Discretisation of Intrinsic Coordinate Dynamic Models

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1 Differential Equations of Motion in Intrinsic Coordinates

Define position, $\mathbf{r}(t)$, velocity, $\mathbf{v}(t)$, and acceleration, $\mathbf{a}(t)$ and a set of unit vectors tangential, normal, and binormal to the motion,

$$\mathbf{e}_T(t) \propto \mathbf{v}(t) \tag{1}$$
$$\mathbf{e}_B(t) \propto \mathbf{v}(t) \times \mathbf{a}(t) \tag{2}$$
$$\mathbf{e}_N(t) = \mathbf{e}_B(t) \times \mathbf{e}_T(t). \tag{3}$$

We'll also need the speed,

$$\dot{s}(t) = |\mathbf{v}(t)|. \tag{4}$$

Begin by looking at how the unit vectors differentiate,

$$\dot{\mathbf{e}}_T(t) = \dot{\psi}(t)\mathbf{e}_N(t) \tag{5}$$
$$\dot{\mathbf{e}}_B(t) = \dot{\phi}(t)\mathbf{e}_N(t) \tag{6}$$
$$\dot{\mathbf{e}}_N(t) = -\dot{\psi}(t)\mathbf{e}_T(t) - \dot{\phi}(t)\mathbf{e}_B(t) \tag{7}$$

where $\dot{\psi}(t)$ is the instantaneous rate of turn, and $\dot{\phi}(t)$ is the instantaneous rate of roll.

Now we can write the acceleration as,

$$\begin{aligned} \mathbf{a}(t) &= \dot{\mathbf{v}}(t) \\ &= \frac{d}{dt}\left(\dot{s}(t)\mathbf{e}_T(t)\right) \\ &= \ddot{s}(t)\mathbf{e}_T(t) + \dot{s}(t)\dot{\mathbf{e}}_T(t) \\ &= \underbrace{\ddot{s}(t)}_{a_T}\mathbf{e}_T(t) + \underbrace{\dot{s}(t)\dot{\psi}(t)}_{a_N}\mathbf{e}_N(t), \end{aligned} \tag{8}$$

The principle upon which the discretisation is based is that a_T and a_N are constant for the duration of a "manoeuvre",

$$\begin{aligned} \ddot{s}(t) &= a_T \\ \dot{s}(t)\dot{\psi}(t) &= a_N. \end{aligned} \tag{9}$$

Alone, this is not sufficient to render the model analytically solvable, but lets see how far we can get. First the speed,

$$\dot{s}(t) = s_0 + a_T t, \tag{10}$$

where $s_0 = s(0)$ is the fixed intial value. Second the rotation,

$$\begin{aligned} \dot{\psi}(t) &= \frac{a_N}{\dot{s}(t)} \\ &= \frac{a_N}{s_0 + a_T t} \\ \Delta\psi(t) &= \int_0^t \dot{\psi}(\tau) d\tau \\ &= \frac{a_N}{a_T} \log\left[\frac{s(t)}{s_0}\right]. \end{aligned} \tag{11}$$

The variable $\Delta\psi(t)$ is the total rotation since the start of the manoeuvre. In general, there is no reason for this rotation to lie within a plane, so $\Delta\psi(t)$ is *not* the same as the angle between $\mathbf{e}_T(0)$ and $\mathbf{e}_T(t)$.

Now we group the unit vector differential equations together into a matrix equation,

$$\underbrace{\frac{d}{dt}\begin{bmatrix} \mathbf{e}_T(t)^T \\ \mathbf{e}_N(t)^T \\ \mathbf{e}_B(t)^T \end{bmatrix}}_{\dot{E}(t)} = \underbrace{\begin{bmatrix} 0 & \dot{\psi}(t) & 0 \\ -\dot{\psi}(t) & 0 & -\dot{\phi}(t) \\ 0 & \dot{\phi}(t) & 0 \end{bmatrix}}_{F(t)} \underbrace{\begin{bmatrix} \mathbf{e}_T^T(t) \\ \mathbf{e}_N^T(t) \\ \mathbf{e}_B^T(t) \end{bmatrix}}_{E(t)}. \tag{12}$$

The roll rate, $\dot{\phi}(t)$ is still not specified by the assumptions we've made so far. In general, this non-homogeneous matrix differential equation has no solution. However, we can find solutions for particular cases.

