Slide show test

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Notation

- $\mathcal{C}$ denotes a code over the finite field $\mathbb{F}_q$.
- The triple of parameters (n, k, r) refers to a code of:
  - length n
  - cardinality q^k
  - locality r
- $[n] := \{1, \ldots, n\}$
- A *restriction* $\mathcal{C}_I$ of the code $\mathcal{C}$ to a subset of coordinates $I \subset [n]$ is the code obtained by removing from each vector the coordinates outside I.

Definition of LRC Codes

Given $a \in \mathbb{F}_q$ consider the sets of codewords of $\mathcal{C}$ with fixed value a at the symbol x_i:

$$\mathcal{C}(i,a) = \{x \in \mathcal{C} : x_i = a\}, \quad i \in [n]$$

Definition 1.1.

A code $\mathcal{C}$ of length n has **locality r** *if $\forall i \in [n]$ there exists a subset $I_i \subset [n] \setminus i, \quad |I_i| \le r$ such that the restrictions of the sets $\mathcal{C}(i,a)$ to the coordinates in I_i for different a are disjoint:*

$$\mathcal{C}_{I_i}(i,a) \cap \mathcal{C}_{I_i}(i,a') = \emptyset, \quad a \neq a'.$$

Maximum rate

Let $\mathcal{C}$ be an (n, k, r) LRC code of cardinality q^k over an alphabet of size q. Then:

Theorem 1.2 (Upper bound on the rate).

The rate of $\mathcal{C}$ satisfies

$$\frac{k}{n} \leq \frac{r}{r+1}$$

Theorem 1.3 (Generalization of Singleton bound).

The minimum distance of $\mathcal{C}$ satisfies

$$d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2$$

A code that achieves the bound with equality will be called an **optimal LRC code**.

Construction of LRC codes

We want to construct a linear (n, k, r)-LRC code. Assume $r|k$ and $(r+1)|n$.
We need:

- $A_1, \ldots, A_{\frac{n}{r+1}}$ disjoint subsets of the field $\mathbb{F}_q$, s.t. $|A_i| = r+1$
- $g(x) \in \mathbb{F}_q[x]$ a polynomial s.t.
  1. $deg(g) = r+1$
  2. g is constant on each set A_i: $g(\alpha) = g(\beta)$ for $\alpha, \beta \in A_i$

We will call g a good polynomial.

Construction of LRC codes

Let $A = \bigcup_{i=1}^{\frac{n}{r+1}} A_i \subset \mathbb{F}_q$, $|A| = n$.
We write now message vectors $a \in \mathbb{F}_q^k$ as $r \times \frac{k}{r}$ matrices.

$$a = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,\frac{k}{r}-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,\frac{k}{r}-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,0} & a_{r-1,1} & \cdots & a_{r-1,\frac{k}{r}-1} \end{pmatrix}$$

Construction of LRC codes

Encoding polynomial

Given the message vector $a \in \mathbb{F}_q^k$, define the **encoding polynomial** as:

$$f_a(x) = \sum_{i=0}^{r-1} x^i \cdot f_i(x)$$

where

$$f_i(x) = \sum_{j=0}^{\frac{k}{r}-1} a_{ij} g(x)^j$$

$$f_a(x) = \begin{pmatrix} x^0 & \dots & x^{r-1} \end{pmatrix} \begin{pmatrix} a_{0,0} & \cdots & a_{0,\frac{k}{r}-1} \\ \vdots & \ddots & \vdots \\ a_{r-1,0} & \cdots & a_{r-1,\frac{k}{r}-1} \end{pmatrix} \begin{pmatrix} g(x)^0 \\ \vdots \\ g(x)^{\frac{k}{r}-1} \end{pmatrix} =$$

$$= \begin{pmatrix} x^0 & \dots & x^{r-1} \end{pmatrix} \begin{pmatrix} f_0(x) \\ \vdots \\ f_{r-1}(x) \end{pmatrix}$$

The codeword for $a \in \mathbb{F}_q^k$ is found as the evaluation vector of f_a at all the points of A.

