

# The Rao-Blackwellised Variable Rate Particle Smoother for Conditionally Linear-Gaussian Models

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**Abstract**—The abstract goes here.

**Index Terms**—

## I. INTRODUCTION

**I**N model-based schemes for probabilistic inference, some unknown quantity is treated as a random process which evolves over time according to a dynamic model. This latent state is observed at a discrete set of times via another random process modelling the measurement mechanism. Using the two models and applying Bayes' rule, inference of the hidden state can be made from the observations.

For simple models with linear dynamics and Gaussian-distributed random variables, optimal analytic inference algorithms exist, including the Kalman filter [1] and Rauch-Tung-Striebel (RTS) smoother [2]. For nonlinear, non-Gaussian models, such analytic solutions do not exist, and it is often necessary to employ numerical approximations, including the particle filter [3] and particle smoother [4], [5] (see [6], [7] for a thorough introduction to particle methods).

Commonly, the unknown quantity under consideration is continually varying – e.g. the position of a moving object – but is modelled as a discrete-time random process synchronous with the observations. This leads to the standard discrete-time hidden Markov model. Such “fixed rate” models are poorly suited to quantities with occasional discontinuities in their evolution; for example, the price of a financial asset which may display large jumps at random times between periods of diffusion-like behaviour, or the kinematic state of a manoeuvring vehicle which may have sudden jumps in the acceleration when turns begin or end. In such cases, a “variable rate” model may be more appropriate, in which the state evolution is dependent upon a set of unknown changepoints.

In a variable rate model, the set of changepoints and associated parameters are modelled as a marked point process (MPP) [8], conditional upon which the state evolves according to some benign dynamics. In [9], [10], the conditional state evolution is modelled as deterministic, while in [11], [12] a conditionally linear-Gaussian state model is considered.

The posterior distribution of the changepoint sequence is analytically intractable, but may be estimated using a particle filter. When the conditional state dynamics are linear-Gaussian, the efficiency of this filter may be greatly improved by using

the method of Rao-Blackwellisation (see, e.g. [13], [14]). Rather than targeting the entire posterior distribution over changepoints and states, the particle filter is used to estimate only the nonlinear changepoint sequence, after which Kalman filtering and smoothing may be used to infer the linear states. This is the Rao-Blackwellised variable rate particle filter (RBVRPF) of [11], [12].

In this paper, a new algorithm is described for use with conditionally linear-Gaussian variable rate models for estimation of the smoothing distribution, i.e. the distribution over the sequence of changepoints and linear states given all the observations. This uses a similar derivation to that of the fixed rate Rao-Blackwellised particle smoother of [15]. The new algorithm is called the Rao-Blackwellised variable rate particle smoother (RBVRPS).

We review the structure of conditionally linear-Gaussian variable rate models in section II and revise the RBVRPF in section III. The new smoothing algorithm is presented in section IV with supporting simulations in section V.

## II. CONDITIONALLY LINEAR-GAUSSIAN VARIABLE RATE MODELS

We consider a general model from time 0 to  $T$ , between which observations,  $\{y_1 \dots y_N\}$ , are made at times  $\{t_1 \dots t_N\}$ . The linear state at these times is written as  $\{x_1 \dots x_N\}$ . During this period, an unknown number of changepoints,  $K$ , occur at times  $\{\tau_1 \dots \tau_K\}$ , each with associated changepoint parameters,  $\{u_1 \dots u_K\}$ . These are the marks of the MPP. Discrete sets containing multiple values over time will be written as, e.g.  $y_{1:n} = \{y_1 \dots y_n\}$ .

The objective for particle approximations will be to estimate the sequence of changepoints. This will be denoted as  $\theta = \{K, \tau_{1:K}, u_{1:K}\}$ . In addition, at a particular time  $t_n$ , the sequence will be divided into past  $\theta_n^- = \{K_n^-, \tau_{n,1:K_n^-}, u_{n,1:K_n^-}\}$ , and future  $\theta_n^+ = \{K_n^+, \tau_{n,1:K_n^+}, u_{n,1:K_n^+}\}$ , where  $K_n^-$  is the number of changepoints before  $t_n$ , at times  $\tau_{n,1:K_n^-}$  with parameters  $u_{n,1:K_n^-}$ , and similarly for the future.

Note that  $\theta_n^-$  not only conveys information about where changepoints occur, but also where they do not. i.e. It is implicit that  $\tau_{n,K_n^-+1}^- > t_n$ .

Assuming a Markovian changepoint sequence, a conditionally linear-Gaussian variable rate model can now be expressed by the following system equations:

$$\{\tau_k, u_k\} \sim p(u_k|\tau_k, \tau_{k-1}, u_{k-1})p(\tau_k|\tau_{k-1}) \quad (1)$$

$$x_n = A_n(\theta_n^-)x_{n-1} + w_n \quad (2)$$

$$y_n = C_n(\theta_n^-)x_n + v_n. \quad (3)$$

The random variables  $w_n$  and  $v_n$  have a zero-mean Gaussian distribution with covariance matrices  $Q_n$  and  $R_n$  respectively. The changepoint density will be constructed such that  $P(\tau_k < \tau_{k-1}) = 0$ . The factorisation of (1) may be relaxed, but is assumed throughout as it simplifies the subsequent derivations.

