sinai, or dynamical entropy. Inequivalent quantum extensions of the Kolmogorov-Sinai entropy have also been proposed.[14–18]

In Ref. 19, a relation was established between the quantum algorithmic complexity as formulated in Ref. 12, that we shall refer to as Gacs complexity in the following, and the quantum dynamical entropies of the shift automorphism on quantum spin chains as formulated by Connes, Narnhofer and Thirring (CNT-entropy)[14] and by Alicki and Fannes (AF-entropy).[15] A quantum spin chain is a one-dimensional lattice with d-level quantum systems at each site and the lattice translations or shift-automorphisms are the simplest possible dynamics on such chains. For ergodic translation invariant states ω on quantum spin chains, the CNT-entropy equals the von Neumann entropy density $s(\omega)$, while the AF-entropy equals $s(\omega) + \log d$.

In Ref. 19, the extra term $\log d$ is given an informational interpretation in terms of the Gacs complexity per spin within the Alicki-Fannes setting. There, the limit rate was obtained starting from increasingly large, but finite-dimensional sub-chains and using the formulation in Ref. 12 that concerns arbitrary, but finite number of spins. As a consequence, a constraint had to be imposed in Ref. 19 on the growth of the classical complexity of finite-size density matrices; namely, that it be slower than the size of the sub-chain. Instead, in this paper, we construct the Gacs complexity operator directly for the infinite dimensional quantum in such a way that it reduces to the finite dimensional Gacs complexity operator when restricted to finite portions of the chain. In this way, one is able to remove the limitation mentioned above which proves to be unnecessary as showed in an example of quantum spin-chain with finite Alicki-Fannes entropy equalling the Gacs complexity rate, while finite-size density matrices have Kolmogorov complexities diverging faster than n.

II. COMPUTABILITY, KOLMOGOROV COMPLEXITY, AND UNIVERSAL PROBABILITY

We start by reviewing the notions of computability, Kolmogorov complexity, and universal semi-probability.

For completeness we give a brief introduction to computability theory based on Ref. 20. Let us denote the input variables by $X_1, X_2, \ldots, X_n$ and the output variables by Y. In most cases we also need local variables $Z_1, Z_2, \ldots, Z_k$. In general, instructions in a program have labels. A program is a finite list of instructions of the following 3 types:

$V \to V + 1$: Increase by 1 the value of the variable V.
$V \to V - 1$: If the value of V is zero leave it unchanged; otherwise decrease it by 1.
IF $V \neq 0$ **GOTO L**: if the value of V is nonzero, perform the instruction with label L; otherwise proceed to the next instruction in the list.

A program halts when there is no instruction to be executed. A function $f : \mathbb{N}^n \to \mathbb{N}$ is called partially computable if there exists a program P which, for each $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$, halts on input $(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$ if and only if $f(x_1, x_2, \ldots, x_n)$ is defined and its output Y is equal to $f(x_1, x_2, \ldots, x_n)$. Indeed, computable functions are associated with programs which halt on every input values, while partially computable functions are associated with programs which may not halt on certain input values.

Example 2.1. The following program computes the function $f(x, y) = x + y$:

	$Y \to X_1$
	$Z_1 \to X_2$
[B]	If $Z_1 \neq 0$ *GOTO A*
	GOTO E
[A]	$Z_1 \to Z_1 - 1$
	$Y \to Y + 1$
	GOTO B

where GOTO L is an abbreviation for

$$Z_2 \to Z_2 + 1,$$
$$IF\ Z_2 \neq 0\ GOTO\ L.$$

Moreover, since there is no label E, the command GOTO E forces the program to halt. Of course, the symbols X_1, X_2 denote input variables, Z_1 a local variable, Y an output variable while A, B, E, L are labels.

It can be proved[20] that there is a one to one correspondence $n \to P_n$ between $\mathbb{N}$ and the set of all programs. The function computed by the program P_n will be denoted by ϕ_n. The mapping $n \to \phi_n$ is an enumeration of partially computable functions. By a universal partially computable function we mean a partially computable function $\phi(x_1, x_2, \ldots, x_k, n)$ such that for each n the function $\phi(x_1, x_2, \ldots, x_k, n) = \phi_n(x_1, x_2, \ldots, x_k)$.

A predicate $STP^{(m)}(x_1, x_2, \ldots, x_m, n, t)$ is a binary function which is 1 if the program P_n halts after t or fewer steps on input $(x_1, x_2, \ldots, x_m)$, while it is 0 if the program P_n does not halt in t or fewer steps. For more details on the subject see Ref. 20.

Let $h : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a function. Then, for each n, $h_n : \mathbb{N} \to \mathbb{R}$ is defined as follows: $h_n(x) = h(n, x)$.

A function $g : \mathbb{N} \to \mathbb{R}$ is called lower semi-computable if there exists a computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that the sequence f_n is an increasing sequence and $\lim_{n\to\infty} f_n = g$.

A function $\mu : \mathbb{N} \to \mathbb{R}$ is called a (semi-computable) semi-measure if it is a positive semi-computable function such that $\Sigma_x \mu(x) \leq 1$.

A function $h : \mathbb{N} \to \mathbb{R}$ is called upper semi-computable if $-h$ is lower semi-computable and it will be called computable if it is lower and upper semi-computable.

A semi-computable semi-measure μ is called universal if for any semi-computable semi-measure ν there exists a constant $C_\nu > 0$ such that for each $x \in \mathbb{N}$, $C_\nu \nu(x) \leq \mu(x)$.

Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ be a computable function increasing in n and let $\psi : \mathbb{N} \to \mathbb{Q}$ be defined by

$$\psi(x) = \lim_{n\to\infty} f(n, x) \quad \text{for all} \quad x \in \mathbb{N}.$$

Let $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ be defined by $\phi(n, x) = f(n, x)$ if $x \leq n$, otherwise 0. Then, ϕ_n is an increasing sequence of computable functions and $\lim_{n\to\infty} \phi_n = \psi$. Let $\alpha : \mathbb{Q} \to \mathbb{N}$ be an injection[24] and let $\Phi : \mathbb{N} \to \mathbb{N}$ be defined as follows: $\Phi(n) = 2^{\alpha(\phi(n,0))} \times 3^{\alpha(\phi(n,1))} \times \ldots \times p_n^{\alpha(\phi(n,n))}$, where p_n is the nth prime number. Then, Φ is a computable function that we can use to define ψ. In this way, we will represent all necessary semi-computable quantities that appear in the following like semi-computable semi-measures, semi-computable Hilbert space vectors and semi-density matrices by computable functions on $\mathbb{N}$.

The set $\bigcup_{n\geq 0}\{0, 1\}^n$ of all binary strings of finite length will be denoted by $\{0, 1\}^*$. The relation $\mathbf{x}(\mathbf{i}^n) = \sum_{j=0}^{n-1} i_j 2^j$ where $\mathbf{i}^{(n)} = i_0 i_1 \ldots i_{n-1} \in \{0, 1\}^n$ defines a one-to-one correspondence between $\{0, 1\}^*$ and $\mathbb{N}$. A function $f : \{0, 1\}^* \to \{0, 1\}^*$ is called (partially computable) computable if the function $\mathbf{x} \circ f \circ \mathbf{x}^{-1} : \mathbb{N} \to \mathbb{N}$ is (partially computable) computable.

