Bezout's Theorem, mixed volumes and mixed multiplicities

Ngo Viet Trung

Institute of Mathematics, VAST, Hanoi

Bezout's theorem

Maclaurin (1720): Two polynomials f, g in two variables with no common factor have at most $\deg f \cdot \deg g$ common zeros.

Bezout (1764): Let $f_1, ..., f_n$ be n polynomials of degree $d_1, ..., d_n$ in n variables having finitely many common zeros. Then

$$\text{number common zeros} \leqslant d_1 \cdots d_n.$$

The bound is attained by generic polynomials of degree $d_1, ..., d_n$.

Generic means that the coefficients can be varied in an open non-empty subset of the parameter space.

Sparse polynomials

A polynomial f is full if it contains all monomials of degree $\deg f$.

It is sparse if it doesn't contain all monomials of degree $\leqslant \deg f$.

Phenomenon: Bezout's theorem does not give a sharp bound for sparse polynomials!

Example: The sparse polynomials

$$f(x, y) = xy + ax + b$$
$$g(x, y) = xy + cy + d$$

have 2 common zeros, whereas 4 is Bezout's bound.

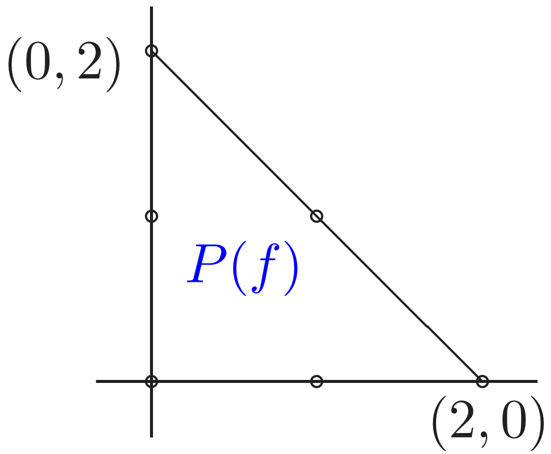
The reason lies in the combinatorial geometry of the sparse polynomials

Newton polytope

$$P(f) := \text{convex hull of the exponents of the monomials of } f$$

Example 1: f is a full polynomial of degree 2

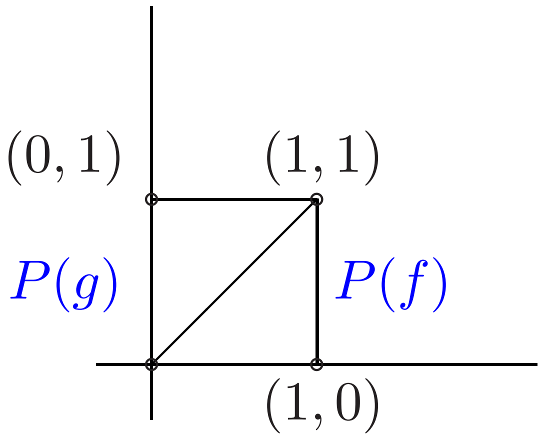
$P(f) = \text{triangle } (2,0), (0,2), (0,0)$



Example 2: $f = xy + \alpha x + \beta, \;\; g = xy + \mu y + \nu$

$P(f) = \text{triangle } (1,1),(1,0),(0,0)$

$P(g) = \text{triangle } (1,1),(0,1),(0,0)$



Minkowski sum

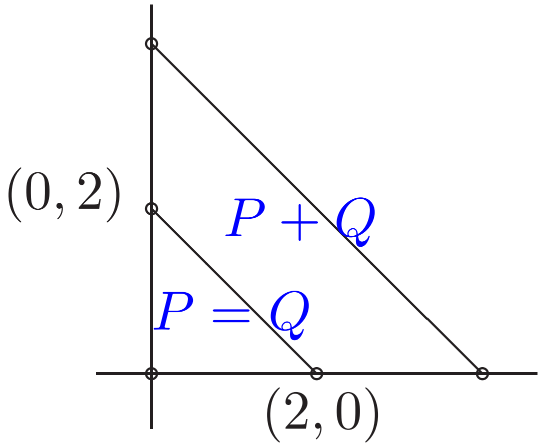
P, Q polytopes in $\mathbb{R}^n$

$$P + Q := \{u + v \mid \; u \in P, \; v \in Q\}$$

(convex hull of the sums of the vertices of P, Q)

