

Problem sheet 2

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1 Nonrational Pentagon

Given the incidences we have given, we should show that the inner pentagon is regular, then we can use <http://mathworld.wolfram.com/Pentagon.html> to show the construction cannot be realised with rational coordinates.

2 6.9: enumerate 4-polytopes with 7 vertices

To enumerate all 4-polytopes with 7-vertices, we started considering only these with a Gale diagram without special points and without coinciding points. For finding these polytopes we wrote an algorithm.

We start considering all possible Gale diagrams, this is a list of 7 +’s and –’s. First we filtered all diagrams that couldn’t come from a polytope. For doing this, we used Theorem 6.19 in Ziegler, which says that a Gale diagram relates to a polytope \iff every cocircuit has at least 2 positive elements. This we can check by, for every index, checking the plusses after the index and the minuses before the index and making sure they sum up to at least 2.

Then we filtered out the isomorphism, by creating the graph of the facet structure and check whether these are isomorphic. For checking this we used a graphlibrary.

Finally we find 5 non-isomorphic Gale diagrams:

$[0, 0, 1, 1, 0, 0, 1], [0, 0, 1, 0, 1, 0, 1], [0, 1, 0, 1, 0, 0, 1], [0, 1, 0, 0, 1, 0, 1], [1, 0, 1, 0, 1, 0, 1]$

3 6.15: 2 different 2-neighborly 4-polytopes

We can easily check that both polytopes are 2-neighborly, since removing any two points from the Gale diagram and then taking the convex hull of the positive and of the negative points, we find that these intersect. Thus the two removed points form an edge. I checked this manually, but I haven't written this down, using symmetries you should have to check around 30 intersections.

If we write down all facets from the first Gale diagram, using the fact that circuits define cofacets. And also from $C_4(8)$, using Gale's evenness criterium, we find that exactly the same structure appears (even the naming is already correct). If we write down the facets from the second Gale diagram, we note there are differences, since their are different convex hulls (we also get the intersection of a triangle and a point).

To make sure that the second graph differs from the first, we used the sagemath graph library (a python-based mathematical computer language) to compare the graphs of the facetstructure. I generated images for both graphs: see figure 1 and figure 2. Using the isomorphism-function of the graph library, we find that both graphs are different.

4 6.17: analyse gale diagram

We can find the facets by finding circuits in the Gale diagram. For example the convex hull of $\{1\}$ and $\{4\}$ intersect, so 14 is a cofacet and 235678 a facet of P . To check that this is an octahedron, we lift 14 to infinity, and put the rest on one line, we get as Gale diagram a line with following points: $2^-, 3^+6^+, 7^-8^-, 5^+$. If we look at the facet structure of the polytope with this Gale diagram, we see the polytope is 4-regular, has 2 vertices 2 and 5 which are neighbours of 3, 6, 7, 8 and that $3 \sim 7, 6 \sim 8, 7 \not\sim 8$ and $3 \not\sim 6$. This is exactly the incidence structure of an octahedron.

For example we also find the cofacet 3457 which results in a facet 1268. Because this facet has 4 vertices in dimension 3, it is a simplex. We find some other simplices: 1267, 3457, 3458.

We also find facets: 14568, 14567, 12347, 12348. To be a square pyramid they need a special point (a zero-point which is non-plus and non-negative), but I don't know how to get one.

If we have the same circuits, we have the same intersections in our Gale diagram so both 7 and 8 should lie on the line 23 and on the line 56, since these lines are different and the intersection is a single point, they must lie on the same point.

