Anonymous Threshold Signatures

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Goals of the work

- Find efficient anonymous threshold signature scheme with compact signature

Examples

Case 1 Toll pricing system that gives discount if there are at least three passengers in vehicle.

Case 2 E-voting system where each candidate needs certain amount of signatures to get to next round.

Case 3 Advertising company pays website holders for showing ads, but only pay on the amount of distinct users receiving the ads.

Requirements for these schemes:

- All anonymous
- For case 1: compact signatures, not excessively complex to compute. Could be interactive.
- For case 2: Non interactive.
- For case 3: Compact and non interactive.

PKE scheme

A public key encryption scheme $PKE = (KG, \mathcal{E}, \mathcal{D})$ consists of three probabilistic and polynomial time algorithms:

- Key generation KG:
  - Input: Security parameter
  - Output: Pair (sk, pk) of secret and public keys.
- Encription $\mathcal{E}$:
  - Input: Plaintext m
  - Output: Ciphertext $c = \mathcal{E}_{pk}(m)$
- Decryprtion $\mathcal{D}$:
  - Input: Ciphertext c
  - Output: Plaintext $m = \mathcal{D}_{sk}(c)$

For any pair (sk, pk) and any plaintext m, it must hold

$$m = \mathcal{D}_{sk}\left(\mathcal{E}_{pk}(m)\right)$$

Homomorphic PKE

Definition 2.1 (Homomorphic PKE).

Let $\mathcal{M}$ be the set of plaintexts s.t. it is closed under an operation $\bullet$. Let $\mathcal{C}$ be the set of ciphertexts s.t. it is closed under an operation $\circ$. A PKE scheme $(KG, \mathcal{E}, \mathcal{D})$ has the homomorphic property if

$$\mathcal{D}_{sk}(\mathcal{E}_{pk}(m_1) \circ \mathcal{E}_{pk}(m_2)) \;=\; m_1 \bullet m_2 \quad \forall m_1, m_2 \in \mathcal{M}.$$

Remark 2.2.

If we write $\mathcal{M}$ additively and $\mathcal{C}$ multiplicativelly, for $a \in \mathbb{Z}^+$ we have:

$$\mathcal{D}_{sk}\left(\mathcal{E}_{pk}(m)^a\right) = a \cdot m$$

Oblivious Polynomial Evaluation

Oblivious Polynomial Evaluation is a protocol involving a sender who knows a polynomial $P \in \mathbb{F}[x]$ and a receiver who knows a value $\alpha \in \mathbb{F}$. At the end of the protocol, the receiver learns $P(\alpha)$ and the sender learns nothing.

	Sender	Receiver
Input	$P \in \mathbb{F}[x]$	$\alpha \in \mathbb{F}$
Output	-	$P(\alpha)$

Bilinear Pairings

Let G_1 and G_2 be two cyclic groups of prime order q. We write them multiplicativelly.

Problem name	Input	Output
Decisional DH (DDH)	$g, g^a, g^b, g^c \in G_1$	TRUE iif $c = ab$
Computational DH (CDH)	$g, g^a, g^b \in G_1$	g^{ab}
Decisional Co-DH (co-DDH)	$g, g^b \in G_1$ $h, h^a \in G_2$	TRUE iif $a = b$
Computational Co-DH (co-CDH)	$g \in G_1$ $h, h^a \in G_2$	g^a

Bilinear Pairings

Definition 2.3 (Bilinear map).

Let G_T be an additional group s.t. $|G_1| = |G_2| = |G_T|$.
A **bilinear map** *is a map $e : G_1 \times G_2 \to G_T$ s.t.:*

- *Is bilinear:* $\forall u \in G_1$, $\forall v \in G_2$, $\forall a, b \in \mathbb{Z}$,

$$e(u^a, v^b) = e(u, v)^{ab}$$

- *Is non-degenerate:* $e(g_1, g_2) \neq 1$.