It will be useful to define the survivor function in the same manner as [10].

$$\begin{aligned} S(\tau_k, t) &= P(\tau_{k+1} > t | \tau_k) \\ &= 1 - \int_{\tau_k}^t p(\xi | \tau_k) d\xi \end{aligned} \quad (4)$$

The tasks of filtering and smoothing may now be considered as estimation of the posterior distributions of  $\theta$  and  $x_{1:N}$ .

### III. THE RAO-BLACKWELLISED VARIABLE RATE PARTICLE FILTER

The Rao-Blackwellised Variable Rate Particle Filter (RBVRPF) was first described in [11], and in [12] it was developed for use in a financial prediction algorithm. The objective of the algorithm is to sequentially estimate the distribution of the changepoint sequence,  $p(\theta_n^- | y_{1:n})$ , at each time  $t_n$ . The linear state filtering distribution,  $p(x_n | \theta_n^-, y_{1:n})$ , can then be estimated by a Kalman filter [1], [16].

The posterior changepoint distribution may be expanded using Bayes' rule.

$$p(\theta_n^- | y_{1:n}) \propto p(y_n | \theta_n^-, y_{1:n-1})p(\theta_n^- | y_{1:n-1}) \quad (5)$$

This distribution cannot be calculated analytically. Instead, a particle filter can be used to approximate the distribution numerically.

A particle filter is an algorithm used to approximate a filtering distribution with a discrete set of weighted samples drawn from that distribution using sequential importance sampling. In this case, each particle will consist of a changepoint sequence between 0 and  $t$ .

$$\hat{p}(\theta_n^- | y_{1:n}) = \sum_j w_n^{(j)} \delta_{\theta_n^{-(j)}}(\theta_n^-) \quad (6)$$

where  $\delta_x(X)$  is a dirac probability mass at the point  $X = x$ .

The particle filter is a recursive algorithm. At the  $n$ th step, a particle,  $\theta_n^{-(i)}$ , is first resampled from those approximating the filtering distribution at the  $(n-1)$ th step, using an appropriately chosen set of proposal weights.

$$q(\theta_{n-1}^- | y_{1:n-1}) = \sum_j v_{n-1}^{(j)} \delta_{\theta_{n-1}^{-(j)}}(\theta_{n-1}^-) \quad (7)$$

The particle is then extended from  $\theta_{n-1}^-$  to  $\theta_n^-$  by sampling from a proposal,  $q(\theta_n^- | \theta_{n-1}^- | \theta_{n-1}^-)$ . The sequence extension,

$\theta_n^- \setminus \theta_{n-1}^-$ , is the set of changepoints which occur between times  $t_{n-1}$  and  $t_n$ , which may be empty or may contain a number of new changepoints. Finally, the particle is weighted according to the ratio of the target and proposal densities.

$$\begin{aligned} w_n^{(i)} &= \frac{p(\theta_n^{-(i)} | y_{1:n})}{q(\theta_{n-1}^- | y_{1:n-1})q(\theta_n^- \setminus \theta_{n-1}^- | \theta_{n-1}^-)} \\ &\propto \frac{p(y_n | \theta_n^-, y_{1:n-1})p(\theta_n^- \setminus \theta_{n-1}^- | \theta_{n-1}^-)p(\theta_{n-1}^- | y_{1:n-1})}{q(\theta_{n-1}^- | y_{1:n-1})q(\theta_n^- \setminus \theta_{n-1}^- | \theta_{n-1}^-)} \\ &= \frac{w_{n-1}^{(i)}}{v_{n-1}^{(i)}} \times \frac{p(y_n | \theta_n^-, y_{1:n-1})p(\theta_n^- \setminus \theta_{n-1}^- | \theta_{n-1}^-)}{q(\theta_n^- \setminus \theta_{n-1}^- | \theta_{n-1}^-)} \end{aligned} \quad (8)$$

The normalisation may be enforced by scaling the weights so that they sum to 1. This is the RBVRPF of [11], [12].

Now consider terms of (8) in turn. First,  $p(y_n | \theta_n^-, y_{1:n-1})$  is the predictive likelihood estimated by the Kalman filter, which will be a Gaussian distribution.

$$p(y_n | \theta_n^-, y_{1:n-1}) = \mathcal{N}(y_n | \mu_n, S_n) \quad (9)$$

The mean and variance are given by the following standard recursions (dependence on  $\theta_n^-$  suppressed for clarity).