Let x and y be two elements of $\{0, 1\}^*$ we say that x is a prefix of y if there is an element $z \in \{0, 1\}^*$ such that $xz = y$, xz denoting the concatenation of x and z. A subset $S \subseteq \{0, 1\}^*$ is called prefix-free if no element of S is a prefix of another elements. A partially computable function on $\{0, 1\}^*$ is called prefix-free if its domain is a prefix-free subset of $\{0, 1\}^*$. In the following by a (prefix)(universal) Machine we mean a (prefix-free)(universal) partially computable function.

These tools allow for a simple proof of the following theorem.

Theorem 2.2. *There exists a universal semi-computable semi-measure.*

Proof. The proof follows by showing that set of semi-computable semi-measures can be enumerated and is similar to the proofs of Theorems 4.4 and 4.6, with necessary modifications, to which we refer the reader (see also Refs. 12 and 7). □

In algorithmic complexity theory as developed by Kolmogorov, Solomonoff, and Chaitin, one attributes to a binary string $\mathbf{i}^{(n)} = i_1 i_2 \cdots i_n \in \{0, 1\}^n$ of length n a complexity $K(\mathbf{i}^{(n)})$ measured by the length of any shortest program p^* (another binary string of length $\ell(p^*)$) in the domain of a prefix-free universal partially computable function or equivalently, that read by a prefix Universal Turing Machine (UTM) $\mathcal{U}$, would output $\mathbf{i}^{(n)}$, $\mathcal{U}[p^*] = \mathbf{i}^{(n)}$,

$$K(\mathbf{i}^{(n)}) = \min\left\{\ell(p) \ : \ \mathcal{U}[p] = \mathbf{i}^{(n)}\right\} . \tag{1}$$

The prefix property means that if $\mathcal{U}$ halts on a program p it does not continue to read on if another program q is appended to p; in other words, no halting program can be used as prefix to a halting program.

Given a universal semi-measure μ on $\{0, 1\}^*$, it turns out[7] that it is related to the Kolmogorov complexity by the following relation,

$$\left|K(\mathbf{i}^{(n)}) + \log_2 \ \mu(\mathbf{i}^{(n)})\right| \leq C \quad \forall \mathbf{i}^{(n)} \in \{0, 1\}^* , \tag{2}$$

where $C > 0$ is a constant which does not depend on the binary string $\mathbf{i}^{(n)}$.

Suppose one has an (algorithmically) computable probability distribution $\nu^{(n)} = \{\nu(\mathbf{i}^{(n)})\}$ on $\Omega_2^n = \{0, 1\}^n$; then, the Shannon entropy is close to the average Kolmogorov complexity, in the sense that

$$\left|H(\nu^{(n)}) - \sum_{\mathbf{i}^{(n)} \in \Omega_2^n} \nu(\mathbf{i}^{(n)}) \, K(\mathbf{i}^{(n)})\right| \leq K(\nu^{(n)}) , \tag{3}$$

where $K(\nu^{(n)})$ is the complexity of the computable probability distribution.[7] Brudno[13] proved that, for ergodic sources, the difference disappears if one considers the rates:

$$h(\nu) = \lim_{n \to +\infty} \frac{1}{n} H(\nu^{(n)}) = k(\mathbf{i}) := \lim_{n \to +\infty} \frac{1}{n} K(\mathbf{i}^{(n)}) , \tag{4}$$

for almost all sequences $\mathbf{i}$ with respect to the measure ν, that is the set of sequences where the equality fails has measure 0.

Remark 2.3. Actually, Brudno proved much more: indeed, his theorem establishes a relation between the entropy rate and the algorithmic information per unit step of almost all trajectories with respect to ergodic states of dynamical maps on generic probability spaces.

III. QUANTUM SPIN CHAINS

A quantum spin chain is the C^*-algebra that arises from the norm completion of local quantum spin algebras of the tensor product form:

$$M_{[-n,n]} = \underbrace{M_d(\mathbb{C}) \otimes M_d(\mathbb{C}) \otimes \cdots M_d(\mathbb{C})}_{2n+1 \ times} = M_d^{\otimes 2n+1} = M_{d^{2n+1}}(\mathbb{C}) . \tag{5}$$

The interpretation is straightforward: one is dealing with a one-dimensional lattice each site of which supports a d-level quantum system (or d-dimensional spin): in the norm-topology (the norm is the one which coincides with the standard matrix-norm on each local algebra) the limit $n \to \ +\infty$ of the nested sequence $M_{[\,-n,\,n]}$ gives rise to the norm-complete infinite dimensional algebra,

$$\mathcal{M} := \lim_{p} M_{[-p,\,p-1]} , \tag{6}$$

that describes an infinite quantum spin lattice, that is a quantum spin chain. In the following we shall consider $d = 2$, namely, a chain of 2-level quantum spins, or spin 1/2 particles, or in the modern jargon, qubits.

Any local spin operator, say $A \in M_{[\,-n,\,n]}$, is naturally embedded into $\mathcal{M}$ as

$$M_{[-n,n]} \ni A \mapsto \mathbb{1}_{-n-1]} \otimes A \otimes \mathbb{1}_{[n+1} \in \mathcal{M} , \tag{7}$$

where $\mathbb{1}_{-n-1]}$ stands for the infinite tensor products of 2×2 identity matrices up to site $-n-1$, while $\mathbb{1}_{[n+1}$ stands for the infinite tensor product of infinitely many identity matrices from site $n+1$ onwards. In this way, the local algebras are sub-algebras of the infinite one sharing a same identity operator.

The simplest dynamics on such quantum spin chains is given by the right shift:

$$\Theta[M_{[-n,n]}] = M_{[-n+1,n+1]}\,, \quad \Theta[\mathbb{1}_{-n-1]} \otimes A \otimes \mathbb{1}_{[n+1}] = \mathbb{1}_{-n]} \otimes A \otimes \mathbb{1}_{[n+2}\,. \tag{8}$$

Any state ω on $\mathcal{M}$ is a positive, normalized linear functional whose restrictions to the local sub-algebras are associated with density matrices $\rho_{[-n,n]}$, namely, with positive matrices in $M_{[-n,n]}(\mathbb{C})$ such that $\mathrm{Tr}_{[-n,n]}\rho_{[-n,n]} = 1$:

$$M_{[-n,n]} \ni A \mapsto \omega(A) = \mathrm{Tr}_{[-n,n]}\Big(\rho^{(n)}\,A\Big)\,. \tag{9}$$

The degree of mixedness of such density matrices is measured by the von Neumann entropy,

$$S(\rho_{[-n,n]}) = -\mathrm{Tr}_{[-n,n]}(\rho_{[-n,n]}\log\rho_{[-n,n]}) = -\sum_j r^j_{[-n,n]}\log r^j_{[-n,n]}\,, \tag{10}$$

where $0 \le r^j_{[-n,n]} \le 1$, $\sum_j r^j_{[-n,n]} = 1$ are the eigenvalues of $\rho_{[-n,n]}$. Notice that the von Neumann entropy is nothing but the Shannon entropy of the spectrum of $\rho_{[-n,n]}$ which indeed amounts to a discrete probability distribution.

