PROOF OF THEOREM 5.4

1. First definitions and comparisons

Let Γ denote the category of finite based sets and based maps, and k a fixed ground field.

Definition 1.1. A *left* Γ*-module* is a functor $\Gamma \to k - mod$, while a *right* Γ*-module* is a functor $\Gamma^{op} \to k - mod$.

Definition 1.2. Given a DGA Λ and a Λ-module R, the *(graded) Loday functor* is the left Γ-module

$$\mathcal{L}(\Lambda, R)\colon \Gamma \to k - mod, \qquad [n] \mapsto \Lambda^{\otimes_R n}.$$

Definition 1.3. Let Lie^*_m denote the dual Lie representation. We define a functor Ξ from left Γ-modules to bicomplexes by

$$\Xi(F)_{p,q} = \mathcal{B}\left(\mathrm{Lie}^*_{q+1}, \Sigma_{q+1}, F(q+1)\right) = \mathrm{Lie}^*_{q+1} \otimes k\left[\Sigma_{q+1}^{\times p}\right] \otimes F(q+1).$$

One differential comes from the bar complex and is easy; the other is more complicated.

Definition 1.4. Given a DGA Λ and a Λ-module R, the Γ-*cohomology* of Λ over R is the bigraded abelian group

$$H\Gamma^{*,*}(\Lambda, R) \coloneqq H^* \mathrm{Tot}\,\mathrm{Hom}^*_R\left(\Xi(\mathcal{L}(\Lambda, R)), R\right).$$

The above clarifies the notation from both of [RW02, Rob03] by resolving the ambigous placements of Ξ and also identifing the bidegrees.

Proposition 1.5 ([Rob03, Cor 3.7] or [Rob18, Cor. 3.17])**.** *There is an isomorphism* $\pi_*(F) \cong H_* \mathrm{Tot}\Xi(F)$ *natural in left* Γ*-modules* F.

We actually need the *dual* of this, which I am having trouble formulating. Here is a guess, which really doesn't make sense, as R is not defined for an arbitrary F

Claim 1.6. *We have an isomorphism*

$$\pi^* \mathrm{Hom}_R(F, R) \cong H^* \mathrm{Tot}\,\mathrm{Hom}_R(\Xi(F), R)$$

natural in left Γ*-modules* F.

Lemma 1.7 ([Rob18, §4.9])**.** *A homotopy ring spectrum* $V = (V, \mu, \eta)$ *determines a graded right* Γ*-module*

$$n \mapsto \pi_* \mathsf{End}(V)(n) = V^*(V^{\wedge n})$$

Remark 1.8. Suppose V satisfies a universal coefficient theorem:

$$V^*(V^{\wedge k}) \simeq \mathrm{Hom}_{V_*}\left((V_* V)^{\otimes k}, V_*\right). \tag{1.9}$$ {UCT_EQ}

Then we have an isomorphism of graded right Γ-modules

$$\pi_* \mathsf{End}(V) \equiv \mathrm{Hom}^*_{\mathcal{V}_*}\left(\mathcal{L}(V_* V, V_*), V_*\right).$$

COMP_COR **Corollary 1.10.** *For* V *satisfing* (1.9)*, we have an isomorphism between* Γ*-homology of* $V_* V$ *and the cohomotopy of* $\pi_* \mathsf{End}(V)$.

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Proof. We have

$$\begin{aligned}
H\Gamma^{*,*}(V_*V, V_*) &= H^*\operatorname{Tot}\operatorname{Hom}^*_{V_*}\left(\Xi\mathcal{L}(V_*V, V_*), V_*\right)\\
&= \pi^* \operatorname{Hom}^*_R\left(\mathcal{L}(V_*V, V_*), V_*\right)\\
&= \pi^*\pi_*\mathsf{End}(V)\\
&= \pi^*\left(V^*(V^{\wedge\bullet})\right). \qquad (1.11)
\end{aligned}$$

{COMP_EQ}

□

2. Results

2.1. Version 1.

Theorem 2.1 ([Rob18, Thm. 4.13])**.** *Let V be a homotopy commutative ring spectrum. Then the obstruction to lifting an n-stage lives in $\pi^n\left(V^{2-n}(V^{\wedge\bullet})\right)$.*

Corollary 2.2. *If V is homotopy commutative and satisfies (1.9), then the obstruction lives in $H\Gamma^{n,2-n}(V_*V, V_*)$.* □

2.2. Version 2.

Proposition 2.3 ([Rob03, Prop. 5.4])**.** *For a homotopy commutative ring spectrum satisfying (1.9), the obstruction lives in $H^n\operatorname{Tot}\operatorname{Hom}^{2-n}_{V_*}(\Xi\mathcal{L}(V_*V, V_*, V_*))$.*

Proof. For each $1 \le m \le n+1$, we have an obstruction in

$$\begin{aligned}
V^1(\nabla^{n+1}/(\nabla^n \cup \partial\nabla^{n+1}) \wedge_{\Sigma_m} V^{\wedge m}) &= \operatorname{Hom}^{2-n}_{V_*}(Lie^*_m \otimes V_*[\Sigma_m^{n-m+1}] \otimes \mathcal{V}_*V^{\otimes m}, V_*)\\
&= \operatorname{Hom}^{2-n}_{V_*}\left(\Xi_{n-m+1,m-1}\mathcal{L}(V_*V, V_*), V_*\right).
\end{aligned}$$

These combine to form an n-cochain in $\operatorname{Tot}\operatorname{Hom}^{2-n}_{V_*V}(\Xi\mathcal{L}(V_*V, V_*), V_*)$.

so this is problem if the number of Σ_m-s is wrong, as we expect

the rest of the proof... □

Corollary 2.4. *The obstruction lives in $H\Gamma^{n,2-n}(V_*V, V_*) = \pi^n\left(V^{2-n}(V^{\wedge\bullet})\right)$.* □

3. Questions

Question 3.1. *Which side of (1.11) is more conceptual? Easier to compute? Useful?*

References

Rob03 [Rob03] A. Robinson, *Gamma homology, Lie representations and E_∞ multiplications*, Invent. Math. **152** (2003), no. 2, 331–348. doi:10.1007/s00222-002-0272-5 1, 2

Rob18 [Rob18] ———, *E_∞ obstruction theory*, Homology Homotopy Appl. **20** (2018), no. 1, 155–184. doi:10.4310/HHA.2018.v20.n1.a10 1, 2

RW02 [RW02] A. Robinson and S. Whitehouse, *Operads and Γ-homology of commutative rings*, Math. Proc. Cambridge Philos. Soc. **132** (2002), no. 2, 197–234. doi:10.1017/S0305004102005534 1