# A brief survey of König-Egerváry graphs

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# Preliminaries

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- $\mu$ is the size of a maximum matching.
- $\tau$ is the size of a minimum cover.
- $\mu \leq \tau$

# The theorem

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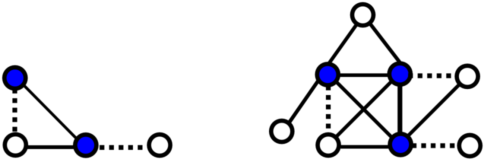

Dotted edges are in $M$. Blue vertices are in $C$.

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- Let $n = |V(G)|$.
- Let $\alpha$ denote the size of a maximum independent set.
- For all graphs, $\alpha + \tau = n$
- A graph is K-E if and only if $\alpha + \mu = n$.

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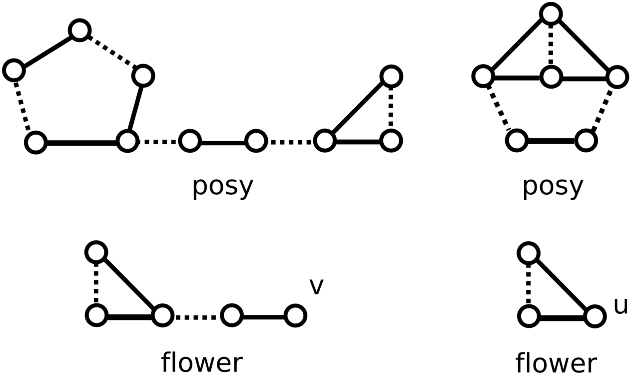
- A *flower* is an alternating path of even length that connects an exposed vertex with the tip of a blossom.
- A *posy* is two blossoms with tips joined by an odd-length alternating path whose first and last edges belong to $M$.

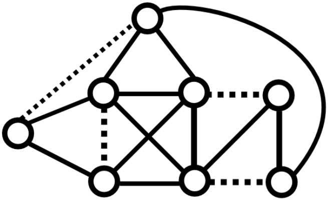
## Theorem

*(Deming, Sterboul) A graph is K-E if and only if all of its maximum matchings are both posy- and blossom-free.*



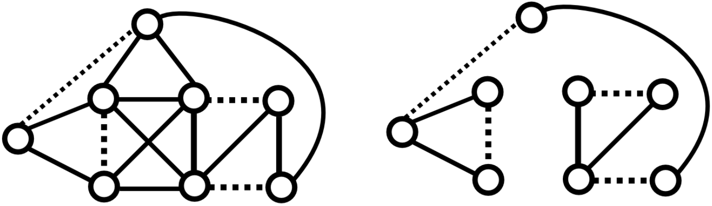

Labeled vertices are not matched.

# Finding flowers



Is this graph K-E?

# Finding flowers



Not K-E.

# What's wrong with posies?

- Assume $G$ has a perfect matching. Then $\mu = n/2$.
- If $G$ is K-E then:
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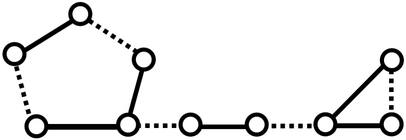
- If $G$ is K-E then:
  1. $\tau = \mu = n/2$;
  2. Because $\alpha + \mu = n$ we know that $\alpha = n/2$.
- Exactly one vertex from each matched edge is in the minimum cover $C$ and the other is in the maximum independent set $S$.

# What's wrong with posies?

Goal: One vertex from each matched edge in $S$, the other in $C$.



Red vertices in $S$. Blue vertices in $C$.

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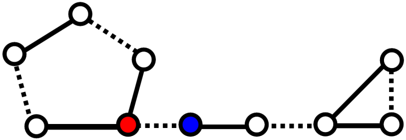

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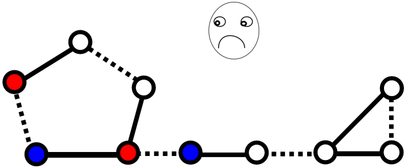

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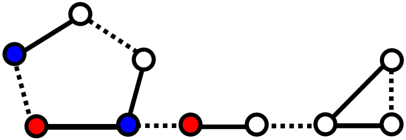

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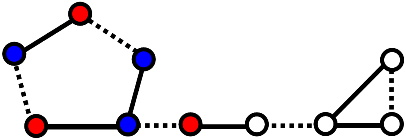

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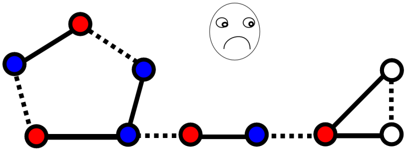

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  2. The vertices of a flower or posy.

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- $M$ can be found in polynomial time using Edmonds' Algorithm.
- Deming's algorithm runs in polynomial time.

# Proposed by graffiti.pc

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*Conjecture:* A graph is K-E if and only if $\alpha = \alpha'$.

# Some definitions

- The *neighbor difference* for an independent set $J$ is $|J| - |N(J)|$.
- An independent set $I_c$ is a *critical independent set* if $|I_c| - |N(I_c)| \geq |J| - |N(J)|$ for every independent set $J$.

