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PhD Research Plan

LOCALLY RECOVERABLE CODES APPLICATIONS

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CHAPTER 1

CONTEXT AND MOTIVATION

In recent years the explosion in the volumes of data being stored online has resulted in distributed storage systems transitioning to erasure coding based schemes in order to ensure reliability with low storage overheads. On such a massive scale, unreachable or failed servers are no longer an exception but a regular occurrence and recovery from such events has to be done efficiently.

In recent years Locally Recoverable Codes (LRC) emerged as the codes of choice for many such scenarios and have been implemented in a number of large scale systems ([8], [11]).

Classical erasure correcting codes guarantee that data can be recovered if a bounded number of codeword coordinates is erased. However recovering data typically involves accessing all surviving coordinates. LRC codes have the property that a symbol of the codeword can be recovering accessing few other symbols of the codeword (called the *recovering set*).

Symbols can have more than one recovering set. Having more than one recovering set is beneficial in practice because it enables more users to access a given portion of data, thus enhancing data availability in the system.

Data storage applications require codes with small redundancy, low locality for information coordinates, large distance, and low locality for parity coordinates.

CHAPTER 2

STATE OF THE ART

2.1 Definition of LRC codes

Consider a linear $[n, k, d]_q$ code $\mathcal{C} \subset \mathbb{F}_q^n$, where q is a prime power. We say that the i-th coordinate of $\mathcal{C}$ has locality r, if the value at this coordinate can be recovered from accessing some other r coordinates of $\mathcal{C}$. We say that the code $\mathcal{C}$ has locality r if every symbol of the codeword $x \in \mathcal{C}$ can be recovered from a subset of r other symbols of x.

Definition 2.1 (LRC Codes)**.** A code $\mathcal{C} \subset \mathbb{F}_q^n$ is a *locally recoverable code* (LRC) with locality r if for every $i \in [n]$ there exists a subset $\mathcal{R}_i \subset [n] \setminus \{i\}$, $|\mathcal{R}_i| \leq r$ and a map ϕ_i such that for every codeword $\mathbf{x} \in \mathcal{C}$ we have

$$\mathbf{x}_i = \phi_i(\{\mathbf{x}_j,\ j \in \mathcal{R}_i\}) \tag{2.1}$$

This definition can be also rephrased as follows. Given $a \in \mathbb{F}_q$ consider the sets of codewords

$$\mathcal{C}(i, a) = \{x \in \mathcal{C} : x_i = a\}, \quad i \in [n]$$

The code $\mathcal{C}$ is said to have locality r if for every $i \in [n]$ there exists a subset $\mathcal{R}_i \subset [n] \setminus \{i\}$, $|\mathcal{R}_i| \leq r$ such that the restrictions of the sets $\mathcal{C}(i, a)$ to the coordinates in $\mathcal{R}_i$ for different a are disjoint:

$$\mathcal{C}_{I_i}(i, a) \cap \mathcal{C}_{I_i}(i, a') = \emptyset, \quad a \neq a' \tag{2.2}$$

The subset I_i is called a *recovering set* for the symbol x_i.

Definition 2.2 (t-LRC Codes)**.** A code $\mathcal{C}$ is said to have t disjoint recovering sets if for every $i \in [n]$ there are pairwise disjoint subsets $R_i^1, ..., R_i^t \subset [n] \setminus \{i\}$ such that for all $j = 1, ..., t$ and every pair of symbols $a, a' \in \mathbb{F}_q$, $a \neq a'$

$$\mathcal{C}(i, a)_{R_i^j} \cap \mathcal{C}(i, a')_{R_i^j} = \emptyset \tag{2.3}$$

For linear LRC codes, the relation between a symbol i and its recovering set I_i is linear. Thus, any symbol in $I_i \cup \{i\}$ can be recovered from the remaining symbols. We then call $I_i \cup \{i\}$ a *repair group*.

2.2 Bounds on parameters of LRC codes

Gopalan et al. proved in [4] the following bounds:

Theorem 2.3. *Let $\mathcal{C}$ be an (n, k, r) LRC code. The rate of $\mathcal{C}$ satisfies*

$$\frac{k}{n} \leq \frac{r}{r+1} \tag{2.4}$$

The minimum distance of $\mathcal{C}$ satisfies:

$$d \leq n - k - \left\lceil \frac{k}{r} \right\rceil + 2 \tag{2.5}$$

Theorem 2.4 ([10, 15])**.** *For (n, k, r, t) LRC codes with $t \geq 2$ disjoint recovering sets:*

$$d \leq n - k + 2 - \left\lceil \frac{t(k-1)+1}{t(r-1)+1} \right\rceil \tag{2.6}$$

We will refer to codes attaining the bound 2.5 (the bound 2.6 in case $t \geq 2$) as optimal LRC codes.

In [12], Tamo, Barg, and Frolov find many new bounds on the distance and rate of LRC codes as well as assymptotic bounds.

