Problem sheet 2

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December 11, 2016

1 Nonrational Pentagon

Given the incidences we have given, we should show that the inner pentagon is regular, then we can use http://mathworld.wolfram.com/Pentagon.html to show the construction cannot be realised with rational coordinates.

2 6.9: enumerate 4-polytopes with 7 vertices

Things we can use:

a) Theorem 6.19 in Ziegler says that a Gale diagram represents a polytope iff every co-circuit has at least 2 positive elements.

b) Their is a cyclic symmetry $(+,+,-,-,-,-,+) = (+,-,-,-,-,+,-)$

c) We should also consider coinciding points (I don't know if we should think about special points as well (pyramids)).

3 6.15: 2 different 2-neighborly 4-polytopes

We can easily check that both polytopes are 2-neighborly, since removing any two points from the Gale diagram and then taking the convex hull of the positive and of the negative points, we find that these intersect. Thus the two removed points form an edge. I checked this manually, but I haven't written this down, using symmetries you should

have to check around 30 intersections.

If we write down all facets from the first Gale diagram, using the fact that cocircuits define cofacets. And also from $C_4(8)$, using Gale's evenness criterium, we find that exactly the same structure appears (even the naming is already correct). If we write down the facets from the second Gale diagram, we note there are differences, since their are different convex hulls (we also get the intersection of a triangle and a point).

To make sure that the second graph differs from the first, we used the sagemath graph library (a python-based mathematical computer language) to compare the graphs of the facetstructure. I generated images for both graphs: see figure 1 and figure 2. Using the isomorphism-function of the graph library, we find that both graphs are different.
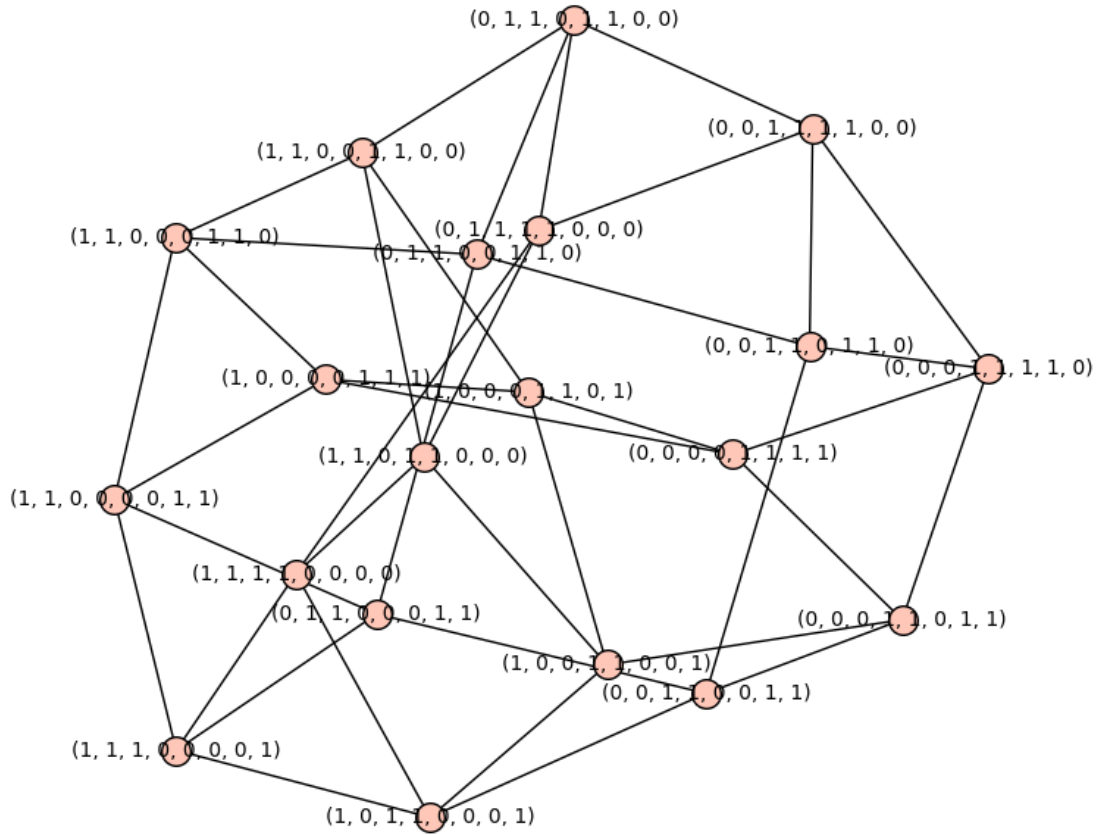

