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Gamma Homology of Group Algebras and of Polynomial Algebras

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ABSTRACT. We explicitly determine Gamma homology in the important examples of group algebras in terms of homology groups of Eilenberg-MacLane spectra. From this result we obtain a complete answer for smooth algebras. In the cases of polynomial algebras and truncated polynomial algebras we prove that the Gamma homology groups do not depend on the module structure on the coefficients. Finally we show that Gamma cohomology of the algebra of cooperations vanishes for Lubin-Tate spectra.

1. Introduction

Gamma homology or Γ-homology (denoted $H\Gamma_*$) is a homology theory for commutative algebras and for E_∞-algebras. It is defined in [**RoWh**] as the homology of a homotopical variant of the cotangent complex, and can be regarded as an analogue for "brave new rings" of André-Quillen homology. For commutative algebras which are flat modules it is isomorphic to topological André-Quillen homology of commutative algebras, as was proven in [**BMC**].

T. Pirashvili and the first author proved in [**PR**] that Γ-homology of a commutative k-algebra A with coefficients in an A-module M is isomorphic to the stable homotopy of a certain spectrum. This spectrum is the realization $|\mathcal{L}|$ of the Γ-module $\mathcal{L}$ due to J.-L. Loday [**L**] which assigns to each finite based set $S_+ = S \sqcup \{0\}$ the A-module $\mathcal{L}(S_+) = M \otimes A^{\otimes S}$. Since $\mathcal{L}$ is a Γ-object of A-modules, $|\mathcal{L}|$ is a module spectrum over the Eilenberg-MacLane spectrum HA.

We use the theorem of [**PR**] to calculate the Γ-homology of commutative group algebra, polynomial algebras and truncated polynomial algebras. These particular algebras are in some sense defined over the sphere spectrum. More precisely, the cotangent complex in these cases is not only a complex of A-modules, but is actually the chain complex of a simplicial set. Therefore its homology, the Γ-homology $H\Gamma_*(A \,|\, k;\, A)$, is a comodule over the relevant dual Steenrod coalgebra $HA_*HA = \pi_*(HA \wedge HA)$, if this is flat over A.

The algebra of cooperations E_*E of a ring spectrum E gives information on its multiplicative structures. In particular with the obstruction theory developed in

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[**Ro**], one can deduce existence and uniqueness of E_∞-structures on ring spectra: if Gamma cohomology of the algebra E_*E vanishes for some homotopy commutative ring spectrum E, then one can refine the given multiplication to a unique E_∞-structure (see [**Ro**, Theorem 5.6]). We prove that Lubin-Tate spectra for Honda formal group laws have this vanishing property. That these spectra possess a unique E_∞-structure is known by work of Goerss and Hopkins [**GH**]; our result gives an elementary proof for this fact.

2. Gamma homology of integral group rings

There is a direct way to prove that Γ-homology of an integral group ring on an abelian group G has a concise description. We will denote the composition in the group G multiplicatively.

Let Γ be the skeleton of the category of finite pointed sets. A functor from Γ to the category of k-modules for a commutative ring k is called Γ-*module.* For a commutative k-algebra A with unit and an A-module M let $\mathcal{L}$ be Loday's Γ-module. On an object $[n] = \{0, 1, \dots, n\}$ this functor is $M \otimes A^{\otimes n}$ with tensor products taken over the ring k. A pointed function $f : [n] \to [m]$ sends a generator $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ with $a_0 \in M$ and $a_i \in A$ for $i \neq 0$ to $b_0 \otimes b_1 \otimes \cdots \otimes b_m$ with $b_i = \prod_{f(j)=i} a_j$ (resp. $b_i = 1$ if $f^{-1}(i) = \emptyset$).

For this section $\mathcal{L}$ will denote the functor for the commutative ring $\mathbb{Z}[G]$ with coefficients in the integers. To give $\mathbb{Z}$ a $\mathbb{Z}[G]$-module structure, we use the standard augmentation $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$ which sends every group element $g \in G$ to 1. We will identify this Γ-module with the one which gives the classical cubical construction of Eilenberg and MacLane. To this end we will use the identification of the first Taylor derivative of Γ-modules as a cubical construction of these functors as it was developed in [**R2**].

