Problem sheet 1

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November 18, 2016

1 a

$P+Q$ is convex if all line segments joining two points of the polytope is contained in the polytope. And we will prove that it happens only if P and Q are convex.

$$P+Q \subset \mathbb{R}^d \quad convex \Longleftrightarrow \forall s_1, s_2 \in P+Q \quad s_1 s_2 \subset P+Q$$

Take two points $s_1, s_2 \in P+Q$. Then, $\exists p_1, p_2 \in P$, $\exists q_1, q_2 \in Q$ such that $s_1 = p_1 + q_1$ and $s_2 = p_2 + q_2$.

Let $r \in s_1 s_2$ be any point of the segment. Then, $\exists \lambda \in [0,1]$ such that $r = \lambda s_1 + (1-\lambda) s_2$.

Expanding it:
$r = \lambda s_1 + (1-\lambda) s_2 = \lambda (p_1 + q_1) + (1-\lambda)(p_2 + q_2) = \lambda p_1 + (1-\lambda) p_2 + \lambda q_1 + (1-\lambda) q_2$

If P and Q are convex, then $p_1 p_2 \subset P$ and $q_1 q_2 \subset Q$, so $\exists p \in P$ s.t. $p = \lambda p_1 + (1-\lambda) p_2$ and $\exists q \in Q$ s.t. $q = \lambda q_1 + (1-\lambda) q_2$.

So $r = p + q \in P+Q \quad \forall r \in s_1 s_2 \Longrightarrow s_1 s_2 \subset P+Q$ for any $s_1, s_2 \in P+Q$
Finally, $P+Q$ is convex if P and Q is convex.

Translating P and Q in $\mathbb{R}^d$ is the same as summing a single point to each. Let $p', q' \in \mathbb{R}^d$ be the points that represent the translation of P and Q. Then $(P+p') + (Q+q') = \{p+q : p \in P+p',\, q \in Q+q'\} = \{p+p'+q+q' : p \in P,\, q \in Q\} = (P+Q) + p' + q'$
So $P+Q$ translates with the composition of the translations of P and Q.

2 b

Lemma 1. *New(fg)=New(f)+New(g) for polynomials* $f(x) = \sum_{i=1}^{m} c_i x_1^{a_{i1}} x_2^{a_{i2}} \dots x_d^{a_{id}}$ *and* $g(x) = \sum_{i=1}^{p} d_i x_1^{b_{i1}} x_2^{b_{i2}} \dots x_d^{b_{id}}$

Proof. We note that $fg(x) = \sum_{i=1}^{m} \sum_{j=1}^{p} c_i d_j x_1^{a_{i1}+b_{j1}} x_2^{a_{i2}+b_{j2}} \dots x_n^{a_{id}+b_{jd}}$.
So New(fg) is the convex hull of points of the form $(a_{i1} + b_{j1}, a_{i2} + b_{j2}, \dots, a_{id} + b_{jd})$. These points are thus of the form $p+q$ with $p \in$ New(f) and $q \in$ New(g), so New$(fg) \subset$ New(f)+New(g).

Now choose a vertex $v \in$ New(f)+ New(g). Then $v = (a_{i1}, a_{i2}, \dots, a_{id})+(b_{j1}, b_{j2}, \dots, b_{jd})$ for some i and j. Then also $c_i \neq 0 \neq d_j$ and therefore $c_i d_j \neq 0$, so $v \in$ New(fg). □

3 c

From the polynomials we can easily derive the points we get to construct the convex hull. For f we find $(0, 0), (0, 1), (1, 1), (1, 0)$ and for $g : (0, 0), (2, 1), (1, 2)$.
I drew a picture in Figure 1. From this picture we can derive that the area$(P) = 1$, the area$(Q) = 4 - 2 - \frac{1}{2} = 1.5$ (we distract from the 2-by-2 square two half 1-by-2 rectangles and half a unit square) and area$(P + Q) = 9 - 2 - \frac{1}{2} = 6.5$ (we distract from the 3-by-3 square two half 1-by-2 rectangles and half a unit square). So $M(P, Q) = 6.5 - 1 - 1.5 = 4$.

