Discretisation of Intrinsic Coordinate Dynamic Models

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1 Differential Equations of Motion in Intrinsic Coordinates

Define position, $\mathbf{r}(t)$, velocity, $\mathbf{v}(t)$, and acceleration, $\mathbf{a}(t)$ and a set of unit vectors tangential, normal, and binormal to the motion,

$$\mathbf{e}_T(t) \propto \mathbf{v}(t) \tag{1}$$

$$\mathbf{e}_B(t) \propto \mathbf{v}(t) \times \mathbf{a}(t) \tag{2}$$

$$\mathbf{e}_N(t) = \mathbf{e}_B(t) \times \mathbf{e}_T(t). \tag{3}$$

We'll also need the speed,

$$\dot{s}(t) = |\mathbf{v}(t)|. \tag{4}$$

Begin by looking at how the unit vectors differentiate,

$$\dot{\mathbf{e}}_T(t) = \dot{\psi}(t)\mathbf{e}_N(t) \tag{5}$$

$$\dot{\mathbf{e}}_B(t) = \dot{\phi}(t)\mathbf{e}_N(t) \tag{6}$$

$$\dot{\mathbf{e}}_N(t) = -\dot{\psi}(t)\mathbf{e}_T(t) - \dot{\phi}(t)\mathbf{e}_B(t) \tag{7}$$

where $\dot{\psi}(t)$ is the instantaneous rate of turn, and $\dot{\phi}(t)$ is the instantaneous rate of roll.

Now we can write the acceleration as,

$$\begin{aligned} \mathbf{a}(t) &= \dot{\mathbf{v}}(t) \\ &= \frac{d}{dt}\left(\dot{s}(t)\mathbf{e}_T(t)\right) \\ &= \ddot{s}(t)\mathbf{e}_T(t) + \dot{s}(t)\dot{\mathbf{e}}_T(t) \\ &= \underbrace{\ddot{s}(t)}_{a_T}\mathbf{e}_T(t) + \underbrace{\dot{s}(t)\dot{\psi}(t)}_{a_N}\mathbf{e}_N(t), \end{aligned} \tag{8}$$

The principle upon which the discretisation is based is that a_T and a_N are constant for the duration of a “manoeuvre”,

$$\begin{aligned}\ddot{s}(t) &= a_T \\ \dot{s}(t)\dot{\psi}(t) &= a_N.\end{aligned} \tag{9}$$

Alone, this is not sufficient to render the model analytically solvable, but lets see how far we can get. First the speed,

$$\dot{s}(t) = s_0 + a_T t, \tag{10}$$

where $s_0 = s(0)$ is the fixed intial value. Second the rotation,

$$\begin{aligned}\dot{\psi}(t) &= \frac{a_N}{\dot{s}(t)} \\ &= \frac{a_N}{s_0 + a_T t} \\ \Delta\psi(t) &= \int_0^t \dot{\psi}(\tau)d\tau \\ &= \frac{a_N}{a_T}\log\left[\frac{s(t)}{s_0}\right].\end{aligned} \tag{11}$$

The variable $\Delta\psi(t)$ is the total rotation since the start of the manoeuvre. In general, there is no reason for this rotation to lie within a plane, so $\Delta\psi(t)$ is *not* the same as the angle between $\mathbf{e}_T(0)$ and $\mathbf{e}_T(t)$.

Now we group the unit vector differential equations together into a matrix equation,

$$\underbrace{\frac{d}{dt}\begin{bmatrix}\mathbf{e}_T(t)^T \\ \mathbf{e}_N(t)^T \\ \mathbf{e}_B(t)^T\end{bmatrix}}_{\dot{E}(t)} = \underbrace{\begin{bmatrix}0 & \dot{\psi}(t) & 0 \\ -\dot{\psi}(t) & 0 & -\dot{\phi}(t) \\ 0 & \dot{\phi}(t) & 0\end{bmatrix}}_{F(t)}\underbrace{\begin{bmatrix}\mathbf{e}_T^T(t) \\ \mathbf{e}_N^T(t) \\ \mathbf{e}_B^T(t)\end{bmatrix}}_{E(t)}. \tag{12}$$

This can then be solved by comparison with the following matrix exponential identity,

$$\begin{aligned}A(t) &= \exp\left(B(t)\right) \\ \dot{A}(t) &= \dot{B}(t)\exp\left(B(t)\right) \\ &= \dot{B}(t)A(t).\end{aligned} \tag{13}$$

Thus, the solution is,

$$\begin{aligned}E(t) &= \exp\left(\int_0^t F(\tau)d\tau + K\right) \\ &= \underbrace{\exp\left(\int_0^t F(\tau)d\tau\right)}_{A(t)}E(0),\end{aligned} \tag{14}$$

where $E_0 = E(0)$ and the second line follows from the intial conditions. We don't know what it is, but for now define a variable for the change in roll,

$$\Delta\phi(t) = \int_0^t \dot{\phi}(\tau)d\tau \tag{15}$$

The matrix integral can now be written as,

$$\begin{aligned} M(t) &= \int_0^t F(\tau)d\tau \\ &= \begin{bmatrix} 0 & \Delta\psi(t) & 0 \\ -\Delta\psi(t) & 0 & -\Delta\phi(t) \\ 0 & \Delta\phi(t) & 0 \end{bmatrix}. \end{aligned} \tag{16}$$

