Discretisation of Intrinsic Coordinate Dynamic Models

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1 Differential Equations of Motion in Intrinsic Coordinates

Define position, $\mathbf{r}(t)$, velocity, $\mathbf{v}(t)$, and acceleration, $\mathbf{a}(t)$ and a set of unit vectors tangential, normal, and binormal to the motion,

$$\mathbf{e}_T(t) \propto \mathbf{v}(t) \tag{1}$$
$$\mathbf{e}_B(t) \propto \mathbf{v}(t) \times \mathbf{a}(t) \tag{2}$$
$$\mathbf{e}_N(t) = \mathbf{e}_B(t) \times \mathbf{e}_T(t). \tag{3}$$

We'll also need the speed,

$$\dot{s}(t) = |\mathbf{v}(t)|. \tag{4}$$

Begin by looking at how the unit vectors differentiate,

$$\dot{\mathbf{e}}_T(t) = \dot{\psi}(t)\mathbf{e}_N(t) \tag{5}$$
$$\dot{\mathbf{e}}_B(t) = \dot{\phi}(t)\mathbf{e}_N(t) \tag{6}$$
$$\dot{\mathbf{e}}_N(t) = -\dot{\psi}(t)\mathbf{e}_T(t) - \dot{\phi}(t)\mathbf{e}_B(t) \tag{7}$$

where $\dot{\psi}(t)$ is the instantaneous rate of turn, and $\dot{\phi}(t)$ is the instantaneous rate of roll.

Now we can write the acceleration as,

$$\begin{aligned}
\mathbf{a}(t) &= \dot{\mathbf{v}}(t) \\
&= \frac{d}{dt}\left(\dot{s}(t)\mathbf{e}_T(t)\right) \\
&= \ddot{s}(t)\mathbf{e}_T(t) + \dot{s}(t)\dot{\mathbf{e}}_T(t) \\
&= \underbrace{\ddot{s}(t)}_{a_T}\mathbf{e}_T(t) + \underbrace{\dot{s}(t)\dot{\psi}(t)}_{a_N}\mathbf{e}_N(t),
\end{aligned} \tag{8}$$

The principle upon which the discretisation is based is that a_T and a_N are constant for the duration of a "manoeuvre",

$$
\begin{aligned}
\ddot{s}(t) &= a_T \\
\dot{s}(t)\dot{\psi}(t) &= a_N .
\end{aligned} \tag{9}
$$

Alone, this is not sufficient to render the model analytically solvable, but lets see how far we can get. First the speed,

$$
\dot{s}(t) = s_0 + a_T t, \tag{10}
$$

where $s_0 = s(0)$ is the fixed intial value. Second the rotation,

$$
\begin{aligned}
\dot{\psi}(t) &= \frac{a_N}{\dot{s}(t)} \\
&= \frac{a_N}{s_0 + a_T t} \\
\Delta\psi(t) &= \int_0^t \dot{\psi}(\tau) d\tau \\
&= \frac{a_N}{a_T} \log \left[\frac{s(t)}{s_0} \right] .
\end{aligned} \tag{11}
$$

The variable $\Delta\psi(t)$ is the total rotation since the start of the manoeuvre. In general, there is no reason for this rotation to lie within a plane, so $\Delta\psi(t)$ is *not* the same as the angle between $\mathbf{e}_T(0)$ and $\mathbf{e}_T(t)$.

Now we group the unit vector differential equations together into a matrix equation,

$$
\underbrace{\frac{d}{dt} \begin{bmatrix} \mathbf{e}_T(t)^T \\ \mathbf{e}_N(t)^T \\ \mathbf{e}_B(t)^T \end{bmatrix}}_{\dot{E}(t)} = \underbrace{\begin{bmatrix} 0 & \dot{\psi}(t) & 0 \\ -\dot{\psi}(t) & 0 & -\dot{\phi}(t) \\ 0 & \dot{\phi}(t) & 0 \end{bmatrix}}_{F(t)} \underbrace{\begin{bmatrix} \mathbf{e}_T^T(t) \\ \mathbf{e}_N^T(t) \\ \mathbf{e}_B^T(t) \end{bmatrix}}_{E(t)} . \tag{12}
$$

In general, this non-homogeneous matrix differential equation has no solution. We can find solutions for particular cases.

2 Planar Motion

The simplest solution results from the assumption $\dot{\phi}(t) = 0$. In this case,

$$
\begin{aligned}
\Delta\phi(t) &= 0 \\
F(t) &= \begin{bmatrix} 0 & \dot{\psi}(t) & 0 \\ -\dot{\psi}(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{13}
$$

For a skew-symmetric matrix with only one non-zero term, $X(t)$, it may be shown that,

$$
\begin{aligned}
Y(t) &= \exp\left(X(t)\right) \\
\dot{Y}(t) &= \dot{X}(t)\exp\left(X(t)\right) \\
&= \dot{X}(t)Y(t).
\end{aligned}
\tag{14}
$$

