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The tree representation of Σ_{n+1}

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Abstract

We show that the space of fully grown n-trees has the homotopy type of a bouquet of spheres of dimension $n-3$ and that the character of the representation of Σ_{n+1} on its only non-trivial reduced homology group is

$$\varepsilon \cdot (\mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} \mathrm{Lie}_n - \mathrm{Lie}_{n+1}).$$

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0. Introduction

We investigate a certain space T_n called the space of fully grown n-trees, and we prove that it has the homotopy type of a bouquet of spheres of dimension $n-3$. The symmetric group Σ_{n+1} acts in a natural way on T_n, so the reduced homology of T_n affords an integral representation of Σ_{n+1}, which we call the tree representation. It has dimension $(n-1)!$. Its restriction to Σ_{n-1} is isomorphic (over the integers) to the regular representation.

We show that the character of the tree representation is

$$\varepsilon \cdot (\mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} \mathrm{Lie}_n - \mathrm{Lie}_{n+1})$$

where ε is the alternating character, and Lie_n is the character of the representation of Σ_n on that component of the free Lie algebra on n generators which is spanned by those words involving each generator exactly once. In the proof we use the Lefschetz fixed-point theorem, and a formula of Brandt [3] to the effect that $\mathrm{Lie}_r = \mathrm{Ind}_{C_r}^{\Sigma_r} \rho_r$, where ρ_r is a faithful linear character of the subgroup generated by an r-cycle in Σ_r.

We became interested in the space T_n because it arises in Γ-homology theory for commutative rings. This is very close to the study of E_∞ structures in homotopy

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theory, where T_n originated [2], and to configuration spaces and free Lie algebras. The connections among these are described in [6]. Restricting the tree representation to Σ_n gives $\varepsilon \cdot \mathrm{Lie}_n$. Our Theorem 3.1 therefore implies a result on the top-dimensional homology of configuration spaces due to Cohen (see [5, Theorem 12.3, p. 302]) which is also proved by completely different methods in [9]. The restriction to Σ_n also yields a known result, due to Stanley [11] on the homology of partition lattices; there is an explicit proof of this in Barcelo [1].

Our results were announced in [10, 13]. Some related results have since been obtained by Hanlon [8].

1. The space of fully grown trees

Definition 1.1. A *tree* is a compact contractible one-dimensional polyhedron Y. There is just one minimal triangulation of Y, namely that one in which every vertex is either an *end* (belongs to just one edge, called a *free* edge) or a *node* (belongs to at least three edges of Y). Non-free edges of Y are called *internal* edges. We shall almost always use this triangulation.

From now on, let $n \geq 2$. We define an *n-tree* to be a tree Y in which

(1) there are exactly $n + 1$ ends, labelled by the integers $0, 1, 2, \dots, n$;

(2) each internal edge α is assigned a length $l(\alpha)$, where $0 < l(\alpha) \leq 1$.

Free edges always have length 1 assigned to them.

An *isomorphism* $Y_1 \to Y_2$ of n-trees is a homeomorphism which is isometric on edges and which preserves the labelling of the ends. The space $\tilde{T}_n$ of isomorphism classes of n-trees is a cubical complex. Two trees belong to the same open cube if they are homeomorphic in a fashion which preserves labels and edges of length 1. The coordinates within each cube are given by the internal edge-lengths. (We use here the fact that a tree has no non-trivial automorphisms.) Topologically, $\tilde{T}_n$ is therefore a finite CW complex.

Since $n \geq 2$, the space $\tilde{T}_n$ has a natural contraction which shrinks all internal edges linearly and simultaneously. Therefore $\tilde{T}_n$ is a cone. Its apex is the tree with no internal edges; its base is the space T_n of trees which are *fully grown* in the sense that at least one internal edge α has the maximum permitted length $l(\alpha) = 1$. (The terminology comes from [2].) The symmetric group Σ_{n+1} acts upon T_n and $\tilde{T}_n$ by permuting the labels $0, 1, \dots, n$.

We now investigate the subcomplex T_n. We recall that a cell complex has *pure dimension r* if every cell is a face of some r-cell.

Proposition 1.2.

