Lecture notes, Week 1

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Abstract

These are lecture notes for MAT244 lectures held in Summer 2021 in the University of Toronto. If you notice any typos, email me at `petr.kosenko@mail.utoronto.ca`.

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1 Introduction to Ordinary Differential Equations

1.1 What are they?

Notation. In our course we will denote the argument by x, and the target function by $y(x)$. By $y'(x)$ we are denoting the usual derivative $\frac{d}{dx}y(x)$, and for $n \geq 1$ the expression $y^{(n)}(x)$ stands for $\frac{d^n}{dx^n}y(x)$ (n-th derivative of y).

Sometimes, x is replaced by t and y is replaced by x, so $y(x)$ turns into $x(t)$. The latter notation is often used in physics and dynamics, as t stands for "time".

Definition 1.1. An **ordinary differential equation** (ODE for short) is a an expression of form

$$F(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0 \tag{1}$$

where $F : \mathbb{R}^{n+1} \to \mathbb{R}$ is a function, and $y(x) : (a, b) \to \mathbb{R}$ is an n times differentiable function defined on an open interval.

If $y(x)$ satisfies 1 for all $x \in (a, b)$, then it is called a **solution** or 1 on (a, b).

Keep in mind that this is a **very** general definition, as we are not imposing any restrictions on F.

Example 1.1. Let $F_1(x, y, z) := z$. Then, if we plug $x, y(x)$ and $y'(x)$ into F, we will end up with

$$F_1(x, y(x), y'(x)) = y'(x) = 0.$$

It is easy to verify that $y(x) = 1$ will be a solution of this ODE, but $y(x) = x$ will not be:

$$F_1(x, x, x') = F(x, x, 1) = 1 \neq 0.$$

Example 1.2. Let $F_2(x, y, z) := z - y$. Then, if we plug $x, y(x)$ and $y'(x)$ into F, we will end up with

$$F_2(x, y(x), y'(x)) = y'(x) - y(x) = 0.$$

It is easy to verify that $y(x) = e^x$ will be a solution of this ODE, but $y(x) = x$ will not be:

$$F_2(x, e^x, (e^x)') = F_2(x, e^x, e^x) = e^x - e^x = 0.$$

$$F_2(x, x, x') = F(x, x, 1) = 1 - x \neq 0.$$

Remark. In this lecture notes I will usually replace $y(x)$ or $y'(x)$ with y and y'. In practice, this does not cause much confusion, as the argument is assumed to be x implicitly.

1.2 Why do we even care about ODE's?

I am not sure if I am able to do a better job then the Chapter 1.1 in the textbook we are using. But, briefly speaking, the idea is that ODE's pop up in a lot of places, because, for example, many physical, biological phenomena are modeled by differential equations. Newton's Second Law serves as one reason for this.

1.3 Classifying ODE's

When constructing an ODE, mathematicians don't use the Definition 1.1, as it is too general for our purposes. First thing we can do, is to restrict ourselves to studying ODE's of form

$$y^{(n)}(x) = G(x, y(x), \ldots, y^{(n-1)}(x)), \tag{2}$$

isolating the highest derivative. If an ODE we study looks like (2), we will call it an **n-th order ODE**.

In this course we will studey the following ODE's, from simple to hard:

1. $y' = f(x)$ – I will call these trivial first-order ODE's, as they can be immedately solved by taking the antiderivative.
2. $y' = f(y)$ – These ODE's are called **autonomous** ODE's, as the RHS does not depend on x.
3. $y' = \dfrac{M(x)}{N(y)}$ – These ODE's are called **separable** ODE's, as they can be quickly solved by "separating the variables":
$$N(y)y' = N(x).$$
4. Next we are going to study linear first order ODE's
$$a_0(x)y + a_1(x)y' = b(x),$$
where a_i and b are all functions (possibly non-constant) of x. If $b = 0$, then the respective ODE is called homogeneous.
If a_0, a_1 and b are constant, then we say that this is a constant-coefficient ODE.
5. In a similar fashion we define linear ODE's of order n:
$$a_0(x)y + a_1(x)y' + \ldots a_{n-1}y^{(n-1)}(x) + a_n y^{(n)}(x) = b(x).$$
6. Moreover, we are going to study linear systems of first-order ODE's, for example,
$$\begin{cases} y_1'(x) = ay_1(x) + by_2(x) \\ y_2'(x) = cy_1(x) + dy_2(x). \end{cases}$$
This, of course, generalizes to n dimensions, as well.
7. Finally, we are going to briefly cover non-linear ODE's and non-linear systems of ODE's as well.

1.4 Slope fields + direction fields

Definition 1.2. A **slope field** is a function, which corresponds a slope to every point in a region in $\mathbb{R}^2$.

A slope field can be defined by a function $k : \mathbb{R}^2 \to \mathbb{R}_{\geq 0} \cup \{\infty\}$, where ∞ stands for the vertical slope, 0 stands for the horizontal line, and non-zero values stand for the tangent of the angle between the slope and the horizontal line.

Definition 1.3. A **direction field** is a function, which corresponds a vector to every point in a region in $\mathbb{R}^2$.

Remark: Sometimes they are called vector fields. Most textbooks do not differentiate between slope fields and direction fields, but one should, because, **technically**, a vector carries more information than just a slope!

If we are given a first-order ODE $y' = f(x, y)$, then we can construct its slope field just by considering the function $(x, y) \to f(x, y)$.

Proposition 1.1. Any solution of $y' = f(x, y)$ is tangent to the slopes, defined by $(x, y) \to f(x, y)$.

The proof is not hard – just recall the geometric definition of the derivative.

I am a bit too lazy to include the pictures here as well, but feel free to play around with Desmos, Geogebra and other sites to understand how slope fields work.

Question: What is a good way to correspond a **direction** field to $y' = f(x, y)$? Do you think there is a way to capture the "speed" of a solution using arrows, not just slopes?

1.5 Wait, when are we going to actually solve some ODE's?

- The simplest ODE's to solve are trivial first-order ODE's:

$$y'(x) = f(x). \tag{3}$$

  Solving (3) amounts to taking the antiderivative:

$$y' = f(x) \Leftrightarrow \int y'dx = \int f(x)dx + C \Leftrightarrow \int dy = \int f(x)dx + C \Leftrightarrow y = \int f(x)dx + C.$$

- Next type on our list are **autonomous** ODEs:

$$y' = f(y). \tag{4}$$

  By definition, an **autonomous** ODE is a first-order ODE with RHS not depending on x. Let us, actually, apply the same idea:

$$y' = f(y) \Leftrightarrow \frac{y'}{f(y)} = 1 \Leftrightarrow \int \frac{y'dx}{f(y)} = x + C.$$

  Once again, we use the change of variables formula to get

$$\int \frac{dy}{f(y)} = x + C.$$

  If we denote $g(y) = \int \frac{y'dx}{f(y)}$, then the general solution can be written down as

$$y = g^{-1}(x + C),$$

  provided that g has a well-defined inverse. Most exercises are constructed in such a way that we can get an explicit solution.

- Now, consider a **separable** ODE

$$y' = \frac{M(x)}{N(y)}.$$

  You should have a guess as to how we are going to solve it:

$$N(y)y' = M(x) \Leftrightarrow \int N(y)dy = \int M(x)dx + C.$$

  And, actually, this is about it: we can't really do better than this without knowing M and N. Solutions, where y can't be expressed in terms of y are called implicit solutions.

Example 1.3. Consider $y' = ky$ for $k \in \mathbb{R} \setminus \{0\}$. Then we can separate the variables to obtain

$$\frac{y'}{y} = k \Leftrightarrow \int \frac{dy}{y} = kx + C \Leftrightarrow \log(|y|) = kx + C \Leftrightarrow |y| = e^C e^{kx}.$$

Keep in mind that we until this point we should keep the absolute value, but $|y| = e^C e^{kx}$ is never zero, so y is a differentiable function which never vanishes, so y cannot change signs, therefore, we can safely get rid of the absolute value. Therefore, the general solution is

$$y(x) = \pm e^C e^{kx}, \quad C \in \mathbb{R}.$$

Remark. Also, keep in mind that $y = 0$ is also a solution! We missed it because we divided by y right away! Therefore, we can replace $\pm e^C$ with C, where C is any real number. It is difficult to keep track of all the absolute values and domains of constants, so it is generally acceptable to write $\int x^{-1} dx = \ln(x) + C$. This works well, provided that $x > 0$.

Example 1.4. Consider $y' = 1 + y^2$. Separating the variables, we get

$$\int \frac{y'}{1+y^2} dx = x + C \Leftrightarrow \int \frac{dy}{1+y^2} = x + C \Leftarrow \arctan(y) = x + C.$$

Therefore, $y(x) = \tan(x + C)$ is a solution, but it is **not** defined everywhere, because $\tan(\pi/2) = \infty$. And this happens despite the fact that $1 + y^2$ is defined everywhere. If you think about it, $1 + y^2$ is a function which grows relatively fast as y increases...

Example 1.5. Consider $y' = -\frac{x}{y}$. Separating the variables, we get

$$yy' = -x \Leftrightarrow \int y dy = -\int x dx + C \Leftrightarrow y^2 + x^2 = 2C.$$

We ended up with the equation for the circle, but, technically, it is still an implicit solution, and there is no way to fix a branch of the square root unless we impose an **initial condition**, in other words, we should set a concrete value to $y(0)$. More on this later in the course.

2 First-order linear equations

2.1 Main definitions

Definition 2.1. A *first-order linear ODE* is an ODE of form

$$A(x)y' + B(x)y = G(x) \tag{5}$$

for some functions $A(x), B(x), C(x)$. Equivalently, we can divide both parts by $A(x)$ and only consider equations

$$y' + p(x)y = g(x), \tag{6}$$

where $p(x) = \frac{B(x)}{A(x)}, q(x) = \frac{C(x)}{A(x)}$.

Remark. In this course we are going to work with (6) (reduced form), so when presented with (5), always divide by $A(x)$.

If $g(x) = 0$, then we call the respective linear ODE **homogeneous**. Otherwise, the ODE is called linear inhomogeneous.