Theorem 2.5. *Let $\mathcal{C}$ be an (n, k, r, t) LRC code with t disjoint recovering sets of size r. Then the rate of C satisfies*

$$\frac{k}{n} \leq \frac{1}{\prod_{j=1}^{t}(1 + \frac{1}{jr})} \tag{2.7}$$

The minimum distance of C is bounded above as follows:

$$d \leq n - \sum_{i=0}^{t} \left\lfloor \frac{k-1}{r^i} \right\rfloor \tag{2.8}$$

$$R_q(r, \delta) \geq 1 - \min_{0<s\leq 1} \left\{ \frac{1}{r+1} \log_q((1 + (q-1)s)^{r+1} + (q-1)(1-s)^{r+1}) - \delta \log_q s \right\} \tag{2.9}$$

2.3 Algebraic Geometric Codes

Let X be a nonsingular irreducible projective curve over $K = \mathbb{F}_q$ with genus g and let $K(X)$ be the function field of X. For a divisor G on X define the vector space $L(G) := \{f \in K(X) | div(f) + G > 0\} \cup \{0\}$.

Assume $P_1, ..., P_n$ are rational points on the curve X and let $D = P_1 + ... + P_n$. Assume G is a divisor on X with rational points and support disjoint from D. Also assume that $2g - 2 < deg(G) < n$.

Definition 2.6. The linear code $\mathcal{C}(D, G)$ over $\mathbb{F}_q$ is the image of the linear map $\alpha : L(G) \to \mathbb{F}_q^n$ where $\alpha(f) = (f(P_1), ..., f(P_n))$.

Theorem 2.7. *The code $\mathcal{C}(D, G)$ has parameters $[n, k, d]_q$ with*

$$n = deg(D), \quad k = deg(G) - g + 1, \quad d \geq d^* = n - deg(G)$$

CHAPTER 3

OPEN PROBLEMS

[5]

3.1 Constructions of binary LRC codes

[14]

3.2 Constructions of codes on algebraic curves

[3, 2, 6]

Let $\varphi : X \to Y$ be a degree-$(r+1)$ morphism of projective smooth irreducible curves over $K = \mathbb{F}_q$. Let $Q_1, ..., Q_s$ be points of $Y(K)$ s.t. for each Q_i there are $r+1$ points $P_{i,0}, ..., P_{i,r}$ in $X(K)$ that map to Q_i.

The map φ induces an injection of function fields $\varphi^* : K(Y) \hookrightarrow K(X)$ that makes $K(X)$ a degree-$(r+1)$ extension of $K(Y)$. Let $e_1, ..., e_r$ be elements of $K(X)$ that are l.i. over $K(Y)$ and whose sets of poles are disjoint from the $P_{i,j}$, and let $f_1, ..., f_t$ be elements of $K(Y)$ that are l.i. over K and whose sets of poles are disjoint from the Q_i.

Given a vector $\mathbf{a} = (a_{i,j}) \in K^{r \times t}$ we define

$$f_{\mathbf{a}} := \sum_{i=1}^{r} e_i \sum_{j=1}^{t} a_{i,j} \varphi^* f_j$$

Let D be the smallest effective divisor on X so that each product $e_i \varphi^* f_j$ lies in $L(D)$, and let $\delta := deg(D)$. Define the code $\mathcal{C} := \{f_{\mathbf{a}}(P_{i,j}) \mid a \in K^{r \times t}\} \subseteq K^{s \times (r+1)}$. If $\delta < s(r+1)$ then the code $\mathcal{C}$ is linear of dimension $k = rt$, length $n = s(r+1)$, and minimum distance d at least $s(r+1) - \delta$.

3.3 Bounds on codes with locality

The bound 2.8 gives the following assymptotic rate bound for LRC codes:

$$R_q^{(t)}(r,\delta) \leq \frac{r^t(r-1)}{r^{t+1}-1}(1-\delta), \quad 0 \leq \delta \leq 1 \tag{3.1}$$

The lower bound 2.7 appears to be far from tight. Tamo and Barg in [13] said they believe that the rate $\left(\frac{r}{r+1}\right)^t$ is the largest possible for a LRC-t code as long as t is not too large (e.g. $t \in O(\log n)$). This rate can be achieved constructing a t-fold power of the binary $(r+1, r)$ single-parity-check code.

Theorem 2.5 is proved applying probabilistic method techniques on the properties of a graph. The problem of optimizing the bound of the rate of LRC-t codes will be studied, and one of the ways could be following a similar proof considering some restrictions that were not considered in [12].

Also, existence of t-LRC codes with arbitrary t and r seems to be a difficult problem.

3.4 List decoding of LRC codes

A code of length n is called (τ, ℓ)-list decodable if the Hamming sphere of radius τ centered at any vector v of length n always contains at most ℓ codewords $c \in \mathcal{C}$. It was shown by Johnson in [9] that any code of length n and distance d is (τ_J, ℓ)-list decodable where $\tau_J = n - \sqrt{n(n-d)}$ is the Johnson radius and $\ell \in poly(n)$.

It was recently shown by Holzbaur and Wachter-Zeh in [7] that the list decoding radius of certain LRC codes exceed the Johnson radius and give a general list decoding algorithm. The complexity of the algorithm is polynomial in n when the number of repairing groups is constant, otherwise it grows exponentially.

This problem will be studied to research for other families of LRC codes that could be list decoded beyond the Johnson radius.

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