In the above expressions $\mathrm{Tr}_{[-n,n]}$ stands for the trace computed with respect to any orthonormal basis of the Hilbert space $\mathbb{H}_{[-n,n]} = (\mathbb{C}^2)^{\otimes 2n+1}$ onto which A linearly acts. Let $|i\rangle \in \mathbb{C}^2$, $i = 0, 1$ be a chosen orthonormal basis in $\mathbb{C}^2$; then, a natural orthonormal basis in $\mathbb{H}_{[-n,n]}$ will consist of tensor products of single spin basis vectors:

$$|\mathbf{i}_{[-n,n]}\rangle = \bigotimes_{j=-n}^{n} |i_j\rangle = |i_{-n}i_{-n+1}\cdots i_n\rangle\,, \tag{11}$$

namely, its elements are indexed by binary strings $\mathbf{i}_{[-n,n]} \in \{0,1\}^{2n+1}$. By going to the limit of an infinite chain, a corresponding representation Hilbert space is generated by orthonormal vectors again denoted by $|\mathbf{i}_{[-n,n]}\rangle$ where n arbitrarily varies and every $\mathbf{i}_{[-n,n]}$ is now a binary sequence in $\{0,1\}^{\mathbb{Z}}$ where all $i_k \notin [-n,n]$ are chosen equal to 0. We shall denote by $\mathbf{i}$ such binary strings, by $\mathbf{\Omega}$ their set and by $|\mathbf{i}\rangle$ the corresponding orthonormal vectors which form the so-called standard basis of $\mathbb{H}$.

Remark 3.1. While all irreducible representations of finite size quantum spin chains as operators on a Hilbert space are unitarily equivalent, in the case of infinitely may spins there are infinitely many inequivalent such representations.[21] *What we are considering here is one of these, a particular case of the so-called Gelfand-Naimark-Segal (GNS) construction:*[21] *it is created acting with finitely many spin flips* $|0\rangle\mapsto|1\rangle$ *on the GNS cyclic state represented by all spins being in the state* $|0\rangle$. *By choosing* $i_k = 1$ *outside any finite interval* $[-n, n]$ *one gets another representation of the same algebra* $\mathcal{M}$. *However, the new representation is inequivalent to the previous one as there is no unitary operator mapping one Hilbert space into the other as it should flip infinitely many spins.*

From (9), a compatibility relation immediately follows; namely,

$$\omega(A \otimes \mathbb{1}_n) = \mathrm{Tr}_{[0,n]}\rho_{[0,n]}(A \otimes \mathbb{1}_n) = \omega(A) = \mathrm{Tr}_{[0,n-1]}(\rho_{[0,n-1]}A) \qquad \forall A \in M_{2^n}(\mathbb{C})\,,$$

so that

$$\mathrm{Tr}_n\rho_{[0,n]} = \rho_{[0,n-1]}\,. \tag{12}$$

On the other hand, if

$$\omega(\mathbb{1}_0 \otimes A) = \mathrm{Tr}_{[0,n]}(\rho_{[0,n]}\mathbb{1}_0 \otimes A) = \omega(A) = \mathrm{Tr}_{[0,n-1]}(\rho_{[0,n-1]}A) \qquad \forall A \in M_{2^n}(\mathbb{C})\,.$$

That is if ω is a translationally invariant state, then

$$\rho_{[0,n-1]} = \mathrm{Tr}_0\rho_{[0,n]}\,, \quad \forall n\,. \tag{13}$$

To any translationally invariant state ω on a quantum spin chain there remains associated a well-defined von Neumann entropy rate (see for instance Ref. 22):

$$s(\omega) = \lim_{n\to+\infty} \frac{1}{n} S(\rho_{[0,n-1]}) = -\lim_{n\to+\infty} \frac{1}{n} \mathrm{Tr}_{[0,n-1]} \left(\rho_{[0,n-1]} \log \rho_{[0,n-1]}\right) . \tag{14}$$

A. Alicki-Fannes entropy

The main idea behind the Alicki-Fannes entropy[15,16] is as follows. Given a quantum spin chain described by the C^*-algebra $\mathcal{M}$, equipped with the shift automorphisms and a translational invariant state $\omega = \omega \circ \Theta$, let a finite partition of unit be any finite collection

$$\mathcal{X} = \{X_i\}_{i=1}^m , \quad \sum_{i=1}^m X_i^\dagger X_i = 1 , \quad X_i \in \mathcal{M} , \tag{15}$$

of local operators X_i belonging to some local algebra $M_{[-n,n]}$. By means of ω and $\mathcal{X}$ we can define a $m \times m$ density matrix $\rho[\mathcal{X}]$ with entries

$$\rho_{ij}[\mathcal{X}] = \omega(X_j^\dagger X_i) = \mathrm{Tr}_{[-n,n]}\left(\rho^{(2n+1)} X_j^\dagger X_i\right) , \tag{16}$$

and von Neumann entropy

$$S(\rho[\mathcal{X}]) = -\mathrm{Tr}(\rho[\mathcal{X}] \log \rho[\mathcal{X}]) . \tag{17}$$

In order to introduce the dynamics into the game, we let the partitions of unit transform under the action of the dynamical map Θ in (8),

$$\Theta^\ell[\mathcal{X}] = \{\Theta^\ell[X_i]\}_{i=1}^m . \tag{18}$$

Then, we refine the partitions of unit from $\ell = 0$ to $\ell = n - 1$ as follows

$$\mathcal{X}^{(n)} = \{X_{i^{(n)}}\} , \quad X_{i^{(n)}} = \Theta^{(n-1)}[X_{i_{n-1}}] \cdots \Theta[X_{i_1}] X_{i_0} , \quad i^{(n)} = i_0 i_1 \ldots i_{n-1} , \tag{19}$$

with $i^{(n)}$ an m-nary string in $\Omega_m^{(n)}$. The refined sets $\mathcal{X}^{(n)}$ are still partitions of unit according to (15); one can thus associate to them density matrices $\rho[\mathcal{X}^{(n)}]$ acting on $\mathbb{C}^{m^n}$ with entries as in (16) and von Neumann entropies $S[\mathcal{X}^{(n)}]$ as in (17).

The corresponding entropy rate is defined by

$$h_\omega^{AF}(\Theta, \mathcal{X}) = \limsup_{n\to\infty} \frac{1}{n} S[\mathcal{X}^{(n)}] . \tag{20}$$

The Alicki-Fannes entropy of $(\mathcal{M}_2, \omega, \Theta)$ is then defined by

$$h_\omega^{AF}(\Theta) = \sup_{\mathcal{X}} h_\omega^{AF}(\Theta, \mathcal{X}) , \tag{21}$$

namely, as the supremum over all possible finite partitions $\mathcal{X}$ of unit by local operators of the quantum spin chain.

Remark 3.2. The lim sup *in (20) has to be used for the sequence of density matrices $\rho[\mathcal{X}^{(n)}]$ is not a stationary one.*[15,16] *In fact, while consistency holds as tracing $\rho[\mathcal{X}^{(n)}]$ over the nth factor yields the density matrix corresponding to the first $n - 1$ factors,* $\mathrm{Tr}_n \rho[\mathcal{X}^{(n)}] = \rho[\mathcal{X}^{(n-1)}]$, *stationarity does not; indeed, in general,* $\mathrm{Tr}_1 \rho[\mathcal{X}^{(n)}] \neq \rho[\mathcal{X}^{(n-1)}]$.