2.2 How to solve linear homogeneous ODEs?

Consider the following ODE:

$$y' + p(t)y = 0 \tag{7}$$

First of all, we notice that any such ODE is separable:

$$y' + p(t)y = 0 \Leftrightarrow y' = -p(x)y,$$

and we can proceed as usual:

$$y' = -p(x)y \Leftrightarrow \int \frac{dy}{y} = -\int p(x)dx + C \Leftrightarrow \ln(y) = -\int p(x)dx + C.$$

By exponentiating both sides, we obtain the general solution for (25):

$$y(x) = Ce^{-P(x)}, \tag{8}$$

where $P(x) = \int p(x)dx$ is **an** antiderivative of $p(x)$ (keep in mind that any particular antiderviative will work). In these notes we will use $P(x)$ a lot, to simplify some of the formulas.

Example 2.1. Consider $y' - 4xy = 0$. We can use (8), by noticing that $p(x) = -4x$, and

$$P(x) = -2x^2, \quad y(x) = Ce^{--2x^2} = Ce^{2x^2}.$$

Let us verify that this is, indeed, a solution, by noticing that chain rule gives $(e^{2x^2})' = 4xe^{2x^2}$.

Example 2.2. In this example we will consider an **initial value problem**:

$$\begin{cases} y' - \sin(x)y = 0 \\ y(0) = 1. \end{cases}$$

To solve an initial value problem, we need to find the solution, which satisfies the ODE and which attains the certain value at a particular point. Simply put, we need to find the constant C.

But you should always start by finding the general solution:

$$y' = \sin(x)y \Leftrightarrow \ln(y) = -\cos(x) + C \Leftrightarrow y(x) = Ce^{-\cos(x)}.$$

We still want $y(0) = 1$, so we get

$$1 = y(0) = Ce^{-\cos(0)} = Ce^{-1} \Rightarrow C = e.$$

Therefore, $C = e$ and $y(x) = e^{1-\cos(x)}$ is the solution we want.

2.3 Linear inhomogeneous ODEs

In this subsection we are going to learn how to solve a general ODE of form (6). There are two popular methods to do so.

2.3.1 Integrating factors

Definition 2.2. A function $\mu(x)$ (sometimes also denoted by $I(x)$) is called an integrating factor if we can multipy both sides (6) by μ to get the following ODE:

$$(\mu(x)y(x))' = g(x)\mu(x). \tag{9}$$

Let us first talk about why we even want to reduce our ODE to (9): the reason is simple. By integrating both parts, we get

$$\mu(x)y(x) = \int g(x)\mu(x)dx + C \Leftrightarrow y(x) = \frac{\int g(x)\mu(x)dx + C}{\mu(x)}. \tag{10}$$

All right, this seems simple, but how to find such μ? The idea is to understand what causes the LHS of (6) to transform like this. So, we have

$$\mu(x)y' + \mu(x)p(x)y = g(x)\mu(x).$$

If $\mu(x)p(x) = \mu'(x)$, then we could have used the product rule:

$$\mu(x)y' + \mu'(x)y = (\mu(x)y(x))'.$$

However, we can treat $\mu(x)p(x) = \mu'(x)$ as a differential equation and solve for μ, and it is another separable ODE. Solving in, we get

$$\mu(x) = C'e^{P(x)}.$$

Remark. The choice of C' does not matter, because for all $C' \in \mathbb{R}$ we have

$$\frac{\int g(x)C'e^{P(x)}dx + C}{C'e^{P(x)}} = \frac{\int g(x)e^{P(x)}dx}{e^{P(x)}} + \frac{C}{C'e^{P(x)}}.$$

Notice that $\frac{C}{C'}$ is still an arbitrary real constant, so WLOG we can assume $C/C' = C$, so we get (10) once again, and we can safely define

$$\mu(x) = e^{P(x)}.$$

Remark. While knowing the formula (10) by heart will prove useful, in quizzes/term tests you will be asked to show your computations, so use this formula only to double-check the answer.

Example 2.3. Let us solve the following initial value problem:

$$\begin{cases} y' + \tan(x)y = \cos(x) \\ y(0) = 1 \end{cases}$$

Step 1. Find $\mu(x)$. We have $p(x) = \tan(x)$, so $P(x) = \int \tan(x)dx = -\ln(\cos(x))$, and we get

$$\mu(x) = e^{-\ln(\cos(x))} = \frac{1}{\cos(x)}.$$

Step 2. By multiplying both sides by $\mu(x)$, we get

$$\frac{y'}{\cos(x)} + \frac{\sin(x)y}{\cos^2(x)} = \left(\frac{y}{\cos(x)}\right) = 1.$$

Integrating both sides yields

$$\frac{y}{\cos(x)} = x + C \Leftrightarrow y(x) = \cos(x)x + C\cos(x).$$

Step 3. Initial condition: let us find C:

$$y(0) = 1 = \cos(0)C = C,$$

so we obtain that $y(x) = (x+1)\cos(x)$ is the solution.

Reamrk. This example shows that integration factor is effective when $\mu(x) = (g(x))^{-1}$.

Example 2.4. Consider

$$y' + \frac{3y}{x} = \frac{e^x}{x^3}.$$

Let us find the integrating factor:

$$P(x) = \int \frac{3}{x}dx = 3\ln(x), \quad \mu(x) = e^{3\ln(x)} = x^3.$$

If we multiply everything by $\mu(x)$, we get

$$(x^3y)' = e^x \Rightarrow x^3y = e^x + C \Rightarrow \frac{e^x + C}{x^3}.$$

2.3.2 Variation of parameters

This method is generally percieved to be more flexible and slightly more intuitive, because the ideas do generalize to higher-order ODEs.

Give (6), we start by solving the respective homogeneous ODE by replacing RHS with zero:

$$y' + p(x)y = 0.$$

We already know how to solve it, in particular, the function $y_1(x) = e^{P(x)}$ solves this ODE. This step is called finding a **particular solution** of the respective homogeneous ODE. However, we want to find the solution to (6), and we **assume** that the true solution can be written as

$$y(x) = u(x)y_1(x).$$

Remark. In my lectures I used $C(x)y_1(x)$, it is more intuitive, but might cause some confusion, because the function $u(x)$ itself is defined up to a constant. When it comes to notation, one desires to be as flexible as possible, but also it should carry appropriate meaning as well and be intuitive. Let us plug this expression for $y(x)$ back into (6):

$$u'(x)y_1(x) + u(x)y_1'(x) + p(x)u(x)y_1(x) = g(x) \Leftrightarrow u'(x)y_1(x) + u(x)\underbrace{(y_1'(x) + p(x)y_1(x))}_{\text{this is zero!}} = g(x),$$

so we end up with

$$u'(x)y_1(x) = g(x). \tag{11}$$

Once again, we can integrate both parts to get $u(x) = \int g(x)(y_1(x))^{-1} + C$. However, plugging this expression in the formula for $y(x)$ yields

$$y(x) = y_1(x)\int g(x)(y_1(x))^{-1} + y_1(x),$$

and $y_1(x)$ is precisely the reciprocal of the integrating factor!

Remark. This shows that the choice between two methods is purely subjective, the underlying arguments are very similar.

Example 2.5. Consider the following ODE:

$$y' - \frac{xy}{x^2+1} = x.$$

Step 1. Solve the respective homogeneous ODE.

$$y' = \frac{xy}{x^2+1} \Leftrightarrow \frac{y'}{y} = \frac{x}{x^2+1} \Rightarrow \ln(y) = \frac{1}{2}\ln(x^2+1) + C \Rightarrow y(x) = C\sqrt{x^2+1}.$$

Step 2. Now we replace C with a function $u(x)$, thus obtaining $y(x) = u(x)\sqrt{x^2+1}$ plug the resulting expression for $y(x)$ back into the initial ODE and we solve for $u(x)$. Due to inevitable cancellations, we can plug this into (11) right away:

$$u'(x)\sqrt{x^2+1} = x \Rightarrow u'(x) = \frac{x}{x^2+1} \Rightarrow u(x) = \sqrt{x^2+1} + C.$$

Therefore, the solution is $y(x) = (\sqrt{x^2+1} + C)\sqrt{x^2+1}$.

3 Bernoulli equations

Now we want to look at the following class of first-order ODEs:

$$y' + p(t)y = g(t)y^n, \tag{12}$$

where $n \in \mathbb{R} \setminus \{0, 1\}$, and $t > 0$ (this will turn out to be important later!). Such ODEs are called **Bernoulli's equations**, and they are not linear ODEs. However, they can reduced to such via the change of variables $z(t) = y(t)^{1-n}$. Applying the chain rule, we know that

$$z' = (1-n)y'y^{-n},$$

and we can multiply both parts of (12) by $(1-n)y^{-n}$ to get

$$(1-n)y'y^{-n} + p(t)(1-n)y^{1-n} = (1-n)g(t) \Leftrightarrow z' + (1-n)p(t)z = (1-n)g(t).$$

Then we can proceed as in Section 1, using either integrating factors or variation of parameters.

Exercise. Variation of parameters works directly for (12) as well. The idea is to consider a particular solution $y_1(t)$ to the homogeneous ODE

$$y' + p(t)y = 0,$$

and then plug $y(x) = u(x)y_1(x)$ into (12). We will get

$$u'(x)y_1(x) + u(x)y_1'(x) + p(t)u(x)y_1(x) = g(t)y^n \Rightarrow u'(x)y_1(x) = g(t)y^n.$$

This is still a separable ODE. Solve it and obtain the general solution for (12).

Example 3.1. Consider the following initial value problem:

$$\begin{cases} y' + y = y^2 \\ y(0) = -1. \end{cases}$$

Let us treat this as a Bernoulli equation with $n = 2$.

Step 1. Consider $z = y^{1-2} = y^{-1}$. Now, solving for z, we get

$$z' - z = -1.$$

This can be solved almost immediately, the solution is $z(t) = 1 + Ce^t$. As $y(t) = \frac{1}{z(t)}$, we get $y(t) = (1 + Ce^t)^{-1}$.

