Comparison of discrete fiber and asymptotic homogenization methods for modeling of fiber-reinforced materials deformations

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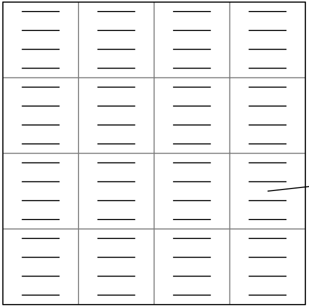
Content

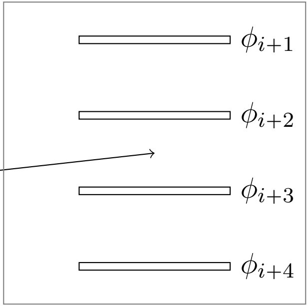
Introduction

- Fiber-reinforced materials are recommended as most of the strongest composite materials
- Numerical modeling of fiber-reinforced materials leads to huge grid size due to fiber size and count
- Discrete fiber method (discerte fracture method): one dimensional fibers
- Asymptotic homogenization method: averaged coarse problem

Model problem



$$\Omega = \Omega_1 \cup \Omega_2$$



$$\Omega_2 = \bigcup_{i=1}^{K} \phi_i$$

- Ω_1 is the main material
- Ω_2 is the fibers

Stress-strain state

$$\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{f}, \quad \boldsymbol{x} \in \Omega,$$

- $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$ is the stress tensor
- $\boldsymbol{C}$ is the elastic tensor
- $\boldsymbol{\varepsilon}$ is the strain tensor
- $\boldsymbol{f}$ is the force source

Voight notation

Stress and elastic tensors

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1112} \\ C_{2211} & C_{2222} & C_{2212} \\ C_{1211} & C_{1222} & C_{1212} \end{pmatrix}$$

Strain tensor

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\,\varepsilon_{12} \end{pmatrix} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \end{pmatrix}.$$

Lame parameters

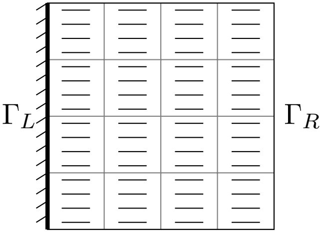
Isotropic materials elastic tensor

$$\boldsymbol{C}_i = \begin{pmatrix} \lambda_i + 2\mu_i & \lambda_i & 0 \\ \lambda_i & \lambda_i + 2\mu_i & 0 \\ 0 & 0 & \mu_i \end{pmatrix}, \quad \boldsymbol{x} \in \Omega_i, \quad i = 1, 2.$$

$$\lambda_i = \frac{E_i \, \nu_i}{(1+\nu_i)(1-2\nu_i)}, \quad \mu_i = \frac{E_i}{2(1+\nu_i)}, \quad \boldsymbol{x} \in \Omega_i, \quad i = 1, 2.$$

- E_i is the Young modulus
- ν_i is the Poisson coefficient

Boundary conditions



Dirichlet condition on the left border

$$\boldsymbol{u} = (0,0), \quad \boldsymbol{x} \in \Gamma_L.$$

Neumann condition on the right border

$$\boldsymbol{\sigma_n} = \boldsymbol{g}, \quad \boldsymbol{x} \in \Gamma_R.$$

$\boldsymbol{\sigma_n} = \boldsymbol{\sigma}\,\boldsymbol{n}$ and $\boldsymbol{n}$ is the normal vector to the border

Finite element approximation

Bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega_1} \boldsymbol{C}_1 \, \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x} + \int_{\Omega_2} \boldsymbol{C}_2 \, \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x},$$

Linear form

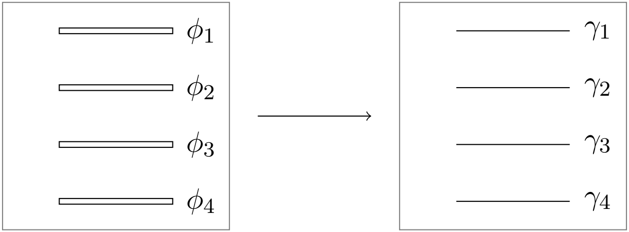
$$L(\boldsymbol{v}) = \int_{\Omega} \boldsymbol{f} \, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_R} \boldsymbol{g} \, \boldsymbol{v} \, \mathrm{d}\boldsymbol{s},$$

Trial and test function spaces

$$V = \widehat{V} = \{\boldsymbol{v} \in H^1(\Omega) : \boldsymbol{v} = (0, 0), \boldsymbol{x} \in \Gamma_L\},$$

H^1 is the Sobolev function space

Discrete fiber approximation



$\Omega_2 = \bigcup_{i=1}^{K} \phi_i$ $\qquad$ $\Gamma_2 = \bigcup_{i=1}^{K} \gamma_i$