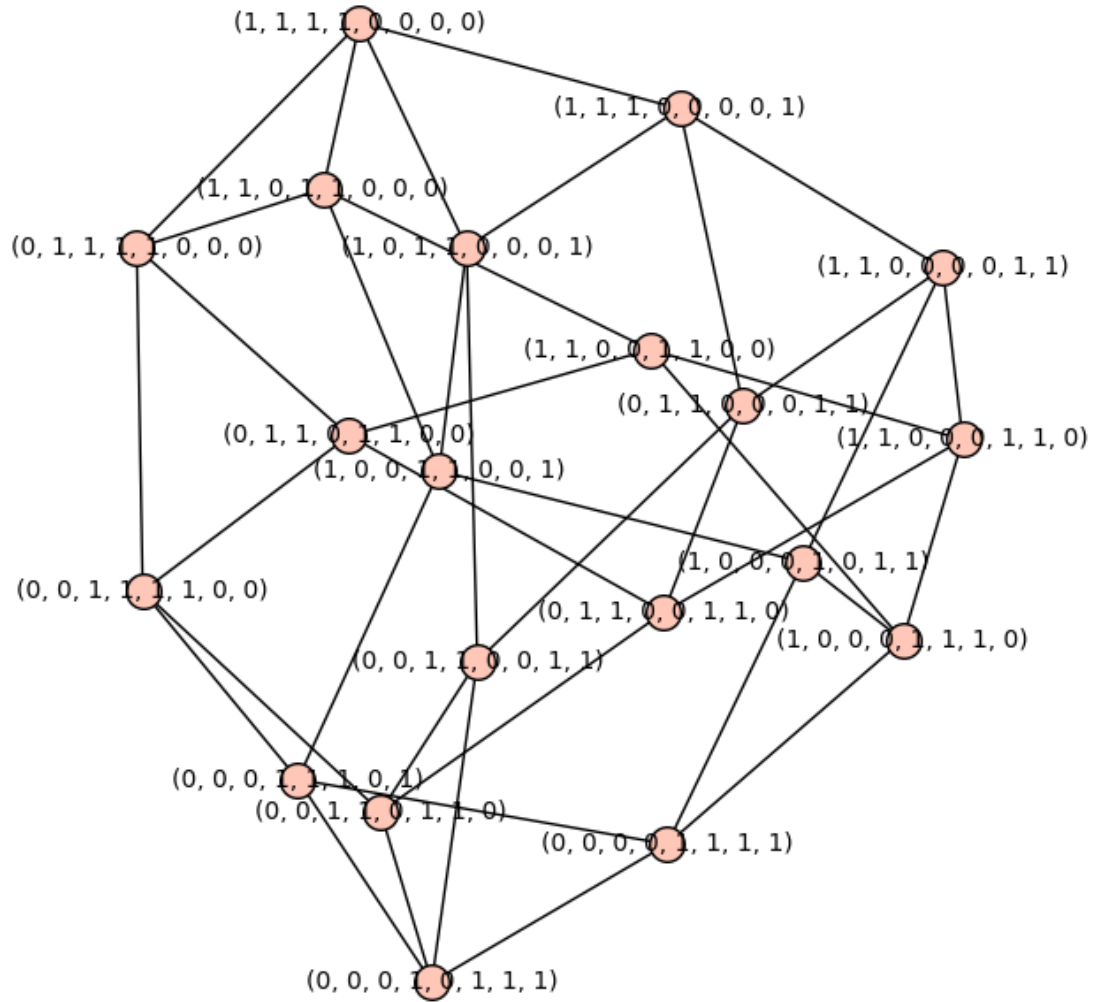Figure 1: The facet-graph of C4(8). Vertices are facets, and edges represent ridges between two facets.

Figure 2: The facet-graph of the second Gale diagram. Vertices are facets, and edges represent ridges between two facets.

4 6.17: analyse gale diagram

We can find the facets by finding cocircuits in the Gale diagram. For example the convex hull of $\{1\}$ and $\{4\}$ intersect, so 14 is a cofacet and 235678 a facet of P. To check that this is an octahedron, we lift 14 to infinity, and put the rest on one line, we get as Gale diagram a line with following points: $2^-, 3^+6^+, 7^-8^-, 5^+$. If we look at the facet structure of the polytope with this Gale diagram, we see the polytope is 4-regular, has 2 vertices 2 and 5 which are neighbours of $3, 6, 7, 8$ and that $3 \sim 78, 6 \sim 78, 7 \not\sim 8$ and $3 \not\sim 6$. This is exactly the incidence structure of an octahedron.

For example we also find the cofacet 3457 which results in a facet 1268. Because this facet has 4 vertices in dimension 3, it is a simplex. We find some other simplices: $1267, 3457, 3458$.

We also find facets: $14568, 14567, 12347, 12348$. To be a square pyramid they need a special point (a zero-point which is non-plus and non-negative), but I don't know how to get one.

If we have the same circuits, we have the same intersections in our Gale diagram so both 7 and 8 should lie on the line 23 and on the line 56, since these lines are different and the intersection is a single point, they must lie on the same point.

XXX Then 2356 lie in the same plane.

Since 2356 must be coplanar, we cannot chose all of them freely and the octahedral facet cannot be prescribed.

XXX The oriented matroid is not rigid, I don't know yet.