Step 2. Initial condition. Now, all is left is to find C. We observe that

$$-1 = y(0) = \frac{1}{1+C} \Rightarrow C = -2.$$

Example 3.2. Let us find the solution for

$$xy' - 4y = x^2\sqrt{y}.$$

Notice that this is not a (reduced?) Bernoulli equation, so we start by dividing both parts by the coefficient at y':

$$y' - \frac{4y}{x} = x\sqrt{y}.$$

Now we can choose $n = \frac{1}{2}$ and set $z = y^{1-n} = y^{1/2}$. The resulting ODE for z looks as follows:

$$z' - \frac{2z}{x} = \frac{x}{2}.$$

Now let us find the integrating factor:

$$P(x) = -2\ln(x), \mu(x) = e^{-2\ln(x)} = \frac{1}{x^2}.$$

Multiplying both sides by $\mu(x)$, we get

$$\frac{z'}{x^2} - \frac{2z}{x^3} = \left(\frac{z}{x^2}\right)' = \frac{1}{2x}.$$

Finally, integration yields

$$z(x) = x^2 \left(\frac{1}{2} \ln(x) + C \right).$$

Now we only have to recover $y(x)$, but $y(x) = z(x)^2$, so

$$y(x) = x^4 \left(\frac{1}{2} \ln(x) + C \right)^2.$$

Warning. Let us try to find the solution which satisfies the condition $y(1) = 4$. Then we get

$$y(1) = 4 = \left(\frac{1}{2} \ln(1) + C \right)^2 = C^2,$$

therefore, $C = \pm 2$. However, this contradicts the existence and uniqueness theorem stated during Week 3, because it is easily checked that the function

$$(x, y) \mapsto x\sqrt{y} + \frac{4y}{x}$$

is continuously differentiable in a small neighborhood of $(1, 4)$. So, where is the problem?

First of all, we should explicitly mention that $y(x) > 0$ for all $x > 0$, or else the square root is not well-defined. This forces us to consider an initial value problem instead:

$$\begin{cases} xy' - 4y = x^2\sqrt{y} \\ y(a) > 0 \end{cases}$$

for some $a > 0$. Moreover, the change of variables $z = y^{1/2}$ forces $z(x) > 0$ for $x > 0$ as well! This means that

$$z(x) = x^2 \left(\frac{1}{2} \ln(x) + C \right) > 0 \Leftrightarrow \frac{1}{2} \ln(x) + C > 0 \Leftrightarrow C > \frac{1}{2} \ln(x^{-1}).$$

In particular, we have

$$C > \frac{1}{2} \ln(a^{-1}).$$

Therefore, we actually end up with a **restriction** on the coefficient C. Therefore, always be careful when considering "dangerous" changes of variables such as $z = \sqrt{y}$.

Example 3.3. Consider

$$y' + 2xy = 2xy^3.$$

This is a Bernoulli equation with $n = 3$. Let us apply variation of parameters directly! For this we would need to find a particular solution of a homogeneous ODE:

$$y' = -2xy \Rightarrow \ln(y) = -x^2 + C,$$

so we choose $y_1(x) = e^{-x^2}$. But now let us consider $y(x) = u(x)e^{-x^2}$. Plugging this guy into our initial ODE, we get

$$u'e^{-x^2} - 2xue^{-x^2} + 2xue^{-x^2} = u'e^{-x^2} = 2xu^3e^{-3x^2} \Rightarrow u' = 2xu^3e^{-2x^2}.$$

Thankfully, we can find the antiderivative of RHS, and we obtain

$$\int \frac{du}{u^3} = -\frac{1}{2}e^{-2x^2} + C \Rightarrow -\frac{1}{2u^2} = -\frac{1}{2}e^{-2x^2} + C.$$

Therefore,

$$u^2 = \frac{1}{e^{-2x^2} - 2C}, \quad y(x) = \pm e^{-x^2} \frac{1}{\sqrt{e^{-2x^2} - 2C}}.$$

4 Homogeneous ODEs (not to be confused with linear homogeneous ODEs)

Definition 4.1. A **homogeneous** ODE is an ODE of the form

$$y' = \varphi\left(\frac{y}{x}\right). \tag{13}$$

Such ODE's can be treated by considering the change of variables $u = \frac{y}{x}$. We have $y' = u'x + u$, therefore, we can transform (13) as follows:

$$u'x + u = \varphi(u) \Rightarrow \frac{u'}{u - \varphi(u)} = \frac{1}{x}.$$

This is a separable ODE, and we already know how to solve them. How3ever, we solved x in terms of u, so we end up with a **parametric solution**:

$$\begin{cases} x = e^{\int \dfrac{du}{u - \varphi(u)}} \\ y = ux. \end{cases}$$

Example 4.1. Consider the following initial value problem:

$$\begin{cases} y' = \dfrac{x^2 + y^2}{x^2 + xy} \\ y(1) = 0. \end{cases}$$

First of all, notice that

$$\frac{x^2 + y^2}{x^2 + xy} = \frac{1 + \frac{y^2}{x^2}}{1 + \frac{y}{x}},$$

so the change $ux = y$ yields

$$u'x + u = \frac{1 + u^2}{1 + u} \Rightarrow u'x = \frac{1 - u}{1 + u} \Rightarrow \int \frac{(1 + u)du}{1 - u} = \ln(x) + C.$$

Using partial fractions, we obtain

$$\int \frac{(1 + u)du}{1 - u} = \int \frac{(2 - 1 + u)du}{1 - u} = \int -1 - \frac{2}{u - 1} du = -u - 2\ln(u - 1),$$

therefore,

$$-u - 2\ln(u - 1) = \ln(x) \Rightarrow x = Ce^{-u - 2\ln(u-1)} = C\frac{e^{-u}}{(u - 1)^2}.$$

From the change of variables we can get an expression for y as well, and we get

$$\begin{cases} x = C\frac{e^{-u}}{(u-1)^2} \\ y = Cu\frac{e^{-u}}{(u-1)^2}. \end{cases}$$

As for the initial conditions, as $y(1) = 0$, so $u \cdot 1 = 0$, therefore $u = 0$, and we have

$$\begin{cases} 1 = C\frac{e^0}{(0-1)^2} \\ 0 = 0, \end{cases}$$

the first equation yields $1 = C$.

5 Existence and Uniqueness for first-order ODEs

Definition 5.1. A solution of a first-order ODE $y' = f(x, y)$ on an interval (a, b) is a differentiable function $y : (a, b) \to \mathbb{R}$ such that for every $x \in (a, b)$ we have

$$y'(x) = f(x, y(x)). \tag{14}$$

In other words, a solution is a function that can be "plugged" into both sides of a differential equation in such a way that the LHS equals the RHS.

The results obtained during Week 2 can be summarized by this theorem:

Theorem 5.1. Let $p(x)$ and $g(x)$ be continuous functions defined on an interval (a, b). Then the inital value problem

$$\begin{cases} y' + p(x)y = g(x) \\ y(x_0) = y_0, \end{cases} \tag{15}$$

where $x_0 \in (a, b)$, admits a **unique** solution on (a, b).

Proof. Well, we already have the explicit solution for this IVP:

$$y(x) = e^{-P(x)} \int_{x_0}^{x} g(x)\mu(x)dx + y_0 e^{-P(x)}, \quad \text{where } P(x) = \int p(x)dx.$$

□

But what about general first-order ODEs? As it turns, there are pretty reasonable conditions we can impose on f in (14) such that the solution of any initial value problem exists and unique. But we need some slight preparation to formulate the main theorem of this lecture.

Definition 5.2. Let $U \subset \mathbb{R}$ be a subset of $\mathbb{R}$. A function $F : U \to \mathbb{R}$ is said to be a **Lipschitz** function if there exists a constant $C > 0$ such that for every $x, y \in U$ we have

$$|F(x) - F(y)| \leq C|x - y|.$$

Example 5.1. Any differentiable function with a bounded derivative on $[a, b]$ is Lipschitz on $[a, b]$. This is a consequence of the Mean Value Theorem: if f is such a function, then for every $x, y \in [a, b]$ with $x < y$ there exists a point $c \in [x, y]$ such that

$$|f(x) - f(y)| = |x - y||f'(c)| \leq |x - y| \sup_{z \in [a,b]} |f'(z)|.$$

In particular, and continuously differentiable function on a **bounded closed** interval is Lipschitz.

However, it is easy to see that there are a lot of natural function with unbounded derivatives which will not be Lipschitz:

- $f(x) = x^2$ is Lipschitz on $[0, 1]$, but is not Lipschitz on the whole real line.
- take $f(x) = 1/x$ on $(0, 1]$ – it is not a Lipschitz function there
- or, for example, $f(x) = \tan(x)$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. It is also not a Lipschitz function.

Theorem 5.2. Consider a (first-order) initial value problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0, \end{cases} \tag{16}$$

where $(x_0, y_0) \in (a, b) \times (c, d) = D$ for some $a, b, c, d \in \mathbb{R}$. Suppose that the following conditions are satisfied:

- the function f is continuous in the domain D

- f is Lipschitz in with respect to y in the domain D. In other words, there exists a constant $C > 0$ such that for every $x \in (a, b)$ and $y_1, y_2 \in (c, d)$ we have

$$|f(x, y_1) - f(x, y_2)| \le C|y_1 - y_2|.$$

Then there exists a $\delta > 0$ such that (16) admits a unique solution in $(x_0 - \delta, x_0 + \delta) \subset (a, b)$. Uniqueness just means that if there are two solutions y_1 and y_2 which solve (16) on $(x_0 - \delta, x_0 + \delta)$, then $y_1(x) = y_2(x)$ for every $x \in (x_0 - \delta, x_0 + \delta)$.

Remark. It is **incredibly vital** to understand that this theorem ensures that the solution is only defined **locally**, in some, potentially, small open neighbourhood of x_0. It does not yield a global solution, unlike in the linear case.

Remark. As we observed before, we can replace the Lipschitz condition with being continuously differentiable in the closure of D.