Definition 2.4 (Gap problem).

A Gap co-Diffie-Hellman (co-GDH) group pair (G_1, G_2) is s.t. co-DDH is easy but co-CDH is hard. When there is an efficient isomorphism $G_1 \cong G_2$ we say G_1 is a Gap group (GDH).

Secret Sharing

$\mathcal{P} := \{P_1, ..., P_n\}$ set of participants.

Definition 2.5 (Monotone Access Structure).

A **Monotone Acess Structure** Γ *is the set of all subsets of* $\mathcal{P}$ *that can recover the secret, which is monotone increasing.*

$$A \in \Gamma, \quad A \subseteq A' \subseteq \mathcal{P} \Rightarrow A' \in \Gamma$$

Secret Sharing

Definition 2.6.

A **perfect secret sharing scheme**, *with respect to a monotone acess structure* Γ *satisfies:*

- *If a subset* $A \in \Gamma$ *of participants pool their shares, then can recover the secret.*
- *If a subset* $A \notin \Gamma$ *of participants pool their shares, they can determine nothing about the secret.*

Definition 2.7.

An **anonymous secret sharing scheme** *is a secret sharing scheme in which the secret can be reconstructed without the knowledge of which participants hold which shares.*

Shamir Secret Sharing

Shamir (1979). Goal: Share a secret $s \in \mathbb{Z}_p$

- $s \in_R \mathbb{Z}_p$ the secret to be shared among $\mathcal{P}$.
- Set $a_0 = s$ and choose $a_1, \dots, a_{t-1} \in_R \mathbb{Z}_p$ with $a_{t-1} \neq 0$
  Set $P(x) = \sum_{i=0}^{t-1} a_i x^i$ polynomial of degree $t-1$
- Choose $\alpha_1, \dots, \alpha_n \in_R \mathbb{Z}_p^*$ all distinct.
- Each participant $P_i \in \mathcal{P}$ is given the share $(\alpha_i, y_i := P(\alpha_i))$
- The secret can be recovered with at least t shares with polynomial interpolation:

$$s = P(0) \leftarrow \sum_{j=1}^{t} y_{i_j} \prod_{k \in [t] \setminus \{j\}} \frac{-\alpha_{i_k}}{\alpha_{i_j} - \alpha_{i_k}}$$

Digital Signatures

A Digital Signature Scheme consists of 3 algorithms:

- Key Generation:
  - Input: Security parameter.
  - Output: Pair (sk, pk) of secret and public keys.
- Sign:
  - Input: Message m, secret key sk.
  - Output: Signature σ on the message m.
- Verify:
  - Input: Message m, signature σ on m, public key pk.
  - Output: TRUE if the signature is valid. Otherwise FALSE.

BLS Signature Scheme

Boneh, Lynn and Shacham (2001)
Let:

- (G_1, G_2) a bilinear group pair of prime order p
- g a generator of G_1
- $e : G_1 \times G_2 \to G_T$ a bilinear pairing.
- $H : \{0,1\}^* \to G_1$ a full-domain hash function.

Key generation: Choose secret key $x \in_R \mathbb{Z}_p$. Set public key $y := g_2^x$.
Sign: Given message $m \in \{0,1\}^*$, compute $h := H(m) \in G_1$ and then compute the signature $\sigma := h^x \in G_1$
Verify: Given public key y, message m and signature σ, compute $h = H(m)$ and verify that $e(\sigma, g_2) = e(h, y)$.

Group Signatures

Allow a member of the group to anonymously sign a message on behalf of the group Properties:

- Unforgeability
- Anonymity

Optional properties:

- Unlinkability
- Traceability

Threshold Digital Signatures

Definition 2.8.