There is a functor $\mathbb{Z}\{\bar{G}\{-\}\}$ from the category Γ to abelian groups defined in [**R2**, 3.2] which sends a finite pointed set $[n]$ to the free abelian group on the set G^n. A pointed map $f : [n] \to [m]$ induces internal multiplication on the generators, i.e.,

$$f_*(g_1, \dots, g_n) = (h_1, \dots, h_m) \qquad \text{with} \qquad h_i = \prod_{f(j)=i} g_j.$$

Note that elements whose index is sent to zero have no effect on the final element. If a preimage of i is empty, the neutral element $1 \in G$ is inserted in the ith place. The first Taylor approximation of $\mathbb{Z}\{\bar{G}\{-\}\}$ is the cubical construction $Q(G)$ of the abelian group G.

LEMMA 2.1. *The functor $\mathcal{L}$ from Γ to abelian groups is isomorphic to the functor $\mathbb{Z}\{\bar{G}\{-\}\}$.*

PROOF. The Loday functor evaluated on an element $[n]$ gives $\mathbb{Z}[G]^{\otimes n} \cong \mathbb{Z}[G^n]$ and the structure maps are given by multiplication and insertion of units. The only point that remains to be checked is the behaviour of $\mathcal{L}$ with respect to maps f with a non-trivial preimage of 0. If $i \neq 0$ is sent to zero then an action of $\mathbb{Z}[G]$ on the integers via the augmentation map is induced. On a generator as above this just results in the disappearance of the variable. $\square$

PROPOSITION 2.2. *Gamma homology of an integral group ring on an abelian group G is isomorphic to the integral homology of the Eilenberg-MacLane spectrum* *of the group G*

$$H\Gamma_*(\mathbb{Z}[G] \,|\, \mathbb{Z}; \mathbb{Z}) \cong H\mathbb{Z}_* HG.$$

PROOF. Gamma homology of $\mathbb{Z}[G]$ with coefficients in the integers is stable homotopy of the functor $\mathcal{L}$ (see [**PR**]) and this in turn is by [**R2**, Theorem 4.5] isomorphic to the homology of the first layer in the Taylor tower which is the cubical construction $Q(\mathcal{L})$ of $\mathcal{L}$. But as the Loday functor $\mathcal{L}$ is isomorphic to the functor $\mathbb{Z}\{\bar{G}\{-\}\}$, we obtain

$$Q(\mathcal{L}) \cong Q(\mathbb{Z}\{\bar{G}\{-\}\}) = Q(G).$$

But it is well-known that the homology of the cubical construction on an abelian group gives the integral homology of the Eilenberg-MacLane spectrum on G

$$H_*(Q(G)) \cong H\mathbb{Z}_* HG.$$

This fact was proved by Eilenberg and MacLane in the fifties; for an easy proof see [**P**]. □

3. The smooth case

From the proof of Proposition 2.2 it is obvious that the statement can be generalized to an arbitrary commutative ring with unit k. We obtain by analogy:

PROPOSITION 3.1. *Gamma homology of an group algebra $k[G]$ for an abelian group G is the k-homology of the Eilenberg-MacLane spectrum of G.*

From this result one can read off a lot of Γ-homology groups. It is known by [**RoWh**, Theorem 6.8 (3)] that Γ-homology vanishes for étale algebras. For a triple-sequence of algebra maps, Γ-homology gives a long exact sequence. Thus we can deduce the following fact:

PROPOSITION 3.2. *If A is a commutative augmented k-algebra and some abelian group algebra $k[G]$ is étale over A, then Γ-homology of A is isomorphic to that of $k[G]$, thus*

$$H\Gamma_*(A \,|\, k; k) \cong Hk_* HG.$$

In particular, Γ-homology of a polynomial algebra $k[x]$ is isomorphic to the k-homology of $H\mathbb{Z}$ because $k[x, x^{-1}] = k[\mathbb{Z}]$ is étale over $k[x]$.

Let k be a noetherian ring and let A be a commutative k-algebra which is essentially of finite type, i.e., A is the localization of a finitely generated k-algebra. Then A is *smooth* if it has étale factorization, i.e., if for every prime ideal $\mathfrak{p}$ in A there is an $f \notin \mathfrak{p}$ such that there is a factorization of the unit map $k \to A_f$ as

$$k \to k[x_1, \ldots, x_m] \xrightarrow{\psi} A_f$$

with ψ an étale map. Thus, for Γ-homology smooth algebras are as good as polynomial algebras.