LRC code

The (n, k, r) LRC code $\mathcal{C}$ is defined as the set of n-dimensional vectors

$$\mathcal{C} = \{(f_a(\alpha), \alpha \in A) : a \in \mathbb{F}_q^k\}$$

Remark 2.1.

$$x \in A_i \Rightarrow \quad g(x) \ constant$$

$$\Rightarrow f_\ell(x) = \sum_{j=0}^{\frac{k}{r}-1} a_{\ell j} g(x)^j \ constant \ in \ A_i$$

$$\Rightarrow deg(f_a(x)) = deg(\sum_{j=0}^{r-1} x^j \cdot f_j(x)) \leq r-1 \ in \ A_i$$

Recovery of the erased symbol

Suppose erased symbol: $\alpha \in A_j$.
Let $(c_\beta, \ \beta \in A_j \setminus \alpha)$ denote the remaining r symbols of the recovering set.
To find the value $c_\alpha = f_a(\alpha)$, find the unique polynomial $\delta(x)$ s.t.

- $\deg(\delta(x)) \leq r$
- $\delta(\beta) = c_\beta \quad \forall \beta \in A_j \setminus \alpha$

This polynomial is:

$$\delta(x) = \sum_{\beta \in A_j \setminus \alpha} c_\beta \prod_{\beta' \in A_j \setminus \{\alpha, \beta\}} \frac{x - \beta'}{\beta - \beta'}$$

Finally, set $c_\alpha = \delta(\alpha)$.

Theorem 2.2.

The linear code $\mathcal{C}$ defined has dimension k and is an optimal (n, k, r) LRC code.

Proof of dimension.

For $i \in \{0, \dots, r-1\}$; $j \in \{0, \dots, \frac{k}{r-1}\}$ the k polynomials $g(x)^j x^i$ all are of distinct degrees, i.e. linearly independent over $\mathbb{F}$.
$\Rightarrow$ The mapping $a \mapsto f_a$ is injective.

$$\deg(f_a(x)) \leq \deg(x^{r-1}) + \deg(g(x)^{\frac{k}{r}-1}) = r - 1 + (r+1)(\frac{k}{r} - 1)$$

$$= k + \frac{k}{r} - 2 \leq n - 2$$

This means that two distinct encoding polynomials give rise to two distinct codevectors. $\Rightarrow$ The dimension of the code is k. $\square$

Proof of optimality.

Since the encoding is linear:

$$d(\mathcal{C}) \geq n - \max_{f_a, a \in \mathbb{F}_q^k} \deg(f_a) = n - k - \frac{k}{r} + 2 \geq n - k - \left\lceil \frac{k}{r} \right\rceil + 2$$

But we have that $d(\mathcal{C}) \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2$. Therefore, we have equality and thus it is an optimal LRC Code. $\square$

Example: (9,4,2) LRC code

We will now construct a $(n = 9, k = 4, r = 2)$ LRC code over the field $\mathbb{F}_q$.

$$q = |\mathbb{F}_q| \geq n \quad \Rightarrow \quad q \geq 9$$

Choose $q = 13$

$$\mathcal{A} = \{A_1 = \{1, 3, 9\}, A_2 = \{2, 6, 5\}, A_3 = \{4, 12, 10\}\}$$

.

$$g(x) = x^3 = \begin{cases} 1 & \text{if } x \in A_1 \\ 8 & \text{if } x \in A_2 \\ 12 & \text{if } x \in A_3 \end{cases}$$

For $a = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \in \mathbb{F}_{13}^4$ define the encoding polynomial:

$$f_a(x) = \begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} 1 \\ x^3 \end{pmatrix} = a_{00} + a_{10}x + a_{01}x^3 + a_{11}x^4$$

