A brief survey of König-Egerváry graphs

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Abstract

König-Egerváry graphs are the class of graphs where the size of a minimum cover equals the size of a maximum matching. In this paper we survey the major results regarding these graphs. We examine various characterizations, combinatorial results relating to independent sets, and the complexity of algorithms pertaining to this class of graphs. We finish by briefly considering a weighted generalization of König-Egerváry graphs.

Contents

1 Introduction

A *matching* of a graph G is a set of edges $M \subseteq E(G)$, no two of which share a common vertex. A *cover* of a graph is a set of vertices $C \subseteq V(G)$ such that every edge of the graph is incident with at least one of the vertices of C. Given a matching, we know that any cover must contain at least one vertex from each matching edge, and, because no two edges of a matching are incident with a common vertex, the vertices of the cover among the ends of matching edges are all distinct. Therefore, all matchings M and all covers C satisfy $|M| \leq |C|$. In 1931, Dénes Kőnig [16] proved that for a bipartite graph the size of a maximum matching equals the size of a minimum cover. Later, in the same year, Jenő Egerváry [11] extended the result to weighted graphs. It is easy to see, however, that the converse does not hold. That is, there are non-bipartite graphs for which we can find an M and a C such that $|M| = |C|$ (see Figure 1). A graph is *König-Egerváry* or *K-E* if it has a matching and cover of equal size. We can think of these graphs as a generalization of the bipartite graphs.

Figure 1: Dotted edges form a maximum matching. Solid vertices form a minimum cover.

This paper is intended to be a brief survey of the major results concerning K-E graphs. In Section 2, we consider various characterizations of K-E graphs. Section 3 covers some combinatorial properties of K-E graphs, with a focus on independent sets. In Section 4, we analyze the complexity of algorithms on K-E graphs. Finally, we briefly look at a weighted extension of K-E graphs.

Throughout the paper, $G = (V, E)$ denotes a simple, connected graph. The size of a maximum matching is denoted by μ. The minimum cover size is denoted by τ. Thus a graph is K-E exactly when $\mu = \tau$. The cardinality of an independent set of maximum size is denoted α and $n = |V(G)|$. Since the complement of a minimum cover is a maximum independent set and vice versa, we will equivalently use that a graph is K-E if and only if $\mu + \alpha = n$. For all other notation, [2] may serve as a standard reference.

2 Characterizations

Subgraph Characterizations

The earliest papers [7, 26] on K-E graphs established a characterization in terms of excluded subgraphs. Before we can state the result, we need a few definitions. Relative to a matching M, a *blossom* is an odd cycle of length $2k + 1$ in which k edges belong to M. The only vertex of this cycle not covered by M is called the *blossom tip*. A *stem* is an alternating path of even length that connects the tip of a blossom with an *exposed* vertex (i.e. a vertex that is not an endpoint of an edge of M). A stem and blossom together make a *flower*. A *posy* is two blossoms with tips joined by an odd-length alternating path whose first and last edges belong to M. It is important to note that the blossoms of a posy need not be disjoint. Figure 2 shows posies and flowers, including a posy with non-disjoint blossoms.

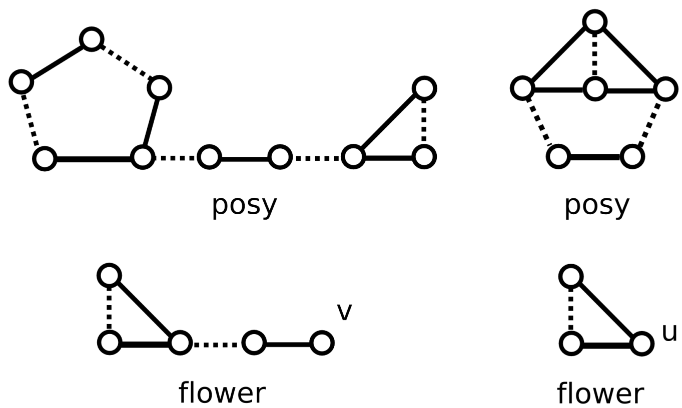

Figure 2: Dotted lines indicate a matched edge. Vertices v, u are exposed by M.

In a bipartite graph the only cycles are even, so any matching is blossom free, hence both flower-free and posy-free. In 1979, Deming and Sterboul characterized K-E graphs as follows.

Theorem 2.1. ([7, 26]) *A graph is K-E if and only if it has a maximum matching that is both flower-free and posy-free.*

Because μ cannot be both equal to and strictly less than τ, if any maximum matching of G is free of flowers and posies, then all maximum matchings are free of flowers and posies. Theorem 2.1 can thus be restated:

Theorem 2.2. ([7, 26]) *A graph is K-E if and only if all of its maximum matchings are both flower-free and posy-free.*

Sterboul used Edmonds' Algorithm [10] to prove this result. Deming's proof, given below, uses a constructive algorithm to produce a maximum independent set S of size $|S| = n - \mu$. If such an S cannot be found, then the algorithm outputs the vertices that form a flower or posy.

The construction of S is based on the following observations.

Proposition 2.3. ([7]) *In a K-E graph with a maximum independent set S and a maximum matching M:*

- *S necessarily contains every vertex exposed by M;*
- *S contains exactly one endpoint of each edge in M.*

Because S is a maximum independent set, its complement $V \backslash S$ is a minimum cover. Then $V \backslash S$ must contain one vertex from each edge in M. Thus, S contains the other vertex of each edge in M and all vertices exposed by M.

We will show presently that if G contains a flower, then we cannot construct an S of the desired size $(n - \mu)$. Let u_0 be the tip of the blossom $u_0, u_1, ..., u_{2k}, u_0$. Then $u_1u_2, u_3u_4, ..., u_{2k-1}u_{2k}$ are the edges of the blossom belonging to M. Suppose that u_0 is in S. Then, to maintain the independence of S, from the edge u_1u_2 we must take u_2 for S. Then we must take vertices u_4, u_6, and so on. But we cannot take vertex u_{2k} as it is adjacent to u_0. Thus we cannot choose an S satisfying Proposition 2.3. Therefore the tip u_0 of the blossom is not in S.

Now consider the stem $u_0, v_1, v_2, ..., v_{2j}$ of the flower in which v_{2j} is exposed by M. Because $u_0v_1 \in M$, we must take one of its vertices to be in S. We are forced to choose v_1. To maintain the independence of S, we must also take $v_3, v_5, ..., v_{2j-1}$. However, since v_{2j} is exposed by M, by Proposition 2.3, we must also take v_{2j} for S, contradicting the independence of S. Therefore this flower prohibits the construction of S. A similar problem is found if G contains a posy.

