Math 135: Project 2 Solutions

Problem 1 [10 pts]

(a) See other file

(b)

	domain	range
sinh :	$(-\infty, \infty)$	$(-\infty, \infty)$
cosh :	$(-\infty, \infty)$	$[1, \infty)$
tanh :	$(-\infty, \infty)$	$(-1, 1)$
sech :	$(-\infty, \infty)$	$(0, 1]$

(c) • For $\cosh(t)$ & $\sinh(t)$, e^t & e^{-t} have domain $(-\infty, \infty)$, while $2 \neq 0$; so they have domain $(-\infty, \infty)$.

• For $\tanh(t) = \dfrac{\sinh(t)}{\cosh(t)} = \dfrac{e^t - e^{-t}/2}{e^t + e^{-t}/2} = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$ &

$$\text{sech}(t) = \frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}},$$

we have e^t & e^{-t} both always positive, so $e^t + e^{-t} > 0$ for all t, so again the domains are $(-\infty, \infty)$.

Problem 2 {10 pts}

(a) • Graphically, the horizontal asymptotes for $y = \tanh(t)$ are $y = -1$ & $y = 1$.

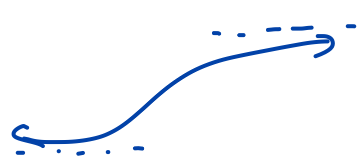
• $$\lim_{t\to\infty} \tanh(t) = \lim_{t\to\infty} \frac{e^t - e^{-t}}{e^t + e^{-t}} = \lim_{t\to\infty} \frac{\cancel{e^t}}{\cancel{e^t}} \left(\frac{1 - e^{-2t}}{1 + e^{-2t}} \right)$$

$$= \lim_{t\to\infty} \frac{1 - e^{-2t}}{1 + e^{-2t}} = \frac{1 - 0}{1 + 0} = \boxed{1}$$

• $$\lim_{t\to -\infty} \tanh(t) = \lim_{t\to -\infty} \frac{e^t - e^{-t}}{e^t + e^{-t}} = \lim_{t\to\infty} \frac{\cancel{e^{-t}}}{\cancel{e^{-t}}} \left(\frac{e^{2t} - 1}{e^{2t} + 1} \right)$$

$$= \lim_{t\to -\infty} \frac{e^{2t} - 1}{e^{2t} + 1} = \frac{0 - 1}{0 + 1} = \boxed{-1}$$

(b) $\tanh(t)$ & $\arctan(t)$ both interpolate btwn two values: -1 & 1 and $-\pi/2$ & $\pi/2$, with general shape



Problem 3 [5 pts]

(a) Graphically:

sinh(t) and tanh(t) are odd,

cosh(t) and sech(t) are even.

(b) • $\sinh(-t) = \dfrac{e^{(-t)} - e^{-(-t)}}{2} = \dfrac{e^{-t} - e^{t}}{2} = \dfrac{-(e^{t} - e^{-t})}{2}$

$= -\sinh(t)$ ✓

• $\cosh(-t) = \dfrac{e^{(-t)} + e^{-(-t)}}{2} = \dfrac{e^{-t} + e^{t}}{2} = \dfrac{e^{t} + e^{-t}}{2} = \cosh(t)$ ✓

• $\tanh(-t) = \dfrac{\sinh(-t)}{\cosh(-t)} = \dfrac{-\sinh(t)}{\cosh(t)} = -\tanh(t)$ ✓

• $\operatorname{sech}(-t) = \dfrac{1}{\cosh(-t)} = \dfrac{1}{\cosh(t)} = \operatorname{sech}(t)$ ✓

Problem 4 [10 pts]

(a) See other file.

(b) $\cosh^2 t - \sinh^2 t$

$$= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2$$

$$= \frac{e^{2t} + 2e^t \cdot e^{-t} + e^{-2t}}{4} - \frac{e^{2t} - 2e^t e^{-t} + e^{-2t}}{4}$$

$$= \frac{\left(\cancel{e^{2t}} + 2 + \cancel{e^{-2t}}\right) - \left(\cancel{e^{2t}} - 2 + \cancel{e^{-2t}}\right)}{4}$$

$$= \frac{4}{4} = 1 \checkmark$$

§B: Calculus of Hyperbolic Functions

Problem 5: [5 pts]

$$\frac{d}{dt}\left(\sinh(t)\right) = \frac{d}{dt}\left(\frac{e^t - e^{-t}}{2}\right) = \frac{e^t - e^{-t}\cdot(-1)}{2}$$

$$= \frac{e^t + e^{-t}}{2} = \cosh(t) \checkmark$$

Problem 6: [15 pts]