This remark can be illustrated by two examples, one of which should be quite familiar to you:

Example 5.2. Consider

$$\begin{cases} y' = 1 + y^2 \\ y(0) = 0. \end{cases}$$

We know that the RHS of the ODE is Lipschitz on any closed interval, but the solution $y(x) = \tan(x)$ is only defined on a particular neighbourhood of 0. Namely, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Example 5.3. Consider

$$\begin{cases} y' = y^{1+\alpha} \\ y(0) = 0 \end{cases}$$

for $\alpha > 0$. This a separable ODE, which can be quickly solved to obtain the solution

$$y(x) = (1 - \alpha x)^{-1/\alpha}.$$

We can easily see that this solution is undefined for $x = \frac{1}{\alpha}$. It is convenient to say that $y(x)$ "blows up" at this point, as $\lim\limits_{x \to \frac{1}{\alpha}^-} = \infty$.

As you can see from these examples, we chosen the functions in RHS in a special way: they all grow "too fast", so the derivative of a solution also grows as fast. Let us formulate a corollary from Theorem 5.2.

Corollary 5.1. Consider the initial value problem 5.2, and suppose that f satisfies all the conditions of the theorem in $D = (a, b) \times \mathbb{R}$, and an additional restriction on growth on f:

$$|f(x, y)| \le M(|y| + 1) \quad \text{for all } x \in (a, b), y \in \mathbb{R}.$$

Then the solution to 5.2 is defined on the whole interval I.

The idea of the proof is to "glue" the solutions on the smaller intervals given by the Theorem 5.2. In fact, this leads to an approach to the interesting problem of determining the **maximal interval of existence** of a solution, but this might be a topic for a different time...

Also, it is interesting to note that if we drop the Lipschitz condition from Theorem 5.2 but keep the continuity, then the solution is still guaranteed to exist locally, but we might get many solutions satisfying the same initial condition!

Example 5.4. Let us look at

$$\begin{cases} y' = \frac{1}{3} y^{2/3} \\ y(0) = 0 \end{cases}$$

This is a classical example of an IVP which violates the conditions of Theorem 5.2. To see why, observe that $y = 0$ is a solution, and, treating the ODE as a separable ODE, we get that $y(x) = (x - C)^3$ is a solution for all $C \in \mathbb{R}$. However, we can "cut and reattach" these solutions to obtain the following "Frankenstein's monster":

$$y(x) := \begin{cases} (x - C_1)^3, & x < C_1, \\ 0, & C_1 \le x \le C_2 \\ (x - C_2)^3 & x > C_2, \end{cases}$$

where $C_1 < 0 < C_2$ are some real constants. I will leave the rigorous verification that this is a solution as an exercise to the reader.

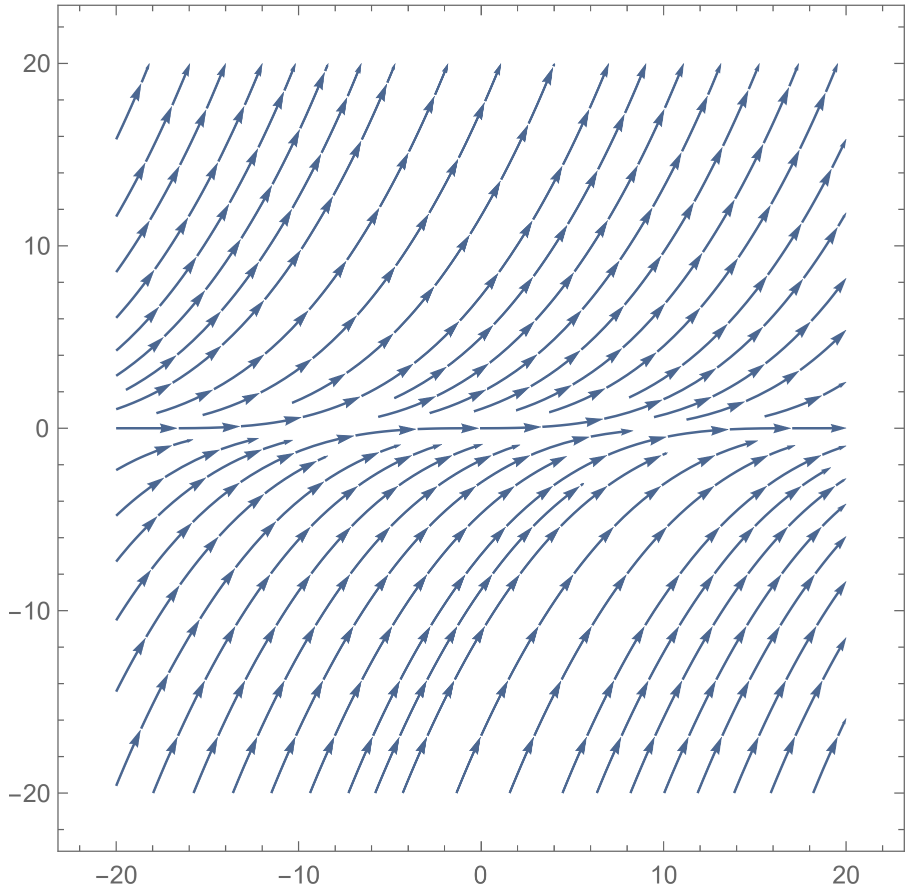

Figure 1: Direction field for $y' = \frac{1}{3}y^{2/3}$: can you see the piecewise defined solutions?

6 Exact equations

We will start this section by breaking the sacred rule of indivisibility of the notation $\frac{dy}{dx}$. In other words, we explicitly state that the following notations are equivalent:

$$M(x, y) + N(x, y)y' = 0 \Leftrightarrow M(x, y)dx + N(x, y)dy = 0.$$

We obtain the second form by multiplying both parts by dx. This notation is somewhat justified by the following operation.

Definition 6.1. For a differentiable function $U : \mathbb{R}^2 \to \mathbb{R}$ we define the **full differential** of U as follows:

$$d(U(x, y)) = \frac{\partial U}{\partial x}(x, y)dx + \frac{\partial U}{\partial y}(x, y)dy. \tag{17}$$

Remark. This definition does require the notion of partial derivatives, but I hope that everyone dealt with multivariate calculus at some point at some point prior to taking MAT244. If not... then, have a few examples:

Example 6.1.

- Let $U(x, y) = xy + 3x^2y^{10}$. All you need to know that partial derivatives only care about their own variable, and regard other variables as constants. And they satisfy all rules of traditional derivatives. So, we get

$$\frac{\partial}{\partial x}(xy + 3x^2y^{10}) = y\frac{\partial}{\partial x}(x) + y^{10}\frac{\partial}{\partial x}(3x^2) = y + 6xy^{10}.$$
$$\frac{\partial}{\partial y}(xy + 3x^2y^{10}) = x\frac{\partial}{\partial y}(y) + 3x^2\frac{\partial}{\partial y}(y^{10}) = x + 30x^2y^9.$$

  Therefore,

$$dU = (y + 6xy^{10})dx + (x + 30x^2y^9)dy.$$

- Let $U(x, y) = e^x \cos(y)$. Then

$$\frac{\partial}{\partial x}(e^x \cos(y)) = e^x \cos(y),$$

  and

$$\frac{\partial}{\partial y}(e^x \cos(y)) = -e^x \sin(y).$$

  So, we end up with

$$dU = e^x \cos(y)dx - e^x \sin(y)dy.$$

Remark. Some authors denote $\frac{\partial U}{\partial x} = U_x$. This seems to be prevalent in partial differential equation textbooks, as writing the $\frac{\partial U}{\partial x}$ takes a bit longer than just putting a subscript (I guess also helps readability?)

Definition 6.2. An equation

$$M(x, y)dx + N(x, y)dy = 0 \tag{18}$$

is called **exact**, if there exists a differentiable function $U : (a, b) \times (c, d) \to \mathbb{R}$ such that

$$dU = M(x, y)dx + N(x, y)dy$$

The point of this definition is that an exact equation can be reduced to

$$dU = 0 \Leftrightarrow U(x, y) = C,$$

and we have obtained an initial solution to (17).

6.1 How do we determine whether an equation is exact?

As we can see, exactness is equivalent to having a function $U(x, y)$ such that

$$\frac{\partial U}{\partial x} = M, \quad \frac{\partial U}{\partial y} = N.$$

But what if we apply $\dfrac{\partial U}{\partial y}$ to the first equality and $\dfrac{\partial U}{\partial x}$ to the second one? We get

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial N}{\partial x}.$$

However, recall that the partial derivatives commute! Therefore, exactness of (18) implies

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \tag{19}$$

for all points in $(a, b) \times (c, d)$.

Remark. Technically, we need to require that the second derivatives are continuous to say that they commute. But in our course the functions M and N will be "nice", and U will always admit continuous second derivatives.

In other words, we have proven that (19) is a **necessary condition** for exactness. But is is sufficient?

6.2 How to solve exact ODEs?

Suppose (18) satisfies (19). Let us try to find $U(x,y)$. Let $(x_0,y_0) \in (a,b)\times(c,d)$. First of all, we recall that $\dfrac{\partial U}{\partial x} = M$, therefore, we can attempt to take the integral with respect to x to recover U via FTC. Thus we obtain

$$\frac{\partial U}{\partial x} = M \Leftrightarrow U(x,y) = \int_{x_0}^{x} M(s,y)ds + \phi(y).$$

But how to find $\phi(y)$? Well, now we can apply $\dfrac{\partial}{\partial y}$ to both parts:

$$U(x,y) = \int_{x_0}^{x} M(s,y)ds + \phi(y) \Leftrightarrow \frac{\partial U}{\partial y} = N = \int_{x_0}^{x} \frac{\partial M}{\partial y}(s,y)ds + \phi'(y).$$

Here we used the fact that $\dfrac{d}{dx}\int f(x,t)dt = \int \dfrac{\partial f}{\partial x}(x,t)dt$ (differentiation under the integral sign). However, this is the point where need to use exactness:

$$N = \int_{x_0}^{x} \frac{\partial M}{\partial y}(s,y)ds + \phi'(y) = \int_{x_0}^{x} \frac{\partial N}{\partial S}(s,y)ds + \phi'(y) = N(x,y) - N(x_0 y) + \phi'(y) \Rightarrow \phi'(y) = N(x_0,y).$$

Finally, we take the integral with respect to y, and we get

$$\phi(y) = \int_{y_0}^{y} N(x_0,y)dy + C_0.$$

Thus, we can formulate the following theorem:

Theorem 6.1. Consider an ODE

$$M(x,y)dx + N(x,y)dy = 0.$$

If M, N are continuously differentiable in $(a,b)\times(c,d)$, and $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$ holds for all $(x,y) \in (a,b)\times(c,d)$, then

$$M(x,y)dx + N(x,y)dy = dU(x,y),$$

where

$$U(x,y) = \int_{x_0}^{x} M(s,y)ds + \int_{y_0}^{y} N(x_0,y)dy + C_0,$$

and $U(x,y) = C$ will be a general (implicit) solution to the ODE.