Note that this is true because $F(t_1)F(t_2) = F(t_2)F(t_1)$, which is also true if $X(t)$ is independent of t, but not in general. Using this identity, the solution is,

$$
\begin{aligned}
E(t) &= \exp\left(\int_0^t F(\tau)d\tau + K\right) \\
&= \underbrace{\exp\left(\int_0^t F(\tau)d\tau\right)}_{A(t)} E_0,
\end{aligned}
\tag{15}
$$

where $E_0 = E(0)$ and the second line follows from the intial conditions. By diagonalisation, the matrix exponential is given by,

$$
A(t) = \begin{bmatrix} \cos(\Delta\psi(t)) & \sin(\Delta\psi(t)) & 0 \\ -\sin(\Delta\psi(t)) & \cos(\Delta\psi(t)) & 0 \\ 0 & 0 & 0 \end{bmatrix}
\tag{16}
$$

This describes motion within the plane spanned by $\mathbf{e}_T(0)$ and $\mathbf{e}_N(0)$. If $\mathbf{e}_B(0) = \mathbf{k}$, then this reduces to the 2D model, in which velocity may be expressed in terms of speed and heading.

The velocity and position are given by,

$$
\mathbf{v}(t) = \dot{s}(t)\left[\cos(\Delta\psi(t))\mathbf{e}_T(0) + \sin(\Delta\psi(t))\mathbf{e}_N(0)\right]
\tag{17}
$$

$$
\begin{aligned}
\mathbf{r}(t) &= \int_0^t \mathbf{v}(\tau)d\tau \\
&= \mathbf{r}(0) + \frac{1}{a_N^2 + 4a_T^2} \\
&\quad \times \left[\left[(2a_T\cos(\Delta\psi(t)) + a_N\sin(\Delta\psi(t)))\,\dot{s}(t)^2 - 2a_T\dot{s}(0)^2\right]\mathbf{e}_T(t)\right. \\
&\quad \left. + \left[(-a_N\cos(\Delta\psi(t)) + 2a_T\sin(\Delta\psi(t)))\,\dot{s}(t)^2 + a_N\dot{s}(0)^2\right]\mathbf{e}_N(t)\right].
\end{aligned}
\tag{18}
$$

3 Constant Roll

I thought it was possible to extend the model by making an assumption about the roll rate, e.g.,

$$
\Delta\phi(t) = r_R t.
\tag{19}
$$

However, this actually renders (12) unsolvable, so we're stuffed.

The most promising approach appears to be transforming $F(t)$, using an eigen-decomposition or the skew-symmetric decomposition,

$$F(t) = U(t)\Sigma(t)U(t)^T \tag{20}$$

$$U(t) = \begin{bmatrix} \frac{\dot{\psi}(t)}{h(t)} & 0 & \frac{-\dot{\phi}(t)}{h(t)} \\ 0 & 1 & 0 \\ \frac{\dot{\phi}(t)}{h(t)} & 0 & \frac{\dot{\psi}(t)}{h(t)} \end{bmatrix} \tag{21}$$

$$\Sigma(t) = \begin{bmatrix} 0 & h(t) & 0 \\ -h(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{22}$$

$$h(t) = \sqrt{\dot{\psi}^2 + \dot{\phi}^2}. \tag{23}$$

4 Initialisation

We need a procedure for specifying the initial normal and binormal unit vectors. This can be done by introducing a new parameter, ϕ_0, to indicate the angle between the binormal vector and the vertical plane containing the velocity vector.

Perpendicular to the tangential vector:	$\mathbf{e}_{T,0} \cdot \mathbf{e}_{B,0} = 0$
Unit magnitude:	$\lvert\mathbf{e}_{B,0}\rvert = 1$
Angle ϕ_0 to the vertical plane:	$\mathbf{e}_{B,0} \cdot \frac{\mathbf{e}_{T,0} \times \mathbf{k}}{\lvert\mathbf{e}_{T,0} \times \mathbf{k}\rvert} = \sin(\phi_0)$

where $\mathbf{k}$ is the z axis unit vector. Solving these, it may be shown that if $\mathbf{e}_{T,0} = [e_{T1,0}, e_{T2,0}, e_{T3,0}]^T$, then the binormal unit vector is given by,

$$\mathbf{e}_{B,k} = \begin{bmatrix} \frac{e_{T2,0}\sin(\phi_0) - e_{T1,0}e_{T3,0}\cos(\phi_0)}{\sqrt{e_{T1,0}^2 + e_{T2,0}^2}} \\ \frac{-e_{T1,0}\sin(\phi_0) - e_{T2,0}e_{T3,0}\cos(\phi_0)}{\sqrt{e_{T1,0}^2 + e_{T2,0}^2}} \\ \cos(\phi_0)\sqrt{e_{T1,0}^2 + e_{T2,0}^2} \end{bmatrix}. \tag{24}$$

Finally, the normal unit vector is given by,

$$\mathbf{e}_{N,0} = \mathbf{e}_{B,0} \times \mathbf{e}_{T,0} \tag{25}$$

5 Gravity

It would be nice to include gravity, but it does not fit easily into the framework. It can be approximated over short time by just adding a gravity term to the velocity and position equations.