E.g. $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. $f_a(x) = 1 + x + x^3 + x^4$

$$c = (f_a(1), f_a(3), f_a(9), f_a(2), f_a(6), f_a(5), f_a(4), f_a(12), f_a(10))$$

$$= (4, 8, 7, 1, 11, 2, 0, 0, 0)$$

$$\delta(x) = \sum_{\beta \in A_j \setminus \alpha} c_\beta \prod_{\beta' \in A_j \setminus \{\alpha, \beta\}} \frac{x - \beta'}{\beta - \beta'}$$

$$(f_a(1), f_a(3), f_a(9), f_a(2), f_a(6), f_a(5), f_a(4), f_a(12), f_a(10))$$

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$$(\cancel{f_a(1)}, f_a(3), f_a(9), f_a(2), f_a(6), f_a(5), f_a(4), f_a(12), f_a(10))$$

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$$1 \in A_1 = \{1, 3, 9\}$$

$$\delta(x) = \sum_{\beta \in A_j \setminus \alpha} c_\beta \prod_{\beta' \in A_j \setminus \{\alpha, \beta\}} \frac{x - \beta'}{\beta - \beta'}$$

$$(\cancel{f_a(1)}, f_a(3), f_a(9), f_a(2), f_a(6), f_a(5), f_a(4), f_a(12), f_a(10))$$

$$(\cancel{4}, 8, 7, 1, 11, 2, 0, 0, 0)$$

$$1 \in A_1 = \{1, 3, 9\}$$

$$\Rightarrow \delta(x) = c_3 \frac{x - 9}{3 - 9} + c_9 \frac{x - 3}{9 - 3} = 2x + 2$$

$$\delta(x) = \sum_{\beta \in A_j \setminus \alpha} c_\beta \prod_{\beta' \in A_j \setminus \{\alpha, \beta\}} \frac{x - \beta'}{\beta - \beta'}$$

$$(\cancel{f_a(1)}, f_a(3), f_a(9), f_a(2), f_a(6), f_a(5), f_a(4), f_a(12), f_a(10))$$

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$$\Rightarrow \delta(x) = c_3 \frac{x - 9}{3 - 9} + c_9 \frac{x - 3}{9 - 3} = 2x + 2$$

$$\delta(1) = 4$$

$$\delta(x) = \sum_{\beta \in A_j \setminus \alpha} c_\beta \prod_{\beta' \in A_j \setminus \{\alpha, \beta\}} \frac{x - \beta'}{\beta - \beta'}$$

$$(f_a(1), f_a(3), f_a(9), \xcancel{f_a(2)}, f_a(6), f_a(5), f_a(4), f_a(12), f_a(10))$$

$$(4, 8, 7, \xcancel{1}, 11, 2, 0, 0, 0)$$

$$\delta(x) = \sum_{\beta \in A_j \setminus \alpha} c_\beta \prod_{\beta' \in A_j \setminus \{\alpha, \beta\}} \frac{x - \beta'}{\beta - \beta'}$$

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$$2 \in A_2 = \{2, 6, 5\}$$

$$\delta(x) = \sum_{\beta \in A_j \setminus \alpha} c_\beta \prod_{\beta' \in A_j \setminus \{\alpha, \beta\}} \frac{x - \beta'}{\beta - \beta'}$$

$$(f_a(1), f_a(3), f_a(9), \xcancel{f_a(2)}, f_a(6), f_a(5), f_a(4), f_a(12), f_a(10))$$

$$(4, 8, 7, \xcancel{1}, 11, 2, 0, 0, 0)$$

$$2 \in A_2 = \{2, 6, 5\}$$

$$\Rightarrow \delta(x) = c_6 \frac{x - 5}{6 - 5} + c_5 \frac{x - 6}{5 - 6} = 9x + 9$$

$$\delta(x) = \sum_{\beta \in A_j \setminus \alpha} c_\beta \prod_{\beta' \in A_j \setminus \{\alpha, \beta\}} \frac{x - \beta'}{\beta - \beta'}$$

$$(f_a(1), f_a(3), f_a(9), \cancel{f_a(2)}, f_a(6), f_a(5), f_a(4), f_a(12), f_a(10))$$