(1) *T_n can be triangulated as a simplicial complex of pure dimension $n - 3$.*

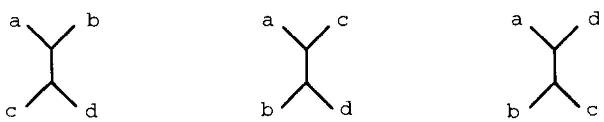
(2) *Every $(n - 4)$-simplex of T_n is a face of precisely three $(n - 3)$-simplices.*

Proof. The simplices of this new structure on T_n are as follows. The vertices are those trees with just one internal edge (which therefore has length 1). From a general tree $t \in T_n$ we can obtain various vertices v_α by stretching one internal edge α to length 1, and shrinking all other internal edges to zero. These v_α are the vertices of the simplex in which t lies: the barycentric coordinates of t are proportional to the lengths of the corresponding internal edges. Since such data uniquely determine a fully grown tree, we have a triangulation of T_n. (The original cell structure on T_n is recovered by subdividing each simplex into cubes according to which barycentric coordinate is greatest.)

Each simplex of T_n corresponds to a *shape* of fully grown tree (i.e. an equivalence class under label-preserving homeomorphism). The maximal simplices correspond to the trees with the most internal edges. These have just three edges at each node. Consideration of the Euler characteristic shows that they have $n-2$ internal edges and so give $(n-3)$-dimensional simplices of T_n.

Any lower-dimensional simplex of T_n corresponds to a tree-shape containing a node where more than three edges meet. One can separate such a node into two, introducing a new edge. This process can be continued until a shape is reached which corresponds to an $(n-3)$-simplex having the original simplex as a face. Therefore T_n has pure dimension $n-3$.

Let σ be an $(n-4)$-simplex of T_n. It corresponds to a tree-shape containing one node ε of order four, all other nodes having order three. The node ε can be separated in three ways



corresponding to the three ways of pairing off the subtrees which meet at ε. The resulting shapes give the three $(n-3)$-simplices having σ as a face. □

As the above proposition might suggest, the complex T_n, shares many (but not all) of the properties of buildings [12]. In this it resembles the split buildings of Charney [4].

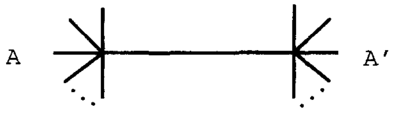
Definitions 1.3. Let $i \in \{0, 1, \ldots, n\}$ be one of the labels of a tree $t \in T_n$. The *entry node* of i is the first node one meets on entering the tree at the end labelled i. It is connected to i by an edge.

Let A be any subset of the set $\{0, 1, \ldots, n\}$ of labels. We denote by U_A the set of trees $t \in T_n$ in which all $i \in A$ have the same entry node. Evidently, U_A is a subcomplex of T_n. From now on we exclude, usually without further mention, the trivial cases $|A| \leq 1$ and $|A| \geq n$, which give respectively the whole of T_n and the empty set.

Lemma 1.4. *The U_A are a family of contractible sets which cover T_n. Every non-trivial intersection of sets in this family is contractible, or empty.*

Proof. In every shape of tree there are two labels with the same entry node. Therefore the sets $U_{\{i,j\}}$ already cover T_n.

Any single U_A (where, as usual, $2 \le |A| \le n-1$) is a conical subset of the closed star of the vertex



and therefore can be linearly contracted to that vertex. (If $A = \{i,j\}$ consists of two elements, then U_A is precisely the closed star.)

Now suppose $\mathscr{A}$ is a subfamily such that $\bigcap_{A \in \mathscr{A}} U_A$ is non-empty. Take a tree belonging to a simplex of maximal dimension in this intersection. Shrink all the internal edges, except for one adjoining the entry node of the labels in some (at least) $A \in \mathscr{A}$. The whole intersection is a conical subset of the closed star of this vertex, and is therefore contractible. □

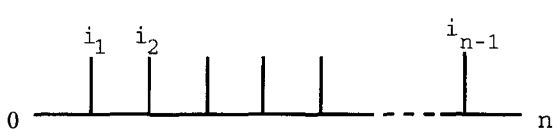
Theorem 1.5. *There is a homotopy equivalence*

$$T_n \simeq \bigvee_{(n-1)!} S^{n-3}$$

and the representation of the subgroup Σ_{n-1} of label permutations on $H_{n-3}(T_n)$ is isomorphic to the regular representation.