THEOREM 3.3. *Let A be smooth and augmented over k. Then Γ-homology of A consists of as many copies of $Hk_* H\mathbb{Z}$ as the dimension of the module of Kähler differentials $\Omega^1_{A|k} \otimes_A k$*

$$H\Gamma_*(A \,|\, k; k) \quad \cong \bigoplus_{\dim_k(\Omega^1_{A|k} \otimes_A k)} Hk_* H\mathbb{Z} \quad \cong \quad \Omega^1_{A|k} \otimes_A Hk_* H\mathbb{Z}.$$

PROOF. We consider the triple $k \to k[x_1, \dots, x_m] \to A_f$. The localized algebra A_f relative to A and the algebra A_f relative to the polynomial algebra both have vanishing Γ-homology, because they are étale. Hence Γ-homology of A relative to k coincides with the one for $k[x_1, \dots, x_m]$ relative to k. As Γ-homology is additive [**RoWh**, Theorem 6.8 (2)] it gives m copies of $Hk_*H\mathbb{Z}$. The number of generators m of the polynomial algebra $k[x_1, \dots, x_m]$ is determined by the module of Kähler differentials which is isomorphic to $H\Gamma_0(A \,|\, k; k)$. $\square$

The calculation of Γ-homology of polynomial algebras allows us to show that a spectral sequence which was developed in [**RoWh**, 4.5] for the calculation of Γ-homology of E_∞-algebras becomes stationary at the E^2-level in the case of a polynomial algebra over $\mathbb{F}_2$. For a prime p let us denote by $\mathcal{S}'$

$$\mathcal{S}' = \begin{cases} (H\mathbb{F}_2)_*H\mathbb{Z} & = \quad \mathbb{F}_2[\xi_1^2, \xi_2, \xi_3, \dots] \text{ if } p = 2 \\ (H\mathbb{F}_p)_*H\mathbb{Z} & = \quad \mathbb{F}_p[\xi_1, \xi_2, \xi_3, \dots] \otimes \Lambda_{\mathbb{F}_p}(\tau_1, \tau_2, \dots) \text{ if } p \text{ is odd.} \end{cases}$$

Here $\Lambda_{\mathbb{F}_p}$ denotes the exterior algebra and the degree of ξ_i is $2^i - 1$ for $p = 2$ and $2p^i - 2$ for odd primes, whereas the degree of τ_i is $2p^i - 1$.

COROLLARY 3.4. *The spectral sequence of* [**RoWh**, 4.5]

$$E^2_{p,q} \cong \mathsf{Untor}^R_{p,q}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow H\Gamma_{p+q}(\mathbb{F}_2[t] \,|\, \mathbb{F}_2;\, \mathbb{F}_2)$$

collapses with $E^2 = E^\infty$. *Here* Untor^R *denotes torsion products of 1-allowable unstable modules over the Dyer-Lashof algebra R.*

PROOF. The calculation in [**M**, 3.3] shows that the E^2-term has the same Poincaré series as the abutment $\mathcal{S}' = \mathbb{F}_2[\xi_1^2, \xi_2, \xi_3, \dots]$. (The very similar example given in 3.3.12 of [**M**] makes this virtually explicit.) All the differentials must therefore be zero. $\square$

This Corollary verifies a conjecture made in [**RoWh**], and is consistent with results in an otherwise-defined E_∞-homology theory which are quoted in unpublished work of Igor Kriz.

4. Independence of module structures

The result in this part is that Γ-homology for smooth and truncated polynomial algebras is independent of the given module structure on the coefficients. Suppose first that A is a polynomial algebra. By additivity of Γ-homology [**RoWh**, 6.8(2)] it suffices to consider the case $A = k[t]$, and we take any $k[t]$-module M as the coefficients. For the algebra $k[t]$ there is a description of Loday's Γ-module $\mathcal{L}$ given in [**R1**]. Let L be the reduced free k-module functor defined (as in [**PR**]) by $L(S_+) = k^{S_+}/k \cong k^S$ and let Sym be the symmetric algebra functor over k. Then there is an isomorphism of functors

$$\mathcal{L}(S_+) \cong M \otimes_{k[t]} \mathrm{Sym}(k^{S_+}).$$

where the action identifies $k[t]$ with the summand of the symmetric algebra generated by the basepoint. As an k-module $\mathcal{L}(S_+)$ is isomorphic to $M \otimes \mathrm{Sym}(k^S)$, but this tensor product is twisted in the sense that whenever a morphism maps something to the basepoint $+$, an action of t on M is induced.