2 Solutions to Matrix Differential Equations

It is well known that the matrix exponential solves the matrix differential equation (12) when $F(t)$ is constant with respect to time. In fact the class of problems for which this solution holds is larger. Consider the time derivative of the matrix exponential,

$$
\begin{aligned}
X(t) &= \exp\{Q(t)\} = \sum_{n=0}^{\infty} \frac{1}{n!} Q(t)^n \\
\dot{X}(t) &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i=1}^{n} Q(t)^{n-i} \dot{Q}(t) Q(t)^{i-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{n!} \underbrace{\sum_{i=1}^{n} \dot{Q}(t) Q(t)^{n-1}}_{n\dot{Q}(t)Q(t)^{n-1}} \qquad (13) \\
&= \dot{Q}(t) \sum_{m=0}^{\infty} \frac{1}{m!} Q(t)^m \\
&= \dot{Q}(t) \exp\{Q(t)\}.
\end{aligned}
$$

Thus, the solution to (12) follows by equating $\dot{Q}(t)$ and $F(t)$ and using the initial condition,

$$
\begin{aligned}
E(t) &= \exp\left(\int_0^t F(\tau) d\tau + K\right) \\
&= \underbrace{\exp\left(\int_0^t F(\tau) d\tau\right)}_{A(t)} E_0. \qquad (14)
\end{aligned}
$$

In (13), the condition required is that $Q(t)$ should commute with its derivative, i.e. $Q(t)\dot{Q}(t) = \dot{Q}(t)Q(t)$. From this it is straightforward to show that,

$$
\begin{aligned}
Q(t)\dot{Q}(t) &= \dot{Q}(t)Q(t) \\
\int_0^t F(\tau) d\tau F(t) &= F(t) \int_0^t F(\tau) d\tau \\
\int_0^t F(\tau) F(t) d\tau &= \int_0^t F(t) F(\tau) d\tau \\
F(\tau) F(t) &= F(t) F(\tau). \qquad (15)
\end{aligned}
$$

So the matrix exponential solves the matrix differential equation when $F(t)$ commutes with itself with a time offset.

Is this necessary and sufficient? Or just sufficient? Ask a mathematician.

3 Planar Motion

The simplest solution results from the assumption $\dot{\phi}(t) = 0$. In this case,

$$
\begin{aligned}
\Delta\phi(t) &= 0 \\
F(t) &= \begin{bmatrix} 0 & \dot{\psi}(t) & 0 \\ -\dot{\psi}(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (16)
\end{aligned}
$$

Testing the sufficient condition,

$$F(t_1)F(t_2) = F(t_2)F(t_1) = \begin{bmatrix} -\dot{\psi}(t_1)\dot{\psi}(t_2) & 0 & 0 \\ 0 & -\dot{\psi}(t_1)\dot{\psi}(t_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{17}$$

Hence the equation is solved by,

$$E(t) = A(t)E_0 \tag{18}$$

$$A(t) = \exp\left\{\int_0^t F(\tau)d\tau\right\} \tag{19}$$

This may be solved by diagonalisation.

$$F(t) = \begin{bmatrix} \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{a_N i}{s(t)} & 0 & 0 \\ 0 & \frac{a_N i}{s(t)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{20}$$

After some tedious algebra, the matrix exponential is given by,

$$A(t) = \begin{bmatrix} \cos(\Delta\psi(t)) & \sin(\Delta\psi(t)) & 0 \\ -\sin(\Delta\psi(t)) & \cos(\Delta\psi(t)) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{21}$$

This describes motion within the plane spanned by $\mathbf{e}_T(0)$ and $\mathbf{e}_N(0)$. If $\mathbf{e}_B(0) = \mathbf{k}$, then this reduces to the 2D model, in which velocity may be expressed in terms of speed and heading.