Example 1: $P = P(f), \; Q = P(g)$

where f, g are full polynomials of degree 2

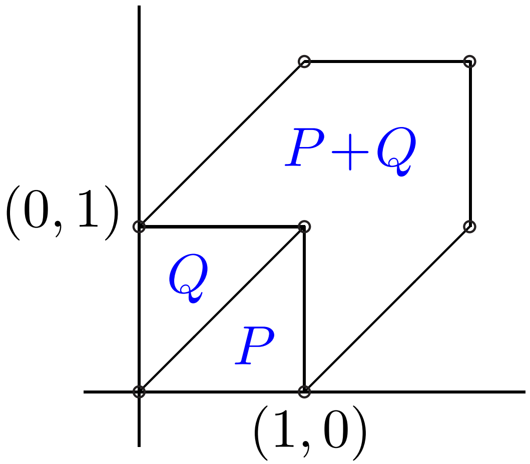


Example 2: $P = P(f)$, $Q = P(g)$

where

$f = xy + \alpha x + \beta$

$g = xy + \mu y + \nu$



Mixed area

$$\text{mixed-area}(P,Q) := \text{area}(P+Q) - \text{area}(P) - \text{area}(Q)$$

Example: $P = P(f),\ Q = P(g)$

If f, g are full polynomials of degree 2, then

$$\text{mixed-area}(P,Q) = 8 - 2 - 2 = 4$$

This is Bezout's bound !

If $f = xy + \alpha x + \beta,\ \ g = xy + \mu y + \nu$, then

$$\text{mixed-area}(P,Q) = 3 - \frac{1}{2} - \frac{1}{2} = 2$$

This is the number of common zeros of f, g !

Mixed volume

$P_1, ..., P_n$ polytopes in $\mathbb{R}^n$

$$MV(P_1, ..., P_n) := \sum_{j=1}^{n} \sum_{1 \leqslant i_1 < ... < i_j \leqslant n} (-1)^{n-j} V(P_{i_1} + \cdots + P_{i_j})$$

where V is the Euclidian volume function

Example: mixed-area if $n = 2$

1. Positivity: $MV(P_1, ..., P_n) \geq 0$
2. Monotony: $MV(P_1, ..., P_n) \leqslant M(Q_1, ..., Q_n)$ if $P_i \subseteq Q_i$

Bernstein's theorem

Bernstein (1975): Let $f_1, ..., f_n$ be n polynomials in n variables having finitely many common zeros in $(\mathbb{C}^*)^n$, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Then

$$\text{number of common zeros in } (\mathbb{C}^*)^n \leqslant MV(P(f_1), ..., P(f_n)).$$

The bound is attained for generic polynomials $f_1, ..., f_n$ with fixed Newton polytopes

Example: $f = xy + \alpha x + \beta, \;\; g = xy + \mu y + \nu$

$$\text{number of common zeros} \leqslant MV(P(f), P(g)) = 2$$

Bezout's theorem as a special case

Let f, g be arbitrary polynomials of degree r, s in two variables with no common factors. Change of coordinates:

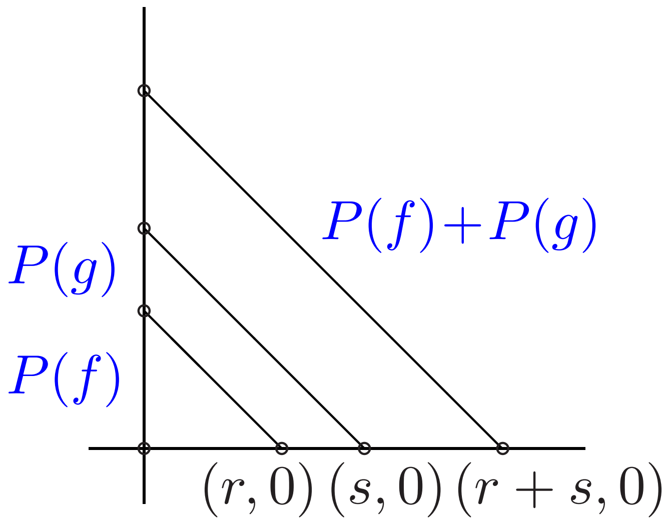
number of common zeros = number of common zeros in $(\mathbb{C}^*)^n$

By Bernstein's theorem we only need to show that

$$MV(P(f), P(g)) \leq rs$$

By monotony we may assume that f, g are full.

Then $P(f), P(g), P(f) + P(g)$ are isosceles right triangles



Therefore, $MV(P(f), P(g)) = \dfrac{(r+s)^2}{2} - \dfrac{r^2}{2} - \dfrac{s^2}{2} = rs$

Proofs of Bernstein's theorem

Bernstein: combinatoric (mysterious)

Teissier: intersection theory (hard)

Question: Is there any algebraic proof for Bernstein's theorem?

