Discretisation of Intrinsic Coordinate Dynamic Models

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1 Differential Equations of Motion in Intrinsic Coordinates

Define position, $\mathbf{r}(t)$, velocity, $\mathbf{v}(t)$, and acceleration, $\mathbf{a}(t)$ and a set of unit vectors tangential, normal, and binormal to the motion,

$$
\begin{align}
\mathbf{e}_T(t) &\propto \mathbf{v}(t) \tag{1}\\
\mathbf{e}_B(t) &\propto \mathbf{v}(t) \times \mathbf{a}(t) \tag{2}\\
\mathbf{e}_N(t) &= \mathbf{e}_B(t) \times \mathbf{e}_T(t). \tag{3}
\end{align}
$$

We'll also need the speed,

$$
\dot{s}(t) = |\mathbf{v}(t)|. \tag{4}
$$

Begin by looking at how the unit vectors differentiate,

$$
\begin{align}
\dot{\mathbf{e}}_T(t) &= \dot{\psi}(t)\mathbf{e}_N(t) \tag{5}\\
\dot{\mathbf{e}}_B(t) &= \dot{\phi}(t)\mathbf{e}_N(t) \tag{6}\\
\dot{\mathbf{e}}_N(t) &= -\dot{\psi}(t)\mathbf{e}_T(t) - \dot{\phi}(t)\mathbf{e}_B(t) \tag{7}\\
& \tag{8}
\end{align}
$$

where $\dot{\psi}(t)$ is the instantaneous rate of turn, and $\dot{\phi}(t)$ is the instantaneous rate of roll.

Now we can write the acceleration as,

$$
\begin{align}
\mathbf{a}(t) &= \dot{\mathbf{v}}(t) \nonumber\\
&= \frac{d}{dt}\left(\dot{s}(t)\mathbf{e}_T(t)\right) \nonumber\\
&= \ddot{s}(t)\mathbf{e}_T(t) + \dot{s}(t)\dot{\mathbf{e}}_T(t) \nonumber\\
&= \underbrace{\ddot{s}(t)}_{a_T}\mathbf{e}_T(t) + \underbrace{\dot{s}(t)\dot{\psi}(t)}_{a_N}\mathbf{e}_N(t), \tag{9}
\end{align}
$$

The principle upon which the discretisation is based is that a_T and a_N are constant for the duration of a "manoeuvre",

$$\begin{aligned}\ddot{s}(t) &= a_T \\ \dot{s}(t)\dot{\psi}(t) &= a_N.\end{aligned} \tag{10}$$

Alone, this is not sufficient to render the model analytically solvable, but lets see how far we can get. First the speed,

$$\dot{s}(t) = s_0 + a_T t, \tag{11}$$

where $s_0 = s(0)$ is the fixed intial value. Second the rotation,

$$\begin{aligned}\dot{\psi}(t) &= \frac{a_N}{\dot{s}(t)} \\ &= \frac{a_N}{s_0 + a_T t} \\ \Delta\psi(t) &= \frac{a_N}{a_T} \log\left[\frac{s(t)}{s_0}\right].\end{aligned} \tag{12}$$

The variable $\Delta\psi(t)$ is the total rotation since the start of the manoeuvre. In general, there is no reason for this rotation to lie within a plane, so $\Delta\psi(t)$ is *not* the same as the angle between $\mathbf{e}_T(0)$ and $\mathbf{e}_T(t)$.

Now we group the unit vector differential equations together into a matrix equation,

$$\underbrace{\frac{d}{dt}\begin{bmatrix}\mathbf{e}_T(t)^T \\ \mathbf{e}_N(t)^T \\ \mathbf{e}_B(t)^T\end{bmatrix}}_{\dot{E}(t)} = \underbrace{\begin{bmatrix}0 & \dot{\psi}(t) & 0 \\ -\dot{\psi}(t) & 0 & -\dot{\phi}(t) \\ 0 & \dot{\phi}(t) & 0\end{bmatrix}}_{F(t)}\underbrace{\begin{bmatrix}\mathbf{e}_T^T(t) \\ \mathbf{e}_N^T(t) \\ \mathbf{e}_B^T(t)\end{bmatrix}}_{E(t)}. \tag{13}$$

This can then be solved by comparison with the following matrix exponential identity,

$$\begin{aligned}A(t) &= \exp\left(B(t)\right) \\ \dot{A}(t) &= \dot{B}(t)\exp\left(B(t)\right) \\ \dot{A}(t) &= \dot{B}(t)A(t)\end{aligned} \tag{14}$$

Thus, the solution is,

$$\begin{aligned}E(t) &= \exp\left(\int_0^t F(\tau)d\tau + K\right) \\ &= \underbrace{\exp\left(\int_0^t F(\tau)d\tau\right)}_{A(t)} E(0),\end{aligned} \tag{15}$$

where $E_0 = E(0)$ and the second line follows from the intial conditions. We don't know what it is, but for now define a variable for the change in roll,

$$\Delta\phi(t) = \int_0^t \dot{\phi}(\tau) d\tau \tag{16}$$

The matrix integral can now be written as,

$$\begin{aligned} M(t) &= \int_0^t F(\tau) d\tau \\ &= \begin{bmatrix} 0 & \Delta\psi(t) & 0 \\ -\Delta\psi(t) & 0 & -\Delta\phi(t) \\ 0 & \Delta\phi(t) & 0 \end{bmatrix}. \end{aligned} \tag{17}$$

To summarise,

$$\begin{aligned} E(t) &= A(t)E(t) \\ A(t) &= \exp\left(M(t)\right) \\ M(t) &= \begin{bmatrix} 0 & \Delta\psi(t) & 0 \\ -\Delta\psi(t) & 0 & -\Delta\phi(t) \\ 0 & \Delta\phi(t) & 0 \end{bmatrix} \\ \Delta\psi(t) &= \frac{a_N}{a_T} \log\left[\frac{s(t)}{s_0}\right] \\ \Delta\phi(t) &= \int_0^t \dot{\phi}(\tau) d\tau \\ \dot{s}(t) &= s_0 + a_T t \end{aligned}$$

2 Planar Motion

3 Constant Roll