UPC - Discrete and algorithmic geometry

Problem sheet 2

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November 28, 2016

1 Matousek

1.1 2

1.1.1 Problem description

We will use Lemma 5.1.2 (Duality preserves incidences) ii) in Matousek: Let p be a point of $\mathbb{R}^d$ distinct from the origin and let h be a hyperplane in $\mathbb{R}^d$ not containing the origin. Let h^- stand for the closed half-space bounded by h and containing the origin, while h^+ denotes the other closed half-space bounded by h. That is, if $h = \{x \in \mathbb{R}^d : \langle a, x\rangle = 1\}$, then $h^- = \{x \in \mathbb{R}^d : \langle a, x\rangle \leq 1\}$. Then $p \in h^- \iff D_0(h) \in D_0(p)^-$.

Let us consider a pentagon which contains the origin. Let $v_i = D_0(l_i)$, where l_i is the line containing the side $a_i a_{i+1}$. Then the points dual to the lines intersecting the pentagon $a_1 a_2 \dots a_5$ fill exactly the exterior of the convex pentagon $P_{ex} = v_1 v_2 \dots v_5$.

1.2 solution

So take a point p outside P_{ex}. Because P_{ex} is convex there exists an edge $v_i v_{i+1}$ (we assume here that $v_{5+1} = v_1$) with supporting line h, such that p lies in the halfplane h^+ (not containing the origin). Because duality preserves incidences we find $D_0(h) \in D_0(p)^+$. Now since $D_0(h) = D_0(D_0(a_{i+1})) = a_{i+1}$, we find that the line segment $[0, a_{i+1}] \cap D_0(p) \neq \varnothing$. So $D_0(p)$ intersects P_{ex}.

Why is $v_i v_{i+1} = D_0(a_{i+1})$?

$$\begin{aligned} v_i &= D_0(a_i a_{i+1}), & v_j &= D_0(a_{i+1}a_{i+2}) \\ \implies D_0(v_i) &= a_i a_{i+1}, & D_0(v_j) &= a_{i+1}a_{i+2} \\ \implies D_0(v_i)D_0(v_j) &= a_{i+1} \\ \implies D_0(a_{i+1}) &= v_i v_j \end{aligned}$$

Now analogous to the first part we can take a point p inside P_{ex} then we find $v_i v_{i+1}$ with a supporting line h such that $p \in h^-$. We find $D_0(h) = a_{i+1} \in D_0(p)^-$ So that $D_0(p)$ intersects the line a_{i+1} outside the pentagon. Because $D_0(p)$ is perpendicular to a_{i+1} it will never intersect the convex polygon

1.3 3

$$X^* = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1, \forall x \in X\}$$
$$X^{**} = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1, \forall x \in X^*\}$$

Now because $\forall x \in X, \forall y \in X^* : \langle x, y\rangle \leq 1 \implies x \in X^{**}$. Also clearly $0 \in X^{**}$. Since X^{**} closed and convex we find $conv(X \cup 0) \subset X^{**}$.
The separation theorem says that for a closed set Z: $conv(Z) = \bigcap$(all closed halfspaces that contain Z). So $conv(X \cup 0)$ is the intersection of all closed halfspace that contain 0 and X.

2 $C_4(7)$

2.1 (a)

We will first calculate the f-vector of $C_4(7)$.

- $f_0 = 7$
- $f_1 = \binom{7}{2} = 21$, because $C_4(7)$ is neighborly
- $f_3 = 14$, we counted this in class, using Gale's evenness criterium.

Define $h_k = \sum_{i \geq k}^{d} f_i(-1)^{(i-k)}\binom{k}{i}$, with $f_{-1} = f_d = 1$. Then the Dehn-Sommerville equations learn us that $h_i = h_{d-i}$.
We find $h_0 = f_0 - f_1 + f_2 - f_3 + f_4 = 7 - 21 + f_2 - 14 + 1 = f_2 - 27$ and $h_4 = f_4 = 1$.
Now $h_0 = h_4 \implies f_3 = 28$.
So $f(C_4(7)) = (7, 21, 28, 14)$. And we find $f(C_4(7)^{\Delta}) = (14, 28, 21, 7)$