Algorithm 1 [Deming's Algorithm]

INPUT: a graph G=(V,E) and a maximum matching M
OUTPUT: either an independent set S satisfying $|S| = n - \mu$ or a set of vertices forming a posy or flower.

```
S ← ∅
H ← G
while H ≠ ∅
    if H contains a vertex z exposed by M
        color z red
        FLAG ← 1
    else
        select edge e ∈ E(H) ∩ M
        color one of e's vertices red, the other blue
        FLAG ← 0
    while a red vertex u is adjacent to uncolored vertex v
        let vw be an edge in M with w uncolored
        (if no such vw exists, then since uv ∉ M we
        have found an M-augmenting path from z to v which
        contradicts that M is maximum)
        color v blue
        color w red
        p(v) ← u
        p(w) ← v
        if two red vertices are adjacent
            if FLAG = 0
               find x, the first node in the predecessor chain p
               erase all colors
               color x blue
               color its matched vertex red
               FLAG ← 2
            else if FLAG = 1
               a flower has been found, output the vertices of p
               stop
            else if FLAG = 2
               a posy has been found, output the vertices of p
               stop
    place all red vertices in S
    delete all colored vertices from H
```

If G contains no posies or flowers, then Algorithm 1 constructs an S with $|S| = n - \mu$. The algorithm stops when H is empty and the desired S has been found or when a flower or posy has been found. Its method is straightforward; red vertices are those to be added to S. The $FLAG$ setting keeps track of where the chain of red vertices began. If the chain began at an exposed vertex, then it's possible that the chain will form a flower. If all of the exposed vertices have already been added to S, then there is only the possibility of finding a posy. $FLAG$ is set to 0 in this case. Note that the algorithm then finds an edge belonging to M and arbitrarily colors one of its vertices red and the other blue. It is possible, however, that this assignment turns out to be a poor choice. The predecessor function is used to 'remember' the initial assignment. If the graph cannot be correctly colored, the algorithm sets $FLAG$ to 2, erases the colors and tries the opposite assignment. Should this fail as well, the vertices of the predecessor function must form a posy. See Figure 3 for an example of a K-E graph in which the algorithm fails to find a maximum independent set on its first pass.

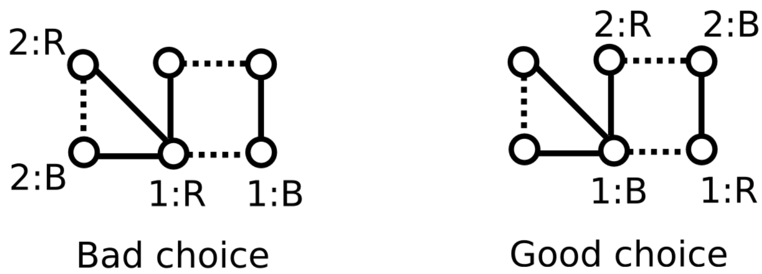

Figure 3: Noted are the assigned colors of the vertices and the order in which they were colored.

Observe that if a graph and a maximum matching form a flower, then the graph does not have a perfect matching, as the end of the stem is exposed by M. Deming [7] showed that a graph G can be extended to a graph G' with a perfect matching with the following device. Let X be the set of all vertices exposed by a maximum matching M. For each $x \in X$, add a new vertex x' and new edges xx' and all $x'y$ where $y \in N(x)$, the neighborhood of x. Let X' be the set of all new vertices x' and $A = \{xx' : x \in X, x' \in X'\}$. Then, clearly, $M' = M \cup A$ is a perfect matching.

Further, given S', a maximum independent set of G', we can easily construct a maximum independent set of G. For each $x' \in S'$ where $x' \in X'$, replace x' by its associate $x \in X$. Since $N(x) = N(x')$ for all $x \in X, x' \in X'$, the resulting set is a maximum independent set in G. Therefore $\alpha(G') = \alpha(G)$.

Proposition 2.4. ([7]) *G' is K-E if and only if G is K-E.*

Proof. Observe that $n(G') = n(G) + |A|$ and $|M'| = |M| + |A|$. However, $\alpha(G') = \alpha(G)$. So $n(G') - \alpha(G') = \mu(G')$ if and only if $n(G) - \alpha(G) = \mu(G)$. $\square$

Theorem 2.5. ([7]) *A graph G is K-E if and only if for every maximum matching M the extension G' with perfect matching M' is posy-free.*

Korach et al. [17] later strengthened this result. They showed that G is K-E if and only if the extension G' does not contain any of the forbidden subgraphs found in Figure 4 on p. 12. It is important to note that in Figure 4 the solid lines represent M-alternating paths that start and end with non-matching edges, while the starred lines represent M-alternating paths that start and end with matching edges.

It's easy to see that if a graph contains one of these subgraphs, then it is not K-E. We see a familiar posy and flower. The lone blossom is a flower, with stem length 0. We can identify a posy in both the Even Mobius Prism and the Odd Prism. In the Even Mobius Prism, $1, 1', 2', 2, 3, 3', ..., 2k', 1$ is a blossom, as is $1', 1, 2, 2', 3', 3..., 2k, 1'$. These blossoms are joined by the odd-length M-alternating path $1, 1'$. In the Odd Prism, $1, 2, 2', 3', 3, ..., 2k + 1', 2k + 1, 1$ is a blossom, as is $1', 2', 2, 3, 3'..., 2k + 1, 2k + 1', 1'$. These blossoms are joined by the odd-length M-alternating path $1, 1'$.

We know that if a graph contains one of the forbidden subgraphs then it is not K-E. The strength of the result is the converse. That is, if a graph is not

K-E, then it must contain one of the forbidden subgraphs. In other words, the Even and Odd Prisms illustrate the manner in which the blossoms of a posy may intersect.

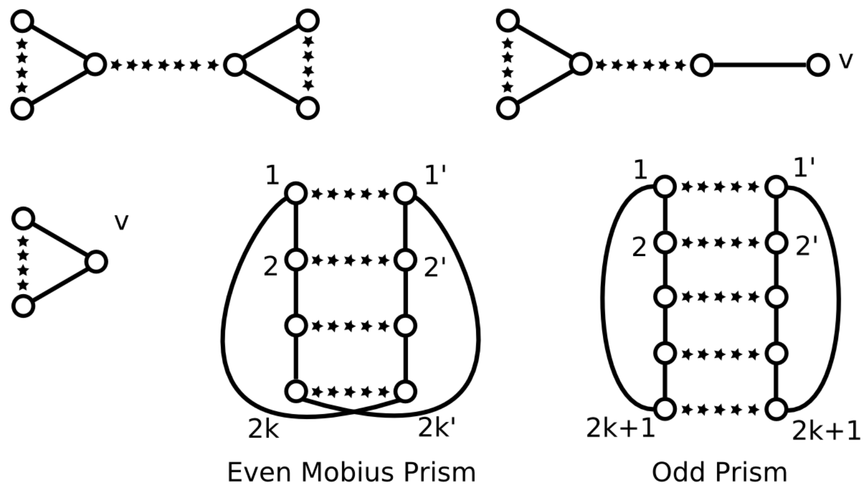

Figure 4: The forbidden subgraphs.