As a concrete example consider a set of 4 matrix units $U_{ij} \in M_2(\mathbb{C})$ such that $U_{ij}^\dagger = U_{ji}$, $U_{ij}U_{k\ell} = \delta_{jk} U_{i\ell}$, and $\sum_{i=1}^2 U_{ii} = 1$. Dividing them by $\sqrt{2}$ one gets a partition of unit

$$\mathcal{U} = \left\{\frac{U_{ij}}{\sqrt{2}}\right\}_{i,j=1,2} \in M_2(\mathbb{C}) ,$$

the simplest choice being

$$U_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \quad U_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} , \quad U_{12} = U_{21}^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

The refined partition that results after n applications of the right shift is

$$\mathcal{U}^{(n)} = \left\{ \frac{U_{i^{(n)}j^{(n)}}}{2^{n/2}} \right\} , \; U_{i^{(n)}j^{(n)}} = U_{i_0 j_0} \otimes U_{i_1 j_1} \otimes \cdots U_{i_{n-1} j_{n-1}} \in M_2^{\otimes n}(\mathbb{C}) = M_{[0,n-1]} . \tag{22}$$

The associated density matrices $\rho[\mathcal{U}^{(n)}] \in M_{4^n}(\mathbb{C})$ have entries and von Neumann entropy given by

$$\begin{aligned} \frac{1}{2^n}\mathrm{Tr}\left(\rho^{(n)}\, U^{\dagger}_{i^{(n)}j^{(n)}} U_{k^{(n)}\ell^{(n)}}\right) &= \frac{1}{2^n}\mathrm{Tr}\left(\rho^{(n)}\, U_{j_0 i_0} U_{k_0 \ell_0} \otimes U_{j_1 i_1} U_{k_1 \ell_1} \otimes \cdots\right) \\ &= \frac{1}{2^n}\mathrm{Tr}\left(\rho^{(n)}\, \delta_{i_0 k_0}\, U_{j_0 \ell_0} \otimes \delta_{i_1 k_1}\, U_{j_1 \ell_1} \otimes \cdots\right) = \frac{\mathbf{1}}{2^n} \otimes \rho^{(n)}, \end{aligned} \tag{23}$$

$$S\Big(\rho[\mathcal{U}^{(n)}]\Big) = S(\rho^{(n)}) + n . \tag{24}$$

The last equality in (23) comes from the fact that $\mathrm{Tr}\Big(\rho^{(n)} U_{i^{(n)}j^{(n)}}\Big)$ are the matrix elements of $\rho^{(n)}$ with respect to the orthonormal basis defined by the choice of matrix units. Then, entropy rate and the Alicki-Fannes entropy are readily computed to be[15,16]

$$h^{AF}_{\omega}(\Theta) = h^{AF}_{\omega}(\Theta, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} S(\rho[\mathcal{U}^{(n)}]) = s(\omega) + 1 . \tag{25}$$

Remark 3.3. With respect to the AF-quantum dynamical entropy, the quantum dynamical entropy introduced by Connes, Narnhofer, and Thirring[14] has a more complicated construction due essentially to the fact that it is based on a classical symbolic modeling with superimposed quantum corrections.[22,23] Consider a quantum spin chain endowed with a translation invariant state with sufficiently fast decaying correlations, the simplest case being the tensor product of a same density matrix ρ, that is,

$$\omega(A_1 \otimes A_2 \otimes \cdots A_n) = \prod_{\ell=1}^{n} \mathrm{Tr}(\rho A_\ell) .$$

In such cases, the CNT-entropy of the shift equals the von Neumann entropy rate (14), that is,

$$h^{CNT}_{\omega}(\Theta) = s(\omega) \qquad (= -\mathrm{Tr}\rho\log\rho \quad \text{in the product state case}) . \tag{26}$$

The origin of the difference by $1 = \log 2$ between the AF-entropy and the entropy rate (which equals the CNT-entropy) lies in that the AF-entropy accounts for measurement-like disturbances on the state of the quantum chain. In quantum mechanics generic measurement processes on a system in a state described by density matrix ρ are identified by partitions of unit $\mathcal{X} = \{X_i\}$ such that the state is changed by a measurement process as follows:

$$\rho \mapsto \sum_i X_i\, \rho\, X_i^{\dagger} . \tag{27}$$

Suppose

$$M_{2^n}(\mathbb{C}) \ni \rho^{(n)} = \sum_i r_i^{(n)}\, |r_i^{(n)}\rangle\langle r_i^{(n)}| \tag{28}$$

is the spectral decomposition of a local state for n qubits described by the local algebra $M_{[0,\,n-1]}$; any such mixed state can be purified, that is, transformed into a projector, by coupling $M_{[0,\,n-1]}$ to itself and by doubling $\rho^{(n)} := \rho_{[0,\,n-1]}$ into

$$\mathbb{C}^{2^n} \otimes \mathbb{C}^{2^n} = \mathbb{C}^{4^n} \ni |\sqrt{\rho^{(n)}}\rangle = \sum_i \sqrt{r_i^{(n)}}\, |r_i^{(n)}\rangle \otimes |r_i^{(n)}\rangle . \tag{29}$$

Given the refined partition of unit $\mathcal{U}^{(n)}$ in (22), one further amplifies the Hilbert space from $\mathbb{C}^{4^n}$ to $\mathbb{C}^{4^n} \otimes \mathbb{C}^{4^n}$ and constructs the following vector state

$$\mathbb{C}^{4^n} \otimes \mathbb{C}^{4^n} \ni |\Psi[\mathcal{U}^{(n)}]\rangle = \sum_i \sum_{(k^{(n)}\ell^{(n)})} \sqrt{r_i^{(n)}}\, U_{k^{(n)}\ell^{(n)}} |r_i^{(n)}\rangle \otimes |r_i^{(n)}\rangle \otimes |k^{(n)}\ell^{(n)}\rangle \,, \tag{30}$$

where the vectors $|k^{(n)}\ell^{(n)}\rangle$ indexed by pairs of binary strings in Ω_2^n form an auxiliary orthonormal basis in $\mathbb{C}^{4^n}$ of cardinality $2^n \times 2^n$.

One thus sees that $|\Psi[\mathcal{U}^{(n)}]\rangle$ is the vector state of a three-partite system consisting of the n qubits, system I, a copy of the latter, system II, and a copy of the first two, system III. From the projection $P = |\Psi[\mathcal{U}^{(n)}]\rangle\langle\Psi[\mathcal{U}^{(n)}]|$, by tracing over the first two systems, respectively, over the last one, one obtains the following marginal states on $M_{[0,\,n-1]} \otimes M_{[0,\,n-1]}$,

$$\mathrm{Tr}_{I,II}(P) = \rho[\mathcal{U}^{(n)}] \,, \qquad \text{respectively,} \tag{31}$$

$$\mathrm{Tr}_{III}(P) = \sum_{(k^{(n)}\ell^{(n)})} U_{k^{(n)}\ell^{(n)}} \otimes 1\, |\sqrt{\rho^{(n)}}\rangle\langle\sqrt{\rho^{(n)}}|\, U^{\dagger}_{k^{(n)}\ell^{(n)}} \otimes 1 = R[\mathcal{U}^{(n)}] \,. \tag{32}$$

Since the latter states are marginal density matrices of a pure state, they have the same spectrum and thus the same von Neumann entropy (see for instance Ref. 22):

$$S\Big(\rho[\mathcal{U}^{(n)}]\Big) = S\Big(R[\mathcal{U}^{(n)}]\Big) = S(\rho^{(n)}) + n \,.$$

Thence, the entropy associated to ω and to the partition of unit $\mathcal{U}^{(n)}$, that is, $\rho[\mathcal{U}^{(n)}]$, is also the entropy of the state $R[\mathcal{U}^{(n)}]$ which results from the action of the POVM $\{U^{\dagger}_{k^{(n)}\ell^{(n)}} \otimes 1\}$ on the purified state $|\sqrt{\rho^{(n)}}\rangle\langle\sqrt{\rho^{(n)}}|$.

