Discrete Morse Theory

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May 26, 2017

1 Monomial Ideals

Monomial ideals are those ideals generated by monomials. For each monomial ideal there exists a unique minimal generating set of monomials.

Definition 1.1. Let $G = (V(G), E(G))$ be a finite simple graph. The *edge ideal* associated to G is the monomial ideal

$$I(G) = \langle x_i x_j | \{x_i, x_j\} \in E(G) \rangle \subseteq R = k[x_1, \dots, x_n]$$

The *cover ideal* is the monomial ideal

$$J(G) = \bigcap_{\{x_i, x_j\} \in E(G)} \langle x_i, x_j \rangle \subseteq R = k[x_1, \dots, x_n]$$

As edge ideals are square-free, we can apply the theory of Stanley Reissner ideals and simplicial complexes. [6]

Definition 1.2. A simplicial complex on $V = \{x_1, \dots, x_n\}$ is a subset Δ of the power set of V (i.e. $\Delta \subseteq \mathcal{P}(V)$) such that

(i) if $F \in \Delta$, and $G \subseteq F$, them $G \in \Delta$.

(ii) $\{x_i\} \in \Delta$ for all i.

Definition 1.3. The *Stanley-Reisner ideal* associated to a simplicial complex Δ is the square-free monomial ideal

$$I_\Delta = \langle x_W | W \notin \Delta \rangle$$

Where $x_W := \prod_{x_i \in W} x_i$, for $W \subseteq V(G)$.

For any simplicial complex Δ, the primary decomposition of I_Δ can be described in terms of the *facets* of Δ. Recall that $F \in \Delta$ is a facet if F is a face that is maximal under inclusion. For each facet F, let

$$P_F = \langle x_i | x_i \notin F \rangle$$

Theorem 1.4. *([6, Theorem 3.1.34]) Let Δ be a simplicial complex with facets $F_1, \ldots, F_t$. Then*

$$I_\Delta = P_{F_1} \cap P_{F_2} \cap \cdots \cap P_{F_t}$$

Definition 1.5. A subset $W \subseteq V(G)$ is a *vertex cover* if $W \cap e \neq \emptyset$ for all $e \in E(G)$. A vertex cover W is a *minimal vertex cover* if no proper subset of W is a vertex cover.

Corollary 1.6. *([6, Corollary 3.1.35]) Let $W_1, \ldots, W_t$ be the minimal vertex covers of G, and set $\langle W_i \rangle = \langle x_j | x_j \in W_i \rangle$. Then*

$$I(G) = \langle W_1 \rangle \cap \cdots \cap \langle W_t \rangle$$

Definition 1.7. Let I be a square-free monomial ideal with primary decomosition

$$I = \langle x_{1,1}, x_{1,2}, \ldots, x_{1,s_1} \rangle \cap \langle x_{2,1}, x_{2,2}, \ldots, x_{2,s_2} \rangle \cap \cdots \cap \langle x_{t,1}, x_{t,2}, \ldots, x_{t,s_t} \rangle$$

The *Alexander Dual* of I, denoted $I^\vee$, is the square-free monomial ideal

$$I^\vee = \langle x_{1,1}x_{1,2} \cdots x_{1,s_1}, x_{2,1}x_{2,2} \cdots x_{2,s_2}, \ldots, x_{t,1}x_{t,2} \cdots x_{t,s_t} \rangle$$

Corollary 1.8. *([6, Corollary 3.1.38]) Let G be a graph. Then $I(G)^\vee = J(G)$.*

2 Free Resolutions

If $I \subseteq R$ an ideal, then a *free resolution* of R/I is an exact sequence

$$\mathbb{F} : \ldots \xrightarrow{\phi_n} F_n \xrightarrow{\phi_{n-1}} F_{n-1} \ldots \xrightarrow{\phi_0} F_0 \to R/I \to 0$$

where each of the of the F_i is a free R-module.

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables with coefficients in a field k. Let $I \subseteq R$ be a monomial ideal generated by monomials $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha \in \mathbb{Z}^n_{\geq 0}$. We denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\epsilon_1, \ldots, \epsilon_n$ will be the natural basis of $\mathbb{Z}^n$. Given the set of generators of a monomial ideal $I = \langle m_1, \ldots, m_r \rangle$, we will consider the monomials $m_\sigma := \operatorname{lcm}(m_i | \sigma_i = 1)$ for any $\sigma \in \{0, 1\}^r$.

