

Discretisation of Intrinsic Coordinate Dynamic Models

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1 Differential Equations of Motion in Intrinsic Coordinates

Define position, $\mathbf{r}(t)$, velocity, $\mathbf{v}(t)$, and acceleration, $\mathbf{a}(t)$ and a set of unit vectors tangential, normal, and binormal to the motion,

$$\mathbf{e}_T(t) \propto \mathbf{v}(t) \tag{1}$$

$$\mathbf{e}_B(t) \propto \mathbf{v}(t) \times \mathbf{a}(t) \tag{2}$$

$$\mathbf{e}_N(t) = \mathbf{e}_B(t) \times \mathbf{e}_T(t). \tag{3}$$

We'll also need the speed,

$$\dot{s}(t) = |\mathbf{v}(t)|. \tag{4}$$

Begin by looking at how the unit vectors differentiate,

$$\dot{\mathbf{e}}_T(t) = \dot{\psi}(t)\mathbf{e}_N(t) \tag{5}$$

$$\dot{\mathbf{e}}_B(t) = \dot{\phi}(t)\mathbf{e}_N(t) \tag{6}$$

$$\dot{\mathbf{e}}_N(t) = -\dot{\psi}(t)\mathbf{e}_T(t) - \dot{\phi}(t)\mathbf{e}_B(t) \tag{7}$$

where $\dot{\psi}(t)$ is the instantaneous rate of turn, and $\dot{\phi}(t)$ is the instantaneous rate of roll.

Now we can write the acceleration as,

$$\begin{aligned} \mathbf{a}(t) &= \dot{\mathbf{v}}(t) \\ &= \frac{d}{dt} (\dot{s}(t)\mathbf{e}_T(t)) \\ &= \ddot{s}(t)\mathbf{e}_T(t) + \dot{s}(t)\dot{\mathbf{e}}_T(t) \\ &= \underbrace{\ddot{s}(t)}_{a_T} \mathbf{e}_T(t) + \underbrace{\dot{s}(t)\dot{\psi}(t)}_{a_N} \mathbf{e}_N(t), \end{aligned} \tag{8}$$

The principle upon which the discretisation is based is that a_T and a_N are constant for the duration of a “manoeuvre”,

$$\begin{aligned}\ddot{s}(t) &= a_T \\ \dot{s}(t)\dot{\psi}(t) &= a_N.\end{aligned}\tag{9}$$

Alone, this is not sufficient to render the model analytically solvable, but lets see how far we can get. First the speed,

$$\dot{s}(t) = s_0 + a_T t,\tag{10}$$

where $s_0 = s(0)$ is the fixed initial value. Second the rotation,

$$\begin{aligned}\dot{\psi}(t) &= \frac{a_N}{\dot{s}(t)} \\ &= \frac{a_N}{s_0 + a_T t} \\ \Delta\psi(t) &= \int_0^t \dot{\psi}(\tau) d\tau \\ &= \frac{a_N}{a_T} \log \left[\frac{s(t)}{s_0} \right].\end{aligned}\tag{11}$$

The variable $\Delta\psi(t)$ is the total rotation since the start of the manoeuvre. In general, there is no reason for this rotation to lie within a plane, so $\Delta\psi(t)$ is *not* the same as the angle between $\mathbf{e}_T(0)$ and $\mathbf{e}_T(t)$.

Now we group the unit vector differential equations together into a matrix equation,

$$\underbrace{\frac{d}{dt} \begin{bmatrix} \mathbf{e}_T(t)^T \\ \mathbf{e}_N(t)^T \\ \mathbf{e}_B(t)^T \end{bmatrix}}_{\dot{E}(t)} = \underbrace{\begin{bmatrix} 0 & \dot{\psi}(t) & 0 \\ -\dot{\psi}(t) & 0 & -\dot{\phi}(t) \\ 0 & \dot{\phi}(t) & 0 \end{bmatrix}}_{F(t)} \underbrace{\begin{bmatrix} \mathbf{e}_T^T(t) \\ \mathbf{e}_N^T(t) \\ \mathbf{e}_B^T(t) \end{bmatrix}}_{E(t)}.\tag{12}$$

In general, this non-homogeneous matrix differential equation has no solution. We can find solutions for particular cases.