The velocity and position are given by,

$$\begin{aligned} \mathbf{v}(t)^T &= \dot{s}(t)\mathbf{e}_T(t)^T \\ &= \dot{s}(t)\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} E(t) && (22) \\ &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \dot{s}(t)A(t)E_0 && (23) \\ \mathbf{r}(t)^T &= \int_0^t \mathbf{v}(\tau)^T d\tau \\ &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \int_0^t \dot{s}(\tau)A(\tau)d\tau E_0 \\ &= \frac{1}{a_N^2 + 4a_T^2} \begin{bmatrix} 2a_T & a_N & 0 \end{bmatrix} \begin{bmatrix} \zeta_1(t) & \zeta_2(t) & 0 \\ \zeta_2(t) & -\zeta_1(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} E_0 \\ \zeta_1(t) &= \cos(\Delta\psi(t))\dot{s}(t) - \dot{s}_0 \\ \zeta_2(t) &= \sin(\Delta\psi(t))\dot{s}(t). \end{aligned}$$

The unpleasant intermediate integration and factorisation used here is,

$$\begin{aligned} \int_0^t \dot{s}(\tau)A(\tau)d\tau &= \begin{bmatrix} \frac{1}{a_N^2+4a_T^2}\left\{2a_T\left[\cos(\Delta\psi(t))\dot{s}(t) - \dot{s}_0\right] + a_N \sin(\Delta\psi(t))\dot{s}(t)\right\} & \frac{1}{a_N^2+4a_T^2}\left\{-a_N\left[\cos(\Delta\psi(t))\dot{s}(t) - \dot{s}_0\right] + 2a_T \sin(\Delta\psi(t))\dot{s}(t)\right\} & 0 \\ \frac{1}{a_N^2+4a_T^2}\left\{a_N\left[\cos(\Delta\psi(t))\dot{s}(t) - \dot{s}_0\right] - 2a_T \sin(\Delta\psi(t))\dot{s}(t)\right\} & \frac{1}{a_N^2+4a_T^2}\left\{2a_T\left[\cos(\Delta\psi(t))\dot{s}(t) - \dot{s}_0\right] + a_N \sin(\Delta\psi(t))\dot{s}(t)\right\} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \frac{1}{a_N^2 + 4a_T^2} \begin{bmatrix} 2a_T & a_N & 0 \\ aN & -2a_T & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\Delta\psi(t))\dot{s}(t) - \dot{s}_0 & \sin(\Delta\psi(t))\dot{s}(t) & 0 \\ \sin(\Delta\psi(t))\dot{s}(t) & -(\cos(\Delta\psi(t))\dot{s}(t) - \dot{s}_0) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

4 Constant Roll Rate

At one point I thought, erroneously, that it was possible to extend the model by assuming the roll rate to be constant,

$$\Delta\phi(t) = r_R t. \tag{24}$$

However, I think this actually renders (12) unsolvable, so we're stuffed.

The most promising approach appears to be transforming $F(t)$, using an eigen-decomposition or the skew-symmetric decomposition,

$$F(t) = U(t)\Sigma(t)U(t)^T \tag{25}$$

$$U(t) = \begin{bmatrix} \frac{\dot\psi(t)}{h(t)} & 0 & \frac{-\dot\phi(t)}{h(t)} \\ 0 & 1 & 0 \\ \frac{\dot\phi(t)}{h(t)} & 0 & \frac{\dot\psi(t)}{h(t)} \end{bmatrix} \tag{26}$$

$$\Sigma(t) = \begin{bmatrix} 0 & h(t) & 0 \\ -h(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{27}$$

$$h(t) = \sqrt{\dot\psi^2 + \dot\phi^2}. \tag{28}$$

5 Proportional Roll Rate

We can get an analytic solution if we assume the following form for the roll rate,

$$\Delta\phi(t) = \frac{a_R}{\dot s(t)}. \tag{29}$$

This allows the matrix in the differential equation to be factorised as,

$$F(t) = \frac{1}{\dot s(t)} \underbrace{\begin{bmatrix} 0 & a_N & 0 \\ -a_N & 0 & -a_R \\ 0 & a_R & 0 \end{bmatrix}}_{\bar F}. \tag{30}$$