Characterizations involving independent sets

An independent set of vertices I_c is a *critical independent set* if $|I_c| - |N(I_c)| \geq |J| - |N(J)|$ for every independent set J. A *maximum critical independent set* is a critical independent set of maximum cardinality. The *critical independence number* $\alpha'(G)$ is the cardinality of a maximum critical independent set.

Larson [19] proved that a graph is K-E if and only if $\alpha' = \alpha$, an assertion originally conjectured by the Graffiti.pc project [8]. The following result will be useful in presenting Larson's proof.

Theorem 2.6. ([4]) *If I_c is a critical independent set in G, then there is a maximum independent set J in G such that $I_c \subseteq J$.*

Figure 5 shows two graphs G and H with labeled critical independent sets. For G, $I_c = \{a, b\}$ is a critical independent set and $J = \{a, b, c\}$ is a maximum independent set containing I_c. For H, $A_c = \{x, y, z\}$ is a critical independent set and $B = \{x, y, z\}$ is a maximum independent set containing A_c. Note that H is bipartite, hence is K-E.

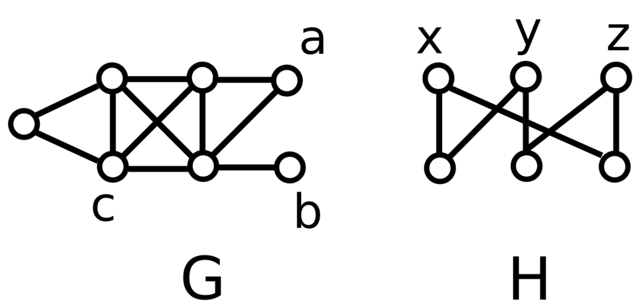

Figure 5: Two graphs with critical independent sets $\{a, b\}$ and $\{x, y, z\}$.

Lemma 2.7. ([18]) *If I_c is a critical independent set of G, then there is a matching of the vertices $N(I_c)$ into a subset of the vertices of I_c.*

Proof. This can be shown by contradiction, with a simple application of Hall's Theorem. Note that one would normally find a matching from some set B into $N(B)$. In this case, however, the problem is to find a matching from $N(I_c)$ into I_c.

Let $H \subseteq G$ with $V(H) = I_c \cup N_G(I_c)$ and $E(H) = \{xy \in E(G) : x \in I_c, y \in N_G(I_c)\}$. Assume there is no matching from $N_G(I_c)$ to I_c. Then by Hall's Theorem, there must be some $J \subseteq N_G(I_c)$ such that $|J| > |N_H(J)|$. Let $X = I_c - N_H(J)$. Then $N_G(X) \subseteq N_G(I_c) - J$ and so $|N_G(I_c)| \geq |N_G(X)| + |J|$. Then

$$\begin{aligned} |X| - |N_G(X)| &> |X| - |N_G(X)| - (|J| - |N_H(J)|) \\ &= (|X| + |N_H(J)|) - (|N_G(X)| + |J|) \\ &\geq |I_c| - |N_G(I_c)|. \end{aligned}$$

This contradicts the hypothesis that I_c is a critical independent set. $\square$

Lemma 2.8. ([19]) *For any graph G with maximum critical independent set I_c, $\alpha = \alpha'$ if and only if $I_c \cup N(I_c) = V(G)$.*

Proof. Suppose that $\alpha = \alpha'$. Let I_c be a maximum critical independent set of G. If $I_c \cup N(I_c) \neq V(G)$, then there must be $v \in V(G) \backslash (I_c \cup N(I_c))$. But since $v \notin N(I_c)$, the set $I_c \cup \{v\}$ is independent. Then $\alpha \geq |I_c \cup \{v\}| = |I_c| + 1 > |I_c| = \alpha'$, a contradiction.

Conversely, suppose that $I_c \cup N(I_c) = V(G)$. By Theorem 2.6, we know that there is a maximum independent set J such that $I_c \subseteq J$. If $I_c \neq J$, then there must be a vertex $v \in J \backslash I_c$. By assumption, $v \in N(I_c)$. But then v is adjacent to a vertex in I_c contradicting the independence of J. Then $J = I_c$ and $\alpha = \alpha'$ $\square$

We are now ready to prove Larson's characterization of K-E graphs.

Theorem 2.9. ([19]) *For any graph, $\alpha = \alpha'$ if and only if $n - \alpha = \mu$.*

Proof. Suppose that $\alpha = \alpha'$. Let I_c be a maximum critical independent set. By Lemma 2.8, $n - \alpha = |N(I_c)|$. Since I_c is independent, $\mu \leq |N(I_c)|$. Lemma 2.7 shows that there is a matching from $N(I_c)$ into I_c. Then, $\mu \geq |N(I_c)|$ and therefore $n - \alpha = |N(I_c)| = \mu$.

Conversely, suppose that $n - \alpha = \mu$. We know $\alpha' \leq \alpha$. Proceeding by contradiction, suppose that $\alpha' < \alpha$. Let I_c be a maximum critical independent set. By Theorem 2.6 there must be a maximum independent set J such that $I_c \subset J$. Because $|V \backslash J| = n - \alpha = \mu$ and J is independent, there is a matching from $V \backslash J$ into J. Then each vertex in $N(J) \backslash N(I_c)$ is matched to a vertex in $J \backslash I_c$. Therefore $|J \backslash I_c| \geq |N(J) \backslash N(I_c)|$. But then

$$\begin{aligned}
|J| - |N(J)| &= (|J \backslash I_c| + |I_c|) - (|N(J) \backslash N(I_c)| + |N(I_c)|) \\
&= (|I_c| - |N(I_c)|) + (|J \backslash I_c| - |N(J) \backslash N(I_c)| \\
&\geq |I_c| - |N(I_c)|.
\end{aligned}$$

But, since $|J| > |I_c|$ this contradicts the fact that I_c is a maximum critical independent set. So $I_c = J$, $|I_c| = |J|$, and $\alpha' = \alpha$. $\square$

Levit and Mandrescu [23] offer up yet another characterization of K-E graphs. While Larson's result (Theorem 2.9) is strictly numerical, this next result shows some structure of the relation between a maximum matching and an independent set. First, some notation. If $A, B \subset V(G)$ and $A \cap B = \emptyset$, then $(A, B) := \{e = ab \in E(G) : a \in A, b \in B\}$.

Theorem 2.10. ([23]) *For a graph, G, the following are equivalent:*

1. *G is K-E;*
2. *Each maximum matching of G is contained in $(S, V\backslash S)$ for some maximum independent set S;*
3. *Every maximum matching of G is contained in $(S, V\backslash S)$ for each maximum independent set S.*

At first glance, this result may seem like it is not new. In proving Theorem 2.2 we saw that given a maximum matching, a maximum independent set necessarily contains one vertex from every matching edge. Theorem 2.10 strengthens that observation. It shows that every maximum independent set S has this property for every maximum matching. Figure 6 shows a non-K-E graph that does not have this property.

