Getting straight the $\cong$'s among Tor^{Γ}, $H_*\Xi$, H_*C_Γ, π_*F, etc.

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0. Definitions

- Γ is the category of pointed finite sets, with all set maps.
- Ω is the category of poined finite sets, with surjections.
- A left Γ-module is a functor $\Gamma \to \mathrm{Mod}_k$ for a fixed commutative ring k. Similarly for Ω-modules.
- t is the trivial Γ-module $S \mapsto \mathrm{Hom}(S, k)$ (you can think of this as $\prod_{s\in S} k$).

For a Γ-module F, we can consider the following:

- $\pi_*^{st}(F) := \mathrm{colim}_{*+n}(F \circ S^n)$, where I'm thinking of S^n as a simplicial set and $F \circ S^n$ refers to the composition $\Delta^{op} \xrightarrow{S^n} \Gamma \xrightarrow{F} \mathrm{Mod}_k$.
- $\pi_*\|F\|$ is the homotopy of Segal's spectrum associated to F.
- The Robinson-Whitehouse complex $C_\Gamma(F)$ (see Pirashvili-Richter) and the complex $\Xi(F)$ (see Robinson, Gamma paper §2.5).
- Given an E_∞-operad $\mathcal{C}$, RW (Definition 2.4) define the notion of a cyclic $\mathcal{C}$-complex, which is essentially a functor $\Gamma \to Ch_k$, along with various operad compatibility maps, e.g. $\mathcal{C}_S \otimes \mathcal{M}_{S\sqcup_{s,t}T} \to \mathcal{M}_T$. One important example is $S \mapsto F(S)$ (where $F(S)$ is regarded as a chain complex concentrated in degree zero).
- Given a $\mathcal{C}$-complex $\mathcal{M}$, RW (§2.8) construct its realization. This is sort of like a geometric realization, but there are more steps. *The more technical work involving $|\mathcal{M}|$ involves showing it is zero for certain $\mathcal{M}$, and for this it suffices to show that $|\mathcal{M}|'$ is zero; that one is somewhat less awful to define.*

1. Various isomorphisms

There is an isomorphism

$$\pi_*^{st}(F) \cong \mathrm{Tor}_*^{\Gamma}(t, F)$$

attributed to Pirashvili in the Hodge paper, which is long and I haven't looked at it. This is used to prove the following three facts, whose proofs all follow the same strategy: compute each side on a specific choice of projective generators. In each case, they show that this is the ground ring in dimension 0, and 0 in dimension > 0. Showing the chain complexes are acyclic involves (painfully) constructing an explicit nullhomotopy.

(1) $\pi_*\|F\| \cong H_*\big|\mathcal{C}\text{-complex that takes } S \mapsto F(S)\big|$ (RW, published version, §7)

(2) $\pi_*^{st}(F) \cong H_* C_\Gamma(F)$ (Pirashvili-Richter §4)

(3) $H_*\Xi(F) \cong$ any of these (Robinson Gamma paper, Corollary 3.7)

Moreover, I think Robinson §3.5 in the E_∞ paper is trying to tell us we have the following relationships

$$\begin{array}{ccccc} \mathrm{Tor}^\Gamma & \cong & \mathrm{Tor}^\Omega & \overset{??}{\cong} & H_* C_\Gamma \\ & & \Vert\wr_{\text{sort of}} & & \\ & & H_*\Xi & & \end{array}$$

which are explained in section 3. Caveats are needed for $H_*\Xi$, and I'm not sure if "??" can be turned into a proof, or is just an intuition.

2. Algebraic properties

The only proof of algebraic properties I've seen is in RW. Setup:

- B is an E_∞ dga;
- $A \subseteq B$;
- M is a B-module.

RW define:

- $\mathcal{K}_K(A; M) = |F|$ for the Γ-module $F : S \mapsto A^{\otimes_K S} \otimes_K M$;
- $\mathcal{K}_K(B/A; M) = \mathcal{K}_K(B; M)/\mathcal{K}_K(A; M)$
- $H\Gamma_*(B/A; M) = H_*\mathcal{K}(B/A; M)$

RW prove:

- **transitivity** (3.4): this essentially just requires $H\Gamma_*$ to be the homology of something of the form $\mathrm{cofib}(F(A) \to F(B))$ for any functor F.
- $\mathcal{K}_K(K; M) \simeq 0$ (Lemma 3.7, Appendix A): this is the hard one, involving an explicit nullhomotopy.
- **base change** (3.9): this follows from the previous fact in a more-or-less formal way. It involves filtering the realization of our $\mathcal{C}$-complex $A^{\otimes(-)} \otimes M$ and observing (using the previous fact) that the quotients are independent of the ground ring. The second part of 3.9 is formal.

Question 1. Can we show $H\Gamma_*(K; M)$ is trivial either using Ξ (this would cut out a step but would probably be very messy), or using a formal argument involving Tor or π_*^{st}?