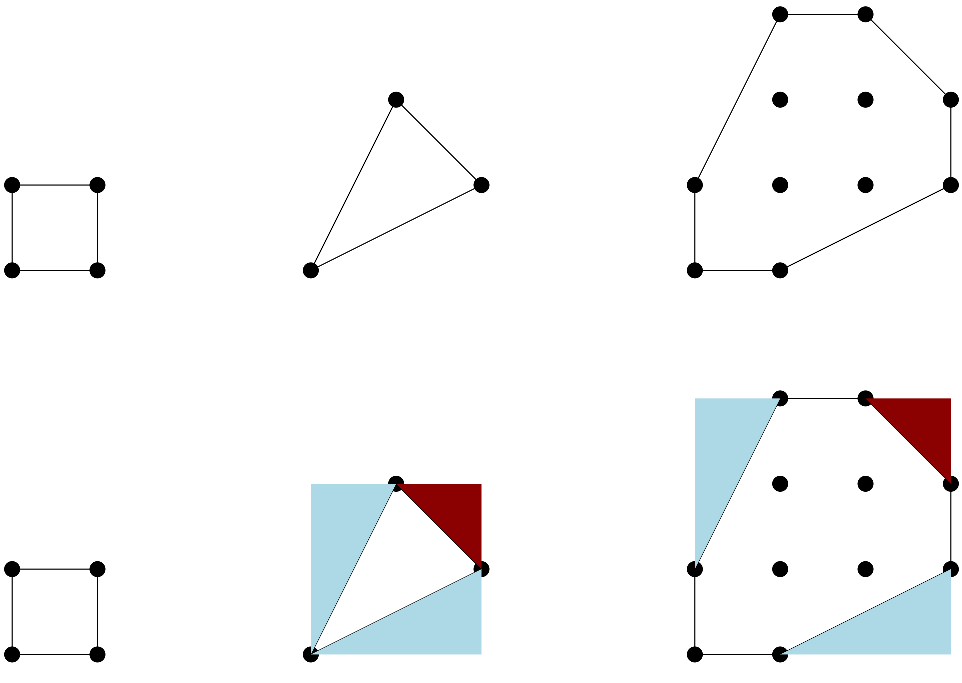

Figure 1: From left to right: P, Q and $P + Q$. The second row is with added triangles to make the calculation of the area easier.

4 d

We can subdivide $P+Q$ into a translate of P, a translate of Q and three parallelograms as seen in Figure 2.

We get $M(P,Q)$ by subtracting the area of P and Q to the area of $P+Q$. $M(P,Q)$ is sum of the area of the remaining three parallelograms.

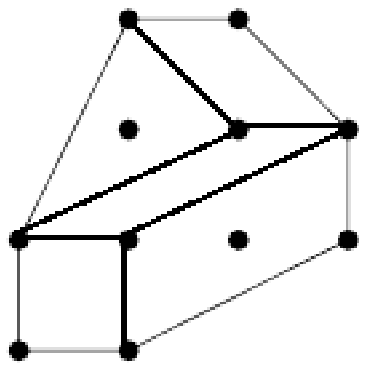

Figure 2: Subdivision of $P+Q$ into a translate of P, a translate of Q and three parallelograms.

5 e

Suppose we have two lattice polygons P and Q. Construct a polynomial f as follows: starting with $f = 0$, for every vertex $p_i = (a_{i1}, a_{i2})$ of P add a term to f of the from $x^{a_{i1}}y^{a_{i2}}$. In this way, $\text{New}(f) = P$. Analogous, construct a g so that $\text{New}(g) = Q$. Note that this construction is fine, since a_{ij} are integers because P, Q are lattices polygons. Because P and Q are polygons, we only have two dimensions and can use Bernstein's Theorem.
Using Bernstein's Theorem we find that M(New(f),New(g)) $= M(P,Q) =$ (number of solutions of the system $f(x,y) = 0, g(x,y) = 0$), which is an integer.

6 f

Suppose we have two general plane algebraic curves f and g of degree d and e and with Newton-polygons $P = \text{New}(f), Q = \text{New}(g)$. For the algebraic plane curve f of degree d, we know that it is of the form $\sum_{i=0}^{d} a_i * x^i * y^{d-i}$ with all $a_i \neq 0$. So that the $P = \text{conv}(\{(i, d-i) | i \in [0, \ldots, d]\}$ which is the triangle through $(0,0), (d,0), (0,d)$. Analogous we find $Q = \Delta\{(0,0), (e,0), (0,e)\}$.

Now we will look at $P+Q$. When calculating $P+Q$ we should only consider the boundaries of P and Q. From the geometric structure (both lie in the corner of the first quadrant) we see that only the hypotenusa of the triangle will be important to define $P+Q$. The points on the hypotenusa of P are $\{(i, d-i)|0 \leq i \leq d, i \in \mathbb{R}\}$ and for $Q : \{(j, e-j)|0 \leq j \leq ein\mathbb{R}\}$. So the points for $P+Q$ will be $\{(i+j, e+d-i-j)|0 \leq i \leq d, 0 \leq j \leq e, i, j \in \mathbb{R}\} = \{(k, e+d-k)|0 \leq k \leq d+e, k \in \mathbb{R}\}$. So we find that $P+Q$ is a right triangle with two legs of length $d+e$.
We find:

$$M(P,Q) = \frac{(d+e)^2}{2} - \frac{d^2}{2} - \frac{e^2}{2} = \frac{d^2+2de+e^2-d^2-e^2}{2} = de$$

Consider now the functions f and g we used in part c). We calculated M(New(f),New(g) $=$ 4. From Bernstein's theorem we can find that they meet in 4 points. We notice that f and g are not general polynomials, but we notice that the points defining New(f) are contained in the Newton polynomial generated by a general polynomial of degree equal to the maximum degree of f. Analogous for G. As such, Bezout's theorem gives an upperbound for the number of meeting points, based on their degree. Here deg(f)*deg(g) $= 2 * 3 = 6$ and $4 \leq 6$.