A (t, n)-**threshold signature scheme** *is a signature scheme in which any set of* t *participants of the group is able to compute a signature on behalf of the group, and any subset of less than* t *participants is unable to compute a valid signature.*

Example of Threshold Signature

Boldyreva (2003)
Setup Algorithm:

- $\mathcal{P} = \{P_i\}$ set of n participants.
- G a Gap group of large prime order $p > n$, and $g \in G$ a generator of the group.
- Choose $sk \in_R \mathbb{Z}_p$ the secret key. Set $pk = g^{sk}$ the public key.
- Set $a_0 = sk$ and choose $a_1, ..., a_{t-1} \in_R \mathbb{Z}_p$ with $a_{t-1} \neq 0$. Set $P(x) = \sum_{i=0}^{t-1} a_i x^i$.
- Choose $\alpha_1, ..., \alpha_n \in_R \mathbb{Z}_p^*$ all distinct.
- Each participant $P_i \in \mathcal{P}$ is given public key $pk_i = \alpha_i$ and secret key $sk_i = P(\alpha_i)$

Sign Algorithm:

- $P = \{P_{i_1}, ... P_{i_t}\}$ set of t participants to sign message m.
- Each P_{i_j} computes partial signature $\sigma_{i_j}(m) = H(m)^{sk_{i_j}}$
- Each P_{i_j} broadcasts the pair $(pk_{i_j}, \sigma_{i_j}(m))$
- The signature σ on m is computed:

$$\sigma(m) = \prod_{P_i \in P} \sigma_i(m)^{\lambda_i^P} = H(m)^{\sum_{P_i \in P} \lambda_i^P sk_i} = H(m)^{sk}$$

where $\lambda_{i_j}^P := \prod_{k \in [t] \setminus \{j\}} \frac{-\alpha_{i_k}}{\alpha_{i_j} - \alpha_{i_k}}$.

Verify Algorithm:

- $e : G \times G \rightarrow G_t$ bilinear pairing.
- σ signature on a message m.
- σ is valid $\Leftrightarrow e(\sigma, g) = e(H(m), pk)$

Anonymity

- Many authors use *anonymity* to refer to the case when the public key of the signer is not disclosed.
- This is not enough for our purposes.

In this work, we are looking for:

- Untraceability: cannot get the public key of the signer from a signature.
- Unlinkability: cannot decide whether two different signatures were signed by the same signer.

Linkable threshold signature schemes are suitable for "one time anonymity".

Single Use: Anonymity

We now consider threshold signature schemes with:

- Non-traceability
- Linkability

We can avoid linkability by newly setting up the scheme after every signature.

Solution with Anonymous Secret Sharing

- Signature scheme with secret key sk.
- Share the secret key among the set of participants using a (t, n)-anonymous threshold secret sharing scheme.
- To compute a signature: a set of t participants recover the secret key and compute the signature.

Solution with Anonymized Threshold BLS Signatures

- Use BLS Threshold Signature Scheme: secret key $sk \in_R \mathbb{Z}_p$ and random polynomial $P(x)$ of degree $t-1$ s.t. $P(0) = sk$
- To reach anonymity with respect to the dealer (who deals the shares) the participants choose random α_i themselves and learn $P(\alpha_i)$ using Oblivious Polynomial Evaluation.

Multiple Use: Anonymity with Non-Linkability

We describe three solutions:

- Constant Size Signature Scheme
- Linkable Group Signature Scheme
- Anonymous Interactive Protocol

Constant Size Anonymous Threshold Signature

Daza et al. (2009)
Setup Algorithm:

- Consider d distinct partitions of the set of participants $\mathcal{P}$ into r parts: $\mathcal{P}^i = \{\mathcal{P}_1^i, ..., \mathcal{P}_r^i\}$.
- For each partition $\mathcal{P}^i, i \in [d]$ set up a (t, r)-Threshold BLS Signature scheme, and give same key pairs to all participants in the same $\mathcal{P}_j^i$

Constant Size Anonymous Threshold Signature

Sign Algorithm

- $\{P_{i_1}, ..., P_{i_t}\}$ set of t participants to sign a message m.
- Signature on m over the i-th signature scheme is attempted. If succeeds, outputs (m, σ, i).
- If signature fails (at least two participants have same secret key), a new signature over a distinct signature scheme is attempted.
- Eventually, the signature will succeed.