Proof. By Σ_{n-1} we here understand the group of permutations of any $n-1$ of the labels, but we may assume without loss of generality that they are $1, 2, \dots, n-1$. We consider the subset $X = \bigcup_{A \subset \{1,2,\dots,n-1\}} U_A$ of T_n. This is covered by the contractible subcomplexes $U_{\{i,j\}}$, $1 \le i,j \le n-1$. Every intersection of these is non-empty, because it contains the vertex pictured above with $A = \{1, 2, \dots, n-1\}$. By Lemma 1.4, each intersection is contractible. Thus X is contractible.

The complement $T_n \backslash X$ consists only of the interiors of the $(n-3)$-dimensional simplices



where $(i_1, i_2, \dots, i_{n-1})$ is any permutation of $(1, 2, \dots, n-1)$. Therefore T_n is obtained from a contractible space by attaching $(n-1)!$ simplices of dimension $n-3$ along their boundaries, and these simplices are regularly permuted by the subgroup Σ_{n-1} of the group of label permutations. It follows that T_n has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-3$ which are permuted in the standard manner by Σ_{n-1}. $\square$

2. The sets of fixed points

In order to find the character of the action of Σ_{n+1} on $H_{n-3}(T_n)$, we investigate the homotopy type of the fixed-point set T_n^g for each $g \in \Sigma_{n+1}$. This of course depends upon the cycle type (conjugacy class) of g. In general T_n^g is not a simplicial subcomplex of T_n, but is certainly a subcomplex of its first barycentric subdivision because Σ_{n+1} acts simplicially on T_n.

If t is a tree representing a point of T_n^g, then the permutation g of labels extends to a unique isometry of t with itself, so that t is a tree with a well-defined action of the cyclic subgroup $\langle g \rangle$. Conversely, any fully grown tree with $\langle g \rangle$-action represents a point of T_n^g. We may therefore regard T_n^g as the space of (fully grown) $\langle g \rangle$-trees.

We introduce the notation

$$U^g_{\{i,j\}} = U_{\{i,j\}} \cap T_n^g$$

for the set of $\langle g \rangle$-trees where i and j have the same entry node. We shall call the set $U^g_{\{i,j\}}$ *balanced* if i and j are distinct and belong to cycles of equal length, or to the same cycle, in the decomposition of the permutation g.

Lemma 2.1. *The balanced sets $U^g_{\{i,j\}}$ cover T_n^g.*

Proof. We know from Lemma 1.4 that the collection of all subsets $U^g_{\{i,j\}}$, balanced or not, covers T_n^g. Suppose that a tree t belongs to $U^g_{\{i,j\}}$, where i belongs to a cycle of length m and j belongs to a cycle of shorter length l. Since t is a $\langle g \rangle$-tree, the entry node of the label i must then be fixed under g^l, and so t also belongs to the set $U^g_{\{i,g^l(i)\}}$, which is balanced. Thus the balanced sets are sufficient. $\square$

Let $\mathscr{A}$ be any family of balanced sets. We introduced the *$\mathscr{A}$-equivalence relation* on the set $\{0, 1, \ldots, n\}$ of labels, defining it to be the smallest equivalence relation which is stable under $\langle g \rangle$, and for which $i \sim j$ whenever $U^g_{\{i,j\}} \in \mathscr{A}$. Alternatively, consider the set of all the labels as a $\langle g \rangle$-set. Then the $\mathscr{A}$-equivalence classes are the elements of the quotient $\langle g \rangle$-set of this by the relations $i \sim j$ for $U^g_{\{i,j\}} \in \mathscr{A}$.

Lemma 2.2. *The intersection of a non-empty family $\mathscr{A}$ of balanced sets is always contractible, except when there is an $\mathscr{A}$-equivalence class containing n or $n+1$ labels.*

In particular, an individual balanced $U^g_{\{i,j\}}$ is contractible except in certain cases where g is an n-cycle or an $(n+1)$-cycle.