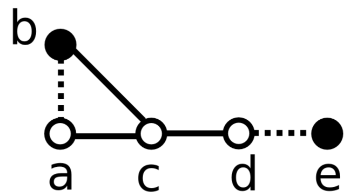

Figure 6: A non-K-E graph with solid vertices indicating a maximum independent set and dotted edges indicating a maximum matching.

3 Combinatorial properties of Independent Sets and Matchings of K-E Graphs

Levit and Mandrescu ([20, 21, 22]) have studied various properties of independent sets of K-E graphs and their connection to maximum matchings. We begin by highlighting a connection between all maximum independent sets and maximum matchings. We will then see that Hall's Marriage Theorem holds for K-E graphs. But first, we must make a brief detour.

Let $\Omega(G) = \{S \subseteq V(G) : S$ is a maximum independent set of $G\}$ and define $core(G) = \bigcap_{S \in \Omega(G)} S$. A graph is *quasi-regularizable* if each edge of G can be replaced by a non-negative number of parallel copies in order to obtain a non-empty regular multigraph. Berge [1] was able to characterize quasi-regularizable graphs.

Theorem 3.1. ([1]) *A graph G is quasi-regularizable if and only if $|S| \leq |N(S)|$ for every independent set S of G.*

Proof. We will prove the necessity of this condition. For a graph G and a subset $A \subseteq V(G)$, define $deg_G(A) := \sum_{v \in A} deg(v)$. If G is quasi-regularizable then form H its regular multigraph extension. Let $S \subseteq V(G)$ be independent. Since H is regular, $deg_H(S) \leq deg_H(N_H(S))$. If every vertex in H has degree a, then $a|S| \leq a|N_H(S)|$. Thus, $|S| \leq |N_H(S)|$. But does the same inequality hold in G? We know that $N_H(S) \subseteq N_G(S)$ since if two vertices are not adjacent in G, then they are not adjacent in H. Therefore $|S| \leq |N_G(S)|$ and since S was arbitrary, the necessity is proved. □

Levit and Mandrescu [20] used Theorem 3.1 to arrive at the following result.

Theorem 3.2. ([20]) *If G is K-E, then the following statements are equivalent:*

1. $\alpha(G) > n/2$;
2. *G has no perfect matchings;*
3. *G is non-quasi-regularizable;*
4. $|core(G)| > |N(core(G))|$.

By Theorem 3.1, it is clear that (4) implies (3). To see that (3) implies (2), we need only to observe that a perfect matching can be considered a regular multigraph extension of G. That (2) implies (1) is clear because $\alpha + \mu = n$ for K-E graphs. That (1) implies (4) is where the strength of Theorem 3.2 lies. The next two lemmas will allow us to complete the proof.

Definition. A graph G is called *α^+-stable* if $\alpha(G+e) = \alpha(G)$ for every edge $e \in E(\overline{G})$.

For example, cycles and odd-length paths are α^+-stable. Observe that a graph may be α^+-stable if it has an isolated vertex. However, a graph with two isolated vertices is not α^+-stable.

Lemma 3.3. ([20]) *If G is an α^+-stable graph with no isolated vertices, then $\alpha(G) \leq n/2$.*

Lemma 3.4. ([20]) *For a graph G, if $H = G[V \backslash (core(G) \cup N(core(G)))]$, then H is α^+-stable and has no isolated vertices. Further, $\alpha(H) = \alpha(G) - |core(G)|$.*

With these two results we can complete the proof of Theorem 3.2, for which it remains to show that (1) implies (4).

Proof. Let

$$H = G[V \backslash (core(G) \cup N(core(G)))]. \tag{1}$$

Assume, for a contradiction, that

$$\alpha(G) > n/2 \tag{2}$$

but

$$|core(G)| \leq |N(core(G))|. \tag{3}$$

By Lemma 3.4, $\alpha(H) = \alpha(G) - |core(G)|$. Now, we show that our assumption implies that $\alpha(H) > |V(H)/2|$:

$$\begin{aligned}
\alpha(H) &= \alpha(G) - |core(G)| \\
&> \frac{|V(G)|}{2} - |core(G)| - \frac{1}{2}(|N(core(G))| - |core(G)|) \\
&= \frac{1}{2}|V(G)| - |core(G)| - |N(core(G))| \\
&= \frac{|V(H)|}{2}, \qquad (4)
\end{aligned}$$

where the relations are justified, respectively, by Lemma 3.4, (2) together with (3), algebra, and (1). Using Lemma 3.4 again, we can see that H is α^+-stable and has no isolated vertices. But Lemma 3.3 implies that $\alpha(H) \leq |V(H)|/2$, and this contradicts (4). This contradiction completes the proof of Theorem 3.2. □

The following result is simply a more pleasing restatement of Theorem 3.2.

Theorem 3.5. ([20]) *If G is K-E, then the following statements are equivalent:*

1. $\alpha(G) = n/2$;
2. *G has a perfect matching;*
3. *G is quasi-regularizable;*
4. $|core(G)| \leq |N(core(G))|$.

As we've seen, K-E graphs enjoy the property that their maximum matchings M are always contained in $(S, V \backslash S)$ for maximum independent sets S (see Theorem 2.10). Based on this analogue of a biparition, Levit and Mandrescu [20] established a 'Hall-type' theorem for K-E graphs.

Theorem 3.6. ([20]) *A K-E graph G has a perfect matching if and only if every independent set S satisfies $|S| \leq |N(S)|$.*

Moreover, in light of Theorem 3.1 and Theorem 3.5, we need only check that the condition holds for $core(G) \subseteq S$.

In [21], Levit and Mandrescu were able to strengthen Theorem 3.5 by replacing the fourth equivalent condition by $|core(G)| = |N(core(G))|$. Again, we will need some preliminary results in order to prove this result.

Lemma 3.7. ([14]) *A graph G is α^+-stable if and only if $|core(G)| \leq 1$.*

To show that this is a necessary condition, suppose that $x, y \in core(G)$ and $x \neq y$. By adding the edge xy to $E(G)$, we have reduced the size of all maximum independent sets by 1. That this condition is also sufficient for α^+-stability is the strength of Lemma 3.7.