3 24-cell

$$P = conv\{\pm e_i \pm e_j : 1 \le i, j \le 4,\ i \ne j\}$$

Notation: we'll note all vertices as pairs (i, j) such that $-4 \le i, j \le 4$ and $i \ne 0 \ne j$ and $i \ne j$. Note that $(i, j) = (j, i)$ as $e_i + e_j = e_j + e_i$.
0-dimmensional faces (vertices): All $4 \cdot \binom{4}{2} = 24$ vertices (i, j)
1-dimmensional faces (edges): Two vertices are adjacent if they share exactly one common vector from the basis. That is:

$$(i, j) \sim (k, l) \iff (i = k, |j| \ne |l|) \vee (i = l, |j| \ne |k|) \vee (j = k, |i| \ne |l|) \vee (j = l, |i| \ne |k|)$$

Nottice that (i, j) and $(i, -j)$ as the segment $(i, j)(i, -j)$ has length 2 unlike the edges that have length $\sqrt{2}$.
So, each vertex has 8 adjacent vertices. As every edge is double counted, the total number of edges is $\frac{24 \cdot 8}{2} = 96$.
2-dimmensional faces (faces): choose any 3 a_1, a_2, a_3 such that $a_1 \sim a_2, a_2 \sim a_3, a_1 \nsim a_3$. So if $a_1 = (i, j)$ with $|i| \ne |j|$, then $a_2 = (j, k)$ with $|i| \ne |k| \ne |j|$ and $a_3 = (k, l)$ with $|l| \ne |i|, |j|, |k|$. The vertex $a_4 = (l, j)$ and $a_5 = (i, k)$ are adjacent to a_1, a_2, a_3. This proves that for any two adjacent edges, either they are in the same triangle, or they are in two adjacent triangles. So all faces are triangles.
Every pair of adjacent vertices has 4 vertices that are adjacent to both. As all faces are triangles and every triangle is counted 4 times for each edge, the total number of triangles is $\frac{96 \cdot 4}{4} = 96$
3-dimmensional faces (cells): choose any face a_1, a_2, a_3.
Then $a_1 = (i, j), a_2 = (j, l), a_3 = (l, i)$ for some i, j, l such that $1 \le |i|, |j|, |l| \le 4$ and $|i| \ne |l| \ne |j| \ne |i|$. Choose $1 \le |k| \le 4$ such that $|k| \ne |i|, |j|, |l|$ (there are only two possibilities: k and $-k$). Consider the vertices (j, k), (i, k) and (l, k). As we can see in Figure 1, the graph of this set of points is 4-regular, where all faces are triangles, there are 8 faces, and each vertex is incident to 4 faces. It is the graph of a regular octahedron. So, all cells are regular octahedrons.

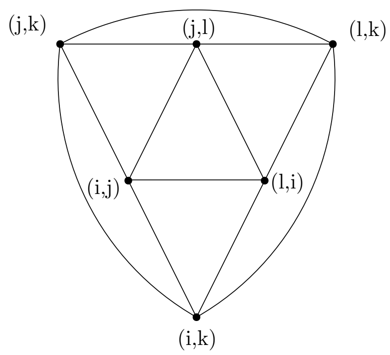

Figure 1: 24cell facet

Given a vertex (i,j), there are 4 possible pairs $|k| \neq |l|$ such that $|k|, |l| \neq |i|, |j|$ and that $(i,j),(i,k),(i,l),(j,k),(j,l),(k,l)$ form an octahedron. So every vertex is in 4 octahedrons. As all cells are counted 4 times, the total number of cells is $\frac{24 \cdot 4}{4} = 24$

<u>Face lattice</u>

A vertex (i,j) is an endpoint of an edge $(i',j')(j',k')$ if and only if $\{i,j\} \subset \{i',j',k'\}$

An edge $(i,j)(j,k)$ is an edge of a face $(i',j')(j',k')(k',i')$ if and only if either $j \in \{i',j',k'\}$ or $\{i,k\} \subset \{i',j',k'\}$

A face $(i,j)(j,k)(k,i)$ is a face of a cell formed by $\{(i',j'),(i',k'),(i',l'),(j',k'),(j',l'),(k',l')\}$ if and only if $\{i,j,k\} \subset \{i',j',k',l'\}$.

<u>Self-polar-duality</u>

We'll now flip the notation. Now, the vertices are noted as $\{i,j,k,l\}$. As seen in the face lattice (two cells share a face if the indices of the face are contained in the indices of both cells at the same time) two vertices $\{i,j,k,l\},\{i',j',k',l'\}$ are adjacent if and only if they share three indices.