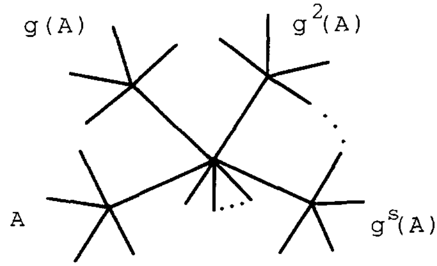


Fig. 1

Proof. Choose any $U^g_{\{i,j\}}$ in the given family $\mathscr{A}$. Let A be the $\mathscr{A}$-equivalence class of i and j. Provided $|A| < n$, there is a vertex v of T_n (illustrated in Lemma 1.4) in which the labels entering at one node are the labels in A. The translates $g^r(v)$ span a simplex, of which the barycentre lies in $\bigcap\mathscr{A}$. The whole of $\bigcap\mathscr{A}$ can be linearly and equivariantly contracted to this barycentre, which appears as given in Fig. 1.

The above demonstration works in particular when $\mathscr{A}$ has a single element $U^g_{\{i,j\}}$. Therefore the last claim of the lemma follows from the observation that an $\mathscr{A}$-equivalence class can then only contain n or $n+1$ elements if g is an n-cycle or an $(n+1)$-cycle and if i, j are labels related by a generator of the cyclic group $\langle g \rangle$. $\square$

We shall need the following result.

Lemma 2.3. *Let Λ_r be the lattice of proper divisors of the integer r. Then the homotopy type of the nerve $N_r = |\Lambda_r|$ is given by*

(1) *$N_r \simeq S^{k-2}$ if r is a product of k distinct primes;*

(2) *$N_r \simeq$ point if r is not square-free.*

Proof. (1) We use induction on the number k of prime factors in r. The induction starts, because Λ_r is empty when $r = 1$ or r is prime. For the inductive step, it suffices to prove that N_{pq} is homeomorphic to the suspension of N_q when p is prime and $(p, q) = 1, q > 1$. In this case $\Lambda_{pq} = \{p, q\} \cup M$, where $M = \{\lambda, p\lambda \mid \lambda \in \Lambda_q\}$. The partially ordered subset M has realization $N_q \times I$, and the divisibility relations involving p and q with elements of M are such that N_{pq} is obtained by attaching a cone on N_q to each end of the cylinder $|M| \approx N_q \times I$, the vertices of the cones being the elements p and q. Therefore $N_{pq} \approx SN_q$, and the induction is complete.

(2) If r is divisible by p^2, where p is prime, then the least common multiple map, $x \mapsto \{p, x\}$, is a deformation retraction onto the sublattice of proper divisors of r divisible by p, which is a cone. $\square$

Theorem 2.4. (1) *If g is a power of an n-cycle or of an $(n+1)$-cycle, and thus has cycle type $(r)^s(1)$ or $(r)^s$, then the fixed-point set T^g_n has the homotopy type of a* *wedge of spheres*

$$T_n^g \simeq \begin{cases} \bigvee_{(s-1)!r^{s-1}} S^{s+k-3} & \text{if } r \text{ is a product of } k \text{ distinct primes,} \\ \text{point} & \text{if } r \text{ is not square-free.} \end{cases}$$

(2) *If g has any other cycle type, then T_n^g is contractible.*

Proof. We first prove (2). Take $\mathscr{A}$ to be the family of all balanced sets $U^g_{\{i,j\}}$. When g is a power of neither an n-cycle nor an $(n+1)$-cycle, there is no $\mathscr{A}$-equivalence class containing n or $n+1$ labels. By Lemma 2.2, all the $U^g_{\{i,j\}}$ and all intersections of them are contractible: therefore their union is contractible. But this union is T_n^g, by Lemma 2.1. Thus (2) is proved.

We turn now to the proof of (1), in the special case when g is an n-cycle or an $(n+1)$-cycle, so that $r = n$ or $r = n+1$. As we remarked at the beginning of Section 2, the cyclic group $\langle g \rangle$ acts by isometries on every tree t in T_n^g. Since it acts transitively upon the labels (or upon all labels except 0), such a tree is necessarily star-shaped: the cyclic action has a single fixed point if $r = n+1$, and fixes only the edge labelled 0 if $r = n$ (see Fig. 2). This implies that T_n^g is homeomorphic to the nerve N_r of the partially ordered set of proper divisors of r. By Lemma 2.3 the special case of (1) is proved.

