Optimal block execution of trades

Petar Maymounkov

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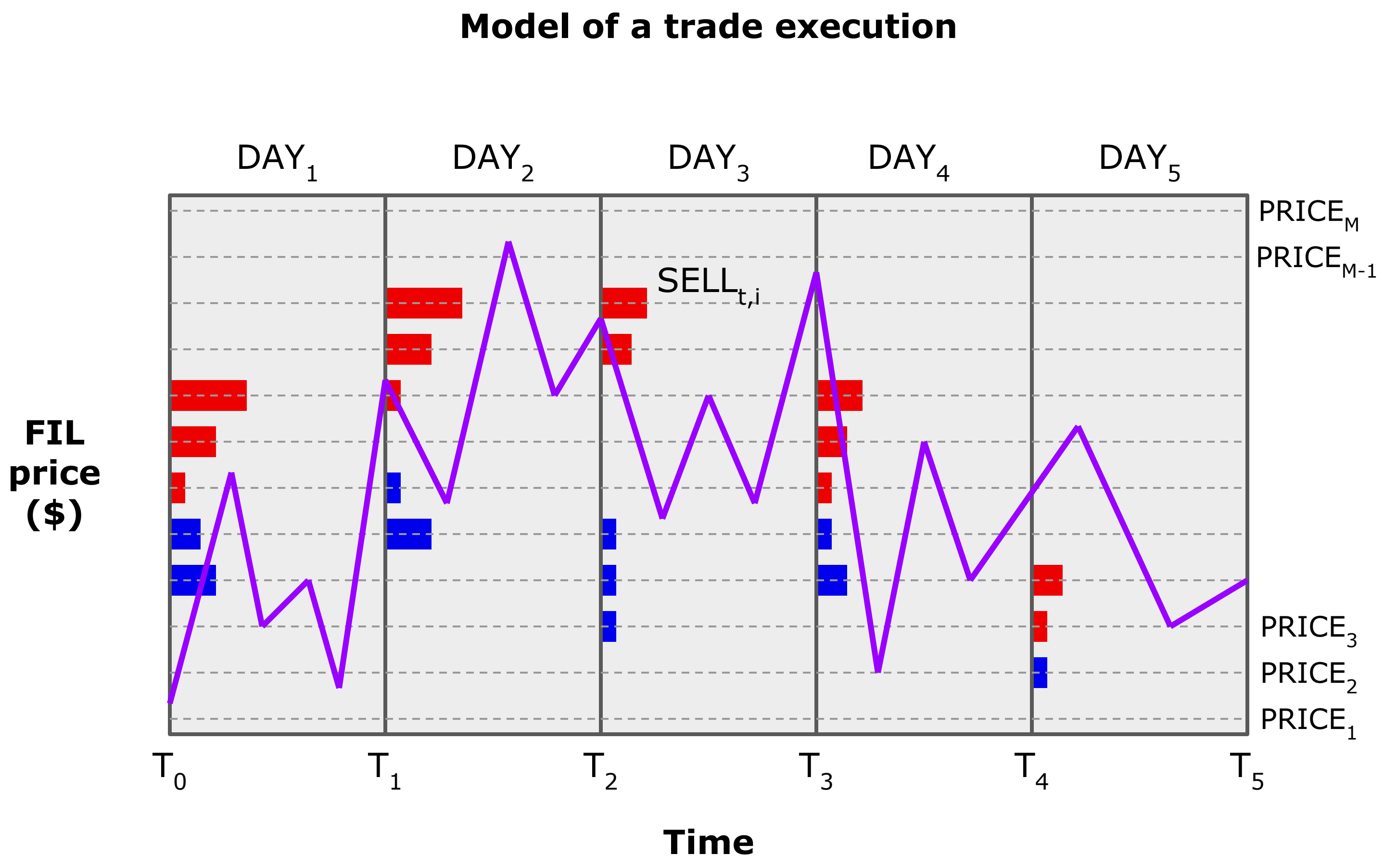
Problem statement, first stab