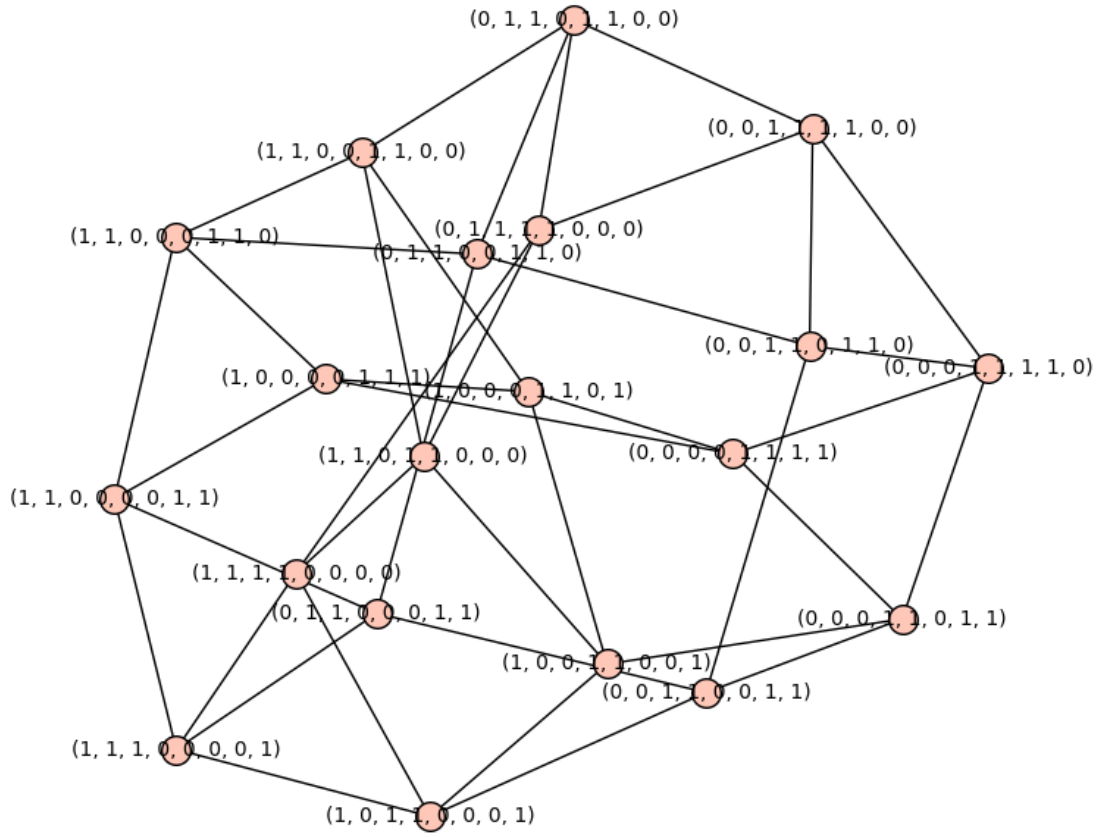


Figure 1: The facet-graph of $C_4(8)$. Vertices are facets, and edges represent ridges between two facets.

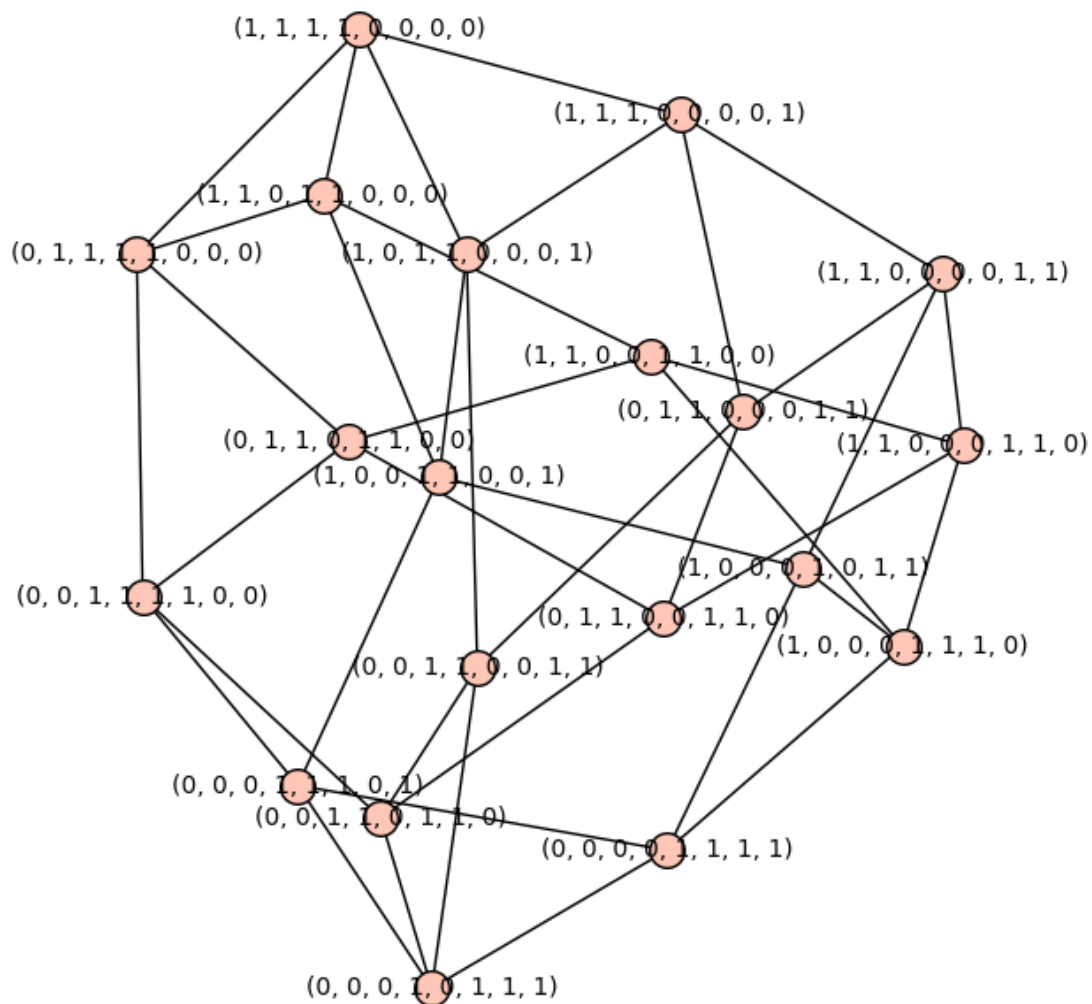


Figure 2: The facet-graph of the second Gale diagram. Vertices are facets, and edges represent ridges between two facets.

XXX Then 2356 lie in the same plane.

Since 2356 must be coplanar, we cannot chose all of them freely and the octahedral facet cannot be prescribed.

XXX The oriented matroid is not rigid, I don't know yet.