To summarise,

$$\begin{aligned} E(t) &= A(t)E(t) \\ A(t) &= \exp\left(M(t)\right) \\ M(t) &= \begin{bmatrix} 0 & \Delta\psi(t) & 0 \\ -\Delta\psi(t) & 0 & -\Delta\phi(t) \\ 0 & \Delta\phi(t) & 0 \end{bmatrix} \\ \Delta\psi(t) &= \frac{a_N}{a_T} \log\left[\frac{s(t)}{s_0}\right] \\ \Delta\phi(t) &= \int_0^t \dot{\phi}(\tau)d\tau \\ \dot{s}(t) &= s_0 + a_T t. \end{aligned}$$

To complete the picture, we need to make an assumption which defines $\dot{\phi}(t)$. The velocity and position can then be recovered using,

$$\mathbf{v}(t) = \dot{s}(t)\mathbf{e}_T(t) \tag{17}$$

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(\tau)d\tau. \tag{18}$$

2 Planar Motion

The simplest solution results from the assumption $\dot{\phi}(t) = 0$. In this case,

$$\begin{aligned} \Delta\phi(t) &= 0 \\ A(t) &= \begin{bmatrix} \cos(\Delta\psi(t)) & \sin(\Delta\psi(t)) & 0 \\ -\sin(\Delta\psi(t)) & \cos(\Delta\psi(t)) & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{19}$$

This describes motion within the plane spanned by $\mathbf{e}_T(0)$ and $\mathbf{e}_N(0)$. If $\mathbf{e}_B(0) = \mathbf{k}$, then this reduces to the 2D model, in which velocity may be expressed in terms of speed and heading.

The velocity and position are given by,

$$\mathbf{v}(t) = \dot{s}(t)\left[\cos(\Delta\psi(t))\mathbf{e}_T(0) + \sin(\Delta\psi(t))\mathbf{e}_N(0)\right] \tag{20}$$

$$\begin{aligned}
\mathbf{r}(t) &= \int_0^t \mathbf{v}(\tau)d\tau \\
&= \mathbf{r}(0) + \frac{1}{a_N^2 + 4a_T^2} \\
&\quad \times \left[\left[(2a_T\cos(\Delta\psi(t)) + a_N\sin(\Delta\psi(t)))\,\dot{s}(t)^2 - 2a_T\dot{s}(0)^2\right]\mathbf{e}_T(t)\right. \\
&\quad \left. + \left[(-a_N\cos(\Delta\psi(t)) + 2a_T\sin(\Delta\psi(t)))\,\dot{s}(t)^2 + a_N\dot{s}(0)^2\right]\mathbf{e}_N(t)\right]. \end{aligned} \tag{21}$$

3 Constant Roll

Things get more complex as soon as we allow a non zero roll rate. Say we introduce another parameter r_R, such that,

$$\Delta\phi(t) = r_R t. \tag{22}$$

Now the expression for $A(t)$ becomes pretty horrendous! Analytic equations exist for $\mathbf{v}(t)$, but they are not integrable, so $\mathbf{r}(t)$ cannot be calculated.

We can make some headway if we impose the condition that either a_T, a_N or r_R should be 0 at all times. The last case has already been addressed in the planar case. For $a_T = 0$, we get some thoroughly horrible, although ultimately tractable equations. For $a_N = 0$, we get very simple equations.

4 Initialisation

We need a procedure for specifying the initial normal and binormal unit vectors. This can be done by introducing a new parameter, ϕ_0, to indicate the angle between the binormal vector and the vertical plane containing the velocity vector.

Perpendicular to the tangential vector:	$\mathbf{e}_{T,0} \cdot \mathbf{e}_{B,0} = 0$
Unit magnitude:	$\lvert\mathbf{e}_{B,0}\rvert = 1$
Angle ϕ_0 to the vertical plane:	$\mathbf{e}_{B,0} \cdot \frac{\mathbf{e}_{T,0}\times\mathbf{k}}{\lvert\mathbf{e}_{T,0}\times\mathbf{k}\rvert} = \sin(\phi_0)$

where $\mathbf{k}$ is the z axis unit vector. Solving these, it may be shown that if $\mathbf{e}_{T,0} = [e_{T1,0}, e_{T2,0}, e_{T3,0}]^T$, then the binormal unit vector is given by,

$$\mathbf{e}_{B,k} = \begin{bmatrix} \frac{e_{T2,0}\sin(\phi_0) - e_{T1,0}e_{T3,0}\cos(\phi_0)}{\sqrt{e_{T1,0}^2 + e_{T2,0}^2}} \\ \frac{-e_{T1,0}\sin(\phi_0) - e_{T2,0}e_{T3,0}\cos(\phi_0)}{\sqrt{e_{T1,0}^2 + e_{T2,0}^2}} \\ \cos(\phi_0)\sqrt{e_{T1,0}^2 + e_{T2,0}^2} \end{bmatrix}. \tag{23}$$

Finally, the normal unit vector is given by,

$$\mathbf{e}_{N,0} = \mathbf{e}_{B,0} \times \mathbf{e}_{T,0} \tag{24}$$

5 Gravity

It would be nice to include gravity, but it does not fit easily into the framework. It can be approximated over short time by just adding a gravity term to the velocity and position equations.