2 Planar Motion

The simplest solution results from the assumption $\dot{\phi}(t) = 0$. In this case,

$$\begin{aligned}\Delta\phi(t) &= 0 \\ F(t) &= \begin{bmatrix} 0 & \dot{\psi}(t) & 0 \\ -\dot{\psi}(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}\tag{13}$$

For a skew-symmetric matrix with only one non-zero term, $X(t)$, it may be shown that,

$$\begin{aligned} Y(t) &= \exp(X(t)) \\ \dot{Y}(t) &= \dot{X}(t) \exp(X(t)) \\ &= \dot{X}(t)Y(t). \end{aligned} \tag{14}$$

Note that this is true because $F(t_1)F(t_2) = F(t_2)F(t_1)$, which is also true if $X(t)$ is independent of t , but not in general. Using this identity, the solution is,

$$\begin{aligned} E(t) &= \exp\left(\int_0^t F(\tau)d\tau + K\right) \\ &= \underbrace{\exp\left(\int_0^t F(\tau)d\tau\right)}_{A(t)} E_0, \end{aligned} \tag{15}$$

where $E_0 = E(0)$ and the second line follows from the initial conditions. By diagonalisation, the matrix exponential is given by,

$$A(t) = \begin{bmatrix} \cos(\Delta\psi(t)) & \sin(\Delta\psi(t)) & 0 \\ -\sin(\Delta\psi(t)) & \cos(\Delta\psi(t)) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{16}$$

This describes motion within the plane spanned by $\mathbf{e}_T(0)$ and $\mathbf{e}_N(0)$. If $\mathbf{e}_B(0) = \mathbf{k}$, then this reduces to the 2D model, in which velocity may be expressed in terms of speed and heading.

The velocity and position are given by,

$$\mathbf{v}(t) = \dot{s}(t) [\cos(\Delta\psi(t))\mathbf{e}_T(0) + \sin(\Delta\psi(t))\mathbf{e}_N(0)] \tag{17}$$

$$\begin{aligned} \mathbf{r}(t) &= \int_0^t \mathbf{v}(\tau)d\tau \\ &= \mathbf{r}(0) + \frac{1}{a_N^2 + 4a_T^2} \\ &\quad \times \left[\left[(2a_T \cos(\Delta\psi(t)) + a_N \sin(\Delta\psi(t))) \dot{s}(t)^2 - 2a_T \dot{s}(0)^2 \right] \mathbf{e}_T(t) \right. \\ &\quad \left. + \left[(-a_N \cos(\Delta\psi(t)) + 2a_T \sin(\Delta\psi(t))) \dot{s}(t)^2 + a_N \dot{s}(0)^2 \right] \mathbf{e}_N(t) \right]. \end{aligned} \tag{18}$$

3 Constant Roll

I thought it was possible to extend the model by making an assumption about the roll rate, e.g.,

$$\Delta\phi(t) = r_R t. \tag{19}$$

However, this actually renders () unsolvable, so we're stuffed.

4 Initialisation

We need a procedure for specifying the initial normal and binormal unit vectors. This can be done by introducing a new parameter, ϕ_0 , to indicate the angle between the binormal vector and the vertical plane containing the velocity vector.

$$\begin{aligned}
 &\text{Perpendicular to the} && \mathbf{e}_{T,0} \cdot \mathbf{e}_{B,0} = 0 \\
 &\text{tangential vector:} && \\
 &\text{Unit magnitude:} && |\mathbf{e}_{B,0}| = 1 \\
 &\text{Angle } \phi_0 \text{ to the} && \\
 &\text{vertical plane:} && \mathbf{e}_{B,0} \cdot \frac{\mathbf{e}_{T,0} \times \mathbf{k}}{|\mathbf{e}_{T,0} \times \mathbf{k}|} = \sin(\phi_0)
 \end{aligned}$$

where \mathbf{k} is the z axis unit vector. Solving these, it may be shown that if $\mathbf{e}_{T,0} = [e_{T1,0}, e_{T2,0}, e_{T3,0}]^T$, then the binormal unit vector is given by,

$$\mathbf{e}_{B,k} = \begin{bmatrix} \frac{e_{T2,0} \sin(\phi_0) - e_{T1,0} e_{T3,0} \cos(\phi_0)}{\sqrt{e_{T1,0}^2 + e_{T2,0}^2}} \\ \frac{-e_{T1,0} \sin(\phi_0) - e_{T2,0} e_{T3,0} \cos(\phi_0)}{\sqrt{e_{T1,0}^2 + e_{T2,0}^2}} \\ \cos(\phi_0) \sqrt{e_{T1,0}^2 + e_{T2,0}^2} \end{bmatrix}. \quad (20)$$

Finally, the normal unit vector is given by,

$$\mathbf{e}_{N,0} = \mathbf{e}_{B,0} \times \mathbf{e}_{T,0} \quad (21)$$

5 Gravity

It would be nice to include gravity, but it does not fit easily into the framework. It can be approximated over short time by just adding a gravity term to the velocity and position equations.