This (obviously) satisfies our condition for a matrix exponential solution. Hence the equation is solved by,

$$E(t) = A(t)E_0 \tag{31}$$

$$\begin{aligned} A(t) &= \exp\left\{\int_0^t F(\tau)d\tau\right\} \\ &= \exp\left\{\bar F \int_0^t \frac{1}{\dot s(t)} d\tau\right\} \end{aligned} \tag{32}$$

This matrix exponential may be found analytically using a modified form of the theorem of [1] (section IV). See appendix A.

$$
\begin{aligned}
A(t) &= \exp\left\{\bar{F}\vartheta(t)\right\} \\
&= I + \frac{\bar{F}}{a_C}\sin(a_C\vartheta(t)) - \frac{\bar{F}^2}{a_c^2}\left[\cos(a_C\vartheta(t)) - 1\right] && (33) \\
a_C &= \sqrt{a_N^2 + a_R^2} && (34) \\
\vartheta(t) &= \int_0^t \frac{1}{\dot{s}(t)} d\tau \\
&= \frac{1}{a_T}\log\left[\frac{\dot{s}(t)}{\dot{s}_0}\right] && (35)
\end{aligned}
$$

The velocity and position are given by,

$$
\begin{aligned}
\mathbf{v}(t)^T &= \dot{s}(t)\mathbf{e}_T(t)^T \\
&= \dot{s}(t)\begin{bmatrix}1 & 0 & 0\end{bmatrix} E(t) && (36) \\
&= \begin{bmatrix}1 & 0 & 0\end{bmatrix} \dot{s}(t)A(t)E_0 && (37) \\
\mathbf{r}(t)^T &= \int_0^t \mathbf{v}(\tau)^T d\tau \\
&= \begin{bmatrix}1 & 0 & 0\end{bmatrix} \int_0^t \dot{s}(\tau)A(\tau)d\tau E_0 \\
\int_0^t \dot{s}(\tau)A(\tau)d\tau &= \left[I + \frac{\bar{F}^2}{a_C^2}\right]\left(s_0 t + \frac{1}{2}a_T t^2\right) \\
&\quad + \frac{\bar{F}}{a_C(a_C^2 + 4a_T^2)}\left[2a_T\zeta_2(t) - a_C\zeta_1(t)\right] && (38) \\
&\quad - \frac{\bar{F}^2}{a_C^2(a_C^2 + 4a_T^2)}\left[a_C\zeta_2(t) + 2a_T\zeta_1(t)\right] && (39) \\
\zeta_1(t) &= \cos(a_C\vartheta(t))\dot{s}(t) - \dot{s}_0 \\
\zeta_2(t) &= \sin(a_C\vartheta(t))\dot{s}(t).
\end{aligned}
$$

Check this integration.

Thus, at last, we have everything we need to calculate $\mathbf{r}(t)$ and $\mathbf{v}(t)$ given initial values (and an initial roll — see next section). This gives us an analytic solution for a type of non-planar trajectory, based upon an additional assumption of the form of the roll rate. The question is, whether it is a realistic assumption. It may not be sensible at all.

The planar model may be recovered as a special case by setting $a_R = 0$ throughout.

6 Initialisation

Assuming we know the initial speed and velocity, we still need a procedure for specifying the initial normal and binormal unit vectors. This can be done by introducing a new parameter, ϕ_0, to indicate the initial angle between the binormal vector and the vertical plane containing the velocity vector.

Perpendicular to the tangential vector:	$\mathbf{e}_{T,0} \cdot \mathbf{e}_{B,0} = 0$
Unit magnitude:	$\lvert\mathbf{e}_{B,0}\rvert = 1$
Angle ϕ_0 to the vertical plane:	$\mathbf{e}_{B,0} \cdot \frac{\mathbf{e}_{T,0} \times \mathbf{k}}{\lvert\mathbf{e}_{T,0} \times \mathbf{k}\rvert} = \sin(\phi_0)$

where $\mathbf{k}$ is the z axis unit vector. Solving these, it may be shown that if $\mathbf{e}_{T,0} = [e_{T1,0}, e_{T2,0}, e_{T3,0}]^T$, then the binormal unit vector is given by,

$$\mathbf{e}_{B,k} = \begin{bmatrix} \frac{e_{T2,0}\sin(\phi_0) - e_{T1,0}e_{T3,0}\cos(\phi_0)}{\sqrt{e_{T1,0}^2 + e_{T2,0}^2}} \\ \frac{-e_{T1,0}\sin(\phi_0) - e_{T2,0}e_{T3,0}\cos(\phi_0)}{\sqrt{e_{T1,0}^2 + e_{T2,0}^2}} \\ \cos(\phi_0)\sqrt{e_{T1,0}^2 + e_{T2,0}^2} \end{bmatrix}. \tag{40}$$