IV. SEMI-COMPUTABLE STATES AND SEMI-COMPUTABLE DENSITY MATRICES

We will regard all quantities of interest as semi-computable semi-measures, semi-computable vectors in a separable infinite dimensional Hilbert space $\mathbb{H}$, semi-computable semi-density matrices on $\mathbb{H}$ and so on, as computable functions from $\mathbb{N}$ into $\mathbb{N}$. Our purpose is to adapt Gacs construction to an infinite quantum spin chain or to its doubling as in the case of the Alicki-Fannes entropy; to this aim we list some useful notations and tools:

(1) Let $\mathbb{H}_n = (\mathbb{C}^2)^{\otimes n}$ be the finite 2^n-dimensional Hilbert space of n-qubits. We will denote by $\mathbb{H}$ the infinite dimensional separable Hilbert space obtained by the closure of the union of the nested infinite sequence $\mathbb{H}_n \subset \mathbb{H}_{n+1}$ with respect to the norm coinciding with the usual Hilbert norm on each $\mathbb{H}_n$.

(2) Let the set Q' be defined as follows:

$$Q' = \{(\varepsilon, p, q) \in \{0,1\} \times \mathbb{N}_+ \times \mathbb{N}_+ |\ p \text{ and } q \text{ are coprime}\} \bigcup \{(0,0,0)\} \,.$$

The mapping $\iota : Q \to Q'$ is defined by

$$\iota(0) = (0,0,0), \quad \iota(\frac{p}{q}) = (0,p,q), \quad \iota(-\frac{p}{q}) = (1,p,q)$$

is bijective and the mapping $\iota' : Q' \to \mathbb{N}$ defined by $(\varepsilon, p, q) \to \langle\varepsilon, \langle p, q\rangle\rangle$, where $\langle x, y\rangle = 2^x(2y+1) - 1$ is injective. We can identify Q with the subset $\iota' \circ \iota(Q)$ of $\mathbb{N}$. Similarly, any finite dimensional rational matrix will be represented by a natural number.

(3) With reference to the indexing of the standard basis in (11), we shall consider the set of all functions from $\mathbb{Z}$ into the set $\{0,1\}$ with finite support and denote it by $\mathbf{\Omega}$. Let $\mathbf{i} \in \mathbf{\Omega}$ and let $\theta : \mathbb{Z} \to \mathbb{Z}$ be the left shift $\theta(n) = n - 1$. Then θ induces the map $(\theta(\mathbf{i}))_n = \mathbf{i}_{n+1}$ on $\mathbf{\Omega}$. The restriction of $\mathbf{i}$ to the subinterval I will be denoted by $\mathbf{i}_I$. Furthermore, let $p, q \in \mathbb{Z}$ and $p \le q$. Assume that the support of $\mathbf{i} \in \mathbf{\Omega}$ is contained in the interval $[p, q] \subset \mathbb{Z}$. Then, $\mathbf{i} = \mathbf{0}]\mathbf{i}_{[p,q]}[\mathbf{0}$, where $\mathbf{0}] = \mathbf{i}_{p-1]}$ and $[\mathbf{0} = \mathbf{i}_{[q+1}$ are infinite sequences of 0's.

(4) The map $\mathbf{\Omega} \to \mathbb{N} \times \mathbb{N}$ that associates to $\mathbf{i} \in \mathbf{\Omega}$ the integers

$$(x = \sum_{k<0} \mathbf{i}_k 2^{-k} \ , \ y = \sum_{k\geq 0} \mathbf{i}_k 2^k)$$

is bijective. Therefore, the following two maps

$$\mathbf{\Omega} \ni \mathbf{i} \mapsto \eta(\mathbf{i}) = \langle x, y\rangle \ , \quad \nu(\mathbf{i}) = y - sign(x)[(\langle x, y\rangle + 1)/2 + y] \ , \tag{33}$$

where $sign(x) = 0$ if $x = 0$ otherwise $sign(x) = 1$ are bijections between $\mathbf{\Omega}$ and $\mathbb{N}$, respectively, $\mathbb{Z}$. Then, the inverse mapping

$$\zeta : (\mathbf{i}, \mathbf{j}) \mapsto \nu^{-1}(\eta(\mathbf{j}) - sign(\eta(\mathbf{i}))[(\langle \eta(\mathbf{i}), \eta(\mathbf{j})\rangle + 1)/2 + \eta(\mathbf{j})]) \tag{34}$$

identifies $\mathbf{\Omega} \times \mathbf{\Omega}$ with $\mathbf{\Omega}$.

(5) Let Σ be the power set of $\mathbf{\Omega}$. For $A \in \Sigma$, let $\mu(A) = \#(A)$. Given the measure space $(\mathbf{\Omega}, \Sigma, \mu)$, by the identification of $\mathbf{\Omega}$ with $\mathbb{Z}$, the Hilbert space $L^2(\mathbf{\Omega}, \Sigma, \mu)$ consists of the square-summable functions $f : \mathbb{Z} \mapsto \mathbb{C}$, i.e., $\sum_{x\in\mathbb{Z}} |f(x)|^2 < \infty$. For any $\mathbf{i} \in \mathbf{\Omega}$, consider the function $\delta_{\mathbf{i}}$ defined by

$$\delta_{\mathbf{i}}(\mathbf{i}) = 1 \ , \qquad \delta_{\mathbf{i}}(\mathbf{j}) = 0 \quad \forall\, \mathbf{j} \neq \mathbf{i} \ .$$

The set of these functions which is in one-to-one correspondence with $\mathbf{\Omega}$ is a Hilbert basis for $L^2(\mathbf{\Omega}, \Sigma, \mu)$ and for each $\mathbf{i} \in \mathbf{\Omega}$, $\delta_{\mathbf{i}}$ will be denoted by $|\mathbf{i}\rangle$. Therefore, the representation Hilbert space $\mathbb{H}$ for the quantum spin chain is isomorphic to $L^2(\mathbf{\Omega}, \Sigma, \mu)$.

(6) The mapping ζ identifies $\mathbb{H} \bigotimes \mathbb{H}$ with $\mathbb{H}$. Furthermore, the set of all elements $\mathbf{i} \in \mathbf{\Omega}$ with support included in $[-n, n]$ will be denoted by $\mathbf{\Omega}_{[-n,n]}$. The subspaces of $L^2(\mathbf{\Omega}, \Sigma, \mu)$ generated by $\mathbf{\Omega}_{[-n,n]}$, namely, $L^2(\mathbf{\Omega}_{[-n,n]})$ are isomorphic to the local quantum spin Hilbert spaces $\mathbb{H}_{[-n,n]} = \mathbb{C}^{\otimes 2n+1}$. The corresponding orthogonal projections from $\mathbb{H}$ onto $\mathbb{H}_{[-n,n]}$ will be denoted by P_n, and the canonical injection from $\mathbb{H}_{[-n,n]}$ into $\mathbb{H}$ will be denoted by $\mathbf{i}_n$. In the following we will identify $\mathbb{H}_{[-n,n]}$ with the subspace $\mathbf{i}_n(\mathbb{H}_{[-n,n]})$ of the Hilbert space $\mathbb{H}$.