Answer: yes (joint work with Jugal Verma, ITT, Mumbay)

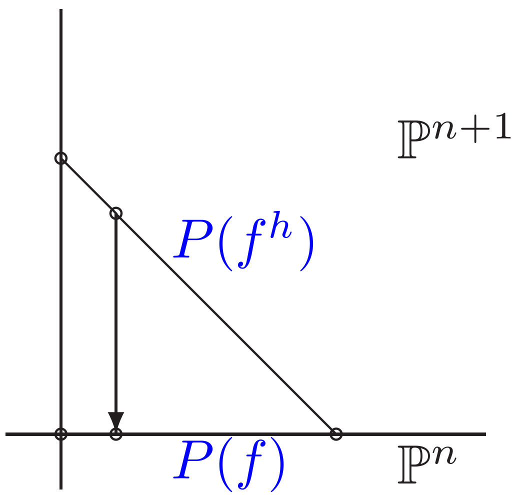
Main tool: Multiplicity theory

Homogenization

$f \in \mathbb{C}[x_1, ..., x_n] \;\longrightarrow\; f^h := x_0^{\deg f} f(x_1/x_0, ..., x_n/x_0)$

$(\alpha_1, ..., \alpha_n) \in \mathbb{C}^n \;\longrightarrow\; (1, \alpha_1, ..., \alpha_n) \in \mathbb{P}^n_{\mathbb{C}}$ is a bijective map between the zeros of f in $(\mathbb{C}^*)^n$ and the zeros of f^h in $\mathbb{P}^n_{\mathbb{C}^*}$

$P(f^h) \subset \mathbb{R}^{n+1}$ can be identified with $P(f) \subset \mathbb{R}^n$ by projection:



Homogeneous Bernstein's theorem

Theorem (homogeneous version): Let $f_1, ..., f_n$ be n homogeneous polynomials of degree $d_1, ..., d_n$ in $n+1$ variables having finitely many common zeros in $\mathbb{P}^n_{\mathbb{C}^*}$. Then

$$\text{number of common zeros in } \mathbb{P}^n_{\mathbb{C}^*} \leqslant MV(P(f_1), ..., P(f_n))$$

$$(P(f_1), ..., P(f_n) \text{ embedded in } \mathbb{R}^n \text{ by projection}).$$

This bound is attained by generic homogeneous polynomials.

Sketch of proof:

number of common zeros $\leftrightarrow$ mixed multiplicity $\leftrightarrow$ mixed volume

Multiplicity

$S = \bigoplus_{t \in \mathbb{N}} S_t$ standard graded algebra over a field k

Hilbert: $\dim_k S_t$ is given by a polynomial $P_S(t)$ for $t \gg 0$

Let $r = \deg P_S(t)$. We may write

$$P_S(t) = \frac{1}{r!} e(S) t^r + \{\text{terms of degree} < r\}$$

Then $e(S)$ is a positive integer, called the multiplicity of S

Theorem: Let $f_1, ..., f_m \in k[x_0, ..., x_n]$ be homogeneous polynomials with a finite set of common zeros in $\mathbb{P}^n_k$. Let $S = k[x_0, ..., x_n]/(f_1, ..., f_m)$. Then

$$\text{number of common zeros of } f_1, ..., f_m \text{ in } \mathbb{P}^n_k \leq e(S)$$

Mixed multiplicities

$S = \bigoplus_{u \in \mathbb{N}^{n+1}} S_u$ standard multigraded algebra over a field k

Van der Waerden: $\dim_k S_u$ is given by a polynomial $P_S(u)$ for $u \gg 0$

Let $r = \deg P_S(u)$. We may write

$$P_S(u) = \sum_{a \in \mathbb{N}^{n+1}, |a|=r} \frac{1}{a!} e_\alpha(S) u^a + \{\text{terms of degree} < r\}$$

where $a = (\alpha_0, ..., \alpha_n)$, $a! = \alpha_0! \cdots \alpha_n!$, $u^a := u_0^{\alpha_0} \cdots u_n^{\alpha_n}$.

Then $e_a(S)$ are non-negative numbers, called the mixed multiplicities of S.

Number of common zeros versus mixed multiplicity

$R := k[x] = k[x_0, ..., x_n]$

$f_1, ..., f_n$ homogeneous polynomials in R with finite set of common zeros in $\mathbb{P}^n_{k^*}$

$I_0 := (x_0, ..., x_n)$ and $I_j := (x^a |\ x^a$ monomial of $f_j)$

$S := R[I_0 t_0, ..., I_n t_n]$ ($\mathbb{N}^{n+1}$-graded algebra)

Theorem: $\deg P_S(u) = n$ and

number of common zeros of $f_1, ..., f_n$ in $\mathbb{P}^n_{k^*} \leq\ e_{(0,1,...,1)}(S)$

Equality holds if $\overline{k} = k$ and $f_1, ..., f_n$ are generic.