$$(4, 8, 7, \cancel{1}, 11, 2, 0, 0, 0)$$

$$2 \in A_2 = \{2, 6, 5\}$$

$$\Rightarrow \delta(x) = c_6 \frac{x-5}{6-5} + c_5 \frac{x-6}{5-6} = 9x + 9$$

$$\delta(2) = 1$$

$$\delta(x) = \sum_{\beta \in A_j \setminus \alpha} c_\beta \prod_{\beta' \in A_j \setminus \{\alpha, \beta\}} \frac{x - \beta'}{\beta - \beta'}$$

$$(f_a(1), f_a(3), f_a(9), f_a(2), f_a(6), f_a(5), \xcancel{f_a(4)}, f_a(12), f_a(10))$$

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$$4 \in A_3 = \{4, 12, 10\}$$

$$\delta(x) = \sum_{\beta \in A_j \setminus \alpha} c_\beta \prod_{\beta' \in A_j \setminus \{\alpha,\beta\}} \frac{x - \beta'}{\beta - \beta'}$$

$$(f_a(1), f_a(3), f_a(9), f_a(2), f_a(6), f_a(5), \xcancel{f_a(4)}, f_a(12), f_a(10))$$

$$(4, 8, 7, 1, 11, 2, \xcancel{0}, 0, 0)$$

$$4 \in A_3 = \{4, 12, 10\}$$

$$\Rightarrow \delta(x) = c_{12}\frac{x - 10}{12 - 10} + c_{10}\frac{x - 12}{10 - 12} = 0$$

$$\delta(x) = \sum_{\beta \in A_j \setminus \alpha} c_\beta \prod_{\beta' \in A_j \setminus \{\alpha, \beta\}} \frac{x - \beta'}{\beta - \beta'}$$

$$(f_a(1), f_a(3), f_a(9), f_a(2), f_a(6), f_a(5), \cancel{f_a(4)}, f_a(12), f_a(10))$$

$$(4, 8, 7, 1, 11, 2, \cancel{0}, 0, 0)$$

$$4 \in A_3 = \{4, 12, 10\}$$

$$\Rightarrow \delta(x) = c_{12} \frac{x - 10}{12 - 10} + c_{10} \frac{x - 12}{10 - 12} = 0$$

$$\delta(4) = 0$$

Example of LRC-2 code

Let $\mathbb{F} = \mathbb{F}_{13}$, $A = \mathbb{F} \setminus \{0\}$
$\mathcal{A} = \{\{1,5,12,8\}, \{2,10,11,3\}, \{4,7,9,6\}\}$
$\mathcal{A}' = \{\{1,3,9\}, \{2,6,5\}, \{4,12,10\}, \{7,8,11\}\}$
$f_a(x) = a_0 + a_1 x + a_2 x^4 + a_3 x^6$

$$a = (1,1,1,1) \quad \longrightarrow \quad c = (4,8,7,5,2,6,2,2,2,3,9,1)$$

As already seen: $\delta(x) = 2x + 2$; $\delta(1) = 4$.

$$\delta'(x) = c_5 \frac{x-12}{5-12}\frac{x-8}{5-8} + c_{12}\frac{x-5}{12-5}\frac{x-8}{12-8} + c_8 \frac{x-5}{8-5}\frac{x-12}{8-12}$$

$$= 6 \cdot 5 \cdot (x^2 + 6x + 5) + 2 \cdot 7 \cdot (x^2 + 1) + 9 \cdot 1 \cdot (x^2 + 9x + 8)$$

$$= x^2 + x + 2 \quad \longrightarrow \quad \delta'(1) = 4$$

Assume every coordinate i has t disjoint recovering sets $\mathcal{R}_i^1, \ldots, \mathcal{R}_i^t$, each of size r, where $\mathcal{R}_i^j \subset [n] \setminus i$.

Definition

The **recovery graph** of a (n, k, r, t) LRC code $\mathcal{C}$ is a directed graph $G = (V, E)$ where:

- $V = [n]$. (Vertices $\leftrightarrow$ coordinates of $\mathcal{C}$).
- $(i, j) \in E \iff j \in \mathcal{R}_i^\ell$ for some $\ell \in [t]$.
  There is an edge $i \to j$ if j is in a recovering set of i.