Discrete fiber approximation (2D)

Bilinear form

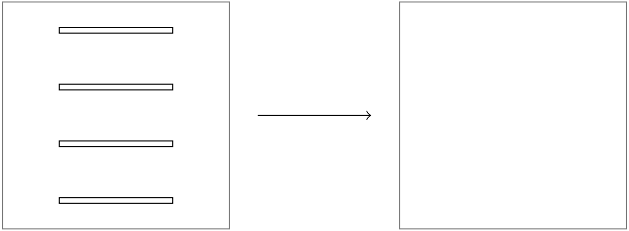
$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{C}_1\,\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} +$$
$$\int_{\Gamma_2} d\,(\lambda_2 + 2\mu_2 - \lambda_1 - 2\mu_1)(\nabla \boldsymbol{u}_{\boldsymbol{\tau}}\,\boldsymbol{\tau})(\nabla \boldsymbol{v}_{\boldsymbol{\tau}}\,\boldsymbol{\tau})\mathrm{d}\boldsymbol{s},$$

Linear form

$$L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_R} \boldsymbol{g}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},$$

- d is thickness of fibers
- $\boldsymbol{u}_{\boldsymbol{\tau}} = \boldsymbol{u}\,\boldsymbol{\tau}$ and $\boldsymbol{\tau}$ is the tangent vector to a fiber line
- Function spaces same as in FEM

Asymptotic homogenization approximation



$\boldsymbol{C}_i, i = 1, 2$ $\qquad\qquad$ $\boldsymbol{C}^*$

$\boldsymbol{C}^*$ is the effective elastic tensor

Asymptotic homogenization approximation

Average of the stress tensor

$$\langle \boldsymbol{\sigma} \rangle = \langle \boldsymbol{C}\, \boldsymbol{\varepsilon} \rangle = \boldsymbol{C}^* \langle \boldsymbol{\varepsilon} \rangle,$$

Average

$$\langle \psi \rangle = \frac{\int_\omega \psi \,\mathrm{d}\boldsymbol{x}}{\int_\omega \mathrm{d}\boldsymbol{x}},$$

ω is a periodic domain.

Periodic problems

$\boldsymbol{u}^k$ is the solution with $\boldsymbol{f}^k, k = 1, 2, 3$

- $\boldsymbol{f}^1 = -\nabla \cdot \boldsymbol{C}\,\boldsymbol{\varepsilon}((x_1, 0))$,
- $\boldsymbol{f}^2 = -\nabla \cdot \boldsymbol{C}\,\boldsymbol{\varepsilon}((0, x_2))$,
- $\boldsymbol{f}^3 = -\nabla \cdot \boldsymbol{C}\,\boldsymbol{\varepsilon}((x_2/2, x_1/2))$.

The effective elastic tensor

- $C^*_{ij11} = \langle \sigma^1_{ij} \rangle, \quad ij = 11, 22, 12,$
- $C^*_{ij22} = \langle \sigma^2_{ij} \rangle, \quad ij = 11, 22, 12,$
- $C^*_{ij12} = \langle \sigma^3_{ij} \rangle, \quad ij = 11, 22, 12.$

$$\boldsymbol{\sigma}^k = \boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{u}^k), \quad k = 1, 2, 3$$

Coarse problem

Bilinear form

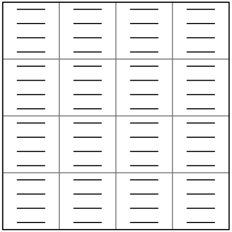
$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{C}^* \, \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{x}$$

Linear form

$$L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \, \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} + \int_{\Gamma_R} \boldsymbol{g} \, \boldsymbol{v} \, \mathrm{d}\boldsymbol{s}.$$

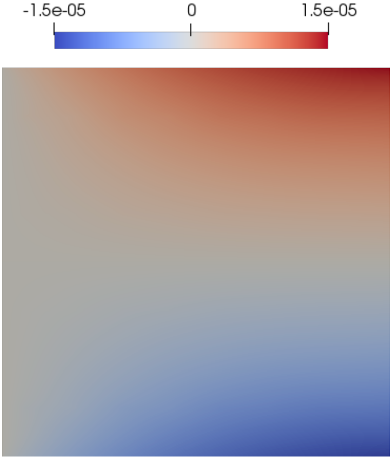
We don't compute a higher order solution of the asymptotic homogenization method