(a)
- $$\frac{d}{dt}\left(\cosh(t)\right) = \frac{d}{dt}\left(\frac{e^t + e^{-t}}{2}\right) = \frac{e^t + e^{-t}\cdot(-1)}{2}$$

$$= \frac{e^t - e^{-t}}{2} = \boxed{\sinh(t)}$$

- $$\frac{d}{dt}\left(\tanh(t)\right) = \frac{d}{dt}\left(\frac{\sinh(t)}{\cosh(t)}\right) = \frac{\frac{d}{dt}(\sinh(t))\cdot\cosh(t) - \sinh(t)\frac{d}{dt}(\cosh(t))}{\cosh^2(t)}$$

$$= \frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t} = \frac{1}{\cosh^2 t}$$

$$= \boxed{\operatorname{sech}^2 t}$$

(b) $y = \sinh(t)$ is a solution to the 2nd-order differential equation

$$y'' = y.$$

Problem 7 {10 pts}

$$\sinh(t) = \frac{3}{4}$$

$$\frac{e^t - e^{-t}}{2} = \frac{e^t - 1/e^t}{2} = \frac{\frac{e^{2t}-1}{e^t}}{2} = \frac{3}{4}$$

$$\frac{e^{2t}-1}{e^t} = \frac{3}{2}$$

$$e^{2t} - 1 = \frac{3}{2}e^t$$

$$e^{2t} - \frac{3}{2}e^t - 1 = 0$$

$$e^t = \frac{\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 4(1)(-1)}}{2(1)} = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} + 4}}{2}$$

$$= \frac{\frac{3}{2} \pm \sqrt{\frac{25}{4}}}{2} = \frac{\frac{3}{2} \pm \frac{5}{2}}{2} = \frac{3 \pm 5}{4}$$

- $e^t = \frac{3+5}{4} = 2 \implies \boxed{t = \ln(2)}$
- $e^t = \frac{3-5}{4} = \frac{-1}{4}$ X IMPOSSIBLE

Problem 8 [15 pts]

(a) $\frac{d}{dt}\left(\sinh(t)\right) = \cosh(t) = \frac{e^t + e^{-t}}{2}$

$\cdot\ e^t > 0,\ e^{-t} > 0 \Rightarrow \cosh(t) > 0$

$\Rightarrow \frac{d}{dt}\left(\sinh(t)\right) > 0$

$\Rightarrow \sinh(t)$ always increasing

(b) $y = \sinh(t) = \frac{e^t - e^{-t}}{2} = \frac{e^{2t} - 1/e^t}{2}$

$2y = \frac{e^{2t} - 1}{e^t}$

$2ye^t = e^{2t} - 1$

$e^{2t} - 2ye^t - 1 = 0$

$\Rightarrow e^t = \frac{+2y \pm \sqrt{(-2y)^2 - 4(1)(-1)}}{2(1)} = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$

$e^t = y \pm \sqrt{y^2 + 1}$, but $y - \sqrt{y^2+1} < 0$ and $e^t > 0$, so this can't happen

$\Rightarrow \boxed{t = \ln\left(y + \sqrt{y^2 + 1}\right)}$

Problem 9 [20 pts]

(a) $y = \sinh^{-1}(t) \iff \sinh(y) = t$ $\Big\downarrow \frac{d}{dt}$

$$\cosh(y) \cdot \frac{dy}{dt} = 1$$

$$\frac{dy}{dt} = \frac{1}{\cosh(y)}$$

$$\left\{\begin{array}{l} 1 = \cosh^2(y) - \sinh^2(y) = \cosh^2(y) - t^2 \\ \cosh^2(y) = 1 + t^2 \\ \cosh(y) = + \sqrt{1+t^2} \end{array}\right.$$

(cosh(y) > 0 always)

$$\Rightarrow \boxed{\frac{dy}{dt} = \frac{d}{dt}\left(\sinh^{-1}(t)\right) = \frac{1}{\sqrt{1+t^2}}}$$

(5) $\sinh^{-1}(t) = \ln\left(t+\sqrt{t^2+1}\right)$

$$\frac{d}{dt}\left(\sinh^{-1}(t)\right) = \frac{1}{t+\sqrt{t^2+1}} \cdot \frac{d}{dt}\left(t+\sqrt{t^2+1}\right)$$

$$= \left(\frac{1}{t+\sqrt{t^2+1}}\right)\left(1+\frac{1}{2\sqrt{t^2+1}}\cdot 2t\right)$$

$$= \frac{1}{t+\sqrt{t^2+1}} + \frac{t}{\left(t+\sqrt{t^2+1}\right)\left(\sqrt{t^2+1}\right)}$$

$$= \frac{\cancel{\sqrt{t^2+1}+t}}{\cancel{\left(t+\sqrt{t^2+1}\right)}\left(\sqrt{t^2+1}\right)}$$

$$= \boxed{\frac{1}{\sqrt{t^2+1}}}$$