A $\mathbb{Z}^n$-graded free resolution of R/I is an exact sequence of free $\mathbb{Z}^n$-graded modules, where the i-th term is of the form

$$F_i = \bigoplus_{\alpha \in \mathbb{Z}^n} R(-\alpha)^{\beta_{i,\alpha}}$$

We say that $\mathbb{F}$ is *minimal* if each of the modules F_i has minimum possible rank; in this case the ranks are the *Betti numbers* of R/I and these form a set of invariants.

2.1 Cellular Resolutions

A CW-complex X is a topological space obtained by attaching cells of increasing dimensions to a discrete set of points $X^{(0)}$. Let $X^{(i)})$ denote the set of i-cells of X and consider the set of all cells $X^{(*)} := \bigcup_{i\geq 0} X^{(i)}$. Then we can view $X^{(*)}$ as a poset with the partial order given by $\sigma' \leq sigma$ if and only if σ' is contained in the closure of σ. We can also give a $\mathbb{Z}^n$-graded structure to X by means of an order preserving map $gr : X^{(*)} \longrightarrow \mathbb{Z}^n_{\geq 0}$.

We say that the free resolution is *cellular* (or is a *CW-resolution*) if there exusts a $\mathbb{Z}^n$-graded CW-complex (X, gr) such that, for all $i \geq 1$:

- there exists a basis $\{e_\sigma\}$ of F_i indexed by the $(i-1)$-cells of X, such that if $e_\sigma \in R(-\alpha)^{\beta_{i,\alpha}}$ then $gr(\sigma) = \alpha$, and
- the differential $d_i : F_i \longrightarrow F_{i-1}$ is given by
$$e_\sigma \mapsto \sum_{\sigma \geq \sigma' \in X^{(i-1)}} [\sigma : \sigma'] \mathbf{x}^{gr(\sigma)-gr(\sigma')} e_{\sigma'}, \quad \forall \sigma \in X^{(i)}$$
where $[\sigma : \sigma']$ denotes the coefficient of σ' in the image of σ by the differential map in the cellular homology of X.

From now on, we will the denote the free resolution as $\mathbb{F}_\bullet = \mathbb{F}_\bullet^{(X,gr)}$. If X is a simplicial complex, we say that the free resolution is *simplicial.*

2.1.1 Taylor Resolution

The Taylor resolution is a simplicial free resolution given by the full simplicial complex on the set of vertices, where each vertex is a generator of the ideal.

Construction: Let $I = \langle m_1, \ldots, m_r \rangle \subseteq R$ be a monomial ideal. Consider the full simplicial complex on r vertices, X_{Taylor}, whose faces are labelled by $\sigma \in \{0,1\}^r$ or, equivalently, by the corresponding monomials m_σ. We have a natural $\mathbb{Z}^n$-grading on X_{Taylor} by assigning $gr(\sigma) = \alpha \in \mathbb{Z}^n$ where $\mathbf{x}^\alpha = m_\sigma$. The *Taylor resolution* is the simplicial resolution $\mathbb{F}_\bullet^{(X_{\text{Taylor}},gr)}$.

Example 2.1. Let $I = \langle x_1^2, x_1x_2, x_2^3x_3 \rangle$. The Taylor resolution of R/I is:

$$\mathbb{F}_\bullet^{(X_{\text{Taylor}},gr)} : \quad 0 \to R^1(-6) \to \begin{matrix} R^1(-3) \\ \bigoplus \\ R^1(-5) \\ \bigoplus \\ R^1(-6) \end{matrix} \to \begin{matrix} R^2(-2) \\ \bigoplus \\ R^1(-4) \end{matrix} \to R/I \to 0$$

$$0 \to R[x_1^2x_2^3x_3] \xrightarrow{\begin{pmatrix} x_1 \\ -1 \\ x_2^2x_3 \end{pmatrix}} \begin{matrix} R[x_1x_2^3x_3] \\ \bigoplus \\ R[x_1^2x_2^3x_3] \\ \bigoplus \\ R[x_1^2x_2] \end{matrix} \xrightarrow{\begin{pmatrix} x_1 & x_1^2 & 0 \\ -x_2^2x_3 & 0 & x_1 \\ 0 & -x_2^3x_3 & -x_2 \end{pmatrix}} \begin{matrix} R[x_2^3x_3] \\ \bigoplus \\ R[x_1x_2] \\ \bigoplus \\ R[x_1^2] \end{matrix} \xrightarrow{\begin{pmatrix} x_2^3x_3 & x_1x_2 & x_1^2 \end{pmatrix}} R/I \to 0$$