Finally, the normal unit vector is given by,

$$\mathbf{e}_{N,0} = \mathbf{e}_{B,0} \times \mathbf{e}_{T,0} \tag{41}$$

7 Gravity

It would be nice to include gravity, but it does not fit easily into the framework. It can be linearly approximated over short time by just adding a gravity term to the velocity and position equations.

A Matrix Exponential of a class of Skew-Symmetric Matrices

Consider a matrix of the form,

$$A = \begin{bmatrix} 0 & a & 0 \\ -a & 0 & -b \\ 0 & b & 0 \end{bmatrix}. \tag{42}$$

It can be shown that for $n > 2$,

$$A^n = -(a^2 + b^2)A^{n-2}. \tag{43}$$

Proof is by induction. In both the following cases,

$$A^{n-2} = \begin{bmatrix} 0 & a_{n-2} & 0 \\ -a_{n-2} & 0 & -b_{n-2} \\ 0 & b_{n-2} & 0 \end{bmatrix}$$
$$A^{n-2} = \begin{bmatrix} c_{n-2} & 0 & d_{n-2} \\ 0 & e_{n-2} & 0 \\ d_{n-2} & 0 & f_{n-2} \end{bmatrix},$$

then by calculation,

$$A^n = A^2 A^{n-2} = -(a^2+b^2)A^{n-2}. \tag{44}$$

The first two powers are given by,

$$A^1 = \begin{bmatrix} 0 & a & 0 \\ -a & 0 & -b \\ 0 & b & 0 \end{bmatrix}$$
$$A^2 = \begin{bmatrix} -a^2 & 0 & -ab \\ 0 & -(a^2+b^2) & 0 \\ -ab & 0 & -b^2 \end{bmatrix}. \tag{45}$$

Thus we can write any power of A as,

$$A^n = \begin{cases} \left[-(a^2+b^2)\right]^{\frac{n-2}{2}} A^2 & n \text{ even} \\ \left[-(a^2+b^2)\right]^{\frac{n-1}{2}} A & n \text{ odd} \end{cases} \tag{46}$$

Now consider the matrix exponential of such a matrix multiplied by a scalar function,

$$\begin{aligned}
&\exp\{Ax(t)\} = \sum_{n=0}^{\infty} \frac{1}{n!}(Ax(t))^n \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} A^n x(t)^n \\
&= I + \sum_{n=1:n \text{ odd}}^{\infty} \frac{1}{n!} \left[-(a^2+b^2)\right]^{\frac{n-1}{2}} Ax(t)^n + \sum_{n=1:n \text{ even}}^{\infty} \frac{1}{n!} \left[-(a^2+b^2)\right]^{\frac{n-2}{2}} A^2 x(t)^n \\
&= I + \frac{A}{\sqrt{a^2+b^2}} \sum_{n=1:n \text{ odd}}^{\infty} (-1)^{\frac{n-1}{2}} \frac{1}{n!} \left[\sqrt{a^2+b^2}x(t)\right]^n - \frac{A^2}{a^2+b^2} \sum_{n=1:n \text{ even}}^{\infty} (-1)^{\frac{n}{2}} \frac{1}{n!} \left[\sqrt{a^2+b^2}x(t)\right]^n \\
&= I + \frac{A}{\sqrt{a^2+b^2}} \sin(\sqrt{a^2+b^2}x(t)) - \left[\cos(\sqrt{a^2+b^2}x(t)) - 1\right] \qquad (47)
\end{aligned}$$

References

[1] D. Bernstein and W. So, “Some explicit formulas for the matrix exponential,” *Automatic Control, IEEE Transactions on*, vol. 38, no. 8, pp. 1228 –1232, aug 1993.