In the remaining case of (1), when g has cycle type $(r)^s$ or $(r)^s(1)$ with $s > 1$, we select one of the r-cycles in the decomposition of g. Without loss of generality we may take this cycle to be $(12 \ldots r)$. Let $\mathscr{B}$ be the family of balanced sets $U^g_{\{i,j\}}$ in which i and j belong to cycles *other than* the selected one. It follows as above from Lemma 2.2 that $\bigcup\mathscr{B}$ is contractible: but $\bigcup\mathscr{B}$ is no longer than whole of T_n^g.

Consider any tree t is the complement $T_n^g \backslash \bigcup\mathscr{B}$. As $t \notin \bigcup\mathscr{B}$, the action of $\langle g \rangle$ must be free on the entry nodes of the elements of each non-selected r-cycle; and only the

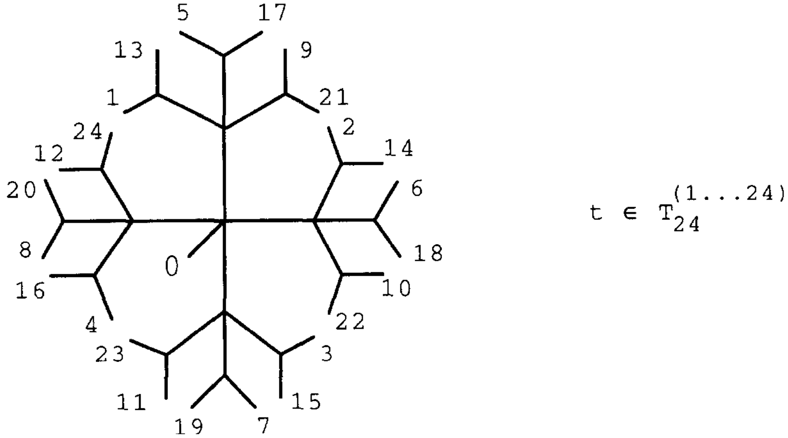


Fig. 2

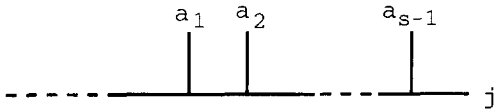


Fig. 3

entry nodes of elements of $\{1, 2, \ldots, r\}$ may coincide with those of any element of another cycle. It follows that t must have the following form: it is a star-shaped tree from the space N_r of proper divisors of r, on each end of which has been attached a frond (see Fig. 3) with $j \in \{1, 2, \ldots, r\}$, and each a_i chosen from a different non-selected r-cycle. By virtue of the $\langle g \rangle$-action on t, any one frond (say that with $j = 1$) determines all the others, so that there are just $(s-1)!\,r^{s-1}$ possible patterns of fronds. (If $r = n$, the label 0 enters at the fixed node.)

Since the fronds introduce $s - 1$ new barycentric coordinates, we conclude that T_n^g is obtained from the contractible space $\bigcup \mathscr{B}$ by attaching $(s-1)!\,r^{s-1}$ copies of the pair $(\Delta^{s-2}, \partial\Delta^{s-2}) * N_r$, where N_r is again the nerve of the partially ordered set of proper divisors of r. Knowing the homotopy type of N_r as above, we deduce that

$$T_n^g \simeq \begin{cases} \bigvee_{(s-1)!\,r^{s-1}} S^{s+k-3} & \text{if } r \text{ is a product of } k \text{ distinct primes,} \\ \text{point} & \text{if } r \text{ is not square-free.} \end{cases}$$

The proof of the theorem is now complete. □

Corollary 2.5. *The Euler characteristic of the fixed-point set T_n^g is given by the formula*

$$\chi(T_n^g) - 1 = \begin{cases} (s-1)!\,(-r)^{s-1}\mu(r) & \text{if } g \text{ has cycle type } (r)^s \text{ with } r > 1, \text{ or } (r)^s(1), \\ 0 & \text{if } g \text{ has any other cycle type,} \end{cases}$$

where μ is the standard Möbius function.