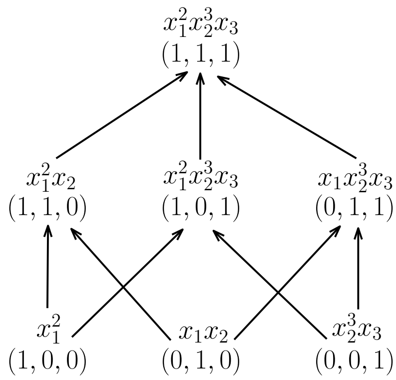

Figure 1: The simplicial complex associated to the taylor resolution

In this example, the Taylor resolution is not minimal. In general, the Taylor resolution is far from being minimal.

2.1.2 Scarf Complex

Unfortunately, the Taylor resolution is usually not minimal. The nonminimality is visible in the nonzero scalars in the differential maps, which occur whenever there exist faces F and G with the same multidegree such that G is in the boundary of F. It is tempting to try to simply remove the nonminimality by removing all such faces; the result is the *Scarf complex.*

Construction: Let $I = \langle m_1, \dots, m_r \rangle \subseteq R$ be a monomial ideal. Let X_{Taylor} be the full simplex on $\{m_1, \dots, m_r\}$, and let X_{Scarf} be the simplicial subcomplex of X_{Taylor} consisting of the faces with unique multidegree,

$$X_{\text{Scarf}} = \{F \in X_{\text{Taylor}} | gr(G) = gr(F) \Longrightarrow G = F\}$$

We say that X_{Scarf} is the *Scarf simplicial complex* of I; the associated algebraic chain complex $\mathbb{F}_\bullet^{(X_{\text{Scarf}}, gr)}$ is called the *Scarf complex* of I. The multidegrees of the faces of X_{Scarf} are called the *Scarf multidegrees* of I.

Example 2.2. Let $I = \langle x_1^2, x_1x_2, x_2^3x_3 \rangle$. The scarf simplicial complex of I is the complex X_{Scarf} in figure 2. The scarf complex of I is the minimal resolution

$$\mathbb{F}_\bullet^{(X_{\text{Scarf}}, gr)}: \quad 0 \to \begin{matrix} R[x_1x_2^3x_3] \\ \bigoplus \\ R[x_1^2x_2] \end{matrix} \xrightarrow{\begin{pmatrix} x_1 & 0 \\ -x_2^2x_3 & x_1 \\ 0 & -x_2 \end{pmatrix}} \begin{matrix} R[x_2^3x_3] \\ \bigoplus \\ R[x_1x_2] \\ \bigoplus \\ R[x_1^2] \end{matrix} \xrightarrow{\begin{pmatrix} x_2^3x_3 & x_1x_2 & x_1^2 \end{pmatrix}} R/I \to 0$$

Example 2.3. $I = \langle x_1x_2, x_1x_3, x_2x_3 \rangle = I(C_3)$ the edge ideal of the 3-cycle. The Scarf simplicial complex of I consists of three disjoint vertices. The Scarf complex of I is the

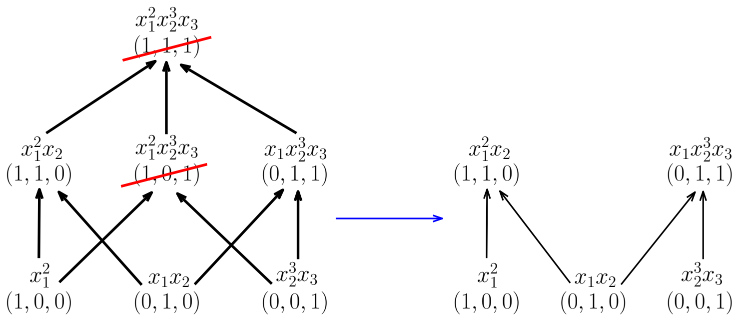

Figure 2: The scarf complex obtained from the Taylor resolution

complex

$$0 \to \begin{matrix} R[x_1x_3] \\ \bigoplus \\ R[x_2x_3] \\ \bigoplus \\ R[x_1x_2] \end{matrix} \xrightarrow{\begin{pmatrix} x_1x_3 & x_2x_3 & x_1x_2 \end{pmatrix}} R/I \to 0$$

