UPC - Commutative Algebra

Discrete Morse Theory

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1 Introduction

2 Monomial Ideals

Monomial ideals are those ideals generated by monomials. For each monomial ideal there exists a unique minimal generating set of monomials.

Definition 2.1. Let $G = (V(G), E(G))$ be a finite simple graph. The *edge ideal* associated to G is the monomial ideal

$$I(G) = \langle x_i x_j | \{x_i, x_j\} \in E(G) \rangle \subseteq R = k[x_1, \dots, x_n]$$

The *cover ideal* is the monomial ideal

$$J(G) = \bigcap_{\{x_i, x_j\} \in E(G)} \langle x_i, x_j \rangle \subseteq R = k[x_1, \dots, x_n]$$

As edge ideals are square-free, we can apply the theory of Stanley Reissner ideals and simplicial complexes. [6]

Definition 2.2. A simplicial complex on $V = \{x_1, \dots, x_n\}$ is a subset Δ of the power set of V (i.e. $\Delta \subseteq \mathcal{P}(V)$) such that

(i) if $F \in \Delta$, and $G \subseteq F$, them $G \in \Delta$.

(ii) $\{x_i\} \in \Delta$ for all i.

Definition 2.3. The *Stanley-Reisner ideal* associated to a simplicial complex Δ is the square-free monomial ideal

$$I_\Delta = \langle x_W | W \notin \Delta \rangle$$

Where $x_W := \prod_{x_i \in W} x_i$, for $W \subseteq V(G)$.

For any simplicial complex Δ, the primary decomposition of I_Δ can be described in terms of the *facets* of Δ. Recall that $F \in \Delta$ is a facet if F is a face that is maximal under inclusion. For each facet F, let

$$P_F = \langle x_i | x_i \notin F \rangle$$

Theorem 2.4. *([6, Theorem 3.1.34]) Let Δ be a simplicial complex with facets $F_1, \ldots, F_t$. Then*

$$I_\Delta = P_{F_1} \cap P_{F_2} \cap \cdots \cap P_{F_t}$$

Definition 2.5. A subset $W \subseteq V(G)$ is a *vertex cover* if $W \cap e \neq \emptyset$ for all $e \in E(G)$. A vertex cover W is a *minimal vertex cover* if no proper subset of W is a vertex cover.

Corollary 2.6. *([6, Corollary 3.1.35]) Let $W_1, \ldots, W_t$ be the minimal vertex covers of G, and set $\langle W_i \rangle = \langle x_j | x_j \in W_i \rangle$. Then*

$$I(G) = \langle W_1 \rangle \cap \cdots \cap \langle W_t \rangle$$

Definition 2.7. Let I be a square-free monomial ideal with primary decomosition

$$I = \langle x_{1,1}, x_{1,2}, \ldots, x_{1,s_1} \rangle \cap \langle x_{2,1}, x_{2,2}, \ldots, x_{2,s_2} \rangle \cap \cdots \cap \langle x_{t,1}, x_{t,2}, \ldots, x_{t,s_t} \rangle$$

The *Alexander Dual* of I, denoted $I^\vee$, is the square-free monomial ideal

$$I^\vee = \langle x_{1,1}x_{1,2} \cdots x_{1,s_1}, x_{2,1}x_{2,2} \cdots x_{2,s_2}, \ldots, x_{t,1}x_{t,2} \cdots x_{t,s_t} \rangle$$

Corollary 2.8. *([6, Corollary 3.1.38]) Let G be a graph. Then $I(G)^\vee = J(G)$.*

3 Free Resolutions

Definition 3.1. If $I \subseteq R$ an ideal, then a *free resolution* of R/I is an exact sequence

$$\mathbb{F} : \ldots \xrightarrow{\phi_n} F_n \xrightarrow{\phi_{n-1}} F_{n-1} \ldots \xrightarrow{\phi_0} F_0 \to R/I \to 0$$

where each of the of the F_i is a free R-module.

$$F_i = \bigoplus_{\alpha \in \mathbb{Z}^n} R(-\alpha)^{\beta_{i,\alpha}}$$

We say that $\mathbb{F}$ is *minimal* if each of the modules F_i has minimum possible rank; in this case the ranks are the *Betti numbers* of S/I and these form a set of invariants.

3.1 Cellular Resolutions

A CW-complex X is a topological space obtained by attaching cells of increasing dimensions to a discrete set of points $X^{(0)}$. Let $X^{(i))}$ denote the set of i-cells of X and consider the set of all cells $X^{(*)} := \bigcup_{i\geq 0} X^{(i)}$. Then we can view $X^{(*)}$ as a poset with the partial order given by $\sigma' \leq sigma$ if and only if σ' is contained in the closure of σ. We can also give a $\mathbb{Z}^n$-graded structure to X by means of an order preserving map $gr : X^{(*)} \longrightarrow \mathbb{Z}^n_{\geq 0}$.

We say that the free resolution is *cellular* (or is a *CW-resolution*) if there exusts a $\mathbb{Z}^n$-graded CW-complex (X, gr) such that, for all $i \geq 1$:

- there exists a basis $\{e_\sigma\}$ of F_i indexed by the $(i-1)$-cells of X, such that if $e_\sigma \in R(-\alpha)^{\beta_{i,\alpha}}$ then $gr(\sigma) = \alpha$, and
- the differential $d_i : F_i \longrightarrow F_{i-1}$ is given by

$$e_\sigma \mapsto \sum_{\sigma \geq \sigma' \in X^{(i-1)}} [\sigma : \sigma'] \mathbf{x}^{gr(\sigma)-gr(\sigma')} e_{\sigma'}, \quad \forall \sigma \in X^{(i)}$$

  where $[\sigma : \sigma']$ denotes the coefficient of σ' in the image of σ by the differential map in the cellular homology of X.

From now on, we will the denote the free resolution as $\mathbb{F}_\bullet = \mathbb{F}_\bullet^{(X,gr)}$. If X is a simplicial complex, we say that the free resolution is *simplicial*.

3.1.1 Taylor Resolution

3.1.2 Scarf Complex

3.1.3 Lyubeznik Resolution

4 Results

In figure 1 we show all neighbourly Gale diagrams we found.

Figure 1: All neighbourly Gale Diagrams, some isomorphic.

Figure 2: The coordinates for all 3 different neighbourly 4-polytopes on 8 vertices up to combinatorial equivalence

([1])

References

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