Lemma 3.8. ([21]) *Any α^+-stable K-E graph has a perfect matching ($\mu = n/2$) or a near-perfect matching ($\mu = (n-1)/2$).*

Proof. Suppose that G is an α^+-stable K-E graph that has neither a perfect matching nor a near-perfect matching. Then there must be at least two vertices exposed by M. By Proposition 2.3, we know that every vertex exposed by M must be in all $S \in \Omega(G)$. But then $|core(G)| \geq 2$ contradicting the fact that G is α^+-stable. $\square$

Lemma 3.9. ([21]) *If G is K-E, then $\bigcap\{V \backslash S : S \in \Omega(G)\} = N(core(G))$.*

Proof. Let $B = \bigcap\{V \backslash S : S \in \Omega(G)\}$ and let $v \in N(core(G))$. Then $v \notin S$ for all $S \in \Omega$ and therefore $N(core(G)) \subseteq B$. Now, let $x \in B$ and let M be a maximum matching of G. By Theorem 2.10, $M \subseteq (S, V \backslash S)$ for all $S \in \Omega(G)$. Let $S \in \Omega(G)$; then $V \backslash S$ forms a minimum vertex cover for G. Further, $|V \backslash S| = |M|$ since G is K-E. Because $x \in B$ there must be an $xy \in M$. By Proposition 2.3 we know that $y \in S$. Since $S \in \Omega(S)$ was arbitrary, this shows that $x \in N(core(G))$ and hence that $B \subseteq N(core(G))$. $\square$

Theorem 3.10. ([21]) *A K-E graph G has a perfect matching if and only if $|core(G)| = |N(core(G))|$.*

This result strengthens (4) of Theorem 3.5. In light of Theorem 3.2, it offers an interesting consequence.

Corollary 3.11. *If $|core(G)| < |N(core(G))|$, then G is not K-E.*

Proof. Let G be K-E. If G contains a perfect matching, then $|core(G)| = |N(core(G))|$. If G does not contain a perfect matching, then $|core(G)| > |N(core(G))|$. $\square$

When looking at α^+-stability, we asked whether the independence number of a graph remained constant with the addition of a new edge. We now turn our attention to edges that, upon deletion, change the independence number.

Definition. An edge $e \in E(G)$ is *α-critical* if $\alpha(G - e) > \alpha(G)$, whereas a vertex $v \in V(G)$ is *α-critical* if $\alpha(G - v) < \alpha(G)$. An edge $e \in E(G)$ is *μ-critical* if $\mu(G - e) < \mu(G)$.

We will work to show that in a K-E graph, α-critical edges are also μ-critical. Before we get there, we need two quick results.

Proposition 3.12. ([22]) *If an edge xy is α-critical, then for any $W \in \Omega(G - xy)$, both $x, y \in W$.*

Proof. If both $x, y \notin W$, then $W \in \Omega(G)$. This violates the condition that $\alpha(G - xy) > \alpha(G)$. Likewise, if $x \in W$ but $y \notin W$, then removing the edge

xy from G will not affect the independence of W in G, so $W \in \Omega(G)$, again violating the condition that $\alpha(G - xy) > \alpha(G)$. Therefore, both $x, y \in W$.

□

Lemma 3.13. ([22]) *No α-critical edge in a graph G has an endpoint in $core(G) \cup N(core(G))$.*

Proof. Let xy be an α-critical edge in G and let $W \in \Omega(G - xy)$. By Proposition 3.12, $x, y \in W$. Then $W \backslash \{x\}$ and $W \backslash \{y\}$ are both maximum independent sets for G. But then $core(G) \subseteq (W \backslash \{x\}) \cap (W \backslash \{y\}) = W \backslash \{x, y\}$. Therefore $x, y \notin core(G)$. If $x \in N(core(G))$, then this would contradict the independence of $W \backslash \{y\}$. By similar reasoning, we may show that $y \notin N(core(G))$. Therefore $x, y \notin core(G) \cup N(core(G))$. □

Theorem 3.14. ([22]) *If G is K-E, then the following are equivalent:*

1. *the edge e is α-critical;*
2. *the edge e is μ-critical and $G - e$ is K-E.*

Proof. If $e = xy$ is an α-critical edge, then by Proposition 3.12, for $W \in \Omega(G - xy)$, we know $x, y \in W$. For such a W, let $S = W \backslash \{x\}$ and M be a maximum matching in G. By Proposition 2.3, since $x \notin S$, x must be matched in M. Again by Proposition 2.3, x must be matched to a vertex in S. Since $N(x) \cap S = \{y\}$, we see x must be matched with y. So we know that the edge xy is μ-critical. Further, since $\alpha(G-e) = \alpha(G)+1$ and $\mu(G-e) = \mu(G)-1$, we see that

$$\alpha(G-e) + \mu(G-e) = \alpha(G) + \mu(G) = n(G) = n(G-e).$$

Therefore $G-e$ is K-E.

Conversely, if e is μ-critical and $G-e$ is K-E, then

$$\alpha(G-e)+\mu(G-e) = n(G-e) = n(G) = \alpha(G)+\mu(G) = \alpha(G)+\mu(G-e)+1.$$

Therefore $\alpha(G-e) = \alpha(G)+1$, and e is α-critical. $\square$

Proposition 3.15. ([22]) *If G is K-E, then every $S \in \Omega(G)$ meets each μ-critical edge in exactly one vertex, and, consequently, every $S \in \Omega(G)$ meets each α-critical edge in exactly one vertex.*

Proof. By definition, a μ-critical edge belongs to every maximum matching. By Proposition 2.3, we know that each $S \in \Omega(G)$ must contain exactly one endpoint of each matching edge. Therefore, every $S \in \Omega(G)$ contains exactly one endpoint of each μ-critical edge. By Theorem 3.14, the same holds for any α-critical edge. $\square$

Let $\eta(G)$ be the number of α-critical edges of G, $\sigma(G) = |\bigcap\{V \backslash S : S \in \Omega(G)\}|$, and $\xi(G)$ be the number of α-critical vertices of G. We quickly verify that $core(G)$ is the set of all α-critical vertices. Let $v \in V(G)$ be an α-critical vertex and suppose that $v \notin core(G)$. Then, there must exist an $S \in \Omega(G)$ such that $v \notin S$. But then removing v from G does not affect the independence of S, contradicting the fact that v is an α-critical vertex. Now, let $u \in core(G)$. If u is not an α-critical vertex, then there is $S \in \Omega(G - u)$ of size $\alpha(G)$. But then S is also a maximum independent set of G, contradicting the fact that $u \in core(G)$. Therefore, $\xi(G) = |core(G)|$.

Proposition 3.16. ([22]) *If G is K-E, then*

1. $\xi(G) + \eta(G) \leq \alpha(G)$;
2. $\sigma(G) + \eta(G) \leq \mu(G)$.