Verify Algorithm:

- Signature (m, σ, i).
- Signature valid $\Leftrightarrow e(\sigma, g) = e(H(m), pk_i)$

Linkable Group Signature Scheme

Chen, Ng and Wang (2011)
Setup Algorithm:

- Participant generates a pair (sk, pk) of secret and public keys.
- Issuer gives the participant a credential that certifies the participant's public key as member of the group.

Sign and Verify Algorithms:

- Based on BLS Signtarue Scheme to verify credentials
- Based on Schnorr Signature Scheme to verify the signature

Threshold Checking Algorithm:

- List of ℓ valid signatures on m.
- Verify received signature σ
- Check if already received same signature, or same signer signed twice.
- If not a duplicate, add σ to the list.
- When $\ell = t$, the threshold is reached, and the signature is the collection $\{\sigma_i\}$ of t valid signatures on m.

Anonymous Interactive Protocol

Set a Threshold BLS Signature Scheme

Each participant P_i owns pair $(\alpha_i, P(\alpha_i))$ of public and secret keys

We propose an improvement on the signing algorithm s.t. the public key is not shared and cannot be obtained

Interactive Protocol

- Interaction between P_i, P_j and additional secure party P_s.
- Goal: given $a \in G$, compute $a^{\frac{-\alpha_j}{\alpha_i - \alpha_j}}$ without sharing α_i, α_j.
- We will write: $a^{\frac{-\alpha_j}{\alpha_i - \alpha_j}} \leftarrow \mathcal{B}(a, P_i, P_j)$

Interactive Protocol

P_s

$P_i \xrightarrow{x_i,\, x_j,\, x_s} \text{BC}$

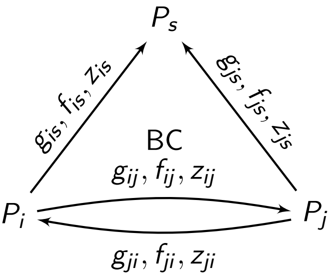
P_j

$$x_i, x_j, x_s \in_R \mathbb{Z}_p^*$$
$$\gamma_{i,0}, \gamma_{i,1}, \gamma_{i,2}, \gamma_{i,3}, \gamma_{i,4} \in_R \mathbb{Z}_p$$
$$g_i(x) \leftarrow \gamma_{i,1} \cdot x + \gamma_{i,0}$$
$$f_i(x) \leftarrow \gamma_{i,2} \cdot x + \alpha_i$$
$$z_i(x) \leftarrow \gamma_{i,4} \cdot x + \gamma_{i,3}$$

$$\gamma_{j,0}, \gamma_{j,1}, \gamma_{j,2}, \gamma_{j,3}, \gamma_{j,4} \in_R \mathbb{Z}_p$$
$$g_j(x) \leftarrow \gamma_{j,1} \cdot x + \gamma_{j,0}$$
$$f_j(x) \leftarrow \gamma_{j,2} \cdot x + \alpha_j$$
$$z_j(x) \leftarrow \gamma_{j,4} \cdot x + \gamma_{j,3}$$

Interactive Protocol

For $k \in \{i, j, s\}$
$g_{ik} \leftarrow g_i(x_k)$
$f_{ik} \leftarrow f_i(x_k)$
$z_{ik} \leftarrow z_i(x_k)$



For $k \in \{i, j, s\}$
$g_{jk} \leftarrow g_j(x_k)$
$f_{jk} \leftarrow f_j(x_k)$
$z_{jk} \leftarrow z_j(x_k)$

