Problem sheet 2

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1 Matousek

1.1 2

1.1.1 Problem description

We will use Lemma 5.1.2 (Duality preserves incidences) ii) in Matousek: Let p be a point of $\mathbb{R}^d$ distinct from the origin and let h be a hyperplane in $\mathbb{R}^d$ not containing the origin. Let h^- stand for the closed half-space bounded by h and containing the origin, while h^+ denotes the other closed half-space bounded by h. That is, if $h = \{x \in \mathbb{R}^d : \langle a, x\rangle = 1\}$, then $h^- = \{x \in \mathbb{R}^d : \langle a, x\rangle \leq 1\}$. Then $p \in h^- \iff D_0(h) \in D_0(p)^-$.

Let us consider a pentagon which contains the origin. Let $v_i = D_0(l_i)$, where l_i is the line containing the side $a_i a_{i+1}$. Then the points dual to the lines intersecting the pentagon $a_1 a_2 \dots a_5$ fill exactly the exterior of the convex pentagon $P_{ex} = v_1 v_2 \dots v_5$.

1.2 solution

So take a point p outside P_{ex}. Because P_{ex} is convex there exists an edge $v_i v_{i+1}$ (we assume here that $v_{5+1} = v_1$) with supporting line h, such that p lies in the halfplane h^+ (not containing the origin). Because duality preserves incidences we find $D_0(h) \in D_0(p)^+$. Now since $D_0(h) = D_0(D_0(a_{i+1})) = a_{i+1}$, we find that the line segment $[0, a_{i+1}] \cap D_0(p) \neq \varnothing$. So $D_0(p)$ intersects $P_e x$.

Why is $v_i v_{i+1} = D_0(a_{i+1})$?

$$
\begin{aligned}
& v_i = D_0(a_i a_{i+1}), && v_j = D_0(a_{i+1}a_{i+2}) \\
\implies & D_0(v_i) = a_i a_{i+1}, && D_0(v_j) = a_{i+1}a_{i+2} \\
\implies & D_0(v_i)D_0(v_j) = a_{i+1} && \\
\implies & D_0(a_{i+1}) = v_i v_j &&
\end{aligned}
$$

Now analogous to the first part we can take a point p inside $P_e x$ then we find $v_i v_{i+1}$ with a supporting line h such that $p \in h^-$. We find $D_0(h) = a_{i+1} \in D_0(p)^-$ So that $D_0(p)$ intersects the line a_{i+1} outside the pentagon. Because $D_0(p)$ is perpendicular to a_{i+1} it will never intersect the convex polygon

1.3 3

$$
\begin{aligned}
X^* &= \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1, \forall x \in X\} \\
X^{**} &= \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1, \forall x \in X^*\}
\end{aligned}
$$

Now because $\forall x \in X, \forall y \in X^* : \langle x, y\rangle \leq 1 \implies x \in X^{**}$. Also clearly $0 \in X^{**}$. Since X^{**} closed and convex we find $conv(X \cup 0) \subset X^{**}$.
The separation theorem says that for a closed set Z: $conv(Z) = \bigcap$(all closed halfspaces that contain Z). So $conv(X \cup 0)$ is the intersection of all closed halfspace that contain 0 and X.

2 $C_4(7)$

2.1 (a)

We will first calculate the f-vector of $C_4(7)$.

- $f_0 = 7$
- $f_1 = \binom{7}{2} = 21$, because $C_4(7)$ is neighborly
- $f_3 = 14$, we counted this in class, using Gale's evenness criterium.

Define $h_k = \sum_{i \geq k}^{d} f_i (-1)^{(i-k)} \binom{k}{i}$, with $f_{-1} = f_d = 1$. Then the Dehn-Sommerville equations learn us that $h_i = h_{d-i}$.
We find $h_0 = f_0 - f_1 + f_2 - f_3 + f_4 = 7 - 21 + f_2 - 14 + 1 = f_2 - 27$ and $h_4 = f_4 = 1$.
Now $h_0 = h_4 \implies f_3 = 28$.
So $f(C_4(7)) = (7, 21, 28, 14)$. And we find $f(C_4(7)^{\Delta}) = (14, 28, 21, 7)$

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