Proof. We offer a proof of (1). See [22] (p. 1689) for a proof of (2). Let $S \in \Omega(G)$. We know that $core(G) \subseteq S$. By Lemma 3.13, we know that no α-critical edge has an endpoint in $core(G)$. By Proposition 3.15, we know that S contains exactly one vertex from each α-critical edge. It follows that $\xi(G) + \eta(G) \leq \alpha(G)$. □

On the Perfect Matching Polytope

The *incidence vector* $\chi_M \in \{0,1\}^E$ of a matching M is the $\{0,1\}$-vector with $\chi_M(e) = 1$ if and only if $e \in M$. The *perfect matching polytope* $PM(G)$ of G

is the convex hull of the set of incidence vectors of the perfect matchings of G. In [9], Edmonds characterized the perfect matching polytope as the set of vectors $\mathbf{x} \in \mathbb{R}^E$ satisfying the following constraints:

$$\text{(1)} \qquad \mathbf{x} \geq 0; \qquad \text{(nonnegativity)}$$

$$\text{(2)} \qquad \sum_{e \in \partial(v)} \mathbf{x}(e) = 1 \text{ for each } v \in V; \qquad \text{(saturation)}$$

$$\text{(3)} \qquad \sum_{e \in \partial(S)} \mathbf{x}(e) \geq 1 \text{ for each odd set } S \subseteq V. \qquad \text{(blossom)}$$

Sometimes the third constraint is implied by the first two. A graph G is called *non-Edmonds* when $PM(G)$ is characterized only by constraints (1) and (2). Kayll [15] proved the following.

Theorem 3.17. ([15]) *K-E graphs are non-Edmonds.*

Two proofs are offered in [15]. In the first proof, we work with a set called the *crossing edges*. Recall that for every maximum matching M of a K-E graph, $M \subseteq (S, V \backslash S)$ when $S \in \Omega(G)$. The *crossing edges* F are defined as those edges in $(S, V \backslash S)$. A doubly stochastic matrix $B = (b_{ij})$ is built having $b_{ij} = \mathbf{x}(ij)$ if $ij \in F$ and 0 otherwise. By the Birkhoff-von Neumann Theorem [27], B can be written as a convex combination of permutation matrices, each having the property that it corresponds to a perfect matching in G. From there, it is straightforward to see that this implies that $\mathbf{x} \in PM(G)$.

For the second proof, we let $\boldsymbol{c} \in \mathbb{R}^E$ be an arbitrary vector and A be the vertex-edge incidence matrix of a K-E graph G. The constraints on the following linear programming problem are precisely Edmonds' constraints (1) and (2).

$$\begin{aligned} \text{maximize:} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to:} \quad & A\boldsymbol{x} = \mathbf{1} \\ & \boldsymbol{x} \geq \mathbf{0}. \end{aligned}$$

Kayll [15] gives a polynomial-time algorithm that takes as input an arbitrary $\boldsymbol{c} \in \mathbb{R}^E$ and produces perfect matching M^* of G and a vector $\boldsymbol{y}^* \in \mathbb{R}^V$ such that, for $\boldsymbol{x}^* := \chi_{M^*}$ the following conditions hold:

$$\boldsymbol{y}^*(u) + \boldsymbol{y}^*(v) \geq \boldsymbol{c}(uv) \text{ for each } uv \in E(G);$$

$$\sum_{u \in V} \boldsymbol{y}^*(v) = \boldsymbol{c}^T\boldsymbol{x}^*.$$

It is shown that $\boldsymbol{y}^*$ is an optimal solution to the dual of the linear programming problem above and therefore $PM(G)$ is characterized by Edmonds' constraints (1) and (2).

4 Algorithmic Aspects

A graph can be tested for the K-E property in polynomial time by using Edmonds' Algorithm [10] to find a maximum matching and then Algorithm 1 to decide whether a graph is K-E. In [25], Mishra et al. explored the complexity of algorithms that search for subgraphs with the K-E property. We will consider the following decision problems.

- K-E VERTEX DELETION SET (KE-VDS): Given a graph G and a positive integer k, does there exist $V' \subseteq V$ such that the induced subgraph $G[V - V']$ is K-E and $|V'| \leq k$?
- MAXIMUM VERTEX INDUCED K-E SUBGRAPH (MVIKES): Given a graph G and a positive integer k, does there exist $V' \subseteq V$ such that $G[V']$ is K-E and $|V'| \geq k$?

Note that for the two algorithms above, we can define analogous algorithms using edge deletion sets or edge induced subgraphs (KE-EDS, MEIKES). It is easy to see that all non-null, non-trivial graphs have K-E deletion sets and K-E subgraphs. Because K_2 is K-E, a K-E edge deletion set and vertex deletion set exist for all such graphs. Similarly, we can find a vertex set and an edge set that induce K_2.

As it is well-known (see, e.g., [12]) that for an arbitrary graph, VERTEX COVER is an NP-complete problem, Mishra et al. [25] used this to show that KE-VDS is NP-complete. Consider this variation on the VERTEX COVER problem.

- ABOVE GUARANTEE VERTEX COVER (g-VC) [25]: Let G be a graph with a maximum matching size μ and k a positive integer. The goal is to decide whether G admits a vertex cover of size at most $\mu + k$.

We will now give a reduction of g-VC to KE-VDS. Recall Deming's extension, Proposition 2.4, in which we construct G' from G, so that G' has a perfect matching and G' is K-E precisely when G is. The following lemma shows that this extension is useful for g-VC as well.

Lemma 4.1. ([25]) *Let G be a graph without a perfect matching and G' be its perfect matching extension. Then G has a vertex cover of size $\mu(G) + k$ if and only if G' has a vertex cover of size $\mu(G') + k$.*

The benefit of this lemma is that for a graph with a perfect matching such as G', we know that $\mu = n/2$.

Lemma 4.2. ([25]) *A graph with a perfect matching has a vertex cover of size at most $n/2 + k$ if and only if it has a K-E vertex deletion set of size at most $2k$.*

Proof. Let G be a graph with a perfect matching M. Let C be a vertex cover of G with $|C| \leq n/2 + k$. Consider the subset $M' \subseteq M$ of matching edges with both endpoints in C. There are at most k of these edges. So $V[M']$ is a K-E deletion set of size at most $2k$.

Conversely let K be a K-E deletion set of size $|K| \leq 2k$. Then $G' = G - K$ is a K-E graph on $n(G) - |K|$ vertices. And G' has a cover C' such that $|C'| = |\mu(G')| \leq (n(G) - |K|)/2$. Then $C' \cup K$ is a vertex cover of G and $|C' \cup K| \leq (n - |K|)/2 + |K| = n/2 + |K|/2 \leq n/2 + k$. □

Theorem 4.3. ([25]) *The* K-E VERTEX DELETION SET problem is NP-complete.

This theorem follows directly from Lemmas 4.1 and 4.2 and the fact that VERTEX COVER is NP-complete.

Theorem 4.4. ([25]) *The* K-E EDGE DELETION SET *problem is NP-complete.*

Although similar in statement to Theorem 4.3, the proof of Theorem 4.4 has an entirely different flavor. Mishra et al. created a polynomial reduction of the NP-complete problem MIN 2-SAT DELETION to the KE-EDS. In the MIN 2-SAT DELETION problem, the goal is to find the minimum number of clauses that must be deleted in order to make a 2-SAT problem satisfiable (see [5] for definition and discussion).

