
Problem sheet 1

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1 a

$P + Q$ is convex if all line segments joining two points of the polytope is contained in the polytope. And we will prove that it happens only if P and Q are convex.

$$P + Q \subset \mathbb{R}^d \text{ convex} \iff \forall s_1, s_2 \in P + Q \quad s_1 s_2 \subset P + Q$$

Take two points $s_1, s_2 \in P + Q$. Then, $\exists p_1, p_2 \in P, \exists q_1, q_2 \in Q$ such that $s_1 = p_1 + q_1$ and $s_2 = p_2 + q_2$.

Let $r \in s_1 s_2$ be any point of the segment. Then, $\exists \lambda \in [0, 1]$ such that $r = \lambda s_1 + (1 - \lambda) s_2$.

Expanding it:

$$r = \lambda s_1 + (1 - \lambda) s_2 = \lambda (p_1 + q_1) + (1 - \lambda) (p_2 + q_2) = \lambda p_1 + (1 - \lambda) p_2 + \lambda q_1 + (1 - \lambda) q_2$$

If P and Q are convex, then $p_1 p_2 \subset P$ and $q_1 q_2 \subset Q$, so $\exists p \in P$ s.t. $p = \lambda p_1 + (1 - \lambda) p_2$ and $\exists q \in Q$ s.t. $q = \lambda q_1 + (1 - \lambda) q_2$.

So $r = p + q \in P + Q \quad \forall r \in s_1 s_2 \implies s_1 s_2 \subset P + Q$ for any $s_1, s_2 \in P + Q$

Finally, $P + Q$ is convex if P and Q is convex.

Translating P and Q in \mathbb{R}^d is the same as summing a single point to each. Let $p', q' \in \mathbb{R}^d$ be the points that represent the translation of P and Q . Then $(P + p') + (Q + q') = \{p + q : p \in P + p', q \in Q + q'\} = \{p + p' + q + q' : p \in P, q \in Q\} = (P + Q) + p' + q'$

So $P + Q$ translates with the composition of the translations of P and Q .

2 b

Lemma 1. $New(fg) = New(f) + New(g)$ for polynomials $f(x) = \sum_{i=1}^m c_i x_1^{a_{i1}} x_2^{a_{i2}} \dots x_d^{a_{id}}$ and $g(x) = \sum_{i=1}^p d_i x_1^{b_{i1}} x_2^{b_{i2}} \dots x_d^{b_{id}}$

Proof. We note that $fg(x) = \sum_{i=1}^m \sum_{j=1}^p c_i d_j x_1^{a_{i1}+b_{j1}} x_2^{a_{i2}+b_{j2}} \dots x_d^{a_{id}+b_{jd}}$. So $New(fg)$ is the convex hull of points of the form $(a_{i1} + b_{j1}, a_{i2} + b_{j2}, \dots, a_{id} + b_{jd})$. These points are thus of the form $p+q$ with $p \in New(f)$ and $q \in New(g)$, so $New(fg) \subset New(f) + New(g)$.

Now choose a vertex $v \in New(f) + New(g)$. Then $v = (a_{i1}, a_{i2}, \dots, a_{id}) + (b_{j1}, b_{j2}, \dots, b_{jd})$ for some i and j . Then also $c_i \neq 0 \neq d_j$ and therefore $c_i d_j \neq 0$, so $v \in New(fg)$. \square

3 c

From the polynomials we can easily derive the points we get to construct the convex hull. For f we find $(0, 0), (0, 1), (1, 1), (1, 0)$ and for $g : (0, 0), (2, 1), (1, 2)$.

I drew a picture in Figure 1. From this picture we can derive that the $area(P) = 1$, the $area(Q) = 4 - 2 - \frac{1}{2} = 1.5$ (we distract from the 2-by-2 square two half 1-by-2 rectangles and half a unit square) and $area(P + Q) = 9 - 2 - \frac{1}{2} = 6.5$ (we distract from the 3-by-3 square two half 1-by-2 rectangles and half a unit square). So $M(P, Q) = 6.5 - 1 - 1.5 = 4$.

4 d

We can subdivide $P + Q$ into a translate of P , a translate of Q and three parallelograms as seen in Figure 2.

We get $M(P, Q)$ by subtracting the area of P and Q to the area of $P + Q$. $M(P, Q)$ is sum of the area of the remaining three parallelograms.

5 e

Suppose we have two lattice polygons P and Q . Construct a polynomial f as follows: starting with $f = 0$, for every vertex $p_i = (a_{i1}, a_{i2})$ of P add a term to f of the form $x^{a_{i1}} y^{a_{i2}}$. In this way, $New(f) = P$. Analogous, construct a g so that $New(g) = Q$. Note that this construction is fine, since a_{ij} are integers because P, Q are lattices polygons. Because P and Q are polygons, we only have two dimensions and can use Bernstein's Theorem.

Using Bernstein's Theorem we find that $M(New(f), New(g)) = M(P, Q) = (\text{number of solutions of the system } f(x, y) = 0, g(x, y) = 0)$, which is an integer.