Numerical simulations

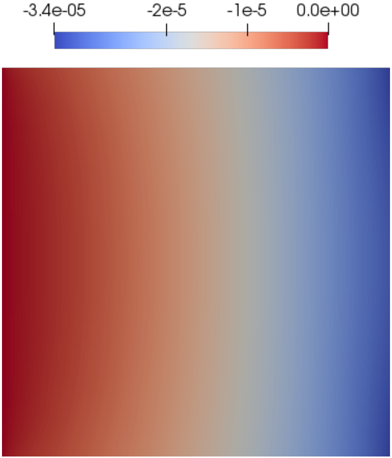


- Ω contains $n \times n$ equal subdomains ω $(n = 4)$
- Each ω contains uniformly distributed $k = K/n^2$ fibers $(k = 4)$
- Fibers size $l \times d$, where $l = 1/2n$, d is the thickness $(l = 1/8)$
- d thickness correlates with grid size
- $\boldsymbol{g} = (0, -10^{-5})$

FEM solution

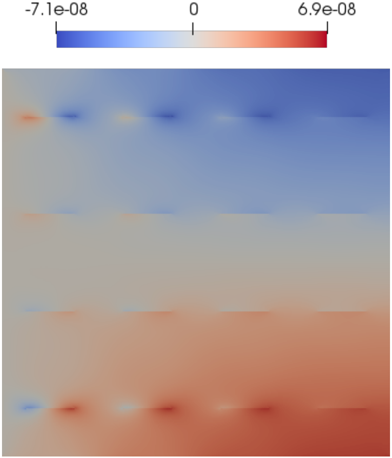


u_1^{fem}

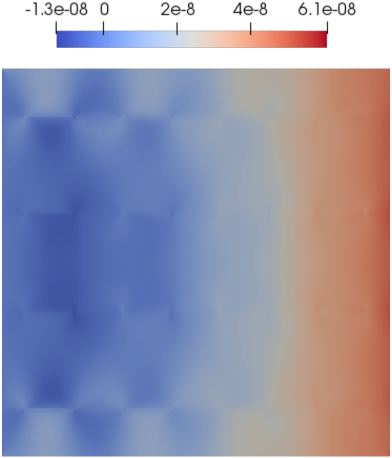


u_2^{fem}

DFM error

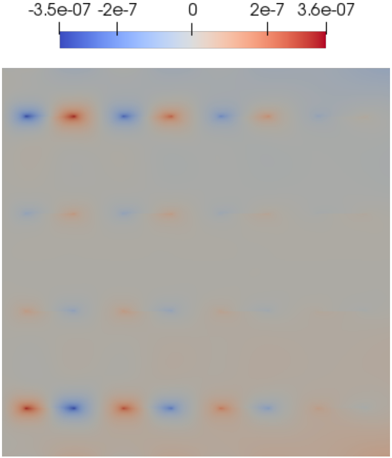


$$u_1^{dfm} - u_1^{fem}$$

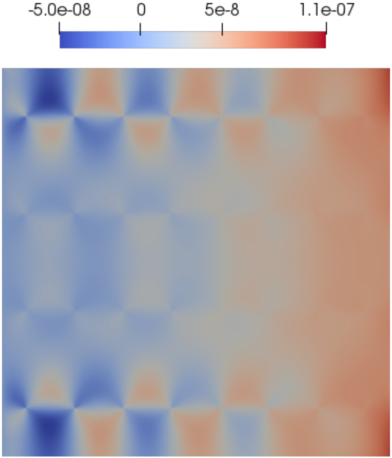


$$u_2^{dfm} - u_2^{fem}$$

AHM error



$u_1^{ahm} - u_1^{fem}$



$u_2^{ahm} - u_2^{fem}$

Relative errors

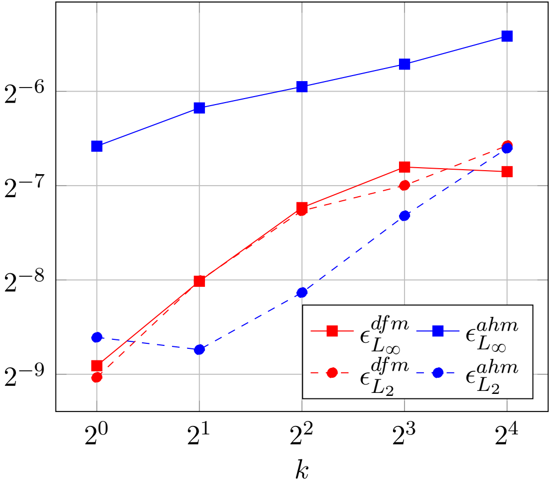
DFM relative error

$$\epsilon_{L_\infty}^{dfm} = \frac{\|\boldsymbol{u}^{dfm} - \boldsymbol{u}^{fem}\|_{L_\infty}}{\|\boldsymbol{u}^{fem}\|_{L_\infty}}, \quad \epsilon_{L_2}^{dfm} = \frac{\|\boldsymbol{u}^{dfm} - \boldsymbol{u}^{fem}\|_{L_2}}{\|\boldsymbol{u}^{fem}\|_{L_2}},$$