Theorem 4.5. ([25]) *Let G have a perfect matching. Let $|E| = m$. Then G has a subgraph with at least $3m/4 + n/8$ edges that is K-E. This subgraph can be found in $O(n + m)$ time.*

Theorem 4.6. ([25]) *A graph with m edges has an edge-induced K-E subgraph of size at least $3m/5$.*

Theorems 4.5 and 4.6 were proved using a randomized algorithm that constructs a K-E subgraph H and considers the expected size of a 'part' of H.

Weighted Extension

In [3], Bourjolly et al. considered a fundamentally different class of graphs. They studied the weighted version of the MINIMUM COVER and MAXIMUM MATCHING problems. We begin by defining these problems. Again, we will only consider simple, connected graphs $G = (V, E)$. Here, each vertex has an integral valued weight associated with it by the weight function $b : V \to \mathbb{Z}^+$. The decision variable $x_i = 1$ if and only if vertex i is in the minimum weight cover.

MINIMUM WEIGHT COVER PROBLEM (I-CP)

$$\text{minimize:} \quad \sum_{i=1}^{n} b_i x_i$$

$$\text{subject to:} \quad x_i + x_j \geq 1 \text{ for every edge } ij \in E$$

$$x_i \in \{0, 1\} \text{ for all } i \in V.$$

For the weighted matching problem, we seek integral valued weights λ_{ij}

on the edges $ij \in E$ so that the sum of these weights on the edges incident to any given vertex i do not exceed b_i.

MAXIMUM WEIGHT MATCHING PROBLEM (I-WMP)

$$\text{maximize:} \qquad \sum_{ij \in E} \lambda_{ij}$$

$$\text{subject to:} \qquad \sum_{j \in N(i)} \lambda_{ij} \leq b_i \text{ for all } i \in V$$

$$\lambda_{ij} \geq 0 \text{ and integral for every edge } ij \in E.$$

Definition. Call a feasible solution λ to I-WMP a *b-matching.* An edge ij is *active* (in λ) if $\lambda_{ij} > 0$. Otherwise, the edge ij is *passive.* When $\Sigma_{j \in N(j)} \lambda_{ij} = b_i$, the vertex i is said to be *saturated* by λ.

It's important to recognize that calling a solution to I-WMP a b-matching is an abuse of the word 'matching'. When all the vertex weights b_i are identically 1, such a solution is a usual matching. However, in a general I-WMP, there is no restriction that a vertex be incident with only one active edge, as we might assume from the word matching.

Definition. Let $v(P)$ be the optimal value of an optimization problem P. A graph is called *b-KE* when $v(\text{I-WMP}) = v(\text{I-CP})$.

We can think of the K-E graphs as the subset of the b-KE graphs when $b_i = 1$ for every vertex i. As a natural first question, we may ask whether a K-E graph is b-KE for all weight functions b. Figure 7 shows that, depending on b, some K-E graphs are not b-KE graphs and some non-K-E graphs are b-KE's.

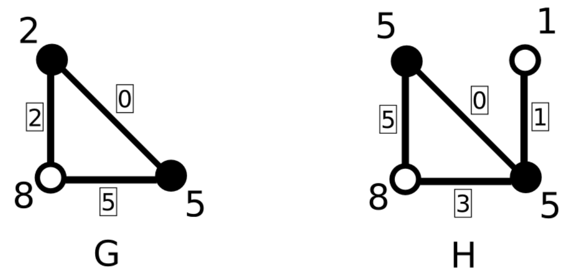

Figure 7: b-weights are denoted by free numbers. λ values are denoted by boxed numbers. A minimum weight cover is denoted by solid vertices. G is not K-E, but is b-KE. H is K-E, but is not b-KE.

The problems C-WMP and C-CP are the continuous relaxations of their respective integer linear programming problems, obtained by dropping the integrality constraints on x and λ. The feasible solutions to C-WMP are called *b-fractional-matchings*. Observe that C-WMP is the dual linear program to C-CP. Thus we get the following chain of inequalities:

$$v(\text{I-WMP}) \leq v(\text{C-WMP}) = v(\text{C-CP}) \leq v(\text{I-CP}).$$

These relations are all equalities when a graph is b-KE. We will see later that $v(\text{C-CP}) = v(\text{I-CP})$ implies that a graph is b-KE, but having $v(\text{I-WMP}) = v(\text{C-WMP})$ does not imply that a graph is b-KE. For example, every vertex in Figure 8 is saturated and therefore $v(\text{I-WMP}) = v(\text{C-WMP})$, but the graph is not b-KE.

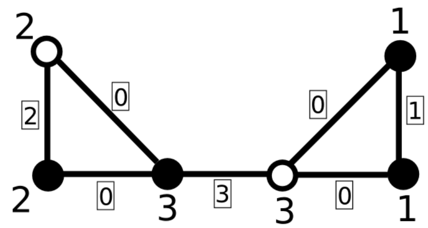

Figure 8: Node b-weights are indicated by free numbers, while λ values are indicated by boxed numbers. Solid vertices indicate a minimum weight cover.

The following result highlights some important properties of b-KE graphs.

Theorem 4.7. ([3]) *The following are equivalent:*

1. (G, b) *is b-KE;*

2. *C-CP has an integral optimal solution;*

3. *For every minimum weight cover C and for every maximum fractional b-matching λ, every edge whose endpoints are both in C is passive in λ;*

4. *There exist a minimum weight cover C and a maximum fractional b-matching λ such that every edge whose endpoints are both in C is passive in λ;*

5. *G has an independent set S such that for every $X \subseteq V \backslash S$, we have $b(X) \leq b(N(X) \cap S)$;*

6. *Given a maximum fractional b-matching λ, there is a subset $C \subseteq V$ such that:*

   (a) *every node of C is saturated by λ;*

   (b) *every edge that is active in λ has exactly one endpoint in C;*

   (c) *every edge that is passive in λ has at least one endpoint in C.*

The statements (3) and (4) follow from (1) by using the Complementary Slackness Theorem (see, e.g., [6]). Statement (6a-c) shows that we can decide whether (G, b) is b-KE without even finding an integral solution to the matching problem. Thus, although it refers to the linear relaxation of WMP, it actually informs about the integer-version of this problem, I-WMP, since the latter is essential for the definition of b-KE. The following property was also shown for the edges active in λ.