$$m_n^- = A_n m_{n-1}^- \quad (10)$$

$$P_n^- = A_n P_{n-1}^- A_n^T + Q_n \quad (11)$$

$$\mu_n = C_n m_n^- \quad (12)$$

$$S_n = C_n P_n^- C_n^T + R_n \quad (13)$$

$$K_n = P_n^- C_n^T S_n^{-1} \quad (14)$$

$$m_n = m_n^- + K_n(y_n - \mu_n) \quad (15)$$

$$P_n = P_n^- - K_n S_n K_n^T \quad (16)$$

Next consider the transition term  $p(\theta_n^- \setminus \theta_{n-1}^- | \theta_{n-1}^-)$ . Properly, any number of new changepoints could occur in the interval  $[t_{n-1}, t_n]$ . However, changepoints will generally be relatively rare events, so the probability of there being more than one between a pair of consecutive observations will be very low. Furthermore, if multiple changepoints did occur between observations, it would be difficult to discriminate them. Therefore, it is assumed throughout that a maximum of one changepoint can occur between  $t_{n-1}$  and  $t_n$ .

The sequence extension,  $\theta_n^- \setminus \theta_{n-1}^-$ , is thus either an empty set or contains one new changepoint. If  $k$  changepoints have already occurred before  $t_{n-1}$ , then the probability of no new changepoint is given by:

$$\begin{aligned} P(\tau_{k+1} > t_n | \tau_k, \tau_{k+1} > t_{n-1}) &= \frac{P(\tau_{k+1} > t_n | \tau_k)}{P(\tau_{k+1} > t_{n-1} | \tau_k)} \\ &= \frac{S(\tau_k, t_n)}{S(\tau_k, t_{n-1})} \end{aligned} \quad (17)$$

The probability of a changepoint occurring is then the complement of this, on account of the assumption of not more than one changepoint. If a changepoint does occur, then it will have the following conditional density:

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1: Set initial particle sufficient statistics,  $\{m_0^{(i)}\}$  and  $\{P_0^{(i)}\}$ 
   according to prior.
2: Initialise particles with  $K_0^- = 0$ .
3: for  $n = 1 \dots N$  do
4:   for  $i = 1 \dots N_F$  do
5:     Sample a changepoint history  $\theta_{n-1}^{-(i)} \sim$ 
        $\sum_j v_{n-1}^{(j)} \delta_{\theta_{n-1}^{-(j)}}(\theta_{n-1}^{-(i)})$ .
6:     Propose  $\theta_n^{-(i)} \setminus \theta_{n-1}^{-(i)} \sim q(\theta_n^- \setminus \theta_{n-1}^- | \theta_{n-1}^{-(i)})$ 
7:     Calculate  $\mu_n^{(i)}$  and  $S_n^{(i)}$  using (10) to (13).
8:     Update state mean and covariance  $m_n^{(i)}$  and  $P_n^{(i)}$ 
       using (14) to (16).
9:     Calculate weight  $w_n^{(i)}$  using (8).
10:   end for
11:   Scale weights such that  $\sum_i w_n^{(i)} = 1$ 
12: end for

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Fig. 1. Rao-Blackwellised Variable Rate Particle Filter

$$p(u_{k+1} | \tau_{k+1}, \tau_k, u_k) p(\tau_{k+1} | \tau_k, t_{n-1} < \tau_{k+1} < t_n) \propto p(u_{k+1} | \tau_{k+1}, \tau_k, u_k) p(\tau_{k+1} | \tau_k) \quad (18)$$

For the most basic “bootstrap” [3] form of RBVRPF,  $\theta_n^- \setminus \theta_{n-1}^-$  may be proposed from the prior by sampling first (17) followed by (18). This leads to the usual simplification of the weight formula.

The choice of proposal weights,  $\{v_{n-1}^{(i)}\}$ , requires particular attention in the design of RBVRPFs. In some models a changepoint may not have an immediate effect on the observations, especially if a jump occurs in some quantity which is only observed via its integral, e.g. if there is a jump in the acceleration of a moving object, yet only the position is measured, the change will not be apparent until several more observations have been made. In the meantime, particles which contain a changepoint at the correct time may all have been removed by the resampling process. A particle pays a debt in transition probability when a changepoint is proposed, and does not see it repaid in likelihood until later. To avoid this loss of good particles, proposal weights should be chosen which preserve a significant number of low-weight particles. One scheme which has been found to work well is described in [9], in which proposal weights are given by:

$$v_{n-1}^{(i)} \propto \max(1, N_F w_{n-1}^{(i)}) \quad (19)$$

where  $N_F$  is the number of filtering particles. The RBVRPF is summarised below.

#### IV. THE RAO-BLACKWELLISED VARIABLE RATE PARTICLE SMOOTHER

Estimating changepoints online is a challenging task because the presence of a change may not be obvious until after it has happened. It is thus expected that a smoothing algorithm will provide significantly improved performance

at changepoint estimation. In this section, the same Rao-Blackwellisation method is used to develop a particle smoother for variable rate models with linear-Gaussian state dynamics (RBVRPS). The derivation follows a similar course to that for the fixed rate Rao-Blackwellised Particle Smoother of [15].