Minkowski formulas

$Q_0, ..., Q_n$ integral polytopes in $\mathbb{R}^n$

$\lambda = (\lambda_0, ..., \lambda_n)$ any vector of positive integers

$\lambda Q := \lambda_0 Q_0 + \cdots + \lambda_n Q_n$

$$SV(\lambda Q) = n! \sum_{a \in \mathbb{N}^{n+1}, |a|=n} \frac{1}{a!} MV(Q_a) \lambda^a$$

(Q_a is the family of α_i copies of Q_i, $i = 0, ..., n$)

$S^\lambda := \bigoplus_{t \geq 0} S_{t\lambda}$ (standard graded algebra)

$$e(S^\lambda) = n! \sum_{a \in \mathbb{N}^{n+1}, |a|=n} \frac{1}{a!} e_a(R) \lambda^a$$

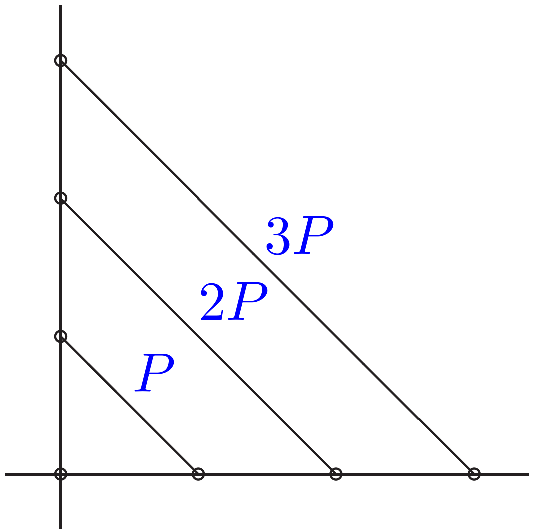
Multiplicity versus volume

P integral polytope on a hyperplane in $\mathbb{R}^{n+1}$

$k[P] := k[x^a |\ a \in P]$ is a standard graded algebra with

$\dim_k k[P]_t$ = number of integral point in tP

Ehrhart (1963): $e(k[P]) = V(P)$



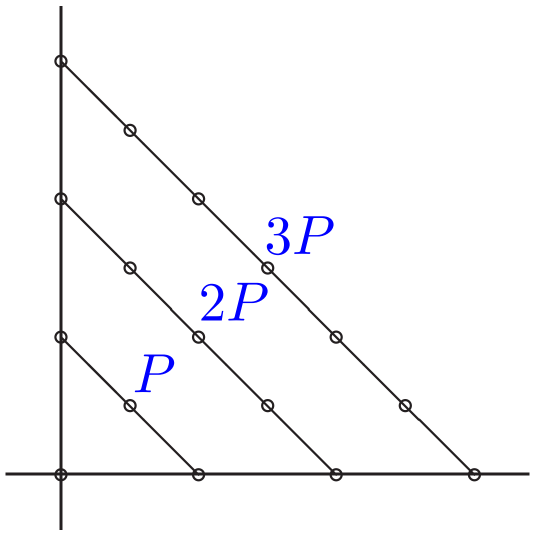
Example: P is the segment with the edges $(2,0),(0,2)$

$\Rightarrow\ k[P]=k[x_1^2,x_1x_2,x_2^2]$

tP is the segment with the edges $(2t,0),(0,2t)$

$\Rightarrow\ \dim k[P]_t = 2t+1$

$\Rightarrow\ e(k[P])=2$



Mixed multiplicity versus mixed volume

Set $Q_0 = P(x_0 + \cdots + x_n)$ and $Q_i = P(f_i)$, $i = 1, ..., n$.

Then $S^\lambda = k[\lambda Q]$

Erhart's formula: $e(S^\lambda) = V(\lambda Q)$

Comparing the two Minkowski formulas $\Rightarrow$

Theorem: $e_a(S) = MV(Q_a)$ for all $a \in \mathbb{N}^{n+1}$ with $|a| = n$

Corollary: $e_{(0,1,...,1)}(S) = MV(Q_1, ..., Q_n)$

$\Rightarrow$ Bernstein's theorem for arbitrary field

Applications

1. Mixed volume is a special case of mixed multiplicity

(consequence: positivity, monotony)

2. Alternate method for computing mixed volume

(computing mixed multiplicity by means of Gröbner Basis)

Open problem

Alexandroff-Fenchel inequality:

$$MV(Q_1, ..., Q_n)^2 \geq MV(Q_1, Q_1, Q_3, .., Q_n) MV(Q_2, Q_2, Q_3, .., Q_n)$$

Question: Does there exists a similar inequality for mixed multiplicities?

References

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