It is not a resolution

Example 2.3 shows that not every monomial ideal is resolved by its Scarf complex. We say that a monomial ideal is *Scarf* if its Scarf complex is a resolution.

Despite that the Scarf complex is not always a resolution, it is always minimal. It is always a subcomplex of a minimal resolution and when it is a resolution, it is minimal.

Theorem 2.4. *[5, Theorem 5.5] If the Scarf complex of I is a resolution, then it is minimal.*

Theorem 2.5. *[5, Theorem 5.6] Let $\mathbb{F}$ be a minimal resolution of I. Then the Scarf complex of I is a subcomplex of $\mathbb{F}$.*

2.1.3 Lyubeznik Resolution

The Taylor resolution is too large and the Scarf complex is too small. We want to construct simplicial resolutions somewhere in between. There are classes of simplicial resolutions which are in general much smaller than the Taylor resolution, yet still manage to always be resolutions. On such class is the class of Lyubeznik resolutions.

Construction: Let $I = \langle m_1, \dots, m_r \rangle \subseteq R$ be a monomial ideal. Fix an order $m_1 \leq \cdots \leq m_r$ on the generating set of I. Consider the simplicial subcomplex $X_{\text{Lyub}} \subseteq X_{\text{Taylor}}$ whose faces of dimension s are labelled by those $\sigma = \epsilon_{i_0} + \cdots + \epsilon_{i_s} \in \{0,1\}^r$ such that, for all $t < s$ and all $j < i_t$

$$m_j \nmid \operatorname{lcm}(m_{i_t}, \dots, m_{i_s})$$

Example 2.6. Let $I = \langle x_1^2, x_1x_2, x_2^3x_3 \rangle$. Independently of the order of the monomials, all faces of dimension 2 will be in the complex, as otherwise it would mean that one of the monomials of the generating set divides another one.

If we fix the ordering $x_1^2 \leq x_1x_2 \leq x_2^3x_3$, the Lyubeznik resolution matches the Taylor resolution ($X_{\text{Lyub},1} = X_{\text{Taylor}}$) because $x_1^2 \nmid x_1x_2^3x_3$. The same happens whith the order $x_2^3x_3 \leq x_1^2 \leq x_1x_2$ as $x_2^3x_3 \nmid x_1^2x_2$. But with the order $x_1x_2 \leq x_1^2 \leq x_2^3x_3$ we have $x_1x_2 \mid x_1^2x_2^3x_3$ and so $\sigma = (1,1,1) \notin X_{\text{Lyub},2}$ and $\sigma' = (0,1,1) \notin X_{\text{Lyub},2}$. With the last order, the Lyubeznik resolution matches the Scarf complex ($X_{\text{Lyub},2} = X_{\text{Scarf}}$).

Remark 2.7. It is unclear how to choose a total ordering on the generators of I which produces a smaller Lyubeznik resolution.

Theorem 2.8. *The Lyubeznik resolutions of I are resolutions.*

3 Morse Discrete Theory

We can use the discrete Morse theory as a method to reduce the number of cells in a CW-complex without changing its homotopy type. In [1] this technique is adapted to the study of cellular resolutions, using the reformulation of discrete Morse theory in terms of acyclic matchings given by Chari in [2] in order to obtain a reduced cellular resolution given a regular one (usually the Taylor resolution).

First, some preliminaries on discrete Morse theory. Consider the directed graph G_X on the set of cells of a regular $\mathbb{Z}^n$-graded CW-complex (X, gr) which edges are given by

$$E_X = \{\sigma \longrightarrow \sigma' \mid \sigma' \leq \sigma, \dim \sigma' = \dim \sigma - 1\}$$

in other words, $\sigma \longrightarrow \sigma'$ is an edge if and only if σ' is a facet of σ.

