UPC - DISCRETE AND ALGORITHMIC GEOMETRY

Problem sheet 2

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1 Matousek

1.1 2

1.1.1 Problem description

We will use Lemma 5.1.2 (Duality preserves incidences) ii) in Matousek: Let p be a point of $\mathbb{R}^d$ distinct from the origin and let h be a hyperplane in $\mathbb{R}^d$ not containing the origin. Let h^- stand for the closed half-space bounded by h and containing the origin, while h^+ denotes the other closed half-space bounded by h. That is, if $h = \{x \in \mathbb{R}^d : \langle a, x\rangle = 1\}$, then $h^- = \{x \in \mathbb{R}^d : \langle a, x\rangle \leq 1\}$. Then $p \in h^- \iff D_0(h) \in D_0(p)^-$.

Let us consider a pentagon which contains the origin. Let $v_i = D_0(l_i)$, where l_i is the line containing the side a_ia_{i+1}. Then the points dual to the lines intersecting the pentagon $a_1a_2 \dots a_5$ fill exactly the exterior of the convex pentagon $P_{ex} = v_1v_2 \dots v_5$.

1.2 solution

So take a point p outside P_{ex}. Because P_{ex} is convex there exists an edge v_iv_{i+1} (we assume here that $v_{5+1} = v_1$) with supporting line h, such that p lies in the halfplane h^+ (not containing the origin). Because duality preserves incidences we find $D_0(h) \in D_0(p)^+$. Now since $D_0(h) = D_0(D_0(a_{i+1})) = a_{i+1}$, we find that the line segment $[0, a_{i+1}] \cap D_0(p) \neq \varnothing$. So $D_0(p)$ intersects P_{ex}.

Why is $v_i v_{i+1} = D_0(a_{i+1})$?

$$\begin{aligned} v_i &= D_0(a_i a_{i+1}), & v_j &= D_0(a_{i+1}a_{i+2}) \\ \implies D_0(v_i) &= a_i a_{i+1}, & D_0(v_j) &= a_{i+1}a_{i+2} \\ \implies D_0(v_i)D_0(v_j) &= a_{i+1} \\ \implies D_0(a_{i+1}) &= v_i v_j \end{aligned}$$

Now analogous to the first part we can take a point p inside P_{ex} then we find $v_i v_{i+1}$ with a supporting line h such that $p \in h^-$. We find $D_0(h) = a_{i+1} \in D_0(p)^-$ So that $D_0(p)$ intersects the line a_{i+1} outside the pentagon. Because $D_0(p)$ is perpendicular to a_{i+1} it will never intersect the convex polygon

1.3 3

$$X^* = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1, \forall x \in X\}$$
$$X^{**} = \{y \in \mathbb{R}^d : \langle x, y\rangle \leq 1, \forall x \in X^*\}$$

Now because $\forall x \in X, \forall y \in X^* : \langle x, y\rangle \leq 1 \implies x \in X^{**}$. Also clearly $0 \in X^{**}$. Since X^{**} closed and convex we find $conv(X \cup 0) \subset X^{**}$.
The separation theorem says that for a closed set Z: $conv(Z) = \bigcap$(all closed halfspaces that contain Z). So $conv(X \cup 0)$ is the intersection of all closed halfspace that contain 0 and X.

2 $C_4(7)$

2.1 (a)

We will first calculate the f-vector of $C_4(7)$.

- $f_0 = 7$
- $f_1 = \binom{7}{2} = 21$, because $C_4(7)$ is neighborly
- $f_3 = 14$, we counted this in class, using Gale's evenness criterium.

Define $h_k = \sum_{i\geq k}^{d} f_i(-1)^{(i-k)}\binom{k}{i}$, with $f_{-1} = f_d = 1$. Then the Dehn-Sommerville equations learn us that $h_i = h_{d-i}$.
We find $h_0 = f_0 - f_1 + f_2 - f_3 + f_4 = 7 - 21 + f_2 - 14 + 1 = f_2 - 27$ and $h_4 = f_4 = 1$.
Now $h_0 = h_4 \implies f_3 = 28$.
So $f(C_4(7)) = (7, 21, 28, 14)$. And we find $f(C_4(7)^{\Delta}) = (14, 28, 21, 7)$

3 24-cell

$$P = conv\{\pm e_i \pm e_j : 1 \leq i,j \leq 4,\ i \neq j\}$$

Half-spaces and hyperplanes

One can easily check that the next 24 half-spaces are valid restrictions for our polytopes and that the intersection of the boundary with the polytope gives 6 vertices.
For $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$:

$$\begin{array}{rcll} x_i & \leq & 1 & \\ -x_i & \leq & 1 & \\ a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 & \leq & 2 & a_1, a_2, a_3, a_4 \in \{-1, 1\} \end{array}$$

The hyperplanes are the same equations with equality.
Any intersection $\{\pm x_i = 1\} \cap \{\pm x_j = 1\}$ (different hyperplanes) gives either 1 point $(i \neq j)$ or it's empty $(i = j)$.
$\{x_i = 1\} \cap \{\sum a_j x_j = 2 : a_i = 1\}$ gives the three vertices of a ridge. 32 different ridges.
$\{-x_i = 1\} \cap \{\sum a_j x_j = 2 : a_i = -1\}$ also gives the three vertices of a ridge. 32 different ridges.
$\{\sum a_j x_j = 2\} \cap \{\sum b_j x_j = 2 : \exists! k, (a_i = b_i, i \neq k) \wedge (a_k = -b_k)\}$ also gives the three vertices of a ridge. As coefficients share only one sign, this gives us $\binom{4}{3} \cdot 2^3 = 32$.
From this intersections, one can easily deduce that two vertices share an edge if they share only one component and with same sign.