Note that $N(i) = \bigcup_{\ell=1}^t \mathcal{R}_i^\ell$

Let C be an (n, k, r, t) LRC code with t dijsoint recovering sets of size r. Then:

Theorem 2.3.

The rate of C satisfies

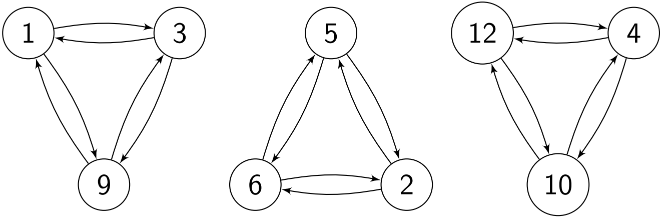
$$\frac{k}{n} \leq \frac{1}{\prod_{j=1}^{t}(1 + \frac{1}{jr})}$$

Theorem 2.4.

The minimum distance of C is bounded above as follows

$$d \leq n - \sum_{i=0}^{t} \left\lfloor \frac{k-1}{r^i} \right\rfloor$$

Recovery graph for the $(9, 4, 2)$-LRC code.
Recall: $\mathcal{A} = \{A_1 = \{1, 3, 9\}, A_2 = \{2, 6, 5\}, A_3 = \{4, 12, 10\}\}$



Color the edges with t distinct colors to differenciate recovering sets.
Let F be a coloring function of the edges:

$$\begin{array}{rccc} F: & E(G) & \longrightarrow & [t] \\ & (i,j) & \longmapsto & \ell \quad \text{iff } j \in \mathcal{R}_i^\ell \end{array}$$

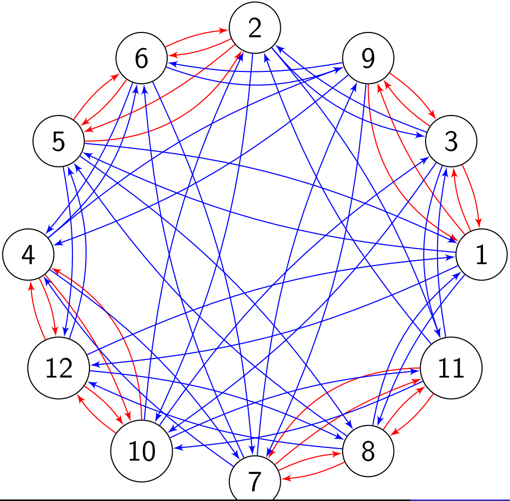
Remark: the out-degree of any vertex $i \in V$ is $\sum_\ell |\mathcal{R}_i^\ell| = tr$, and the edges leaving i are colored in t colors.

Recovery graph for the $(12, 4, \{2, 3\})$-LRC code with edge coloring.

Recall:

$\mathcal{A} = \{\{1, 5, 12, 8\}, \{2, 10, 11, 3\}, \{4, 7, 9, 6\}\}$

$\mathcal{A}' = \{\{1, 3, 9\}, \{2, 6, 5\}, \{4, 12, 10\}, \{7, 8, 11\}\}$



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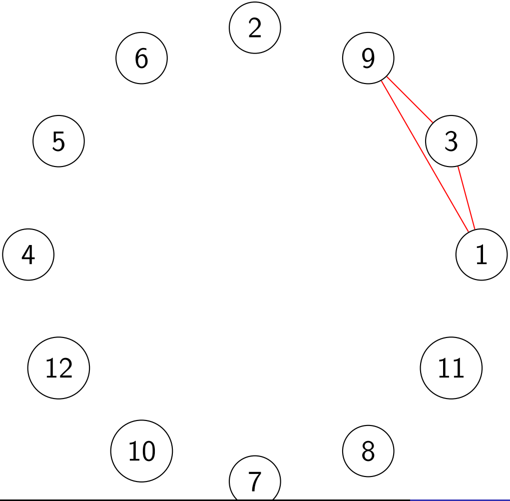