For a given set of edges $\mathcal{A} \subseteq E_X$, denote by $G_X^{\mathcal{A}}$ the graph obtained by reversing the direction of the edges in $\mathcal{A}$, i.e., the directed graph with edges

$$E_X^{\mathcal{A}} = (E_X \setminus \mathcal{A}) \cup \{\sigma' \Longrightarrow \sigma \mid \sigma \longrightarrow \sigma' \in \mathcal{A}\}$$

When each cell of X occurs in at most one edge of $\mathcal{A}$, we say that $\mathcal{A}$ is a *matching* on X. A matching $\mathcal{A}$ is *acyclic* if the associated graph $G_X^{\mathcal{A}}$ is acyclic. Given an acyclic matching $\mathcal{A}$ on X, the $\mathcal{A}$-*critical cells* of X are the cells of X that are not contained in any edge of $\mathcal{A}$. Finally, an acyclic matching $\mathcal{A}$ is *homogeneous* whenever $gr(\sigma) = gr(\sigma')$ for any edge $\sigma \longrightarrow \sigma' \in \mathcal{A}$.

Proposition 3.1. *[1, Proposition 1.2] Let (X, gr) be a regular $\mathbb{Z}^n$-graded CW-complex and $\mathcal{A}$ a homogeneous acyclic matching. Then, there is a (not necessarily regular) CW-complex $X_{\mathcal{A}}$ whose i-cells are in one-to-one correspondence with the $\mathcal{A}$-critical i-cells of X, such that $X_{\mathcal{A}}$ is homotopically equivalent to X, and that inherits the $\mathbb{Z}^n$-graded structure.*

In the theory of cellular resolutions, we have the following consequence.

Theorem 3.2. *[1, Theorem 1.3] Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a monomial ideal. Assume that (X, gr) is a regular $\mathbb{Z}^n$-graded CW-complex that defines a cellular resolution $\mathbb{F}_\bullet^{(X,gr)}$ of R/I. Then, for a homogeneous acyclic matching $\mathcal{A}$ on G_X, the $\mathbb{Z}^n$-graded CW-complex $(X_{\mathcal{A}}, gr)$ supports a cellular resolution $\mathbb{F}_\bullet^{(X_{\mathcal{A}},gr)}$ of R/I.*

Proposition 3.1 and theorem 3.2 imply that, given a resolution from a CW-complex, if we have an acyclic matching on it, there is another CW-complex whose cells are the cells that are in the matching, and is homotopically equivalent.

The problem is that the CW-complex $(X_{\mathcal{A}}, gr)$ that we obtain is not necessarily regular. Therefore, we cannot always iterate the procedure. To overcome this obstacle, we may use *algebraic discrete Morse theory* developed in [3].

4 Pruning algorithms

In this section, we disscuss some pruning algorithms that find a homogeneous acyclic matching to get a smaller resolution.

4.1 Pruned resolution

Our goal is to prune the excess of information in the directed graph $G_{X_{\text{Taylor}}}$ in a very simple way. The algorithm produces a homogeneous acyclic matching $\mathcal{A}_P$ on X_{Taylor}. Using discrete Morse theory, this will provide a free resolution of R/I. It will not be minimal in general, but it will be smaller than the Lyubeznik resolution.

Construction: Given the set of edges $E_{X_{\text{Taylor}}}$, for j from 1 to r, incrementing by 1
Prune the edge $\sigma \longrightarrow \sigma + \epsilon_j$ for all $\sigma \in \{0,1\}^r$ such that $\sigma_j = 0$.

'Prune' means remove the edge if it survived after step $(j-1)$ and $gr(\sigma) = gr(\sigma + \epsilon_j)$. When we remove an edge, we also remove its two vertices and all the edges incident to them.

At the end, we get $\mathcal{A}_P$ the set of edges that have been pruned.

The main result for this algorithm is the following theorem:

Theorem 4.1. *[4, Theorem 3.3] Let $\mathcal{A}_P \subseteq E_{X_{Taylor}}$ be the set of pruned edges obtained using the pruning algorithm. Then $\mathcal{A}_P$ is a homogeneous acyclic matching on X_{Taylor}*

As a consequence, we get the cellular free resolution.