The particles of the final filtering step approximate the distribution,  $p(\theta | y_{1:N})$ , which is desired smoothing distribution. However, in the same manner as the fixed rate filter-smoother of [17], this approximation is likely to be degenerate – the particles all share the same set of changepoints from early times, with variation only appearing for changepoints close to  $T$ . For a good characterisation of the smoothing distribution, it is necessary to rejuvenate the set of particles. This is achieved with a backward pass through the observations in a similar manner to the forward-backward methods described in [5], [15].

The target smoothing distribution may be expanded with Bayes rule.

$$p(\theta | y_{1:N}) = p(\theta_n^+ | y_{1:N}) p(\theta_n^- | \theta_n^+, y_{1:N}) \quad (20)$$

Thus, a particle representing  $p(\theta_n^+ | y_{1:N})$  may be extended backwards by sampling from the backwards conditional distribution,  $p(\theta_n^- | \theta_n^+, y_{1:N})$ , which may be approximated by reweighting the particle of the  $n$ th filtering distribution. The resulting particles are then marginalised by discarding the changepoints which come before  $t_{n-1}$  (which will still be suffering from low particle diversity) to leave a set of samples from  $p(\theta_{n-1}^+ | y_{1:N})$ , and the procedure continues recursively.

If the future changepoints and their parameters,  $\tilde{\theta}_n^+$ , have already been sampled (and may thus be considered fixed), then the backward conditional term may be expressed in terms of the filtering distribution.

$$\begin{aligned} p(\theta_n^- | \tilde{\theta}_n^+, y_{1:N}) &\propto p(\theta_n^-, \tilde{\theta}_n^+ | y_{1:N}) \\ &= \int p(x_n, \theta_n^-, \tilde{\theta}_n^+ | y_{1:N}) dx_n \\ &\propto \int p(y_{n+1:N} | x_n, \theta_n^-, \tilde{\theta}_n^+, y_{1:n}) p(x_n, \theta_n^-, \tilde{\theta}_n^+ | y_{1:n}) dx_n \\ &= \int p(y_{n+1:N} | x_n, \theta_n^-, \tilde{\theta}_n^+) p(x_n | \theta_n^-, y_{1:n}) dx_n p(\tilde{\theta}_n^+ | \theta_n^-) p(\theta_n^- | y_{1:n}) \end{aligned} \quad (21)$$

Finally, the RBVRPF approximation is substituted for the filtering distribution.

$$p(\theta_n^- | \tilde{\theta}_n^+, y_{1:N}) = \sum_i \tilde{w}_n^{(i)} \delta_{\theta_n^{-(i)}}(\theta_n^+) \quad (22)$$

where the backwards conditional weights are given by

$$\tilde{w}_n \propto \int p(y_{n+1:N} | x_n, \theta_n^{-(i)}, \tilde{\theta}_n^+) p(x_n | \theta_n^{-(i)}, y_{1:n}) dx_n p(\tilde{\theta}_n^+ | \theta_n^{-(i)}) \quad (23)$$

As before, normalisation is enforced by scaling the weights so that they sum to 1.

If  $k$  changepoints occur before time  $t_n$ , then the changepoint transition term  $p(\theta_n^+ | \theta_n^-)$  may be expressed as:

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1: Run Rao-Blackwellised particle filter to approximate
    $p(x_n, \theta_{[0, t_n]} | y_{1:n})$  with particles  $\theta_{[0, t_n]}^{(i)}$  and associated
   Gaussian moments  $m_n^{(i)}$  and  $P_n^{(i)}$ .
2: for  $i = 1 \dots N_S$  do
3:   Initialise particle using  $\theta_{[0, T]} \sim \sum_j w_N^{(j)} \delta_{\theta_{[0, T]}^{(j)}}(\theta_{[0, T]})$ 
4:   for  $n = N \dots 1$  do
5:     Backwards Kalman filter: Calculate  $\tilde{m}_n^{-(i)}$  and  $\tilde{P}_n^{-(i)}$ 
       using 26 to 32.
6:     for  $j = 1 \dots N_P$  do
7:        $\tilde{w}_n^{(j)} \propto w_n^{(j)} p(\theta_{[t_n, T]}^{(j)} | \theta_{[0, t_n]}^{(j)}) \mathcal{N}(\tilde{m}_n^{-(j)} | m_n, \tilde{P}_n^{-(j)} + P_n)$ 
8:     end for
9:     Sample  $\tilde{\theta}_{[0, t_n]}^{(i)} \sim \sum_j \tilde{w}_n^{(j)} \delta_{\theta_{[0, t_n]}^{(j)}}(\theta_{[0, t_n]})$ 
10:    Discard  $\tilde{\theta}_{[0, t_{n-1}]}^{(i)}$ 
11:  end for
12: end for

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Fig. 2. Rao-Blackwellised Variable Rate Particle Smoother

$$p(\tilde{\theta}_n^+ | \theta_n^-) = p(\tilde{\tau}_{k+1:K}, \tilde{u}_{k+1:K} | \tau_k, u_k, \tau_{k+1} > t_n) \propto p(\tilde{u}_{k+1} | \tilde{\tau}_{k+1}, \tau_k, u_k) p(\tilde{\tau}_{k+1} | \tau_k). \quad (24)$$