Remark. By swapping x and y in the above argument, we can get a slightly different formula:

$$U(x,y) = \int_{x_0}^{x} M(s,y_0)ds + \int_{y_0}^{y} N(x,t)dt + C_0.$$

As always, use the final formulas only to double-check the correctness of your answers. I will require you to include the full computations while solving quizzes or term tests.

Example 6.2. Let us consider

$$e^x\cos(2y)dx + (9y^2 - 2e^x\sin(2y))dy = 0.$$

Step 1. Check that this ODE is, indeed, exact. Just compute the partial derivatives:

$$\frac{\partial M}{\partial y} = e^x \frac{\partial}{\partial y}\cos(2y) = -2e^x\sin(2y),$$

$$\frac{\partial N}{\partial x} = -2e^x\sin(2y).$$

Good, the partial derivatives are equal. **Step 2.** Find the function U. Let us just repeat the argument we used to find the general formula.

$$U(x,y) = \int e^x \cos(2y)dx = e^x \cos(2y) + \phi(y).$$

Then we find $\phi(y)$:

$$N(x,y) = 9y^2 - 2e^x \sin(2y) = -2e^x \sin(2y) + \phi'(y) \Rightarrow 9y^2 = \phi'(y) \Rightarrow \phi(y) = 3y^3.$$

So, we get

$$U(x,y) = e^x \cos(2y) + 3y^3 + C_0,$$

and the implicit solution is

$$e^x \cos(2y) + 3y^3 = C.$$

We can impose an initial condition $y(0) = 0$ and plug in into the solution to get

$$e^0 \cos(0) + 0 = 1 = C.$$

7 Exact equations and integrating factors

Not every ODE is exact. However, what if we still want to solve it using the techniques described in the previous section?

Definition 7.1. A function $\mu(x,y)$ is called an integrating factor for an ODE

$$Mdx + Ndy = 0$$

if after multiplying both sides by μ the resulting equation becomes exact. In other words,

$$(M(x,y)\mu(x,y))dx + (N(x,y)\mu(x,y))dy = 0$$

is an exact ODE.

This definition does give us a necessary and sufficient condition for the modified ODE to be exact:

$$\frac{\partial M\mu}{\partial y} = \frac{\partial N\mu}{\partial x}.$$

Let us apply the product rule to both parts:

$$M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}. \tag{20}$$

But there is no obvious way to move forward from here: (20) is a partial differential equation which is in no way easier to solve than the initial one.

However, by making some assumptions on the form of μ, we can make the resulting PDE much simpler to solve.

- We can assume that the integrating factor only depends on x, so $\mu(x,y) = \mu(x)$
- We can assume that the integrating factor only depends on y, so $\mu(x,y) = \mu(y)$
- We can assume that the integrating factor only depends on xy, so $\mu(x,y) = \mu(xy)$

7.1 The case of $\mu(x, y) = \mu(x)$

In this case we replace $\mu(x, y)$ with $\mu(x)$ in (20) to get

$$\mu(x)\frac{\partial M}{\partial y} = M\frac{\partial \mu}{\partial y} + \mu(x)\frac{\partial M}{\partial y} = N\mu'(x) + \mu(x)\frac{\partial N}{\partial x}.$$

By moving some terms, we get

$$\mu(x)\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\mu'(x).$$

This is a separable ODE with respect to μ:

$$\frac{\mu'}{\mu} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}.$$

Apply this method if and only if

$$\text{the function } \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \text{ is a function of } x.$$

Example 7.1. Consider

$$(y + 3y^2e^{2x})dx + (1 + 2ye^{2x})dy = 0.$$

Let us find an integrating factor and find the solution satisfying $y(0) = 1$. First of all, we notice that

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 + 6ye^{2x} - 4ye^{2x} = 1 + 2ye^{2x} = N.$$

Therefore,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = 1,$$

and $\frac{\mu'}{\mu} = 1$. So, $\mu(x) = e^x$ will work as an integrating factor.

By multiplying both parts of the ODE by e^x, we get an exact ODE and we can solve it as such:

$$(ye^x + 3y^2e^{3x})dx + (e^x + 2ye^{3x})dy = 0,$$

(no need to check that this ODE is exact)

$$U(x, y) = \int ye^x + 3y^2e^{3x}dx = ye^x + y^2e^{3x} + \phi(y),$$

then we look for $\phi(y)$:

$$N(x, y) = e^x + 2ye^{3x} = e^x + 2ye^{3x} + \phi'(y) \Rightarrow \phi(y) = C_0.$$

So, we get

$$U(x, y) = ye^x + y^2e^{3x} + C_0,$$

and the implicit solution is

$$ye^x + y^2e^{3x} = C.$$

Plugging in $x = 0, y = 1$, we get

$$1 + 1 = C = 2.$$

7.2 The case of $\mu(x,y) = \mu(y)$

In this case we replace $\mu(x,y)$ with $\mu(y)$ in (20) to get

$$M\mu'(y) + \mu(y)\frac{\partial M}{\partial y} = \mu(y)\frac{\partial N}{\partial x}.$$

Once again, it is a separable ODE:

$$\frac{\mu'(y)}{\mu(y)} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}.$$

Apply this method if and only if

$$\text{the function } \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \text{ is a function of } y.$$

Example 7.2. Let us consider

$$(-y\sin(x) + y^3\cos(x))dx + (3\cos(x) + 5y^2\sin(x))dy = 0.$$

We want to find an integrating factor and a solution satisfying $y(\pi/4) = \sqrt{2}$.

First of all, we find the difference $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$:

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\sin(x) + 3y^2\cos(x) - (-3\sin(x) + 5y^2\cos(x)) = 2\sin(x) - 2y^2\cos(x).$$

Therefore,

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2}{y}.$$

So, we have $\log(\mu) = 2\ln(y)$ and $\mu = y^2$. We multiply both sides by y^2 to get

$$(-y^3\sin(x) + y^5\cos(x))dx + (3y^2\cos(x) + 5y^4\sin(x))dy = 0.$$

This is an exact ODE, which we solve as usual:

$$U(x,y) = \int -y^3\sin(x) + y^5\cos(x)dx = y^3\cos(x) + y^5\sin(x) + \phi(y).$$

Then we get

$$N(x,y) = 3y^2\cos(x) + 5y^4\sin(x) = 3y^2\cos(x) + 5y^4\sin(x) + \phi'(y) \Rightarrow \phi = C_0.$$

So, our implicit solution is

$$y^3\cos(x) + y^5\sin(x) = C.$$

Plugging $x = \pi/4, y = \sqrt{2}$, we get

$$2\sqrt{2}\frac{1}{\sqrt{2}} + 4\sqrt{2}\frac{1}{\sqrt{2}} = C = 6.$$

7.3 The case of $\mu(x,y) = \mu(xy)$

In this case we replace $\mu(x,y)$ with $\mu(xy)$ in (20) to get

$$(M \cdot x - N \cdot y)\mu'(xy) = \mu(xy)\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right),$$

because we can apply the chain rule to the composition

$$(x,y) \mapsto xy \mapsto \mu(xy).$$

In other words, we get

$$\frac{\partial}{\partial x}(\mu(xy)) = y\frac{d}{dt}\mu(t)\big|_{t=xy} = y\mu'(xy), \quad \frac{\partial}{\partial y}(\mu(xy)) = x\frac{d}{dt}\mu(t)\big|_{t=xy} = x\mu'(xy)$$

Once again, it is a separable ODE:

$$\frac{\mu'(xy)}{\mu(xy)} = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M \cdot x - N \cdot y}.$$

Apply this method if and only if

$$\text{the function } \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M \cdot x - N \cdot y} \text{ is a function of } xy.$$

Example 7.3. Consider

$$(3y\cos(x+y) - xy\sin(x+y))dx + (3x\cos(x+y) - xy\sin(x+y))dy = 0.$$

Find an integrating factor and a solution satisfying $y(\pi/2) = \pi/2$.

First of all, let us compute $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ first:

$$\frac{\partial N}{\partial x} = 3\cos(x+y) - 3x\sin(x+y) - y\sin(x+y) - xy\cos(x+y),$$

$$\frac{\partial M}{\partial y} = 3\cos(x+y) - 3y\sin(x+y) - x\sin(x+y) - xy\cos(x+y)$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2(y-x)\sin(x+y).$$

Therefore,

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{Mx - Ny} = \frac{2}{xy}.$$

By setting $t = xy$, we get $\frac{\mu'}{\mu} = \frac{2}{t}$, so $\mu = x^2y^2$. Multiplying both sides by x^2y^2, we get

$$(3x^2y^3\cos(x+y) - x^3y^3\sin(x+y))dx + (3x^3y^2\cos(x+y) - x^3y^3\sin(x+y))dy = 0.$$

Then

$$U(x,y) = \int 3xy^2\cos(x+y) - x^2y^2\sin(x+y)dx = x^3y^3\cos(x+y) + \phi(y),$$

which can be obtained by integration by parts. Then we get

$$N(x,y) = 3x^3y^2\cos(x+y) - x^3y^3\sin(x+y) = 3x^3y^2\cos(x+y) - x^3y^3\sin(x+y) + \phi'(y) \Rightarrow \phi'(y) = 0.$$

So, our implicit solution is

$$x^3y^3\cos(x+y) = C.$$

8 Second-order linear ODEs

In this section we are finally moving past first-order ODEs.