Corollary 4.2. *[4, Corollary 3.4] Let $I \subseteq R = k\{x_1, \dots, x_n\}$ be a monomial ideal and $\mathcal{A}_P \subseteq E_{X_{Taylor}}$ be the set of pruned edges obtained using the pruning algorithm. Then, the $\mathcal{Z}^n$-graded CW-complex $(X_{\mathcal{A}_P}, gr)$ supports a cellular free resolution $\mathbb{F}_\bullet^{(X_{\mathcal{A}_P}, gr)}$ of R/I.*

Remark 4.3. The free resolution $\mathbb{F}_\bullet^{(X_{\mathcal{A}_P}, gr)}$, like the Lyubeznik resolution, strongly depends on the order of the generators of the monomial ideal I. In general, it is neither simplicial nor minimal.

Still, a nice feature about this algorithm, is that we do not need to care if the generating set of I is minimal or not. This feature is given by [4, Lemma 3.6].

4.2 Simplicial pruned resolution

The cellular complex $X_{\mathcal{A}_P}$ from the pruned resolution may not be simplicial. If we choose carefully the edges that we prune in the pruning algorithm in order to preserve this property, we will obtain a simplicial free resolution of R/I that, in general, will be bigger than the pruned one.

Construction: Given the set of edges $E_{X_{\text{Taylor}}}$, for j from 1 to r, incrementing by 1

Prune the edge $\sigma \longrightarrow \sigma + \epsilon_j$ for all $\sigma \in \{0,1\}^r$ such that $\sigma_j = 0$.

'Prune' means remove the edge if it survived after step $(j-1)$, $gr(\sigma) = gr(\sigma + \epsilon_j)$ and no face $\tau > \sigma$ survives at this step (j).

At the end, we get $\mathcal{A}_S$ the set of edges that have been pruned.

We have that $\mathcal{A}_S$ is an acyclic matching on X_{Taylor} and the corresponding free resolution $\mathbb{F}_{\bullet}^{(X_{\mathcal{A}_S}, gr)}$ is a simplicial free resolution of the monomial ideal I.

4.3 Example

Let $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6)$ be the edge ideal of a 6-path. The steps formed with the pruning algorithm are the following:

- Step (1): No edge is pruned.
- Step (2): We prune four edges $(1,0,1,0,0) \Leftarrow (1,1,1,0,0)$, $(1,0,1,0,1) \Leftarrow (1,1,1,0,1)$, $(1,0,1,1,0) \Leftarrow (1,1,1,1,0)$ and $(1,0,1,1,1) \Leftarrow (1,1,1,1,1)$.
- Step (3): We prune $(0,1,0,1,0) \Leftarrow (0,1,1,1,0)$ and $(0,1,0,1,1) \Leftarrow (0,1,1,1,1)$.
- Step (4): We prune the edge $(0,0,1,0,1) \Leftarrow (0,0,1,1,1)$.
- Step (5): No edge is pruned.

The betti tables for the Taylor resolution, Lyubeznik resolution and the pruned resolution of I are:

	0	1	2	3	4	5
total :	1	5	10	10	5	1
0 :	1	.	.	.	.	.
1 :	.	5	4	3	2	1
2 :	.	.	6	6	3	.
3 :	.	.	.	1	.	.
4 :	.	.	.	.	.	.
5 :	.	.	.	.	.	.
6 :	.	.	.	.	.	.

	0	1	2	3	4	5
total :	1	5	10	10	5	1
0 :	1	.	.	.	.	.
1 :	.	5	4	3	2	1
2 :	.	.	6	6	3	.
3 :	.	.	.	1	.	.
4 :	.	.	.	.	.	.
5 :	.	.	.	.	.	.
6 :	.	.	.	.	.	.

	0	1	2	3	4
total :	1	5	7	4	1
0 :	1	.	.	.	.
1 :	.	5	4	.	.
2 :	.	.	3	4	1
3 :	.	.	.	.	.
4 :	.	.	.	.	.
5 :	.	.	.	.	.
6 :	.	.	.	.	.

With minimal free resolution:

$$0 \to R^1(-6) \to R^4(-5) \to \begin{matrix} R^4(-3) \\ \bigoplus \\ R^3(-4) \end{matrix} \to R^5(-2) \to R/I \to 0$$

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