Notation: we'll note all vertices as pairs (i,j) such that $-4 \leq i,j \leq 4$ and $i \neq 0 \neq j$ and $|i| \neq |j|$. Note that $(i,j) = (j,i)$ as $e_i + e_j = e_j + e_i$.
0-dimensional faces (vertices): All $4 \cdot \binom{4}{2} = 24$ vertices (i,j)
1-dimensional faces (edges): Two vertices are adjacent if they share exactly one common vector from the basis. That is:

$$(i,j) \sim (k,l) \iff (i = k, |j| \neq |l|) \vee (i = l, |j| \neq |k|) \vee (j = k, |i| \neq |l|) \vee (j = l, |i| \neq |k|)$$

Nottice that (i,j) and $(i,-j)$ as the segment $(i,j)(i,-j)$ has length 2 unlike the edges that have length $\sqrt{2}$.
So, each vertex has 8 adjacent vertices. As every edge is double counted, the total number of edges is $\frac{24 \cdot 8}{2} = 96$.
2-dimensional faces (faces): choose any 3 vertices a_1, a_2, a_3 such that $a_1 \sim a_2, a_2 \sim a_3, a_1 \nsim a_3$. So if $a_1 = (i,j)$ with $|i| \neq |j|$, then $a_2 = (j,k)$ with $|i| \neq |k| \neq |j|$ and $a_3 = (k,l)$ with $|l| \neq |i|, |j|, |k|$. The vertex $a_4 = (l,j)$ and $a_5 = (i,k)$ are adjacent to a_1, a_2, a_3. This proves that for any two adjacent edges, either they are in the same triangle, or they are in two adjacent triangles. So all faces are triangles.
Every pair of adjacent vertices $(i,j), (j,k)$ has 3 other vertices $(i,k), (j,l), (j,-l)$ that are adjacent to both. As all faces are triangles and every triangle is counted 3 times (once per edge), the total number of triangles is $\frac{96 \cdot 3}{3} = 96$
3-dimensional faces (cells): choose any face a_1, a_2, a_3.

Then $a_1 = (i,j), a_2 = (j,l), a_3 = (l,i)$ for some i,j,l such that $1 \leq |i|,|j|,|l| \leq 4$ and $|i| \neq |l| \neq |j| \neq |i|$. Choose $1 \leq |k| \leq 4$ such that $|k| \neq |i|,|j|,|l|$ (there are only two possibilities: k and $-k$). Consider the vertices (j,k), (i,k) and (l,k). As we can see in Figure 1, the graph of this set of points is 4-regular, where all faces are triangles, there are 8 faces, and each vertex is incident to 4 faces. It is the graph of a regular octahedron. So, all cells are regular octahedrons.

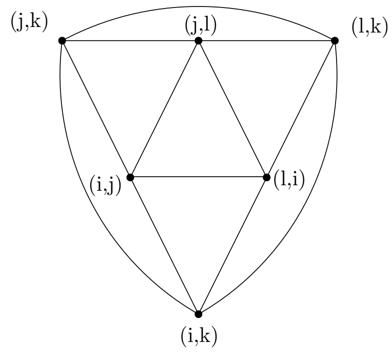

Figure 1: 24cell facet

Each ridge lies in 2 facets. If all facets are octahedra then we count 2 facets per ridge that are counted 8 times (once per each face of the octahedron. That gives us $\frac{96 \cdot 2}{8} = 24$ facets.

Face lattice

A vertex (i,j) is an endpoint of an edge $(i',j')(j',k')$ if and only if $\{i,j\} \subset \{i',j',k'\}$

An edge $(i,j)(j,k)$ is an edge of a face $(i',j')(j',k')(k',i')$ if and only if either $j \in \{i',j',k'\}$ or $\{i,k\} \subset \{i',j',k'\}$

A face $(i,j)(j,k)(k,i)$ is a face of a cell formed by $\{(i',j'),(i',k'),(i',l'),(j',k'),(j',l'),(k',l')\}$ if and only if $\{i,j,k\} \subset \{i',j',k',l'\}$.

Self-polar-duality

Every cell is adjacent to other 8 cells (one for each face). So, each vertex of the dual polytope is adjacent to other 8 vertices.
Every face is adjacent to 12 other faces (three for each edge as seen before). So, each edge of the dual polytope is adjacent to other 12 edges. It is the same as in the primal polytope as each vertex has 8 adjacent vertices, but 4 are common. So $2 \cdot 8 - 4 = 12$
Every edge is adjacent to 12 other edges. So, every face in the dual polytope is adjacent to other 12 vertices. Explained just before.
Every point is adjacent to other 8 points. So, every cell in the dual polytope is adjacent to other 8 cells. Same as in the primal polytope.

The dual polytope has the same number of faces in all dimensions and the same adjacency as the primal polytope. The 24-cell is self-polar-dual.