Definition 8.1. A **second-order linear ODE** is an ODE of form

$$y'' + p(x)y' + q(x)y = g(x), \tag{21}$$

where p, q, g are functions defined on an interval (a, b).

- If $p(x), q(x), g(x)$ are constant functions, then (21) is called a **constant-coefficient** ODE.

- If $g(x) = 0$, then (21) is called a **homogeneous** ODE.

In this week's lectures we are mostly going to discuss homogeneous constant-coefficient ODEs, but let us start by formulating several important facts that we are going to use now:

Theorem 8.1 (superposition principle)**.** If y_1 and y_2 are solutions to

$$y'' + p(x)y' + q(x)y = 0,$$

then $ay_1 + by_2$ is a solution of (21) for any $a, b \in \mathbb{R}$.

This theorem immediately follows from the linearity of derivative.

Theorem 8.2 (existence and uniqueness for second-order ODEs)**.** If $p(x), q(x), g(x)$ are continuous on (a, b), then the solution to initial value problem ($x_0 \in (a, b)$)

$$\begin{cases} y'' + p(x)y' + q(x)y = g(x), \\ y(x_0) = a, \\ y'(x_0) = b, \end{cases}$$

exists, unique, and it is defined everywhere on (a, b).

Remark. Keep in mind that in order for a solution to be unique, we need to fix the values of $y(x_0)$ and $y'(x_0)$. This, together with the superposition principle, suggests that the space of solutions is a two-dimensional vector space over $\mathbb{R}$.

In particular, this remark suggests that while looking for a general solution of a second-order ODE, we should expect to get two constants instead of just one. If we are provided with two functions $y_1(x), y_2(x)$, which generate the whole solution space, we will call them a **fundamental set of solutions**. Equivalently, we require $y_1(x)$ and $y_2(x)$ to be linearly independent.

8.1 Constant-coefficient homogeneous second-order ODEs

As it turns out, even if all functional coefficients are constant, the task of solving a second-order ODE is not that trivial!

Consider

$$y'' + ay' + by = 0. \tag{22}$$

To illustrate the solution scheme, we will try to guess a solution. For example, when does the function $e^{\lambda x}$ solve (21)? Let us just plug this guy into (21):

$$(e^{\lambda x})'' + a(e^{\lambda x})' + be^{\lambda x} = 0 \Leftrightarrow e^{\lambda x}(\lambda^2 + a\lambda + b) = 0.$$

As $e^{\lambda x}$ is never zero, we can cancel it out, so we end up with

$$\lambda^2 + a\lambda + b = 0. \tag{23}$$

Therefore, we deduce that λ has to be the root of (23), which we will call the **characteristic equation** of (22).

So, when approaching such an ODE, one wants to start by writing down the characteristic polynomial (23) of (22), and then finding its roots. As it is always a quadratic equation, we will always have three cases:

1. Suppose that the equation (23) has two distinct real roots λ_1 and λ_2.

   Then we already know that $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$ are the solutions to (22). However, we know that any linear combination of y_1 and y_2 also has to be a solution. Therefore, the general solution to this ODE is
   $$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}. \tag{24}$$
   If we are also given initial conditions $y(x) = a, y'(x) = b$, then we can use them to recover the values of C_1 and C_2. Let us consider a simple example.

Example 8.1. Consider

$$\begin{cases} y'' + 3y' + 2y = 0, \\ y(0) = 1, \quad y'(0) = 0. \end{cases}$$

First of all, we identify the characteristic polynomial:

$$\lambda^2 + 3\lambda + 2 = 0.$$

It is easily seen that $\lambda = -1$, $\lambda = -2$ are the roots of this equation. Therefore, the general solution is

$$y(x) = C_1 e^{-x} + C_2 e^{-2x}.$$

Let us plug in the inital conditions:

$$y'(x) = -C_1 e^{-x} - 2C_2 e^{-2x}, \quad y(0) = 1 = C_1 + C_2, y'(0) = 0 = -C_1 - 2C_2.$$

Let us rewrite this as a 2 by 2 system of linear equations:

$$\begin{cases} C_1 + C_2 = 1, \\ -C_1 - 2C_2 = 0. \end{cases}$$

Solving this system, we obtain $C_1 = 2, C_2 = -1$.

2. Suppose that the equation (23) has two distinct complex roots $\lambda + i\mu$ and $\lambda - i\mu$.

   In other words, (23) has no real roots! There are several ways to overcome this obstacle, but the fastest way to do this is to just consider the **complex solutions**

   $$y_1(x) = e^{(\lambda+i\mu)x}, \quad y_2(x) = e^{(\lambda-i\mu)x}.$$

   If we put our trust in complex differentiation, we can see that these functions do, indeed, solve the ODE. But recall that we are actually looking for real-valued solutions. So, let us combine the Euler's formula and the superposition principle:

   $$y_1(x) = e^{\lambda x}(\cos(\mu x) + i\sin(\mu x)), \quad y_2(x) = e^{\lambda x}(\cos(\mu x) - i\sin(\mu x)).$$

   The superposition principle implies that $\dfrac{y_1 + y_2}{2}$ and $\dfrac{y_1 - y_2}{2i}$ are also the solutions to (22). But

   $$\frac{y_1 + y_2}{2} = e^{\lambda x}\cos(\mu x), \quad \frac{y_1 - y_2}{2i} = e^{\lambda x}\sin(\mu x).$$

   Those who don't really trust in complex numbers can directly verify that these are real solutions to the ODE. So, in the end we get that the general solution in this case can be written as

   $$y(x) = e^{\lambda x}(C_1\cos(\mu x) + C_2\sin(\mu x)).$$

Example 8.2. Consider the IVP

$$\begin{cases} y'' + 4y' + 5y = 0, \\ y(0) = 1, \quad y'(0) = -1. \end{cases}$$

Let us solve the characteristic equation:

$$x^2 + 4x + 5 = 0, \quad x_{12} = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i.$$

So, our two particular solutions are $e^{-2x}\cos(x)$ and $e^{-2x}\sin(x)$, and the general solution is

$$y(x) = C_1 e^{-2x}\cos(x) + C_2 e^{-2x}\sin(x).$$

To find C_1 and C_2, we need to plug in the initial conditions:

$$y(0) = 1 = C_1$$

$$-1 = y'(0) = -2C_1 e^{-2\cdot 0}\cos(0) - C_1 e^{-2\cdot 0}\sin(0) - 2C_2 e^{-2\cdot 0}\sin(0) + C_2 e^{-2\cdot 0}\cos(0) = -2C_1 + C_2 = -2 + C_2.$$

So, $C_2 = 1$.

3. Suppose that the equation (23) has a repeated real root λ. This is another tricky case, because our method only gives us one solution: $x \mapsto e^{\lambda x}$. Once again, there are several ways to derive the second solution, but for now let us try variation of parameters here, and let us assume that there is a solution $y(x) = u(x)e^{\lambda x}$. Plugging this guy into the ODE, we get

$$u''(x)e^{\lambda x} + 2\lambda u'(x)e^{\lambda x} + \lambda^2 u(x)e^{\lambda x} + a(\lambda u(x)e^{\lambda x} + u'(x)e^{\lambda x}) + bu(x)e^{\lambda x} = 0.$$

Dividing by $e^{\lambda x}$, and collecting the terms, we get

$$u(x)\underbrace{(\lambda^2 + a\lambda + b)}_{\text{cancels out!!!}} + au'(x) + 2\lambda u'(x) + u''(x) = au'(x) + 2\lambda u'(x) + u''(x) = 0.$$

Wait a second, but isn't $a + 2\lambda = 0$? So, we can simplify even further to get

$$u''(x) = 0.$$

The last equation is a trivial second-order ODE, which can be solved immediately, so we get $u(x) = C_1 + C_2 x$.

This derivation shows that $y_1(x) = e^{\lambda x}$ and $y_2(x) = xe^{\lambda x}$ form a fundamental set of solutions, so our general solution looks like this:

$$y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}.$$

Example 8.3. Consider

$$\begin{cases} y'' + 2y' + y = 0, \\ y(0) = 2, y'(0) = 5. \end{cases}$$

The characteristic equation

$$\lambda^2 + 2\lambda + 1 = 0$$

has only one root $\lambda = -1$. Therefore, the general solution is

$$y(x) = e^{-x}(C_1 + xC_2),$$

and we can plug in the initial condition to obtain

$$y(0) = C_1 = 2, \quad y'(x) = -e^{-x}(C_1 + xC_2) + C_2 e^{-x},$$

so

$$5 = y'(0) = C_2 - C_1 = C_2 - 2,$$

so $C_2 = 7$.

8.2 General homogeneous second-order ODEs

Now we turn our attention to general ODEs of form

$$y'' + p(x)y' + q(x)y = 0. \tag{25}$$

We will require some additional tools to solve such ODEs.

Definition 8.2. Let $y_1, \dots y_n$ be $n-1$ times differentiable functions in an interval (a, b). Then we define their **Wronskian** as a determinant

$$W[y_1, \dots, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}.] \end{vmatrix}.$$

For example, if $n = 2$, we get that

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

One way to motivate this definition is to look at a fundamental set of solutions for some linear second-order ODE y_1 and y_2. If the initial conditions are $y(x_0) = a$ and $y'(x_0) = b$, then we have to solve the system

$$\begin{cases} C_1 y_1(x_0) + C_2 y_2(x_0) = a \\ C_1 y_1'(x_0) + C_2 y_2'(x_0) = b. \end{cases}$$

But the Cramer's rule immediately yields

$$C_1 = \frac{\begin{vmatrix} a & y_2 \\ b & y_2' \end{vmatrix}}{W[y_1, y_2]}, \quad C_2 = \frac{\begin{vmatrix} y_1 & a \\ y_1' & b \end{vmatrix}}{W[y_1, y_2]}.$$

In fact, we have the following theorem:

Theorem 8.3 (Theorem 3.2.4)**.** The functions $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

if and only if there **exists** a point x_0 such that $W[y_1, y_2](x_0) \neq 0$.