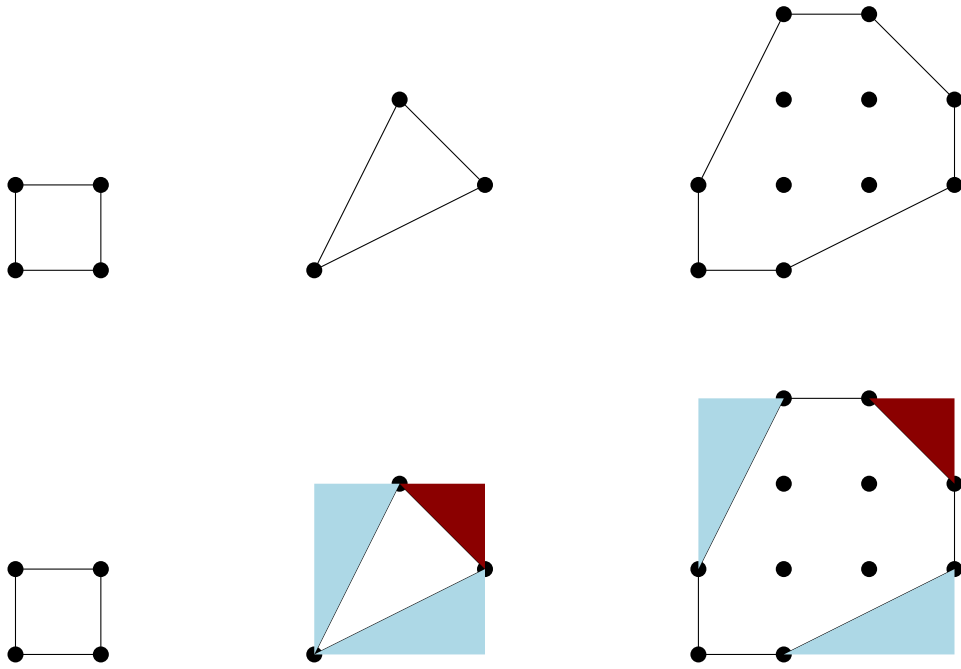


Figure 1: From left to right: P , Q and $P + Q$. The second row is with added triangles to make the calculation of the area easier.

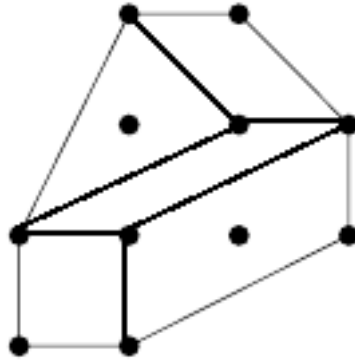


Figure 2: Subdivision of $P + Q$ into a translate of P , a translate of Q and three parallelograms.

6 f

Suppose we have two general plane algebraic curves f and g of degree d and e and with Newton-polygons $P = \text{New}(f)$, $Q = \text{New}(g)$. For the general algebraic plane curve f of degree d , we know that it contains a part of the form $\sum_{i=0}^d a_i * x^i * y^{d-i}$ with all $a_i \neq 0$. All other terms will have their sum of degrees smaller than d , so that the $P = \text{conv}(\{(i, d-i) | i \in [0, \dots, d]\})$ which is the triangle through $(0, 0)$, $(d, 0)$, $(0, d)$. Analogous we find $Q = \Delta\{(0, 0), (e, 0), (0, e)\}$.

Now we will look at $P + Q$. When calculating $P + Q$ we should only consider the boundaries of P and Q . From the geometric structure (both lie in the corner of the first quadrant) we see that only the hypotenusa of the triangle will be important to define $P + Q$. The points on the hypotenusa of P are $\{(i, d-i) | 0 \leq i \leq d, i \in \mathbb{R}\}$ and for $Q : \{(j, e-j) | 0 \leq j \leq e, j \in \mathbb{R}\}$. So the points for $P + Q$ will be $\{(i+j, e+d-i-j) | 0 \leq i \leq d, 0 \leq j \leq e, i, j \in \mathbb{R}\} = \{(k, e+d-k) | 0 \leq k \leq d+e, k \in \mathbb{R}\}$. So we find that $P + Q$ is a right triangle with two legs of length $d + e$.

We find:

$$M(P, Q) = \frac{(d+e)^2}{2} - \frac{d^2}{2} - \frac{e^2}{2} = \frac{d^2 + 2de + e^2 - d^2 - e^2}{2} = de$$

Consider now the functions f and g we used in part c). We calculated $M(\text{New}(f), \text{New}(g)) = 4$. From Bernstein's theorem we can find that they meet in 4 points. We notice that f and g are not general polynomials, but we notice that the points defining $\text{New}(f)$ are contained in the Newton polynomial generated by a general polynomial of degree equal to the maximum degree of f . Analogous for G . As such, Bezout's theorem gives an upperbound for the number of meeting points, based on their degree. Here $\deg(f) * \deg(g) = 2 * 3 = 6$ and $4 \leq 6$.