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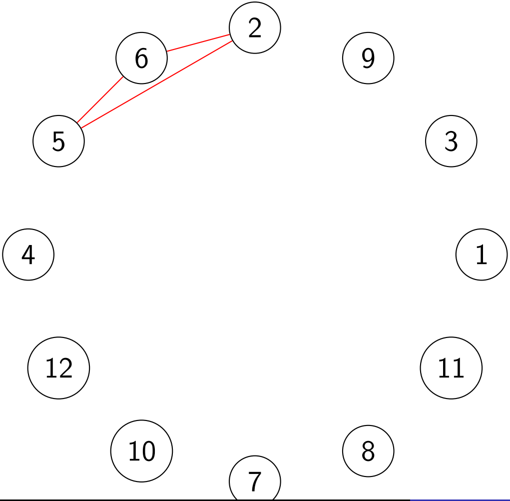

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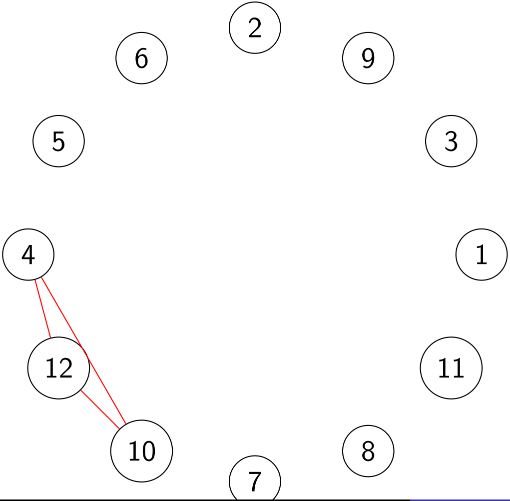

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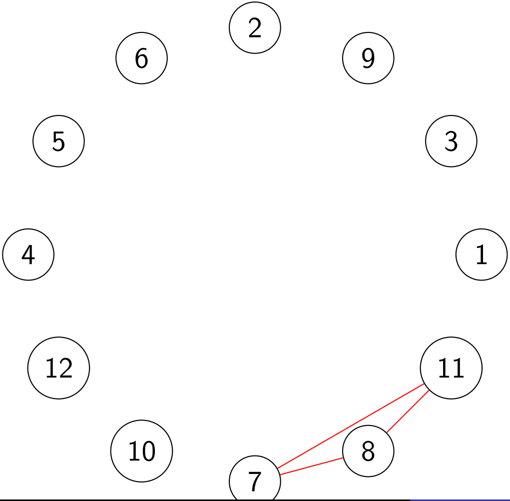

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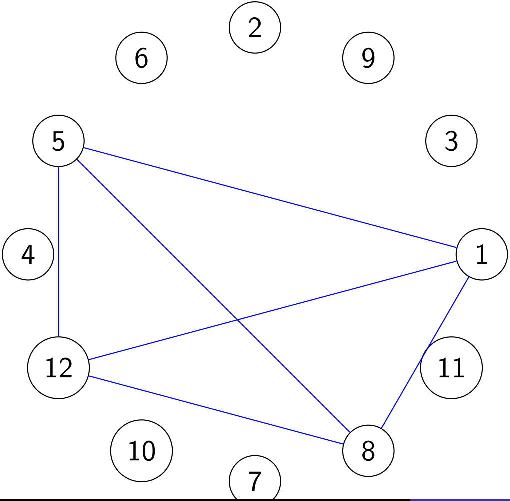

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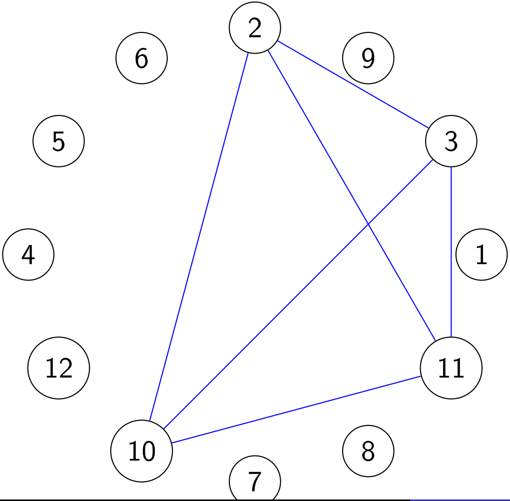