But there are even more useful ways to utilize the Wronskian!

8.3 Abel's formula

Lemma 8.1. Let y_1 and y_2 form a fundamental set of solutions for

$$y'' + p(x)y' + q(x)y = 0.$$

Then

$$\frac{d}{dx} W[y_1, y_2](x) = -p(x) W[y_1, y_2](x).$$

Proof. Observe that

$$\frac{d}{dx} W[y_1, y_2](x) = (y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2') = y_1 y_2'' - y_1'' y_2.$$

Now we recall that

$$y_1'' = -p(x) y_1' - q(x) y_1, \quad y_2'' = -p(x) y_2' - q(x) y_2,$$

because y_1 and y_2 solve the respective homogeneous ODE. Therefore,

$$y_1 y_2'' - y_1'' y_2 = y_1(-p(x) y_2' - q(x) y_2) - y_2(-p(x) y_1' - q(x) y_1) = -p(x) y_1 y_2' + p(x) y_2 y_1' = -p(x) W[y_1, y_2](x).$$

□

Theorem 8.4 (Abel's formula)**.** Let y_1 and y_2 form a fundamental set of solutions for

$$y'' + p(x)y' + q(x)y = 0.$$

Then

$$W[y_1, y_2](x) = C e^{\int -p(x) dx},$$

where the constant C only depends on y_1, y_2.

Proof. The previous lemma implies that the Wronskian has to satisfy the separable equation

$$W' = -p(x)W.$$

Solving for W, we get

$$W[y_1, y_2](x) = Ce^{\int -p(x)dx}.$$

□

Corollary 8.1. Let y_1 be a solution for

$$y'' + p(x)y' + q(x)y = 0.$$

Then the function

$$y_2(x) = y_1(x) \int \frac{W[y_1, y_2](x)}{y_1^2(x)} dx \tag{26}$$

forms a fundamental set of solutions $\{y_1(x), y_2(x)\}$ together with y_1.

Proof. First of all, let us write the definition of the Wronskian,

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1',$$

and divide both parts by y_1^2. We get

$$\frac{W[y_1, y_2]}{y_1^2} = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \left(\frac{y_2}{y_1}\right)'$$

By taking the antiderivative, we get

$$\int \frac{W[y_1, y_2]}{y_1^2} dx = \frac{y_2}{y_1} + C \Leftrightarrow y_2(x) = y_1(x) \left(\int \frac{W[y_1, y_2]}{y_1^2} dx - C \right).$$

The choice of C will not matter as we are looking for a **particular** linearly independent solution. So, might as well just choose $C = 0$. □

Example 8.4. On Wednesday we have shown that if $t^2 + at + b = 0$ has one repeated root $t = \lambda$, then the function $y_1(x) = e^{\lambda x}$ is a solution to the second-order ODE

$$y'' + ay' + by = 0.$$

Let us use the Abel's formula to recover the second solution.

$$W = e^{-ax},$$

and

$$y_2 = e^{\lambda x} \int e^{-ax} e^{-2\lambda x} dx = e^{\lambda x} \int e^{(-a-2\lambda)x} dx.$$

Once again, we remember that $a + 2\lambda = 0$, therefore,

$$y_2(x) = e^{\lambda x} \int e^{(-a-2\lambda)x} dx = e^{\lambda x} \int 1 dx = (x + C)e^{\lambda x}.$$

It doesn't matter which C to choose, so we can fix $y_2(x) = xe^{\lambda x}$. And our general solution is

$$y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}.$$

Remark. When solving a term test or a quiz problem, it is HIGHLY advisable to find the Wronskian separately, then use (26). And check that the resulting y_2 is an actual solution, to make sure you never made a typo!

Example 8.5. We can easily verify that $y_1(x) = e^x$ solves

$$(x-1)y'' - xy' + y = 0.$$

How to find the second solution?

Step 0. Let us reduce the ODE (it is easy to forget this!) We get

$$y'' - \frac{x}{x-1}y' + \frac{y}{x-1} = 0.$$

Therefore, $p(x) = -\dfrac{x}{x-1}$, and

$$W[y_1, y_2](x) = \exp\left(\int \frac{x}{x-1}dx\right) = \exp(x + \ln(x-1)) = (x-1)e^x.$$

So, we can use (26) to get

$$y_2(x) = e^x \int (x-1)e^{-x}dx = e^x((1-x)e^{-x} - (\int -e^{-x})) = -x.$$

Finally, we obtain that the general solution looks like this:

$$y(x) = C_1 e^x - C_2 x.$$

Finally, we can try to solve an inverse problem: let y_1 and y_2 be two linearly independent functions. How to construct an ODE for which these two functions serve as solutions? The answer is quite surprising, we just consider

$$\begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = 0.$$

The idea is that determinant is equal to zero if and only if (y, y', y'') can be expressed as a linear combination of (y_1, y_1', y_1'') and (y_2, y_2', y_2''), but this precisely means that $y(x) = C_1 y_1(x) + C_2 y_2(x)$.

Example 8.6. Let us try $y_1(x) = x^2, y_2(x) = (x+1)$. Then

$$\begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = -(x^2+2x)y'' + (2x+2)y' - 2y = 0.$$

Finally, let us try to find y_2 as like we had only been given y_1. So, we would have to reduce this ODE to the standard form:

$$y'' - \frac{2x+2}{x^2+2x}y' + \frac{2}{x^2+2x}y = 0.$$

So, compute the Wronskian:

$$W[y_1, y_2](x) = \exp\left(\int \frac{2x+2}{x^2+2x}dx\right) = \exp(\ln(x) + \ln(x+2)) = x(x+2).$$

Let us find the second solution:

$$\tilde{y}_2(x) = x^2 \int \frac{x(x+2)}{x^4}dx = x^2\left(-\frac{1}{x^2} - \frac{1}{x}\right) = -x - 1.$$

Well, we got $-y_2$ instead of y_2 (why?). But this is not a big deal, as it still generates the same solution space.

9 Approaching non-homogeneous second-order linear ODEs

We continue our journey through the realm of second-order ODEs. Our goal for this week is to understand how to approach inhomogeneous second-order ODEs:

$$y'' + p(x)y' + q(x)y = g(x), \tag{27}$$

where p, q, g are continuous functions on an interval $I = (a_1, a_2)$, and $g(x) \neq 0$ identically on I.

Once again, we will employ a simple but general fact which you might have seen before:

Lemma 9.1. If y_1 and y_2 are solutions to (27), then the difference $y_1 - y_2$ solves the homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0. \tag{28}$$

Proof.

$$(y_1-y_2)''+p(x)(y_1-y_2)'+q(x)(y_1-y_2) = y_1''+p(x)y_1'+q(x)y_1-(y_2''+p(x)y_2'+q(x)y_2) = g(x)-g(x) = 0.$$

□

Now we combine the existence & uniqueness plus the above lemma to get the following statement:

Theorem 9.1 (superposition principle)**.** Fix a particular solution $y_p(x)$ of (27). Then any solution $y(x)$ of (27) can be written as the sum of a solution y_h of (28) and a y_p.

So, we at least managed to reduct part of the problem to solving a homogeneous ODE. But, how to find a particular solution y_p?

9.1 Undetermined coefficients

Let us concentrate on ODEs of this form:

$$y'' + ay' + by = g(x), \tag{29}$$

where $a, b \in \mathbb{R}$ and $g(x)$ is a linear combination of (polynomials and/or exponential functions and/or trigonometric functions). So, we want $g(x)$ to be as nice as possible.

Keep in mind that the homogeneous part of this ODE is constant-coefficient, and we can consistently find both solutions in the fundamental set.

Example 9.1. Consider an ODE of form

$$y'' + ay' + by = f_0 + f_1 x + \dots f_n x^n,$$

where $b \neq 0$ (otherwise we can reduce to a first-order ODE). Then we guess $y(x) = g_0 + g_1 x + \dots g_n x^n$. Then we get the following system:

$$\begin{cases} bg_n & = f_n, \\ ang_n + bg_{n-1} & = f_{n-1}, \\ n(n-1)g_n + a(n-1)g_{n-1} + bg_{n-2} & = f_{n-2}, \\ \vdots & \\ 2g_2 + ag_1 + bg_0 & = f_0. \end{cases}$$

This system can be always solved for g_n. Keep in mind that in the actual problems the degree of the polynomial in the RHS will usually be low.

For example, consider $y'' - 3y' + 2y = x$. To find a particular solution of this ODE, we try $y(x) = Ax + B$. Plugging this function into the ODE, we get

$$0 - 3A + 2Ax + 2B = x \Rightarrow 2A = 1, 2B - 3A = 0.$$

We can solve the resulting system to get $A = \frac{1}{2}, B = \frac{3}{4}$.

Also, consider $y'' - 3y + 2y = x^2 + 1$. Then, as expected, we look for $y(x) = Ax^2 + Bx + C$. We get

$$2A - 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2 + 1 \Rightarrow \begin{cases} 2A = 1, \\ -6A + 2B = 0, \\ 2A - 3B + 2C = 1. \end{cases}$$

This system can be solved, and we obtain $A = \frac{1}{2}, B = \frac{3}{2}, C = \frac{9}{4}$.

Example 9.2. Consider an ODE of form

$$y'' + ay' + by = Ce^{kx}, \tag{30}$$

where a, b, C are some constants. We want to guess a particular solution of this ODE. Let us try $y(x) = Ae^{kx}$. Plugging this guy into our ODE, we get

$$Ak^2e^{kx} + Aake^{kx} + Abe^{kx} = Ce^{kx}.$$

We get $A = C(k^2 + ak + b)^{-1}$. Let us denote $L(\lambda) = \lambda^2 + a\lambda + b$. Then we get the following statement:

Lemma 9.2. If k is not a root of the characteristic polynomial of $L(\lambda)$, then $y_p(x) = \dfrac{C}{L(k)}e^{kx}$ is a solution of (30).