As a second example, take A to be the truncated polynomial algebra $k[t]/(t^n)$. There is an analogous formula

$$\mathcal{L}(S_+) \cong M \otimes_A \mathrm{Tym}(S_+)$$

where $\mathrm{Tym}(S)$ is notation for the truncated algebra obtained by imposing upon $\mathrm{Sym}\, L(S_+)$ the relations $s^n = 0$, $s \in S$. If k has characteristic p and $n = p^\ell$ these relations are equivalent to $y^{p^\ell} = 0$ for all $y \in \mathrm{Sym}\, L(S_+)$, because the pth power map is a homomorphism.

Our results are consequences of the following theorem.

THEOREM 4.1. *Let k be a commutative ring (with unit), A a commutative k-algebra and M a A-module.*

- *If A is $k[t]$ or $k[t, t^{-1}]$ then*

$$H\Gamma_*(A \,|\, k;\, M) \quad \cong \quad HM_*(H\mathbb{Z}).$$

- *If k has characteristic p, and A is $k[t]/(t^{p^\ell})$ where ℓ is a positive integer, then*

$$H\Gamma_*(A \,|\, k;\, M) \quad \cong \quad HM_*(H(\mathbb{Z}/p^\ell\mathbb{Z})).$$

PROOF. In each case we must calculate the stable homotopy of the corresponding abelian spectrum. The rth term of this spectrum is obtained by applying Loday's Γ-module degreewise to a based simplicial r-sphere $\mathbb{S}^r$ (see [**PR**]). The free k-module functor L there gives the simplicial module of reduced k-chains on $\mathbb{S}^r$.

We first consider the case $A = k[t]$. Here the formula given above for $\mathcal{L}$ indicates that the Loday module evaluated on $\mathbb{S}^r$ gives $M \otimes_{k[t]} \mathrm{Sym}(k^{\mathbb{S}^r})$ and this is the twisted tensor product with M of the symmetric algebra on the reduced chains $C_*(\mathbb{S}^r; k)$. This is isomorphic to the simplicial module of reduced chains with coefficients in M on the infinite symmetric product $SP^\infty\, \mathbb{S}^r$. The Steenrod splitting [**St**, §22]

$$C_*(SP^\infty X) \quad \cong \quad \bigoplus_{i=0}^{\infty} C_*(SP^{i+1} X,\, SP^i X)$$

implies that the face operators in $\mathcal{L}(\mathbb{S}^r)$ are identical with those in $C_*(SP^\infty\, \mathbb{S}^r; M)$; and that, despite the twisting, they are independent of the action of t on M. Therefore for $r > i$

$$\begin{aligned} H\Gamma_i(k[t] \,|\, k;\, M) \quad &\cong \quad \pi_{r+i}\, C_*(SP^\infty\, \mathbb{S}^r;\, M) \\ &\cong \quad H_{r+i}(SP^\infty\, \mathbb{S}^r;\, M) \\ &\cong \quad HM_i(H\mathbb{Z}) \end{aligned}$$

because $SP^\infty\, \mathbb{S}^r$ is an Eilenberg-MacLane complex $K(\mathbb{Z}, r)$ by the Dold-Thom theorem [**DT**, §6].

The case of the Laurent algebra $A = k[t, t^{-1}]$ follows as above from the étaleness of $k[t, t^{-1}]$ over $k[t]$.

The final case is when $A = k[t]/(t^{p^\ell})$. In characteristic p the formula before the statement of the theorem shows $\mathcal{L}(S_+)$ to be isomorphic to the tensor product of the reduced chains on the free $(\mathbb{Z}/p^\ell\mathbb{Z})$-module on the set S twisted with M. Since the free $(\mathbb{Z}/p^\ell\mathbb{Z})$-module functor converts $\mathbb{S}^r$ into $K(\mathbb{Z}/p^\ell\mathbb{Z},\, r)$, the result follows just as in the first case. $\square$