(7) For a quantum chain linear operator T on $\mathbb{H}$, $P_n T P_n$ will be denoted by T_n.

Definition 4.1.

(1) *A vector* $|\psi\rangle = \sum_{\mathbf{i}\in\mathbf{\Omega}} a_{\mathbf{i}}|\mathbf{i}\rangle \in \mathbb{H}$ *will be said elementary if of its expansion coefficients* $a_{\mathbf{i}}$ *with respect to the fixed orthonormal basis* $\{|\mathbf{i}\rangle\}$ *only a finite number is not zero and those are algebraic numbers. In this way, elementary states can be uniquely associated to natural numbers.*

(2) *A state* $|\psi\rangle = \sum_{\mathbf{i}\in\mathbf{\Omega}} a_{\mathbf{i}}|\mathbf{i}\rangle \in \mathbb{H}$, *where* $a_i \in \mathbb{R}$ *will be termed semi-computable if there exists a computable sequence of elementary vectors* $|\psi_n\rangle = \sum_{i\in\mathbb{N}} a_{n,i}|i\rangle$ *and a computable function* $k : \mathbb{N} \to \mathbb{Q}$, *such that* $lim_{n\to\infty} k_n = 0$, *and for each* n, $|a_{\mathbf{i}} - a_{n,i}| \leq k_n$. *Since the set of all computable functions is countable, the set of all semi-computable elements of* $\mathbb{H}$ *are countable.*

(3) *A linear operator* $M_{2^{2n+1}}(\mathbb{C}) \ni T : \mathbb{H}_{[-n,n]} \to \mathbb{H}_{[-n,n]}$ *will be called elementary if the real and imaginary parts of all of its matrix entries are rational numbers. It follows that the elementary operators can be numbered.*

(4) *The linear operator* $T : \mathbb{H} \to \mathbb{H}$ *is a semi-density matrix if* T *is positive and* $0 \leq \mathrm{Tr}(T) \leq 1$.

(5) *Let* $n_1, n_2 \in \mathbb{N}$, *and* $n_1 \leq n_2$. *Let* $T_j : \mathbb{H}_{[-n_j,n_j]} \to \mathbb{H}_{[-n_j,n_j]}, j = 1, 2$, *be two linear operators:* T_2 *will be said to be quasi-greater than* T_1, $T_1 \leq_q T_2$, *if* $P_{n_1} T_2 P_{n_1} - T_1 \geq 0$, *where* P_{n_1} *is the canonical projection from* $\mathbb{H}_{n_2}$ *to* $\mathbb{H}_{n_1}$. *A sequence of linear operators* $T_n : \mathbb{H}_{[-n,n]} \to \mathbb{H}_{[-n,n]}$ *will be called quasi-increasing if for all* $n \geq 1$, $T_{n+1} \geq_q T_n$.

Proposition 4.2. Let T_n *be a quasi-increasing sequence of semi-density matrices on* $\mathbb{H}$. *Then* $\lim_{n\to\infty} T_n$ *converges in the trace-norm to a semi-density matrix.*

Proof. Since the sequence T_n is quasi-increasing, $\mathrm{Tr}(T_n)$ is an increasing sequence and since for every n, $\mathrm{Tr}(T_n) \leq 1$, the sequence converges in trace-norm, $\|X\|_{tr} = \mathrm{Tr}\sqrt{X^\dagger X}$ to an operator T in

the Banach space $T(\mathbb{H})$ of trace-class operators on $\mathbb{H}$, moreover,

$$\mathrm{Tr}(T) = \lim_{n\to+\infty} \mathrm{Tr}(T_n) = \lim_{n\to+\infty} \|T_n\|_{tr} = \|T\|_{tr} \leq 1 \ .$$

Therefore, T must be positive. □

Definition 4.3. A linear operator T on $\mathbb{H}$ *is a semi-computable semi-density matrix, if there exists a computable quasi-increasing sequence of elementary semi-density matrices* $T_n \in B(\mathbb{H}_{[-n,n]}) \subseteq B(\mathbb{H})$ *such that* $\lim_{n\to\infty}\|T - T_n\|_{tr} = 0$.

We now provide an enumeration of the set of all semi-computable semi-density matrices and deduce the existence of a universal semi-computable semi-density matrix.

Theorem 4.4. *The set of all semi-computable semi-density matrices on* $\mathbb{H}$ *can be enumerated.*

Proof. Let $\{\phi_i\}_{i\geq 0}$ be the standard enumeration of all partially computable functions on $\mathbb{N}$. For $n \in \mathbb{N}$, we change ϕ_n into ψ_n which represents a semi-computable semi-density matrix ρ_n on $\mathbb{H}$. Let $\psi_n(0) = 0$. Assume that $\psi_n(x)$ is defined for $0 \leq x \leq t - 1$ and z is the smallest integer number such that $\psi_n(t-1) = \phi_n(z)$. In order to define $\psi_n(t)$, assume that there is a least integer number x_0, $0 \leq x_0 \leq t$, greater than z, satisfying the relation $STP^{(1)}(x_0, n, t) = 1$ and such that $\phi_n(x_0)$ can be interpreted as an elementary semi-density matrix $\rho_n(t)$ strictly quasi-greater than $\rho_n(t-1)$. Then we set $\psi_n(t) = \phi_n(x_0)$. Otherwise, $\psi_n(t) = \psi_n(t-1)$. Clearly, ψ_n is a computable function and by Proposition 4.2, $\lim_{t\to\infty}\rho_n(t)$ is a semi-computable semi-density matrix. Conversely to each semi-computable semi-density matrix on $\mathbb{H}$ there corresponds a computable function $\psi : \mathbb{N} \to \mathbb{N}$ of the above form. □

Definition 4.5. A semi-computable semi-density matrix μ *is called universal if for any semi-computable semi-density matrix* μ *there exists a constant* $C_\rho > 0$ *such that* $C_\rho\,\rho \leq \mu$.

Theorem 4.6. There exists a universal semi-computable semi-density matrix on $\mathbb{H}$.

Proof. Let $\{\mu_i\}_{i\geq 0}$ be the enumeration of all semi-computable semi-density matrices and set

$$\mu = \sum_{k\geq 0} 2^{-k}\mu_k \ . \tag{35}$$

Clearly, μ is a semi-computable semi-density matrix, and for each semi-computable semi-density matrix μ_k we have $2^{-k}\mu_k \leq \mu$. Therefore, μ is a universal semi-computable semi-density matrix. □

In Ref. 12, the notion of universal semi-density matrix was originally formulated with the explicit restriction to finitely many qubits. Then, given any semi-computable semi-density matrix ρ, two quantum algorithmic entropies were introduced:

$$\underline{H}(\rho) = -\log(\mathrm{Tr}(\rho\mu)) \tag{36}$$

and

$$\overline{H}(\rho) = -\mathrm{Tr}(\rho\log\mu) \ . \tag{37}$$

In the following, we shall adopt the second one as a definition of *quantum algorithmic entropy of the state* ρ for it naturally emerges as the mean value with respect to ρ of a quantum complexity operator:

$$\kappa = -\log\mu \ . \tag{38}$$

Definition 4.7. Since Theorem 4.6 establishes the existence of a universal semi-density matrix for an infinite dimensional quantum spin chain, we take (38) with μ *as in (35) as the complexity*

operator of a quantum spin chain and (37) as the Gacs entropy of any density matrix ρ associated to the chain.