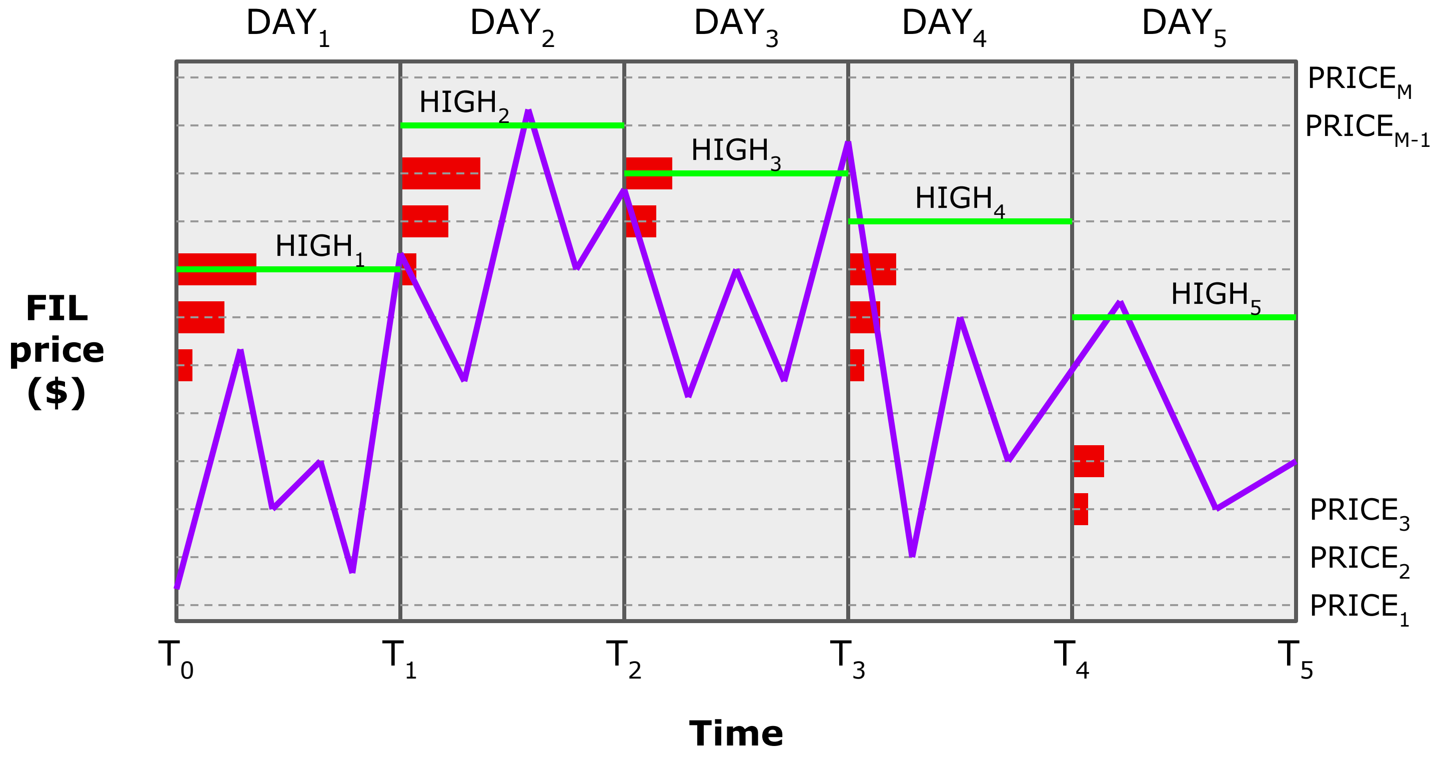
Traditionally, OTC service by brokers to institutional clients.
Electronic version, around 2005.
Now crypto OTC services.

Sell 1 FIL before a deadline for as much USD as possible, given a (probabilistic) model of FIL/USD.

(Market Making is block execution on both sides of the market.)