3. Bar construction

3.1. **Relating C_Γ and bar construction.** The slogan is that the Robinson-Whitehouse complex $C_\Gamma(F)$ is essentially the bar construction for computing $\mathrm{Tor}^\Gamma_*(t, F)$... or, more precisely, Tor in a category isomorphic to the category of Γ-modules. There is a Morita equivalence between left Γ-modules and left Ω-modules, which can be repackaged as the statement

$$\mathrm{Tor}^\Gamma_*(t, F) \cong \mathrm{Tor}^\Omega_*(\varpi, cr(F))$$

(Robinson "E_∞" Theorem 1) where ϖ and $cr(F)$ should be thought of as the corresponding Ω-modules to t and F (but they can be given precise definitions).

In §3.5, Robinson ("E_∞") says that $\mathrm{Tor}^\Omega_*(\varpi, cr(F))$ can be computed by a bar construction

$$B_q(\varpi, \Omega, cr(F)) = \bigoplus_{\substack{\text{chains in } \Omega \\ S_0 \twoheadleftarrow \ldots \twoheadleftarrow S_q}} \varpi(S_0) \otimes k\{\text{all simplices } S_0 \overset{f_1}{\twoheadleftarrow} \ldots \overset{f_q}{\twoheadleftarrow} S_q\} \otimes cr(F)(S_q)$$

$$\overset{\text{def.}}{=} \varpi \bigoplus_{\substack{\underline{1} \twoheadleftarrow \ldots \twoheadleftarrow S_q \\ |S_1| > 1}} k\{\text{all simplices } \underline{1} \overset{f_1}{\twoheadleftarrow} \ldots \overset{f_q}{\twoheadleftarrow} S_q\} \otimes cr(F)(S_q).$$

Typical elements thus look like

$$(S_q \overset{f_q}{\twoheadrightarrow} \ldots \underset{\neq}{\overset{f_1}{\twoheadrightarrow}} \underline{1}) \otimes \kappa. \tag{1}$$

This is almost the Robinson-Whitehouse complex (see PR §2), whose terms look like

$$(\underline{n} \overset{f_q}{\twoheadrightarrow} \ldots \overset{f_1}{\twoheadrightarrow} \underline{1}) \otimes \tau$$

for $\tau \in F(\underline{n})$. If we're trying to compute $\mathrm{Tor}^\Omega(\varpi, cr(F))$, I'm not sure why we're getting $F(\underline{n})$ instead of $cr(F)(\underline{n})$ here. The rest of PR proves $H^\Gamma(F) \cong \pi^{st}_*(F)$ by computing each side on projectives, so I'm not sure if there's a way to work out the details here.

3.2. **Relationship between Ξ and bar complex.** Robinson (§3.5, E_∞ paper) notes that you can break up $(S_q \overset{f_q}{\twoheadrightarrow} \ldots \underset{\neq}{\overset{f_1}{\twoheadrightarrow}} \underline{1})$ as:

- a (possibly empty) string of isomorphisms, followed by
- a (possibly empty) string of surjections starting with a non-isomorphism and ending in a non-isomorphism to $\underline{1}$.

This is useful because

- simplices in the Barratt-Eccles operad look like a string of isomorphisms;[1]

[1] Why? $E\Sigma_n$ is the nerve of the category whose object set $= \Sigma_n$, and there is one morphism between any pair. You can think of this as a subcategory of the category of sets of cardinality n (i.e. the objects are now $\sigma([1, \ldots, n])$ for $\sigma \in \Sigma_n$), with the obvious isomorphism between any pair. Obviously the category of *all* finite sets of cardinality n is much bigger... somehow the idea is that this doesn't matter?

- strings of surjections starting with a non-isomorphism and ending in a non-isomorphism to $\underline{1}$ correspond to simplices in $N(S//\Omega//\underline{1})$ (by definition—see Definition 3). This is good because Lemma 2 says that $N(S//\Omega//\underline{1}) \simeq \partial\widetilde{T}_S$.

So essentially our bar construction looks like "Barratt-Eccles $\otimes \partial\widetilde{T} \otimes cr(F)$" and there is an identification of $cr(F)$ in Corollary 2. Recall this is a complex that computes Tor^Γ; Robinson defines a bigrading on it corresponding to these two pieces, such that taking homology in the $\partial\widetilde{T}$ direction (i.e., the E_1 page of a spectral sequence) results in

$$H_*(\partial\widetilde{T}) \otimes (\text{something related to Barratt-Eccles}) \otimes cr(F)$$

and we know that $H_*(\partial\widetilde{T})$ is the Lie representation. Moreover, Corollary 2 gives a computation of $cr(F)$. That is, Ξ shows up as the E_1 page in a spectral sequence computing Tor^Γ, and under certain circumstances this spectral sequence collapses.