PROOF OF THEOREM 5.4

1. First definitions and comparisons

Let Γ denote the category of finite based sets and based maps, and k a fixed ground field.

Definition 1.1. A *left* Γ*-module* is a functor $\Gamma \to \mathsf{Mod}_k$, while a *right* Γ*-module* is a functor $\Gamma^{op} \to \mathsf{Mod}_k$.

Definition 1.2. Given a DGA Λ and a Λ-module R, the *(graded) Loday functor* is the left Γ-module

$$\mathcal{L}(\Lambda, R)\colon \Gamma \to \mathsf{Mod}_k, \qquad [k] \mapsto \Lambda^{\otimes_R k}.$$

Definition 1.3. Let Lie_m^* denote the dual Lie representation. We define a functor Ξ from left Γ-modules to bicomplexes by

$$\Xi(F)_{p,q} = \mathcal{B}_p\left(\mathrm{Lie}_{q+1}^*, \Sigma_{q+1}, F(q+1)\right) = \mathrm{Lie}_{q+1}^* \otimes k\left[\Sigma_{q+1}^{\times p}\right] \otimes F(q+1).$$

One differential comes from the bar complex and is easy; the other is more complicated.

Definition 1.4. Given a DGA Λ and a Λ-module R, the Γ*-cohomology* of Λ over R is the bigraded abelian group

$$H\Gamma^{*,*}(\Lambda, R) := H^* \operatorname{Tot} \operatorname{Hom}_R^* \left(\Xi(\mathcal{L}(\Lambda, R)), R\right).$$

The above clarifies the notation from both of [RW02, Rob03] by resolving the ambigous placements of Ξ and also identifing the bidegrees.

Proposition 1.5 ([Rob03, Cor 3.7] or [Rob18, Cor. 3.17])**.** *There is an isomorphism*

$$\pi_*(F) \cong H_* \operatorname{Tot} \Xi(F)$$

natural in left Γ*-modules* F*.*

We actually need the dual result, which is not stated in [Rob03].

Proposition 1.6 ([Rob18, Cor 3.17])**.** *For any right* Γ*-module* F*, the stable cohomotopy* $\pi^n(F)$ *is the cohomology of the complex*

$$\pi^*(F) = H^* \operatorname{Tot}_{p,q} \operatorname{Hom}\left(\mathrm{Lie}_{q+1}^* \otimes k[\Sigma_{q+1}]^{\otimes p}, F(q+1)\right). \tag{1.7}$$ {COMP1_EQ}

Lemma 1.8 ([Rob18, §4.9])**.** *A homotopy ring spectrum* $V = (V, \mu, \eta)$ *determines a graded right* Γ*-module*

$$[k] \longmapsto V^*(V^{\wedge k}) = \pi_* \mathsf{End}(V)$$

UCT_REM **Remark 1.9.** If V satisfies a universal coefficient theorem as on the left below,

$$V^*(V^{\wedge k}) \simeq \operatorname{Hom}_{V_*}^*\left((V_*V)^{\otimes k}, V_*\right) \qquad V^*(V^{\wedge \bullet}) \equiv \operatorname{Hom}_{V_*}^*\left(\mathcal{L}(V_*V, V_*), V_*\right) \tag{1.10}$$ {UCT_EQ}

then we have an isomorphism of graded right Γ-modules as on the right.

COMP_COR **Corollary 1.11.** *For* V *satisfing* (1.10)*, we have an isomorphism between* Γ*-homology of* V_*V *and the cohomotopy of* $V^*(V^{\wedge\bullet})$*.*

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Proof. When F^* is the graded Γ-module $\mathrm{Hom}^*_{V_*}(\mathcal{L}(V_*V), V_*), V_*)$, (1.7) implies we have isomorphisms

$$\begin{aligned}\pi^* \mathrm{Hom}^*_{V_*}(\mathcal{L}(V_*V), V_*), V_*) &= H^* \mathrm{Tot}_{p,q} \mathrm{Hom}^*_{V_*}\left(\mathrm{Lie}^*_{q+1} \otimes k[\Sigma_{q+1}]^{\otimes p} \otimes (V_*V)^{\otimes q+1}, V_*\right) \\ &= H^* \mathrm{Tot}\, \mathrm{Hom}^*_{V_*}(\Xi\mathcal{L}(V_*V, V_*), V_*)\,. \qquad (1.12)\end{aligned}$$

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Thus we have

$$\begin{aligned}\pi^* V^*(V^{\wedge\bullet}) &= \pi^* \mathrm{Hom}^*_{V_*}(\mathcal{L}(V_*V, V_*), V_*) \\ &= H^* \mathrm{Tot}\, \mathrm{Hom}^*_{V_*}(\Xi\mathcal{L}(V_*V, V_*), V_*) \\ &= H\Gamma^{*,*}(V_*V, V_*) \qquad (1.13)\end{aligned}$$

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where the first equality is by Remark 1.9, the second is by (1.12), and the third is by definition. □

2. Results

2.1. Version 1. In [Rob18], the main result is of a more general nature, living in stable cohomotopy, which is then specified in the case V satisfies a universal coefficient isomorphism to Γ-cohomology.