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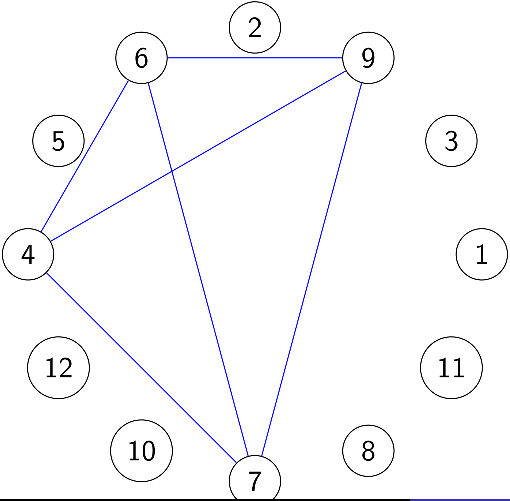

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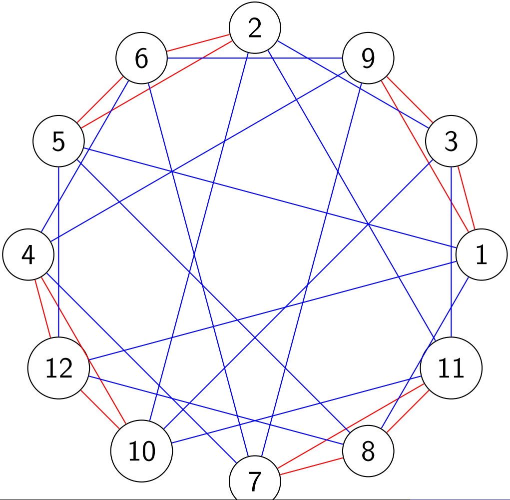

Lemma 2.5.

There exists a subset of the vertices $U \subseteq V$ of size at least

$$|U| \geq n \left(1 - \frac{1}{\prod_{j=1}^{t} \left(1 + \frac{1}{jr} \right)} \right)$$

such that for any $U' \subseteq U$, $G_{U'}$ has at least one vertex $v \in U'$ such that its set of outgoing edges is missing at least one color.

For a given permutation τ of the set of vertices V, define the coloring of some of the vertices.
The color $j \in [t]$ is assigned to v if

$$\tau(v) > \tau(m) \quad \forall m \in \mathcal{R}_v^j$$

If this condition is satisfied for several values of m, the vertex v is assigned any of these colors.
If this condition is not satisfied at all, the vertex v is not colored.

Let U be the set of colored vertices. Consider one of its subsets $U' \subseteq U$.
Assume toward a contradiction that every vertex of $G_{U'}$ has outgoing edges of all t colors.
Choose a vertex $v \in U'$ and construct the following walk: if the path ends at some vertex with color j, choose one of its outgoing edges colored in j.

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \dots \longrightarrow v_\ell$$

We can extend this path indefinitely as every vertex has outgoing edges of all t colors.

Since the graph $G_{U'}$ is finite, there will be a vertex (call it v_1) that is encountered twice.

$$v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow \dots \longrightarrow v_\ell \longrightarrow v_1$$

But:

$$\left.\begin{array}{cc} v_i & \text{color j} \\ (v_i, v_{i+1}) & \text{color j} \end{array}\right\} \implies \tau(v_i) > \tau(v_{i+1})$$

$$\tau(v_1) > \tau(v_2) > \cdots > \tau(v_\ell) > \tau(v_1)$$

Contradiction!!!
Then, there must be a vertex in U' such that its set of outgoing edges is missing one color.