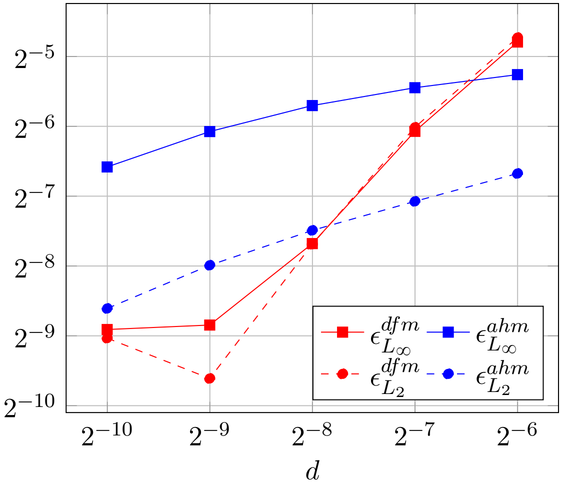
AHM relative error

$$\epsilon_{L_\infty}^{ahm} = \frac{\|\boldsymbol{u}^{ahm} - \boldsymbol{u}^{fem}\|_{L_\infty}}{\|\boldsymbol{u}^{fem}\|_{L_\infty}}, \quad \epsilon_{L_2}^{ahm} = \frac{\|\boldsymbol{u}^{ahm} - \boldsymbol{u}^{fem}\|_{L_2}}{\|\boldsymbol{u}^{fem}\|_{L_2}},$$

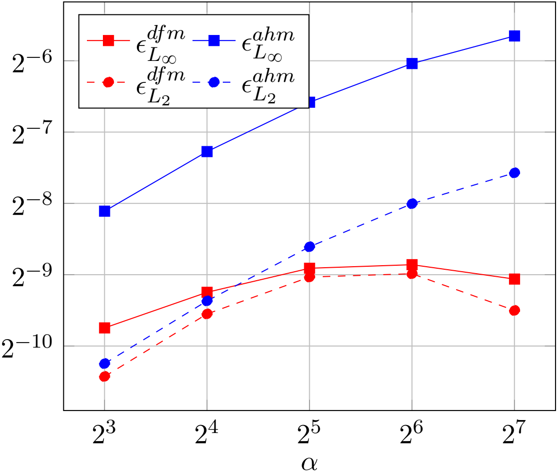
Number of fibers in ω



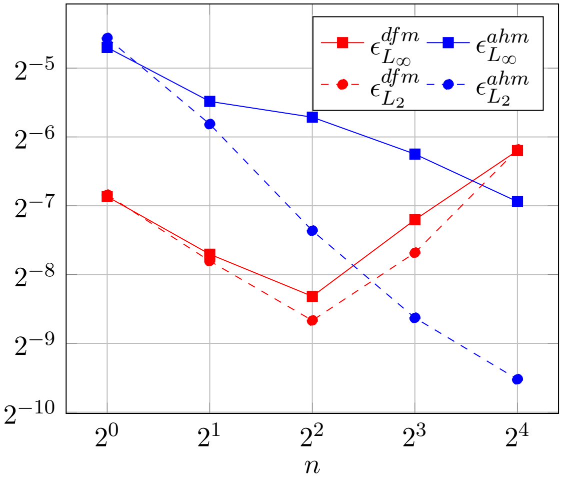
Thickness of fibers



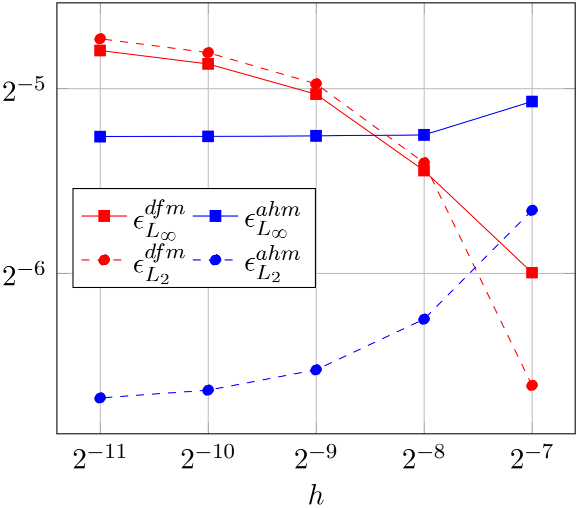
Ratio of Young modulus



Number of subdomains in one direction



Grid step ($d = 1/64, l = 1/8$)



Conclusion

- DFM comparing to AHM showed better accuracy for a large ratio of Young modulus
- DFM is more convenient for thick fibers
- AHM solution is better for a large number of equal subdomains
- Using DFM we can solve on more coarse meshes

Thank you

Thank you for your attention!