Theorem 2.1 ([Rob18, Thm. 4.13])**.** *Let V be a homotopy commutative ring spectrum. Then the obstruction to lifting an n-stage lives in $\pi^n\left(V^{2-n}(V^{\wedge\bullet})\right)$.*

Corollary 2.2. *If V is homotopy commutative and satisfies (1.10), then the obstruction lives in $H\Gamma^{n,2-n}(V_*V, V_*)$.* □

2.2. Version 2. In [Rob03], the obstruction is found in an explicit group, which is defined(?) to be Γ-cohomology, which is then related to stable cohomotopy.

Proposition 2.3 ([Rob03, Prop. 5.4])**.** *For a homotopy commutative ring spectrum satisfying (1.10), the obstruction lives in $H\Gamma^{n,2-n}(V_*V, V_*) := H^n\, \mathrm{Tot}\, \mathrm{Hom}^{2-n}_{V_*}(\Xi\mathcal{L}(V_*V, V_*, V_*))$.*

Proof. Suppose we have a cofiber sequence

$$\nabla^n \cup \partial\nabla^{n+1} \wedge_{\Sigma_m} V^{\wedge m} \longrightarrow \nabla^{n+1} \wedge_{\Sigma_m} V^{\wedge m} \longrightarrow \nabla^{n+1}/(\nabla^n \cup \partial\nabla^{n+1}) \wedge_{\Sigma_m} V^{\wedge m}. \qquad (2.4)$$

{COFIBRE_EQ

Moreover, we have

$$\begin{aligned}\nabla^{n+1}/(\nabla^n \cup \partial\nabla^{n+1}) &\simeq \left(\tilde{T}_m/\partial\tilde{T}_m\right) \wedge \left(E\Sigma_m^{n-m+1}/E\Sigma_m^{n-m}\right) \\ &\simeq \left(\bigvee_{(m-1)!} S^{m-2}\right) \wedge \left(\bigvee_A S^{n-m+1}\right) \\ &\simeq S^{n-1} \wedge ((m-1)! \times A)_+\end{aligned}$$

where $A = \Sigma_m^{n-m+2}$ and $\mathbb{Z}[(m-1)!] = \mathrm{Lie}^*_m$. Thus, by the LES in cohomology associated to (2.4), the obstruction to lifting the n-stage to an $(n+1)$-stage (using that the n-stage automatically lifts to $\partial\nabla^{n+1}$) lives in

$$\begin{aligned}V^1\left(\nabla^{n+1}/(\nabla^n \cup \partial\nabla^{n+1}) \wedge_{\Sigma_m} V^{\wedge m}\right) &= V^{2-n}\left(((m-1)! \times A)_+ \wedge_{\Sigma_m} V^{\wedge m}\right) \\ &= V^{2-n}\left(((m-1)! \times \bar{A})_+ \wedge V^{\wedge m}\right)\end{aligned}$$

where $\Sigma_m \times \bar{A} = A$. By the universal coefficient isomorphism, this group is also given by

$$\mathrm{Hom}^{2-n}_{V_*}(\mathrm{Lie}^*_m \otimes V_*[\bar{A}] \otimes V_*V^{\otimes m}, V_*) = \mathrm{Hom}^{2-n}_{V_*}(\Xi_{n-m+1,m-1}\mathcal{L}(V_*V, V_*), V_*)\,.$$

These combine to form an n-cochain in $\operatorname{Tot}\operatorname{Hom}^{2-n}_{V_*V}(\Xi\mathcal{L}(V_*V, V_*), V_*)$.

so this is an issue if $A = \mathrm{Conf}_{n-m+2}(\Sigma_m)$, so the number of Σ_m-s is wrong

The rest of the proof shows this obstruction class is in fact a well-defined cocycle, using explicit information about Lie^*_m. □

Corollary 2.5. *The obstruction lives in* $\pi^n\left(V^{2-n}(V^{\wedge\bullet})\right)$. □

3. Questions

Question 3.1. *Which side of* (1.13) *is more conceptual? Easier to compute? Useful?*

References

Rob03 [Rob03] A. Robinson, *Gamma homology, Lie representations and E_∞ multiplications*, Invent. Math. **152** (2003), no. 2, 331–348. doi:10.1007/s00222-002-0272-5 1, 2

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