COROLLARY 4.2. *For a prime p, let $A = \mathbb{F}_p[t]/(t^{p^\ell})$, acting upon $\mathbb{F}_p$ through the augmentation. Let $\mathcal{S}$ be the dual Steenrod coalgebra for the prime p. Then*

$$H\Gamma_*(A \,|\, \mathbb{F}_p; \mathbb{F}_p) \quad \cong \quad \begin{cases} \mathcal{S} & \textit{when } \ell = 1 \\ \mathcal{S}' \oplus \mathcal{S}'[-1] & \textit{when } \ell > 1\,. \end{cases}$$

where $\mathcal{S}'$ the comodule $H\mathbb{F}_{p*}(H\mathbb{Z})$ defined in section 3

PROOF. This follows from the second part of Theorem 4.1. When $\ell > 1$, a standard calculation shows that $H\mathbb{F}_{p*}(H(\mathbb{Z}/p^\ell\mathbb{Z}))$ is the direct sum of a copy of $H\mathbb{F}_{p*}(H\mathbb{Z})$ embedded by the surjection $\mathbb{Z} \to \mathbb{Z}/p^\ell\mathbb{Z}$ and a (-1)-shifted copy which is the image of the Bockstein homomorphism. □

By some standard arguments one can prove that Γ-homology of group algebras over fields does not depend on the module structure of the coefficients either. First suppose that the ground field $\mathbb{F}$ is a prime field.

In the characteristic zero case Γ-homology is simply Harrison homology and therefore a natural summand of Hochschild homology. But Hochschild homology of a commutative group algebra is independent of the module structure because it can be identified with group homology with trivial coefficients.

So we may assume $\mathbb{F}$ has finite characteristic p. When the group $G = \langle x \rangle$ is cyclic of order p^ℓ, the group ring is a truncated polynomial algebra of height p^ℓ on $x - 1$; when G is infinite cyclic, the group algebra is a Laurent algebra. When $G = \langle x \rangle$ is cyclic of finite order prime to p, the group ring is étale over $\mathbb{F}$ so that the Γ-homology is zero ([**RoWh**, 6.8(3)]). Every abelian group is the directed colimit of its finitely-generated subgroups; and Γ-homology commutes with directed colimits, so we have the result for all G. Finally the flat base-change result of [**RoWh**, 6.8(1)] allows us to replace the prime field $\mathbb{F}$ by any extension.

5. The solid case

So far we considered algebras which were flat over the ground ring. In these cases Γ-homology is isomorphic to topological André-Quillen homology [**BMC**]. But in the non-flat case Γ-homology has to be defined indirectly [RoWh, §3], because the Loday functor gives a spectrum which in general has the wrong homotopy type. We illustrate this with the example of 'solid' algebras.

DEFINITION 5.1. *A k-algebra A is* solid *if the algebra-multiplication $\mu : A \otimes_k A \to A$ is an isomorphism.*

Solid rings were classified by Bousfield and Kan in [**BK**, Prop.3.1]; they are built out of quotients like $\mathbb{Z}/n\mathbb{Z}$ and localisations $\mathbb{Z}[J^{-1}]$ for some set of primes J.

OBSERVATION 5.2. *Stable homotopy vanishes for solid algebras.*

PROOF. The Γ-module whose stable homotopy we want to calculate is canonically isomorphic to the constant functor. □

For a commutative ring R and an ideal $I \subset R$ the first André-Quillen homology of R/I relative to R is I/I^2 ([**A**, p.75, Proposition 1]), hence it is non-trivial in general. Thus in these cases the stable homotopy is actually smaller than André-Quillen homology.

6. Lubin-Tate spectra for Honda formal group laws.

Let E be a Lubin-Tate spectrum and let E_*E be its commutative algebra of cooperations with its left E_*-module structure induced by the left unit $\eta_L : E \simeq E \wedge \mathbb{S} \xrightarrow{\mathrm{id}\wedge\eta} E \wedge E$. We provide a proof for the fact that Γ-cohomology of E_*E vanishes if E is a Lubin-Tate spectrum for a Honda formal group law. In view of the obstruction theory for commutativity developed by the second author [**Ro**], this yields a unique E_∞-structure on these Lubin-Tate spectra.