Notice that the complexity operator of the quantum chain assigns the following Gacs entropy to a local density matrix $\rho_{[-n,n]}$ on $\mathbb{H}_{[-n,n]}$:

$$\overline{H}(\rho_{[-n,n]}) = \mathrm{Tr}(\rho_{[-n,n]}\, P_n \kappa\, P_n)\,, \tag{39}$$

where P_n projects from the Hilbert space $\mathbb{H}$ on which μ acts onto the finite dimensional Hilbert space $\mathbb{H}_{[-n,n]}$ on which $\rho_{[-n,n]}$ acts.

On the other hand, one could consider the restriction $\mu^{(n)} = P_n\, \mu\, P_n$ of the universal density matrix μ to $\mathbb{H}_{[-n,n]}$; the natural guess is that $\mu^{(n)}$ might indeed be a universal semi-computable semi-density matrix for the local spin algebra $M_{[-n,n]}$.

That is indeed so is proved in the next lemma.

Lemma 4.8. Let T be a universal semi-computable semi-density matrix which is the limit of a computable quasi-increasing sequence of elementary semi-density matrices T_n. Then, for each k, P_kTP_k is a universal semi-computable semi-density matrix on $\mathbb{H}_{[-k,k]}$.

Proof. Clearly, the sequence $P_kT_nP_k$, $n \geq k$ is a computable quasi-increasing sequence of elementary semi-density matrices; moreover,

$$\lim_{n\to\infty} P_k T_n P_k = P_k T P_k\,.$$

Since T is a universal semi-computable semi-density matrix, for each semi-computable semi-density matrix R_k on $\mathbb{H}_{[-k,k]}$, there exists a positive constant C_k such that $T - C_kR_k \geq 0$. Therefore, $P_kTP_k - C_kR_k \geq 0$. □

Then, given a local spin algebra $M_{[-n,n]}$, we obtain the original finite dimensional formulation of Ref. 12. Indeed, given $\mu^{(n)} = P_n\, \mu\, P_n$, its complexity operator will be

$$\kappa^{(n)} = -\log \mu^{(n)}\,, \tag{40}$$

and, given a state $\rho^{(n)} = \rho_{[-n,n]}$ on $\mathbb{H}_{[-n,n]}$, its Gacs algorithmic entropy will be

$$\overline{H}^{(n)}(\rho^{(n)}) = -\mathrm{Tr}(\rho^{(n)}\, \log \mu^{(n)})\,, \tag{41}$$

where the trace is computed on $\mathbb{H}_{[-n,n]}$.

Based on the infinite dimensional formulation of the complexity operator, we can now study the Gacs algorithmic complexity per site of translation invariant states of quantum spin chains and relate it to their von Neumann entropy rate and AF-entropy.

Theorem 4.9. *Let $\rho^{(n)} \in B_1^+(\mathbb{H}_{[-n,n]})$ be a computable sequence of semi-computable density matrices giving rise to a shift-invariant state ω on the quantum spin chain $\mathcal{M}$. Then*

$$\lim_{n\to\infty} \frac{\overline{H}^{(n)}(\rho^{(n)})}{2n+1} = \lim_{n\to\infty} \frac{\overline{H}(\rho^{(n)})}{2n+1} = s(\omega)\,, \tag{42}$$

where $s(\omega)$ is the von Neumann entropy rate in (14). Also, with reference to the Alicki-Fannes entropy and the density matrices $R[\mathcal{U}^{(n)}]$ on the doubled local sub-algebras $M_{[-n,n]}\otimes M_{[-n,n]}$ in (32), it holds that

$$\lim_{n\to\infty} \frac{\overline{H}^{(n)}(R[\mathcal{U}^{(n)}])}{2n+1} = \lim_{n\to\infty} \frac{\overline{H}(R[\mathcal{U}^{(n)}])}{2n+1} = s(\omega) + 1\,. \tag{43}$$

Proof. By normalizing $\mu^{(n)}$ with $\mathrm{Tr}(\mu^{(n)}) \leq 1$ and using that for any two density matrices $\rho_{1,2}$, ρ_2 invertible, $\mathrm{Tr}(\rho_1(\log\rho_1 - \log\rho_2)) \geq 0$,[5] one estimates

$$S(\rho^{(n)}) \leq -\mathrm{Tr}\left(\rho^{(n)}(\log\mu^{(n)} - \log\mathrm{Tr}(\mu^{(n)}))\right) \leq \overline{H}^{(n)}(\rho^{(n)}).$$

Analogously, $S(\rho^{(n)}) \leq \overline{H}(\rho^{(n)})$. Observe that μ on $\mathbb{H}$ and $\mu^{(n)}$ on $\mathbb{H}_{[-n,n]}$ for each n are invertible.

Let $\rho = \sum_{n\geq 2}\rho^{(n)}/n(\log n)^2$. Then, ρ is a semi-computable semi-density matrix. So, there exists $p \in N$ such that $\rho \leq 2^p\mu$. Because of the operator monotonicity of the logarithm, one estimates

$$S(\rho^{(n)}) \leq \overline{H}(\rho^{(n)}) = -\mathrm{Tr}\Big(\rho^{(n)}\ \log\mu\ \Big) \leq p - \mathrm{Tr}(\rho^{(n)}\log\rho\,)$$
$$\leq S(\rho^{(n)}) + p + \log n + 2\log\log n.$$

Since p is independent of n, then clearly we have

$$\lim_{n\to\infty}\frac{\overline{H}(\rho^{(n)})}{2n+1} = s(\omega).$$

On the other hand, $\rho^{(n)} \leq 2^p n(\log n)^2\mu$, and hence

$$\begin{aligned} S(\rho^{(n)}) &\leq \overline{H}^{(n)}(\rho^{(n)}) = -\mathrm{Tr}_{[-n,n]}\Big(\rho^{(n)}\ \log\mu^{(n)}\Big) \\ &= -\mathrm{Tr}_{[-n,n]}\Big(\rho^{(n)}\ \log P_n\,\mu\ P_n\Big) \\ &\leq -\mathrm{Tr}_{[-n,n]}\Big(\rho^{(n)}\ \log P_n\,\rho^{(n)}\ P_n\Big) + p + \log n + 2\log\log n \\ &\leq S(\rho^{(n)}) + p + \log n + 2\log\log n, \end{aligned} \tag{44}$$

where p is independent of n, then

$$\lim_{n\to\infty}\frac{\overline{H}^{(n)}(\rho^{(n)})}{2n+1} = s(\omega).$$

The relations in (43) can be proved in the same way, once one extends the construction of a universal semi-computable semi-density matrix to the case of the C^*-algebra arising from the inductive limit of the nested net of double local sub-algebras $M_{[-n,n]}\otimes M_{[-n,n]}$. This can be done by means of the map in (34). □

In Ref. 19, both the above relations have been proved under the condition that the Kolmogorov complexity rates

$$\lim_{n\to\infty}\frac{K(\rho^{(n)})}{2n+1} = 0 = \lim_{n\to\infty}\frac{K(R[\mathcal{U}^{(n)}])}{2n+1}\,, \tag{45}$$

where $K(\rho^{(n)})$ and $K(R[\mathcal{U}^{(n)}])$ are the classical Kolmogorov complexities of the elementary density matrices $\rho^{(n)}$ and $R[\mathcal{U}^{(n)}]$. These latter quantities can be interpreted as follows. As already mentioned in Sec. IV, elementary density matrices can be numbered; therefore, to each of them there corresponds a natural number whence one can define the Kolmogorov complexity of any such matrix as the Kolmogorov complexity of the string corresponding to the binary expansion of its associated natural number.