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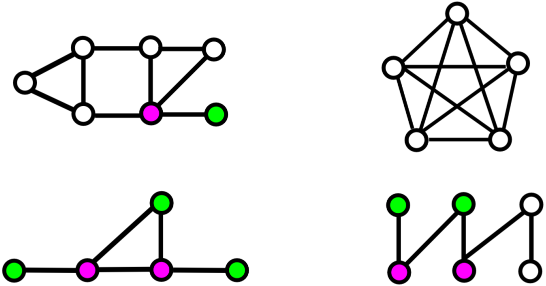

$I_c$ denoted by green vertices. $N(I_c)$ denoted by purple vertices.

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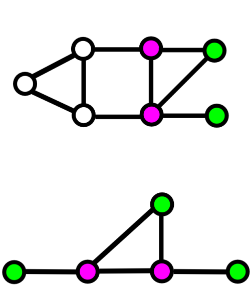

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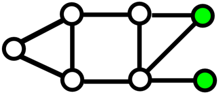
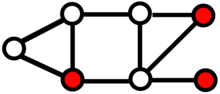
# Preliminary results

**Theorem**

*(Butenko and Trukhanov) If $I_c$ is a critical independent set in $G$, then there is a maximum independent set $J$ in $G$ such that $I_c \subseteq J$.*

- Proved for vertex-weighted graphs.
- Idea: Let $I := (J \cup I_c) \backslash N(I_c)$ . Show that $|I| \geq |J|$.

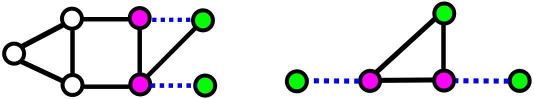




# The Matching Lemma

## Theorem

*(Larson) If $I_c$ is a critical independent set of $G$, then there is a matching of the vertices $N(I_c)$ into a subset of the vertices of $I_c$.*



- Application of Hall's Theorem

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- This contradicts the independence of $J$.

# Theorems to use

1. If $I_c$ is a critical independent set in $G$, then there is a maximum independent set $J$ in $G$ such that $I_c \subseteq J$.

2. If $I_c$ is a critical independent set of $G$, then there is a matching of the vertices $N(I_c)$ into a subset of the vertices of $I_c$.

3. For any graph $G$ with maximum critical independent set $I_c$, $\alpha = \alpha'$ if and only if $I_c \cup N(I_c) = V(G)$.

# A brief extension (Bourjolly et al, 1984)

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Minimum Weight Cover Problem (I-CP)

$$\text{minimize:} \quad \sum_{i=1}^{n} b_i x_i$$

$$\text{subject to:} \quad x_i + x_j \geq 1 \text{ for every edge } ij \in E$$

$$x_i \in \{0,1\} \text{ for all } i \in V.$$

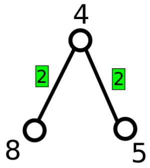
# A brief extension

Maximum Weight Matching Problem (I-WMP)

maximize: $$\sum_{ij \in E} \lambda_{ij}$$

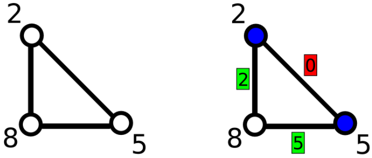
subject to: $$\sum_{j \in N(i)} \lambda_{ij} \leq b_i \text{ for all } i \in V$$

$$\lambda_{ij} \geq 0 \text{ and integral for every edge } ij \in E.$$



# First glance

- A graph is *b-KE* if $v$(I-WMP) $= v$(I-CP)
- What is the connection between b-KE and K-E?



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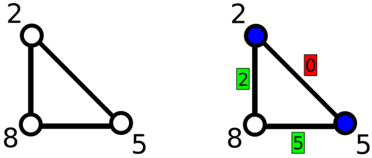


G is b-KE but underlying graph is not K-E.

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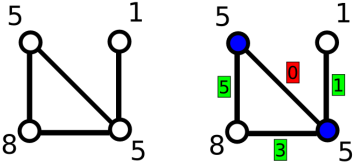

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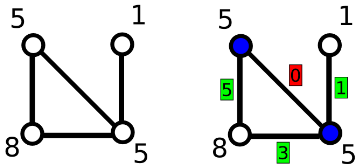
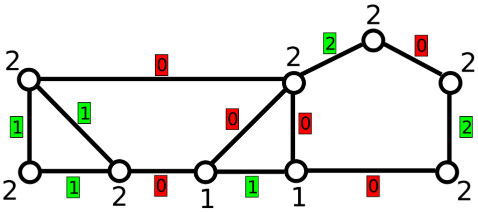



G is not b-KE but underlying graph is K-E.

# b-posies




Solve I-WMP.

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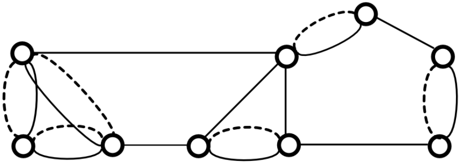

Replace matched edges with weak/strong edges.

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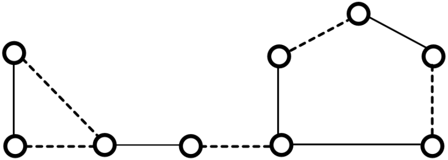

Find a b-posy.