We may note here that on restriction to the subgroup Σ_n this agrees with a result which Hanlon [7] obtained for the partition lattice by other methods.

3. The tree representation

Theorem 3.1. *The integral representation of Σ_{n+1} on the homology group $H_{n-3}(T_n)$ has character*

$$\varepsilon \cdot (\mathrm{Ind}_{\Sigma_n}^{\Sigma_{n+1}} \mathrm{Lie}_n - \mathrm{Lie}_{n+1})$$

where ε is the alternating character, and Lie_n *is the character of the representation of Σ_n on that component of the free Lie algebra on n generators which is spanned by those words involving each generator exactly once.*

Proof. By the Lefschetz fixed-point theorem, the alternating sum of the traces of the homology automorphisms induced by any element $g \in \Sigma_{n+1}$ equals the Euler characteristic of the corresponding fixed-point set T_n^g. If χ is the character of the representation on $H_{n-3}(T_n)$, we have by Corollary 2.5 that

$$(-1)^{n-3}\chi(g)$$
$$= \begin{cases} (s-1)!(-r)^{s-1}\mu(r) & \text{if } g \text{ has cycle type } (r)^s \text{ with } r > 1, \text{ or } (r)^s(1), \\ 0 & \text{if } g \text{ has any other cycle type,} \end{cases}$$

because the only other homology group is the trivial module $H_0(T_n)$.

On the other hand, the induced character formula gives precisely these values for the virtual character

$$\varepsilon \cdot (\mathrm{Ind}_{C_n}^{\Sigma_{n+1}} \rho_n - \mathrm{Ind}_{C_{n+1}}^{\Sigma_{n+1}} \rho_{n+1})$$

where ρ_r is a faithful linear character of the subgroup of Σ_{n+1} generated by an r-cycle, and ε is the alternating character.

Now a result of Brandt [3] for the Lie character implies that $\mathrm{Lie}_r = \mathrm{Ind}_{C_r}^{\Sigma_r} \rho_r$. The theorem thus follows from the formula above and the transitivity of induction. □

References

[1] H. Barcelo, On the action of the symmetric group on the free Lie algebra and the partition lattice, J. Combin. Theory Ser. A 55 (1990) 93–129.

[2] J.M. Boardman, Homotopy structures and the language of trees, Proc. Symp. Pure Math. 22 (1971) 37–58.

[3] A.J. Brandt, The free Lie ring and Lie representations of the full linear group, Trans. Amer. Math. Soc. 56 (1944) 528–536.

[4] R.M. Charney, Homology stability for GL_n of a Dedekind domain, Invent. Math. 56 (1980) 1–17.

[5] F.R. Cohen, T.J. Lada and J.P. May, The homology of iterated loop spaces, Lecture Notes in Mathematics, Vol. 533 (Springer, Berlin, 1976).

[6] F.R. Cohen, Artin's braid groups, classical homotopy theory, and sundry other curiosities, in: Braids (Santa Cruz, CA), Contemporary Mathematics #78 (American Mathematical Society, Providence, RI, 1988) 167–206.

[7] P. Hanlon, The fixed-point partition lattices, Pacific J. Math. 96 (1981) 319–341.

[8] P. Hanlon, Otter's method and the homology of homeomorphically-irreducible k-trees, J. Combin. Theory Ser A., to appear.

[9] G.I. Lehrer and L. Solomon, On the action of the symmetric group on the cohomology of the complement of its reflecting hyperplanes, J. Algebra 104 (1986) 410–424.

[10] C.A. Robinson, The space of fully-grown trees, Sonderforschungsbereich 343, Universität Bielefeld, preprint 92-083, 1992.

[11] R.P. Stanley, Some aspects of groups acting on finite posets, J. Combin. Theory Ser. A 32 (1982) 132–161.

[12] J. Tits, On buildings and their applications, in: Proc. International Congress of Mathematicians, Vancouver (1974) 209–220.

[13] S.A. Whitehouse, Γ-(Co)homology of commutative algebras and some related representations of the symmetric group, Ph.D. Thesis, University of Warwick, 1994.