The second term in (23) is the familiar Kalman filter density  $p(x_n | \theta_n^{-(i)}, y_{1:n}) = \mathcal{N}(x_n | m_n^{(i)}, P_n^{(i)})$ . The first term,  $p(y_{n+1:N} | x_n, \theta_n^{-(i)}, \tilde{\theta}_n^+)$ , is an improper likelihood density, and may be calculated analytically using a backwards Kalman filter, in a similar manner to that used in the two-filter smoother [15], [16], [18]. Such a backwards Kalman filter uses the following recursions. Details are provided in appendix A, and in the aforesaid references.

$$p(y_{n+1:N} | x_n, \theta_n^{-(i)}, \tilde{\theta}_n^+) = Z_n \mathcal{N}(x_n | \tilde{m}_n, \tilde{P}_n) \quad (25)$$

$$\tilde{m}_n^- = A_{n+1}^{-1} \tilde{m}_{n+1} \quad (26)$$

$$\tilde{P}_n^- = A_{n+1}^{-1} (\tilde{P}_{n+1} + Q_{n+1}) A_{n+1}^{-T} \quad (27)$$

$$\tilde{\mu}_n = C_n \tilde{m}_n^- \quad (28)$$

$$\tilde{S}_n = C_n \tilde{P}_n^- C_n^T + R_n \quad (29)$$

$$\tilde{K}_n = \tilde{P}_n^- C_n^T \tilde{S}_n^{-1} \quad (30)$$

$$\tilde{m}_n = \tilde{m}_n^- + \tilde{K}_n (y_n - \tilde{\mu}_n) \quad (31)$$

$$\tilde{P}_n = \tilde{P}_n^- - \tilde{K}_n \tilde{S}_n \tilde{K}_n^T \quad (32)$$

Substituting into (23), the backwards conditional weights are given by:

$$\tilde{w}_n \propto p(\theta_n^+ | \theta_n^{-(i)}) \mathcal{N}(\tilde{m}_n^- | m_n, \tilde{P}_n^- + P_n) \quad (33)$$

Samples of  $\theta_n^-$  may be drawn from the particle distribution of 22. Once sampling has progressed backwards from  $n = N \dots 1$ , a complete particle from the smoothing distribution will have been generated. This procedure may then be repeated until sufficient particles have been obtained. The procedure is summarised below.

## V. SIMULATIONS

### A. Finance Application

The RBVRPS algorithm was tested on the financial time series model of [11], [12], in which prices of an asset are treated as noisy observations of a latent state, which evolves according to a drift-diffusion with occasional jumps.

The latent state is a vector with two elements, the underlying value of the asset, and the trend followed by this value.

$$\mathbf{x}_n = [x_n, \dot{x}_n]^T \quad (34)$$

This evolves continuously according to a drift-diffusion model:

$$d\mathbf{x}_t = \begin{bmatrix} 0 & 1 \\ 0 & -\lambda \end{bmatrix} \mathbf{x}_t dt + \begin{bmatrix} 0 \\ \sigma \end{bmatrix} d\beta(t) \quad (35)$$

where  $\lambda$  introduces a mean regression effect on the trend and  $\beta(t)$  is standard Brownian motion (with unit diffusion constant).

In addition, the state experiences jumps at random times,  $\{\tau_k\}$ . Two types of jumps occur: value jumps, indicated by  $u_k = 1$ , and trend jumps, indicated by  $u_k = 2$ . The magnitudes of jumps are zero-mean Gaussian-distributed with standard deviation  $\sigma_{J1}$  and  $\sigma_{J2}$  respectively. This model may be discretised at the observation times by matrix fraction decomposition (see e.g. [19] *Not sure what to cite here. Simo's thesis? Or the references therein which I've never seen?*). Assuming Gaussian observation noise with standard deviation  $\sigma_y^2$ , the resulting discrete time dynamics are described by the following equations (see B):

$$\mathbf{x}_n = A \mathbf{x}_{n-1} + \mathbf{w}_n \quad (36)$$

$$y_n = C \mathbf{x}_n + v_n \quad (37)$$

where the  $\mathbf{w}_b$  and  $v_n$  are Gaussian random variables with covariance matrixes  $Q_n$  and  $R$  respectively. The time between observations times is denoted  $\Delta t = t_n - t_{n-1}$ .