To show that there exists such U of large carinality, we choose τ uniformly at random among $\mathfrak{S}_n$ and compute $E(|U|)$.
Let $A_{v,j}$ be the event that the equation $\tau(v) > \tau(m) \quad \forall m \in \mathcal{R}_v^j$ holds for the vertex v and the color j. Since all vertices have r outgoing edges for each of the t different colors, the probability $Pr(A_{v,j})$ does not depend on v, we write $A_j := A_{v,j}$.
Recall that $v \in U$ if v is colored with some color ℓ, and is colored with ℓ if A_ℓ happens.

$$Pr(v \in U) = Pr(\bigcup_{j=1}^{t} A_j)$$

Inclusion-exclusion formula:

$$Pr(\bigcup_{j=1}^{t} A_j) = \sum_{\emptyset \neq S \subseteq [t]} (-1)^{|S|-1} Pr(\bigcap_{j \in S} A_j)$$

.

For that, we need, for any set $S \subseteq [t]$ the probability that all the $A_j, j \in S$ occur simultaneously.

$$Pr(\bigcap_{j \in S} A_j) = \frac{1}{|S|\, r + 1}$$

Proof.

For any assignation of values given to $\{v\} \cup (\bigcup_{j \in S} R_v^j)$, only 1 out of $|S|\, r + 1$ is the maximum. $\square$

$$Pr\left(\bigcup_{j=1}^{t} A_j\right) = \sum_{\emptyset \neq S \subseteq [t]} (-1)^{|S|-1} Pr\left(\bigcap_{j \in S} A_j\right)$$

$$= \sum_{\emptyset \neq S \subseteq [t]} (-1)^{|S|-1} \frac{1}{|S|\, r + 1} = \sum_{j=1}^{t} (-1)^{j-1} \binom{t}{j} \frac{1}{|S|\, r + 1}$$

$$= 1 - \frac{1}{\prod_{j=1}^{t} \left(1 + \frac{1}{jr}\right)}$$

Let X_v be the indicator for the event that $v \in U$.

$$E(|U|) = \sum_{v \in V} E(X_v) = \sum_{v \in V} Pr(v \in U) = n \cdot Pr(\bigcup_{j=1}^{t} A_j)$$

$$= n(1 - \frac{1}{\prod_{j=1}^{t}\left(1 + \frac{1}{jr}\right)})$$

Observation: there exists $\tau \in \mathfrak{S}_n$ for which $|U| \geq E(|U|)$.

Proof of Theorem 2.3

Let $U \subseteq [n]$ be the set of vertices of cardinality as in the one constructed in Lemma 2.5.

Claim: every coordinate $i \in U$ can be recovered by accessing the coordinates in $\bar{U} = [n] \setminus U$.

By Lemma 2.5, for any $U' \subseteq U$, $\exists v \in U'$ that is missing one color, say ℓ, in its outgoing edges in G'_U.

$$\Rightarrow \quad \mathcal{R}_v^{\ell} \subseteq \overline{U'} \quad \Rightarrow \quad v \text{ can be recovered from } \overline{U'}$$

Suppose the values of the coordinates in $\bar{U}$ are known.

Consider:

- $U^{(0)} = U$
- $U^{(i+1)} = U^{(i)} \setminus \{v_i\}$
  s.t. $R_{v_i}^{\ell_i} \subseteq \overline{U^{(i)}}$ for some $\ell_i \in [t]$

Every v_i can be recovered from $\overline{U^{(i)}}$ which consists of the known values of $\overline{U}$ and the i previously recovered values $v_0, v_1, ..., v_{i-1}$.

Conclusion: Every coordinate $i \in U$ can be recovered from the coordinates in $\overline{U}$.

From the last claim, we deduce

$$k \leq \left|\overline{U}\right| \leq \frac{n}{\prod_{j=1}^{t}(1 + \frac{1}{jr})}$$

From where we get a bound on the rate

$$\frac{k}{n} \leq \frac{1}{\prod_{j=1}^{t}(1 + \frac{1}{jr})}$$

□