Corollary 4.8. ([3]) *If (G, b) is b-KE, then the subgraph spanned by the active edges of any fractional b-matching must form a bipartite graph.*

Proof. By Theorem 4.7 (6b) every active edge in a fractional b-matching has exactly one endpoint in a covering C. This guarantees that the active edges cannot form an odd cycle, as that would require an active edge with two endpoints in C. □

Bourjolly et al. [3] generalized Theorem 2.2 to weighted graphs. As a simplifying step, they showed that they need only consider a graph that has a perfect fractional b-matching. Unlike Deming's graph extension (Proposition 2.4), they reduced the problem by considering a subgraph, which they call the 'fickle' subgraph. We give the definition in order to highlight its connections to results that appeared in Section 3. But first, we should define the MAXIMUM INDEPENDENT SET (I-ISP) problem. It is equivalent in form to the MINIMUM WEIGHT COVER problem (I-CP) except for an inequality. We'll use C-ISP to denote the continuous relaxation of I-ISP.

MAXIMUM INDEPENDENT SET PROBLEM (I-ISP)

$$\begin{aligned} \text{minimize:} \quad & \sum_{i=1}^{n} b_i x_i \\ \text{subject to:} \quad & x_i + x_j \leq 1 \text{ for every edge } ij \in E \\ & x_i \in \{0,1\} \text{ for all } i \in V. \end{aligned}$$

Let (G, b) be a weighted graph. Let V_1 be the set of all $j \in V$ such that $x_j = 1$ for all optimal solutions of C-ISP. Let V_0 be the set of all $j \in V$ such that $x_j = 0$ for all optimal solutions of C-ISP. It is then shown that $V_0 = N(V_1)$. This can be considered a weighted generalization of Lemma 3.9. The *fickle subgraph* of (G, b) is $(G[V \backslash (V_0 \cup V_1)], b)$. The same writers, in a previous paper [13] showed that the fickle subgraph has a perfect fractional b-matching.

Theorem 4.9. ([3]) *The weighted graph (G, b) is b-KE if and only if its fickle* *subgraph is b-KE.*

Using the fickle subgraph will reduce the size of the problem of deciding whether a graph is b-KE; however, the fickle subgraph may be hard to find as it requires not just one solution to C-ISP, but all solutions.

We now turn to creating an algorithm to decide whether a graph is b-KE. Bourjolly et al. [3] propose a characterization of b-KE graphs that generalizes Deming's [7] and Sterboul's [26] characterization (Theorem 2.2) to weighted graphs. It is assumed that the fractional b-matching problem has been solved. Recall that an edge ij is *active* if $\lambda_{ij} > 0$ and is *passive* if $\lambda_{ij} = 0$. From this b-matching, we build a multigraph H with the properties:

- $V(H) = V(G)$;
- Two vertices of H are linked by a *weak* edge if they were linked by a passive edge in G;
- Two vertices of H are linked by a *weak* edge and a *strong* edge if they were linked by an active edge in G.

We now generalize flowers and posies to the multigraph H. Let Γ be a *simple* odd cycle; that is, each vertex of Γ is incident to exactly two edges of Γ.

- Γ is a *b-blossom* if every vertex of Γ but one is incident to two edges of Γ having different nature. The remaining vertex is the *tip*. The tip can be *weak* or *strong*, depending on the nature of its incident edges.

- A *b-posy* consists of two b-blossoms together with an alternating path connecting their tips such that a weak tip is incident with a strong edge of the path and a strong tip is incident with a weak edge of the path.

- A *b-flower* consists of a b-posy where one of the b-blossoms has been replaced by a single unsaturated vertex.

While this is quite similar to previous definitions of flowers and posies (see p. 5), there are some fundamental differences. As a b-matching does not form a matching in the underlying graph, it is possible for two strong edges to meet at a vertex. Further, as each active edge in G creates a strong and a weak edge in H, the recognition of b-posies is more difficult, as we now have a choice as to which type of edge we take. Further, as the b-blossoms of a b-posy may overlap, we might need to take the strong edge between two vertices in one b-blossom and the weak edge in the other.

Figure 9 on p. 40 demonstrates the process of finding a b-posy. Because every vertex is saturated, we know the b-matching is optimal. The last graph depicts a b-posy, indicating the graph is not b-KE. Note that the two b-blossoms are joined by an alternating path of even length, a configuration not allowed in an unweighted posy.

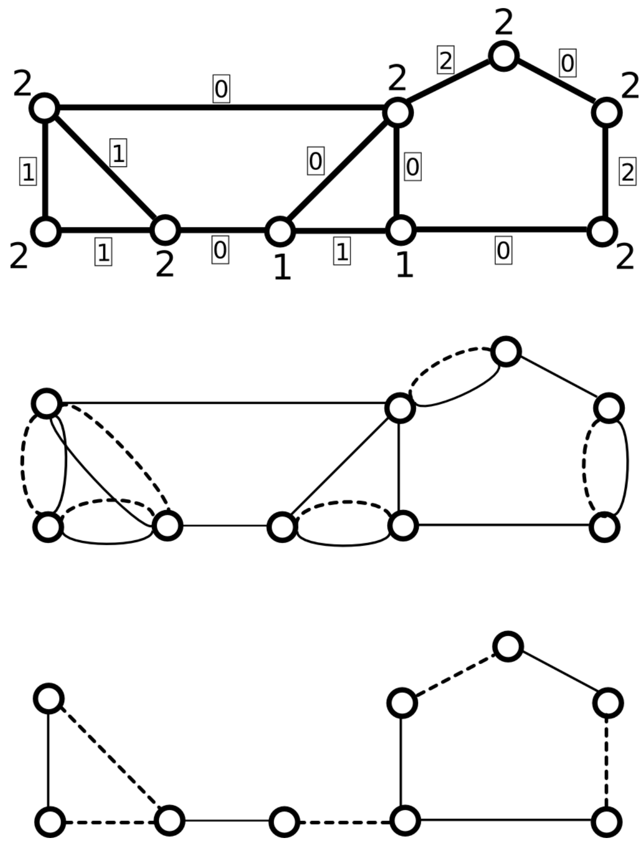

Figure 9: b-weights are indicated by free numbers; λ values are indicated by boxed numbers. In the second and third graphs, strong and weak edges are denoted by dotted edges and solid edges, respectively.

Theorem 4.10. ([3]) *The weighted graph* (G, b) *is b-KE if and only if H has no b-posies or b-flowers.*

The proof follows the method that Deming used to prove Theorem 2.2. Bourjolly et al. gave a direct proof to show that the lack of b-flowers and b-posies is necessary. To show that this is also sufficient, they gave an algorithm that either confirms that the graph is b-KE or identifies a b-posy or b-flower. Aside from the difficulties in choosing which weak or strong edge to take, their algorithm follows a similar pattern to Deming's [7] (see Algorithm 1).

5 Conclusions

In this paper, we examined various characterizations, combinatorial, and algorithmic properties of K-E graphs. König-Egerváry graphs continue to represent an active area of research in mathematics. Many of the articles cited were published within the last decade. This final result (Theorem 4.10), dealing with b-posies on weighted graphs seems to be an attractive generalization of the posy characterization proposed by Deming and Sterboul (Theorem 2.2). It leaves open the question as to what other results that we have considered generalize nicely to weighted graphs.

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