$$A = \begin{bmatrix} 1 & \frac{1}{\lambda}(1 - e^{(-\lambda\Delta t)}) \\ 0 & e^{(-\lambda\Delta t)} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q_n = \begin{cases} Q_{\text{diff}} + Q_{\text{jump}} & \exists k : \tau_k \in [t_{n-1}, t_n] \\ Q_{\text{diff}} & \text{otherwise} \end{cases}$$

$$Q_{\text{jump}} = \begin{cases} \begin{bmatrix} \sigma_{J1}^2 & 0 \\ 0 & 0 \end{bmatrix} & u_k = 1 \\ \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{J2}^2 \end{bmatrix} & u_k = 2 \end{cases}$$

$$Q_{\text{diff}} = \frac{\sigma^2}{2\lambda} \begin{bmatrix} \frac{1}{\lambda^2}(2\lambda\Delta t - (3 - e^{(-\lambda\Delta t)})(1 - e^{(-\lambda\Delta t)})) & \frac{1}{\lambda}(1 - e^{(-\lambda\Delta t)})^2 \\ \frac{1}{\lambda}(1 - e^{(-\lambda\Delta t)})^2 & 1 - e^{(-2\lambda\Delta t)} \end{bmatrix}$$

$$R = [\sigma_y^2]$$

The times between changepoints were assumed to be exponentially distributed, with equal probability of value and trend jumps.

$$p(\tau_k, u_k | \tau_{k-1}, u_{k-1}) = p(u_k) p(\tau_k | \tau_{k-1}) \quad (44)$$

$$p(\tau_k | \tau_{k-1}) = \begin{cases} \frac{1}{\alpha} \exp(\alpha(\tau_k - \tau_{k-1})) & \tau_k > \tau_{k-1} \\ 0 & \tau_k < \tau_{k-1} \end{cases} \quad (45)$$

$$p(u_k) = \begin{cases} 0.5 & u_k = 1 \\ 0.5 & u_k = 2 \end{cases} \quad (46)$$

The algorithms were first tested on artificial data simulated from this model. The following parameters were used:  $\Delta t = 0.0017$ ,  $N = 1000$ ,  $\alpha = 20$ ,  $\lambda = 5$ ,  $\sigma = 0.05$ ,  $\sigma_{J1} = 0.005$ ,  $\sigma_{J2} = 0.05$ ,  $\sigma_y = 0.001$ .

The filter used  $N_P = 100$  particles, and the smoother resampled  $N_S = 100$  sequences. Bootstrap proposals were used for the filter.

An example realisation simulated from the model is shown in figure 3. Running the RBVRPF and RBVRPS algorithms on this example produces particle approximations to the changepoint sequence, approximating  $p(\theta | y_{1:N})$ . There is no way to calculate this distribution analytically, so objectively quantifying the quality of the approximations is challenging. As a first comparison, it is possible to collapse the multi-dimensional particle approximations into a single dimension kernel density estimate of the changepoint times. (This is a similar concept to approximating multi-target state distributions with a probability hypothesis density [20], [21].) The results of this process are shown in figure 4. The filter changepoint kernel density plot demonstrates the degeneracy of the filter estimate. Almost every changepoint estimated is present in every particle. This problem has been solved by the smoother, the results of which show higher kernel density at more obvious jumps. Furthermore, the kernel density for the trend jumps ( $u_k = 2$ ) contains peaks which are broader and of lower magnitude, as expected, because the trend is not observed directly.

It is also possible to compare the algorithms via the accuracy of the state estimates they generate. Here we can compare three options: the filtering results, using the RBVRPF and a Kalman filter for the state estimates; the filter-smoother results, using the final RBVRPF approximation for the changepoint sequence and a Rauch-Tung-Striebel (RTS) smoother for the state estimates; and the smoothing results, using the RBVRPS followed by an RTS smoother. For the example run, these are shown in figure 5. Again, the improved particle diversity of the smoother is apparent.

For quantitative comparison, the algorithms were tested on XX realisations from the model, and the following statistics were calculated for each:

- The number of unique changepoint sequences. This is a measure of particle diversity of the approximation.
- The number of unique changepoint times. Another measure of particle diversity.
- The optimal sub-pattern assignment (OSPA) distance between the maximum a posteriori (MAP) changepoint

