
Problem sheet 1

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1 a

2 b

Lemma 1. $\text{New}(fg) = \text{New}(f) + \text{New}(g)$ for polynomials $f(x) = \sum_{i=1}^m c_i x_1^{a_{i1}} x_2^{a_{i2}} \dots x_d^{a_{id}}$ and $g(x) = \sum_{i=1}^p d_i x_1^{b_{i1}} x_2^{b_{i2}} \dots x_d^{b_{id}}$

Proof. We note that $fg(x) = \sum_{i=1}^m \sum_{j=1}^p c_i d_j x_1^{a_{i1}+b_{j1}} x_2^{a_{i2}+b_{j2}} \dots x_d^{a_{id}+b_{jd}}$. So $\text{New}(fg)$ is the convex hull of points of the form $(a_{i1} + b_{j1}, a_{i2} + b_{j2}, \dots, a_{id} + b_{jd})$. These points are thus of the form $p+q$ with $p \in \text{New}(f)$ and $q \in \text{New}(g)$, so $\text{New}(fg) \subset \text{New}(f) + \text{New}(g)$.

Now choose a vertex $v \in \text{New}(f) + \text{New}(g)$. Then $v = (a_{i1}, a_{i2}, \dots, a_{id}) + (b_{j1}, b_{j2}, \dots, b_{jd})$ for some i and j . Then also $c_i \neq 0 \neq d_j$ and therefore $c_i d_j \neq 0$, so $v \in \text{New}(fg)$. \square

3 c

From the polynomials we can easily derive the points we get to construct the convex hull. For f we find $(0, 0), (0, 1), (1, 1), (1, 0)$ and for g : $(0, 0), (2, 1), (1, 2)$.

I drew a picture in Figure 1. From this picture we can derive that the $\text{area}(P) = 1$, the $\text{area}(Q) = 4 - 2 - \frac{1}{2} = 1.5$ (we distract from the 2-by-2 square two half 1-by-2 rectangles and half a unit square) and $\text{area}(P + Q) = 9 - 2 - \frac{1}{2} = 6.5$ (we distract from the 3-by-3 square two half 1-by-2 rectangles and half a unit square). So $M(P, Q) = 6.5 - 1 - 1.5 = 4$.

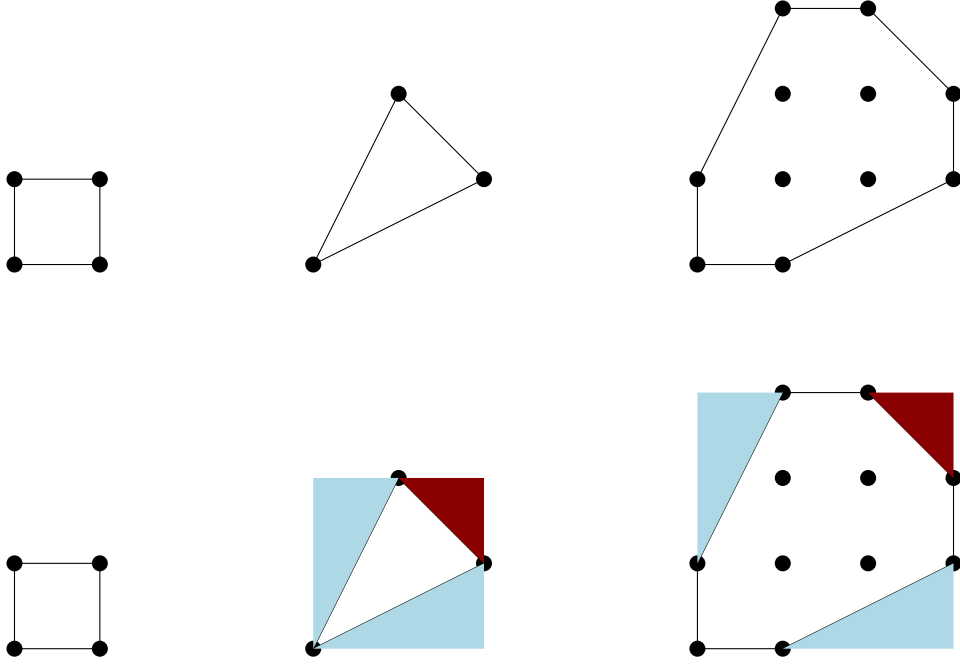


Figure 1: From left to right: P , Q and $P + Q$. The second row is with added triangles to make the calculation of the area easier.

4 d

5 e

Suppose we have two lattice polygons P and Q . Construct a polynomial f as follows: starting with $f = 0$, for every vertex $p_i = (a_{i1}, a_{i2})$ of P add a term to f of the form $x^{a_{i1}}y^{a_{i2}}$. In this way, $\text{New}(f) = P$. Analogous, construct a g so that $\text{New}(g) = Q$. Note that this construction is fine, since a_{ij} are integers because P, Q are lattice polygons. Because P and Q are polygons, we only have two dimensions and can use Bernstein's Theorem.

Using Bernstein's Theorem we find that $M(\text{New}(f), \text{New}(g)) = M(P, Q) = (\text{number of solutions of the system } f(x, y) = 0, g(x, y) = 0)$, which is an integer.

6 f

Suppose we have two general plane algebraic curves f and g of degree d and e and with Newton-polygons $P = \text{New}(f), Q = \text{New}(g)$. For the algebraic plane curve f of degree d , we know that it is of the form $\sum_{i=0}^d a_i * x^i * y^{d-i}$ with all $a_i \neq 0$. So that the $P = \text{conv}(\{(i, d-i) | i \in [0, \dots, d]\})$ which is the triangle through $(0, 0), (d, 0), (0, d)$. Analogous we find $Q = \Delta\{(0, 0), (e, 0), (0, e)\}$.

Now we will look at $P + Q$. When calculating $P + Q$ we should only consider the boundaries of P and Q . From the geometric structure (both lie in the corner of the first quadrant) we see that only the hypotenusa of the triangle will be important to define $P + Q$. The points on the hypotenusa of P are $\{(i, d - i) | 0 \leq i \leq d, i \in \mathbb{R}\}$ and for $Q : \{(j, e - j) | 0 \leq j \leq e, j \in \mathbb{R}\}$. So the points for $P + Q$ will be $\{(i + j, e + d - i - j) | 0 \leq i \leq d, 0 \leq j \leq e, i, j \in \mathbb{R}\} = \{(k, e + d - k) | 0 \leq k \leq d + e, k \in \mathbb{R}\}$. So we find that $P + Q$ is a right triangle with two legs of length $d + e$.

We find:

$$M(P, Q) = \frac{(d + e)^2}{2} - \frac{d^2}{2} - \frac{e^2}{2} = \frac{d^2 + 2de + e^2 - d^2 - e^2}{2} = de$$

Consider now the functions f and g we used in part c). We calculated $M(\text{New}(f), \text{New}(g)) = 4$. From Bernstein's theorem we can find that they meet in 4 points. We notice that f and g are not general polynomials, but we notice that the points defining $\text{New}(f)$ are contained in the Newton polynomial generated by a general polynomial of degree equal to the maximum degree of f . Analogous for G . As such, Bezout's theorem gives an upperbound for the number of meeting points, based on their degree. Here $\deg(f) * \deg(g) = 2 * 3 = 6$ and $4 \leq 6$.