Interactive Protocol

$$\boxed{h_s \leftarrow (g_{is} + g_{js})(f_{is} - f_{js}) + z_{is} + z_{js}}$$

$$\begin{tikzcd}
 & P_s \arrow[d, "h_s"] & \\
 & BC & \\
P_i \arrow[ur, "h_i"'] & & P_j \arrow[ul, "h_j"]
\end{tikzcd}$$

$$\boxed{\begin{array}{c} h_i \leftarrow (g_{ii} + g_{ji})(f_{ii} - f_{ji}) + z_{ii} + z_{ji} \\ h \leftarrow \sum_{k\in\{i,j,s\}} h_k \prod_{\ell\neq k} \frac{-x_\ell}{x_k - x_\ell} \end{array}}$$

$$\boxed{\begin{array}{c} h_j \leftarrow (g_{ij} + g_{jj})(f_{ij} - f_{jj}) + z_{ij} + z_{jj} \\ h \leftarrow \sum_{k\in\{i,j,s\}} h_k \prod_{\ell\neq k} \frac{-x_\ell}{x_k - x_\ell} \end{array}}$$

Interactive Protocol

$$\boxed{A_i \leftarrow a^{\frac{1}{h}(g_{ii}+g_{ji})}} \qquad P_i \xrightarrow{\quad A_i \quad} P_j \qquad \boxed{\begin{array}{c} A_j \leftarrow a^{\frac{1}{h}(g_{ij}+g_{jj})} \\ B \leftarrow A_i^{\frac{-x_j}{x_i - x_j}} A_j^{\frac{-x_i}{x_j - x_i}} \end{array}}$$

Interactive Protocol

$$\boxed{\text{Output: } B'} \qquad P_i \xleftarrow{\quad B' \quad} P_j \qquad \boxed{B' \leftarrow B^{-\alpha_j}}$$

Where

$$B' = a^{\frac{-\alpha_j}{\alpha_i - \alpha_j}}$$

Interactive Protocol

Partial Signature:

- Let $P = \{P_i, P_{j_1}, \dots, P_{j_{t-1}}\}$
- Let $a_0 = H(m)^{s_i}$
- For $k \in [t]$ compute

$$a_k \leftarrow \mathcal{B}(a_{k-1}, P_i, P_{j_k})$$

- $\sigma_i(m) = a_t = H(m)^{s_i \prod_{i \in [t-1]} \frac{-\alpha_{j_k}}{\alpha_{j_i} - \alpha_{j_k}}}$

Signature:

$$\sigma(m) = \prod_{P_i \in P} \sigma_i(m)$$

Anonymous Interactive Protocol

Unlinkability:

- Scheme remains unlinkable while participants honest but curious.
- If an adversary corrupts P_j, P_s she can interpolate α_i from f_{ij}, f_{is}
- To allow an adversary to corrupt up to ℓ participants and still be unlinkable we can extend the algorithm:
  - f_k, g_k polynomials of degree ℓ
  - $2\ell + 1$ participants: P_i,P_j, and $2\ell - 1$ secure parties.

Complexity

- P_i to compute σ_i: $t - 1$ interactions.
- To compute σ: $t(t-1)$ interactions.

Conclusions

Constant Size Anonymous Threshold Signature:

- Compact signature.
- Unlinkability determined by the amount of participants in each part of the partitions.
- Not always a group of t participants can compute a signature, but we can control the probability of not succeeding.

Linkable Group Signature Scheme:

- Any set of t participants can compute a signature.
- Threshold is achieved by collecting t unlinked signatures on the same message.
- Length of the signature grows linearly with t.
- Verification complexity is quadratic on t.

Conclusions

Anonymous Interactive Protocol:

- Unlinkable and untraceable whenever an adversary can corrupt at most one participant.
- Compact signature.
- Requires big amount of interactions: quadratic in t.
- Constant signature verification time (independent of t).

Main problem of finding compact, non-interactive, unlinkable anonymous threshold siganture scheme remains open.