The restriction on the increase of $K(\rho^{(n)})$ and $K(R[\mathcal{U}^{(n)}])$ with n is however not necessary; indeed, by constructing, as done before, an infinite dimensional universal semi-computable semi-density matrix, one can control all universal semi-computable semi-density matrices of the local sub-algebras of the quantum chains, independently of n.

The following example indeed shows an instance of quantum spin chain which does not satisfy the conditions (45) and nevertheless fulfils the conclusions of Theorem 4.9.

Example 4.10. Let P_0 and P_1 be two orthogonal projections in $M_2(\mathbb{C})$ and let $P_{\boldsymbol{i}^{(n)}} = \bigotimes_{j=0}^{n-1} P_{i_j}$ denote the orthogonal projections obtained by tensor products. Let the starting one site density matrix be $\rho_{\{0\}} = \frac{P_0+P_1}{2}$ and assume that $\rho^{(n)} = \rho_{[0,\,n-1]}$ be defined such that its complexity $K(\rho^{(n)}) \geq n^2$. We now recursively construct $\rho^{(n+1)}$ so that on one hand the family of density matrices satisfies the compatibility and translation invariant conditions (12) and (13), *whence*

$$\lim_{n\to\infty}\frac{S(\rho^{(n)})}{n} = s(\omega) < +\infty\,,$$

and, on the other hand, so that $K(\rho^{(n+1)}) \geq (n+1)^2$, *whence*

$$\lim_{n\to\infty} \frac{K(\rho^{(n)})}{n} = +\infty \, .$$

Write $\rho^{(n)} = \sum_{\boldsymbol{i}^{(n)}} a_{\boldsymbol{i}^{(n)}} P_{\boldsymbol{i}^{(n)}}$. *Then, the conditions* (12) and (13) *yield*

$$\mathrm{Tr}_{\{0\}}\rho^{(n+1)} = \mathrm{Tr}_{\{n+1\}}\rho^{(n+1)} = \rho^{(n)} \, ,$$

whence

$$\sum_{\boldsymbol{i}^{(n)}\in\Omega_2^{(n)}} (a_{0\boldsymbol{i}^{(n)}} + a_{1\boldsymbol{i}^{(n)}})P_{\boldsymbol{i}^{(n)}} = \sum_{\boldsymbol{i}^{(n)}\in\Omega_2^{(n)}} (a_{\boldsymbol{i}^{(n)}0} + a_{\boldsymbol{i}^{(n)}1})P_{\boldsymbol{i}^{(n)}} = \sum_{\boldsymbol{i}^{(n)}\in\Omega_2^{(n)}} a_{\boldsymbol{i}^{(n)}} P_{\boldsymbol{i}^{(n)}} \, .$$

Then, because of the orthogonality of the projections $P_{\boldsymbol{i}^{(n)}}$, *it follows that*

$$\begin{aligned}
a_{0\mathbf{i}^{(n-2)}0} + a_{0\mathbf{i}^{(n-2)}1} &= a_{0\mathbf{i}^{(n-2)}}, \\
a_{0\mathbf{i}^{(n-2)}1} + a_{1\mathbf{i}^{(n-2)}1} &= a_{\mathbf{i}^{(n-2)}1}, \\
a_{1\mathbf{i}^{(n-2)}1} + a_{1\mathbf{i}^{(n-2)}0} &= a_{1\mathbf{i}^{(n-2)}}, \\
a_{1\mathbf{i}^{(n-2)}0} + a_{0\mathbf{i}^{(n-2)}0} &= a_{\mathbf{i}^{(n-2)}0} \, ,
\end{aligned}$$

for any of the 2^{n-2} *strings* $\mathbf{i}^{(n-2)} \in \Omega_2^{n-2}$. *In this way, the system of* 2^n *equations can be subdivided into* 2^{n-2} *sub-systems of* 4 *equations each. Let us focus upon the system above defined by the string* $\mathbf{i}^{(n-2)}$; *the values at the right-hand side have been chosen at step* $n-1$. *They are positive, with all the others they sum up to* 1. *Without loss of generality, we may assume they are in decreasing order:* $a_{0\mathbf{i}^{(n-2)}} \geq a_{\mathbf{i}^{(n-2)}1} \geq a_{1\mathbf{i}^{(n-2)}} \geq a_{\mathbf{i}^{(n-2)}0} > 0$. *We can now choose* $a_{1\mathbf{i}^{(n-2)}1} = x_{\mathbf{i}^{(n-2)}}$, *a positive real number such that* $x_{\mathbf{i}^{(n-2)}} \leq a_{1\mathbf{i}^{(n-2)}}$ *with Kolmogorov complexity* $K(x_{\mathbf{i}^{(n-2)}}) \geq n^2$. *Then,*

$$a_{1\mathbf{i}^{(n-2)}0} = a_{1\mathbf{i}^{(n-2)}} - x_{\mathbf{i}^{(n-2)}} \, , \; a_{0\mathbf{i}^{(n-2)}1} = a_{\mathbf{i}^{(n-2)}1} - x_{\mathbf{i}^{(n-2)}} \, , \; a_{0\mathbf{i}^{(n-2)}0} = a_{0\mathbf{i}^{(n-2)}} - a_{\mathbf{i}^{(n-2)}1} + x_{\mathbf{i}^{(n-2)}} \, .$$

Therefore, the coefficients at step n are positive, the sum of all of them is 1 *and they satisfy the desired condition on the increase of the algorithmic complexity of* $\rho^{(n)}$.

V. CONCLUSIONS

In this work we have extended the notions of computability, semi-computability, semi-computable vector states, and semi-computable density matrices to infinite dimensional Hilbert spaces. These extensions are necessary to describe algorithmically by classical Turing machines quantum systems with infinitely many degrees of freedom. In this paper we have applied them to the discussion, from a computer science point of view, of the complexity of quantum spin chains with the shift dynamics.

In classical information theory, Brudno has proved a relation between the Kolmogorov-Sinai dynamical entropy of ergodic time-evolutions and the algorithmic complexity per unit time step of all almost trajectories. In quantum information theory there are different extensions of both the Kolmogrov-Solomonoff-Chatin algorithmic complexity and of the Kolmogorov-Sinai dynamical entropy: their possible relations can be found in Ref. 22.

The techniques developed in this paper have been applied to quantum spin chains. They allowed us to show that the Gacs algorithmic entropy per site of translation invariant states is equal to the von Neumann entropy rate. This could be done by removing an unnecessary condition in a previous proof of the same relations.[19]

The next important step is to try to establish a Brudno-like relation for a dense set of states of the quantum spin chain and for more general state transformations than the space-translations.

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[24] An instance of such an injection is the map $\iota' \circ \iota$ defined in Sec. IV.