REMARK 6.1. *There is one class of spectra for which it is trivial to prove that the obstruction groups vanish: Call a ring spectrum* solid *if the multiplication induces a weak equivalence* $A \wedge A \xrightarrow{\sim} A$. *In these cases, such as Moore spectra on* $\mathbb{Z}[J^{-1}]$ *for a set of primes* J, *the algebra of cooperations is isomorphic to the coefficient ring and the obstruction groups vanish for a trivial reason.*

For a fixed but arbitrary height n the coefficient ring for $E = E_n$ is $E_* = W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$ with the u_i in degree zero and u in degree two. Here $W\mathbb{F}_{p^n}$ denotes the ring of Witt vectors on $\mathbb{F}_{p^n}$. The ring $W\mathbb{F}_{p^n}[[u_1, \dots, u_{n-1}]]$ has a maximal ideal $m = (p, u_1, \dots, u_{n-1})$.

The fact that the Lubin-Tate spectra possess an E_∞-structure is well-known; in [**GH**], Goerss and Hopkins prove a stronger result [**GH**, theorems 7.9,7.10]: They obtain a unique E_∞-structure on each Lubin-Tate spectrum E and they prove that the space of E_∞-maps between two Lubin-Tate spectra has weakly contractible components and π_0 is given by the maps of formal groups laws between the respective formal groups. The aim of the following part is to provide an easy and short proof which uses standard techniques; in particular it gives a proof for [**Ro**, Cor.5.7]. We will prove the following

THEOREM 6.2. *If* $E = E_n$ *is the Lubin-Tate spectrum for the Honda formal group law for height* n, *then* Γ-*cohomology of* E_*E *relative to* E_* *vanishes.*

Reduction to the case $H\Gamma^*(E_*E \mid E_*; E_*/m)$. This is the argument given in Rezk's paper [**Re**] adjusted to the cohomology theory $H\Gamma$.

LEMMA 6.3. *It suffices to show that* $H\Gamma^s(E_*E \mid E_*; E_*/m)$ *vanishes for all* $s \geq 0$*; then* $H\Gamma^s(E_*E \mid E_*; E_*)$ *vanishes in all degrees.*

PROOF. The short exact sequence

$$0 \to m^d/m^{d+1} \to E_*/m^{d+1} \to E_*/m^d \to 0$$

gives a long exact coefficient sequence for $H\Gamma$. As m^d/m^{d+1} is finite dimensional over E_*/m and as we assume that $H\Gamma^*(E_*E \mid E_*; E_*/m)$ vanishes, we obtain that $H\Gamma^s(E_*E \mid E_*; E_*/m^d)$ vanishes for all $s \geq 0$ and all $d \geq 1$. The completeness of E_* with respect to m allows us to write E_* as

$$0 \to E_* \to \prod_{d \geq 1} E_*/m^d \xrightarrow{1-S} \prod_{d \geq 1} E_*/m^d \to 0.$$

This yields a long exact sequence in Γ-cohomology and proves the claim. □

Flat base-change and étaleness. Again, the flat-base-change-arguments are taken from [**Re**]. One should note that there is flat base-change for *co*homology, even if the algebra is only flat and not projective (as E_*E is over E_*) because in the flat-base-change-arguments in [**RoWh**, Theorem 6.8] it is actually proved that a flat base-change of algebras induces a genuine *isomorphism* of complexes and not just a weak-equivalence. We can now proof Theorem 6.2.

PROOF. We consider the map $E_* \to E_*/m$. The flatness of E_*E over E_* yields an isomorphism

$$H\Gamma^q(E_*E \mid E_*; E_*/m) \cong H\Gamma^q(E_*E/m \mid E_*/m; E_*/m).$$

As E_*E/m is isomorphic to $E_0E/m \otimes_{E_0/m} E_*/m$ and $E_0/m \to E_*/m$ is the inclusion of the field $\mathbb{F}_{p^n}$ into the Laurent polynomial algebra $\mathbb{F}_{p^n}[u^{\pm 1}]$ we obtain

$$H\Gamma^q(E_*E/m \mid E_*/m; E_*/m) \cong H\Gamma^q(E_0E/m \mid E_0/m; E_0/m).$$

The structure of E_0E/m is well-known (see for instance [**Re**, 17.4]). If E is the spectrum for the Honda formal group law of height n then

$$E_0E/m = \mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n}[a_0, a_1, \ldots]/(a_0^{p^n-1} - 1, a_1^{p^n} - a_1, \ldots).$$

Thus this algebra consists of tensor products of étale algebras and the claim is proven. □

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