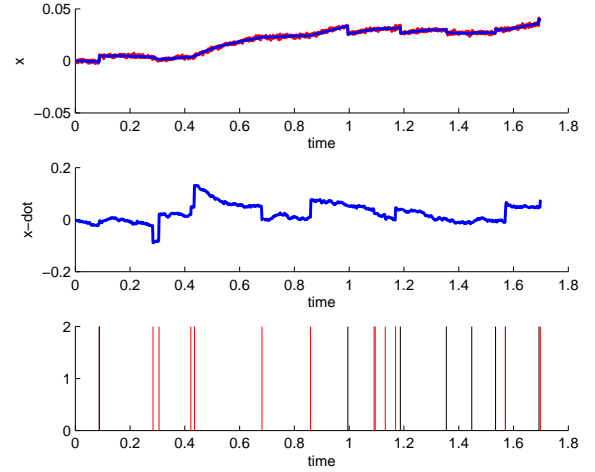


Fig. 3. An example simulated data set

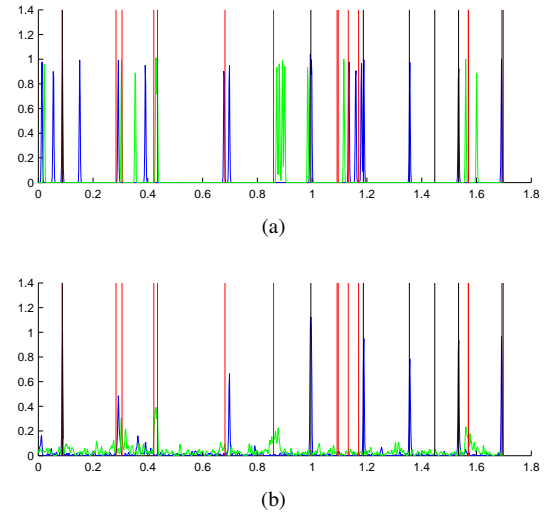


Fig. 4. Filter (a) and smoother (b) kernel density estimates for value (blue) and trend (green) jump times. Correct times overlaid as vertical lines (black=value, red=trend).

sequence and the true sequence. The OSPA is a concept from multiple target tracking introduced in [22] to measure the distance between sets of points of varying cardinality. It is thus well suited to measuring the error between changepoint sequences of varying length. For each approximation, the sequence from the MAP particle was used as a point estimate.

- The root-mean-square error (RMSE) of the MAP state estimate. The MAP state estimate is taken to be the Gaussian mean of the conditional state distribution, from the Kalman smoother or RTS, selected from the particle with the highest posterior probability.
- The root-mean-square error (RMSE) of the mean state estimate. The mean state estimate is the average of the Gaussian means from all the particles.

Note that the MAP particle may be found by calculating the posterior probability of each. The terms required for this calculation will already have been evaluated during the

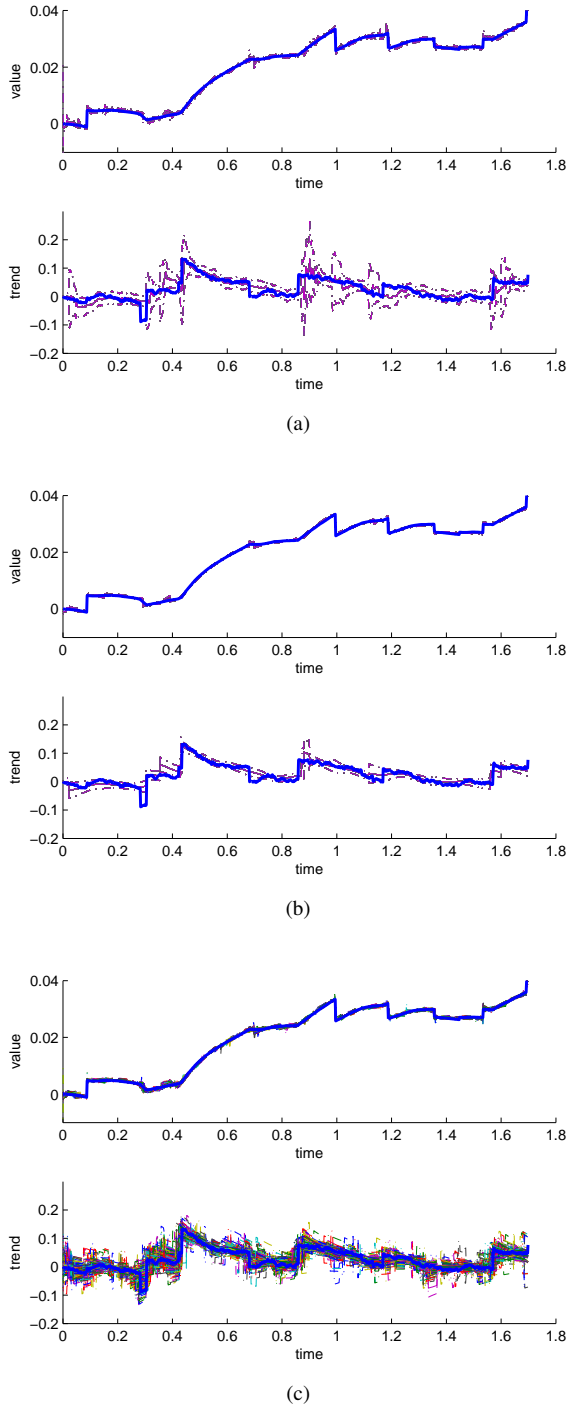


Fig. 5. State estimates using RBVRPF + KF (a) final RBVRPF approximation + RTS smoother (b) and RBVRPS + RTS smoother. Solid lines show means of each particle. Dashed lines show mean  $\pm 2$  standard deviations.

filtering procedure.

$$p(\theta|y_{1:N}) \propto p(y_{1:N}|\theta)p(\theta) \\ S(\tau_K, T) \prod_{k=1}^K p(u_k)p(\tau_k|\tau_{k-1}) \prod_{n=1}^N p(y_n|\theta, y_{1:n}) \quad (47)$$

The results are shown in tables I and II. The OSPA parameters are: exponent  $p = 1$ , threshold  $c = 0.01$ .