Model of a trade execution
DAY$_1$
DAY$_2$
DAY$_3$
DAY$_4$
DAY$_5$
PRICE$_M$
PRICE$_{M-1}$
SELL$_{t,i}$
FIL price ($)
PRICE$_3$
PRICE$_2$
PRICE$_1$
T$_0$
T$_1$
T$_2$
T$_3$
T$_4$
T$_5$
Time

If only selling, HIGH PRICE between syncs determines limit order matches.



Assumptions and objective

Assume given PROB_i: Probability that the next day highest price is PRICE_i.

Let USD_t = dollars earned from selling FIL 1 between T_t and T_{LAST} using strategy.

Objective:

maximize $E[USD_0]$ over the choices of $\overbrace{SELL_{i,t}}^{\text{the unknowns (the strategy)}}$

such that $SELL_{i,t} >= 0$ for all i,t
and $\sum_i SELL_{i,t} <= 1$ for all t

Solution

$$USD_{LAST} = 0$$

$$USD_t = \text{“matched orders between } T_t \text{ and } T_{t+1}\text{”} + \text{“unsold FIL”} \cdot USD_{t+1}$$

$$= \text{“orders with } SELL_{i,t} <= HIGH_t\text{”} + \text{“unsold FIL”} \cdot USD_{t+1}$$

$$= \Sigma_{p:price} \underbrace{[HIGH_t = PRICE_p]}_{\substack{\text{indicator variable:} \\ \text{1 if condition met,} \\ \text{0 otherwise}}} \cdot \Big(\underbrace{\Sigma_{i<=p}\, PRICE_i \cdot SELL_{t,i}}_{\substack{\text{proceeds from} \\ \text{matched sell} \\ \text{orders}}} + \underbrace{(1 - \Sigma_{i<=p} SELL_{t,i})}_{\substack{\text{remaining FIL} \\ \text{(not matched)}}} \cdot USD_{t+1} \Big)$$

$$E[USD_t] = \Sigma_{p:price}\, PROB_p \cdot \big(\Sigma_{i<=p}\, PRICE_i \cdot SELL_{t,i} + (1 - \Sigma_{i<=p} SELL_{t,i}) \cdot E[USD_{t+1}] \big)$$

maximize $E[USD_t]$ over $SELL_{t,i}$ using linear programming

Implementation intermission

github.com/petar/BlockExecExpMax.jl

Discussion and improvements

Caveat: Maximizing expectation, $E[USD_0]$, can produce bi-modal (all-or-nothing) outcomes.

Intuitively, we want to maximize expectation **and** minimize variance.

However, minimizing variance contradicts maximizing expectation.

So, we must embed our preferences in a single objective function.

Sharpe ratio: $E[USD_0]$ / $StdDev[USD_0]$

Weighted: $E[USD_0]$ - $3*StdDev[USD_0]$

Optimization challenges

Maximizing Sharpe (or any variance-based objectives) is **not** linear programming.

maximize_x Sharpe **and** maximize_x Sharpe^2 **find the same x**.

$$\text{Sharpe}^2 = E[USD_0]^2 \,/\, \text{StdDev}[USD_0]^2$$

$$= E[USD_0]^2 \,/\, \text{Var}[USD_0]$$

$$= E[USD_0]^2 \,/\, (\, E[USD_0{}^2] - E[USD_0]^2 \,)$$

Therefore, the Sharpe maximization is **rational** and **quadratic** (in the unknowns $SELL_{t,i}$).

Possible approaches

Linear/quadratic/conic/semidefinite programming find global optima,

but are harder to work with: objective needs to be rewritten cleverly.

Alternatively, one could find an optimum for the objective (e.g. Sharpe) using gradient methods like **(Stochastic) Gradient Descent**. Global optimum is not always found, but no need to rewrite the problem objective.

Takeaway

Every “control” problem can be somehow written as a mathematical optimization problem:

maximize f(x), subject to a(x) >= 0, b(x) >= 0, etc.

With a little effort you can find a decent solution to any optimization problem.

(Use Julia as a one-stop-shop for math.)