	RBVRPF	RBVRPS
mean no. sequences	7.8	100
mean no. jump times	47.8	1076.7
MAP sequence OSPA	$5.36 \times 10^{-3}$	$4.98 \times 10^{-3}$

TABLE I  
CHANGEPOINT SEQUENCE ESTIMATION PERFORMANCE.

	RBVRPF + KF	RBVRPF + RTS	RBVRPS + RTS
mean value estimate RMSE	$5.31 \times 10^{-4}$	$4.48 \times 10^{-4}$	$4.16 \times 10^{-4}$
mean trend estimate RMSE	$2.56 \times 10^{-2}$	$1.79 \times 10^{-2}$	$1.49 \times 10^{-2}$
MAP value estimate RMSE	$5.98 \times 10^{-4}$	$4.48 \times 10^{-4}$	$4.49 \times 10^{-4}$
MAP trend estimate RMSE	$3.13 \times 10^{-2}$	$1.79 \times 10^{-2}$	$1.74 \times 10^{-2}$

TABLE II  
STATE ESTIMATION PERFORMANCE.

The new RBVRPS algorithm outperforms the filter in all respects, both in terms of accuracy and particle diversity.

*Another interesting question is how the posterior distribution over the number of changepoints is expected to behave. The prior is poisson distributed with mean  $\alpha T$ . The means of both the filter and smoother particle posterior are consistently greater than the true number (by about 50%). Is this expected and why?*

Finally, we demonstrate the algorithms on a real financial data set, 1000 data points representing 1.7s of USD-GBP foreign exchange rate data from April 2008, using the same parameters as for the previous simulations. State estimates and jump time kernel density estimates are shown in figure 6. In this example, there is no ground truth against which to judge the results. However, the smoother does a pleasing job of estimating jump times in the same areas that one would if analysing the data “by eye”.

### B. Tracking Application

*Add a tracking application here, in which the changepoints mark manoeuvre start points and there is a turn rate or acceleration parameter for each.*

## VI. CONCLUSION

A new smoothing algorithm has been introduced for variable rate models which have linear-Gaussian system equations conditional on a set of unknown changepoint times. The algorithm employs the method of Rao-Blackwellisation, using a particle filter to estimate the nonlinear changepoint sequence, and Kalman filtering/smoothing methods to estimate the linear state components. The smoothing algorithm is shown to improve particle diversity and accuracy when compared to the filter.

### APPENDIX A

#### DERIVATION OF THE BACKWARDS KALMAN FILTER

Appendix one text goes here.

### APPENDIX B

#### MODEL DISCRETISATION

Appendix two text goes here.



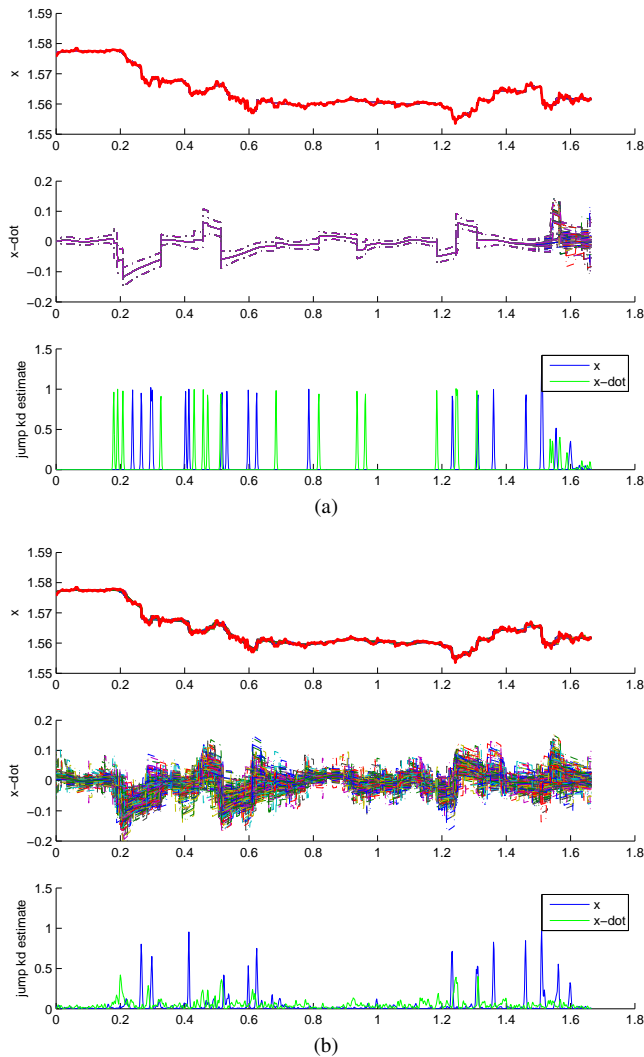


Fig. 6. Filter (a) and smoother (b) state and kernel density estimates. State estimates show mean for each particle (solid) as well as mean  $\pm 2$  standard deviations (dashed). Kernel density estimates for value (blue) and trend (green) jump times.

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Pete Bunch Biography text here.

Simon Godsill Biography text here.