But wait a second, what to do if k is a root of $L(\lambda)$? Then you can check that the above method does not work:

$$Ak^2e^{kx} + Aake^{kx} + Abe^{kx} = 0 = Ce^{kx}.$$

So, we try a familiar trick: we guess $y(x) = Axe^{kx}$. So, we get

$$y(x) = Axe^{kx}, \quad y'(x) = A(e^{kx} + kxe^{kx}), \quad y''(x) = A(2ke^{kx} + k^2xe^{kx}).$$

Plugging this into our ODE, we get

$$A(2ke^{kx} + k^2xe^{kx}) + a(A(e^{kx} + kxe^{kx})) + Abxe^{kx} = Ce^{kx}.$$

We can cancel out e^{kx}, and then collect some terms:

$$A(2k + a) + Ax\underbrace{(k^2 + ka + b)}_{\text{this is zero}} = C \Leftrightarrow A(2k + a) = C.$$

Remark. Observe that here $2k + a = L'(k)$, so $A = \dfrac{C}{L'(k)}$.

Finally, we need to treat the case where k is a double root of the characteristic equation, then we have to try $y(x) = Ax^2e^{kx}$. However, you can check that $y(x) = \frac{C}{2}x^2e^{kx} = \frac{C}{L''(k)}x^2e^{kx}$ works.

For example, let us try $y'' - 3y' + 2y = 5e^{6x}$. We can just plug in Ae^{6x} into the ODE to find A, but the formula we have derived earlier provides the value of A immediately:

$$L(\lambda) = \lambda^2 - 3\lambda + 2, \quad A = \frac{5}{L(6)} = \frac{5}{20} = \frac{1}{4}.$$

Now we consider $y'' - 3y' + 2y = e^x$. Once again, we can just directly plug in Axe^x and solve for A, or use the formula to get

$$L(\lambda) = \lambda^2 - 3\lambda + 2, \quad A = \frac{1}{L'(1)} = -1.$$

Example 9.3. To deal with ODEs of form

$$y'' + ay' + by = Cx^le^{kx},$$

we have to consider:

1. $y(x) = A_0 e^{kx} + A_1 x e^{kx} + \dots A_l x^l e^{kx}$ for some real A_i, if k is not a root of the characteristic equation $L(x)$,

2. $y(x) = x(A_0 e^{kx} + A_1 x e^{kx} + \dots A_l x^l e^{kx})$ for some real A_i, if k is a single root of the characteristic equation $L(x)$,

3. $y(x) = x^2(A_0 e^{kx} + A_1 x e^{kx} + \dots A_l x^l e^{kx})$ for some real A_i, if k is a double root of the characteristic equation $L(x)$.

Example 9.4. When dealing with ODEs of form

$$y'' + ay' + by = C\cos(kx)$$

or

$$y'' + ay' + by = C\sin(kx),$$

you always start with $y(x) = A\cos(kx) + B\sin(kx)$.

For example, consider $y'' - 3y' - 4y = 2\sin(x)$. Let us plug in $y(x) = A\cos(x) + B\sin(x)$. Then we get

$$-A\cos(x) - B\sin(x) + 3A\sin(x) - 3B\cos(x) - 4A\cos(x) - 4B\sin(x) = 2\sin(x) \Rightarrow \begin{cases} 3A - 5B = 2 \\ 5A + 3B = 0, \end{cases}$$

as $\sin(x)$ and $\cos(x)$ are linearly independent. So, we get $A = \frac{3}{17}$ and $B = -\frac{5}{17}$.

Question. Does this method work for any k? How you modify $y(x)$ in case it does not?

Example 9.5. Finally, you might get something like

$$y'' + py' + qy = Ce^{kx}\cos(mx),$$

then you can go into the realm of complex numbers by applying the Euler's formula to $\cos(mx)$, or you try

$$y(x) = Ae^{kx}\cos(mx) + Be^{kx}\sin(mx).$$

Once again, if $k + im$ is a root of the characteristic equation, you might have to modify this formula by multiplying your guess by x.

9.2 Variation of parameters

While the method of undetermined coefficients is nice, it does not apply to non-constant-coefficient ODEs. To treat general second-order ODEs (27), we need to generalize a method we used to solve non-homogeneous linear equations.

The idea is to somehow solve the respective homogeneous ODE (28), thus obtaining a general solution

$$y_g(x) = C_1 y_1(x) + C_2 y_2(x).$$

Then we replace C_1 with $u_1(x)$ and C_2 with $u_2(x)$. In other words, we attempt to find the particular solution in the form of

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Now we have to find u_1 and u_2. Let us plug this expression into our ODE. So, we get

$$y_p'(x) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$
$$y_p''(x) = u_1''y_1 + 2u_1'y_1' + u_1y_1'' + u_2''y_2 + 2u_2'y_2' + u_2y_2''.$$

$$(u_1''y_1 + 2u_1'y_1' + u_1y_1'') + (u_2''y_2 + 2u_2'y_2' + u_2y_2'') + p(x)(u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2') + q(x)(u_1y_1 + u_2y_2) = g(x).$$

Thankfully, $u_1(y_1'' + p(x)y_1' + q(x)y_1) = u_2(y_2'' + p(x)y_2' + q(x)y_2) = 0$, so this is equivalent to

$$u_1''y_1 + 2u_1'y_1' + u_2''y_2 + 2u_2'y_2' + p(x)(u_1'y_1 + u_2'y_2) = g(x).$$

However, solving this is still hard, so let us impose a relation on u_1 and u_2:

$$u_1'y_1 + u_2'y_2 = 0.$$

This causes massive cancellations, as

$$(u_1'y_1 + u_2'y_2)' = u_1''y_1 + u_1'y_1' + u_2''y_2 + u_2'y_2' = 0,$$

so

$$u_1''y_1 + 2u_1'y_1' + u_2''y_2 + 2u_2'y_2' + p(x)(u_1'y_1 + u_2'y_2) = u_1'y_1' + u_2'y_2' = g(x).$$

So, we are left with the following equations:

$$\begin{cases} u_1'y_1 + u_2'y_2 & = 0 \\ u_1'y_1' + u_2'y_2' & = g(x). \end{cases}$$

This is a linear system with respect to u_1' and u_2'. Applying the Cramer's rule, we immediately get

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{W[y_1, y_2]}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{W[y_1, y_2]},$$

this is equivalent to

$$u_1' = \frac{-y_2 g}{W[y_1, y_2]}, \quad u_2' = \frac{y_1 g}{W[y_1, y_2]}.$$

Now it remains to integrate both equalities, so we get

$$u_1(x) = \int \frac{-y_2 g}{W[y_1, y_2]} dx + C_1, \quad u_2(x) = \int \frac{y_1 g}{W[y_1, y_2]} dx + C_2.$$

To sum everything up, we get the following theorem:

Theorem 9.2. If $\{y_1, y_2\}$ form a fundamental set of solutions to (28), then the general solution to (27) can be written as

$$y(x) = y_1(x) \left(\int \frac{-y_2 g}{W[y_1, y_2]} dx + C_1 \right) + y_2(x) \left(\int \frac{y_1 g}{W[y_1, y_2]} dx + C_2 \right).$$

Remark. Keep in mind that if we change the constants, then the two solutions differ precisely by a linear combination of y_1 and y_2, as the superposition principle tells us.

Suppose that we are given some initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Then we need to find the constants C_1 and C_2. This will require us to replace the indefinite integrals with definite ones:

$$y(x) = y_1(x) \left(\int_{x_0}^{x} \frac{-y_2 g}{W[y_1, y_2]} dx + C_1 \right) + y_2(x) \left(\int_{x_0}^{x} \frac{y_1 g}{W[y_1, y_2]} dx + C_2 \right).$$

If we plug in $x = x_0$, we get

$$\begin{cases} y(x_0) = y_0 = C_1 y_1(x_0) + C_2 y_2(x_0), \\ y'(x_0) = y_0' = C_1 y_1'(x_0) + C_2 y_2'(x_0) - y_1(x_0) \dfrac{y_2(x_0) g(x_0)}{W[y_1, y_2](x_0)} + y_2(x_0) \dfrac{y_1(x_0) g(x_0)}{W[y_1, y_2](x_0)}. \end{cases}$$

But the big term cancels out, so we just get

$$\begin{cases} y_0 = C_1 y_1(x_0) + C_2 y_2(x_0), \\ y_0' = C_1 y_1'(x_0) + C_2 y_2'(x_0). \end{cases}$$

Now we can use the Cramer's rule to find C_1 and C_2:

$$C_1 = \frac{\begin{vmatrix} y_0 & y_2(x_0) \\ y_0' & y_2'(x_0) \end{vmatrix}}{W[y_1, y_2](x_0)}, \quad C_2 = \frac{\begin{vmatrix} y_1(x_0) & y_0 \\ y_1'(x_0) & y_0' \end{vmatrix}}{W[y_1, y_2](x_0)}.$$

As always, memorizing these formulas does not make that much sense, because for a particular case it is easier to follow the derivation itself.

Example 9.6. Consider the following initial value problem:

$$\begin{cases} y'' - y = \frac{2}{e^t+1}, \\ y(0) = y'(0) = 0. \end{cases}$$

Step 1. We find the general solution of the respective homogeneous ODE. It is easily seen that the roots of the characteristic equation are ± 1, therefore, we can choose a fundamental set $y_1 = e^t$, $y_2 = e^{-t}$.

Step 2. Now we construct the system of equations for u_1' and u_2':

$$\begin{cases} u_1'e^t + u_2'e^{-t} = 0 \\ u_1'e^t - u_2'e^{-t} = \frac{2}{e^t+1}. \end{cases}$$

Then we get

$$u_1' = e^{-t} - \frac{e^{-t}}{1+e^{-t}}, \quad u_2' = \frac{e^t}{1+e^t}.$$

By taking the antiderivatives, we obtain

$$u_1 = -e^{-t} + \ln(1+e^{-t}) + C_1, \quad u_2 = \ln(e^t+1) + C_2.$$

10 Euler's equations

Definition 10.1. A second-order **Euler's equation** is an ODE of form

$$x^2y'' + pxy' + qy = 0,$$

where $p, q \in \mathbb{R}$, and $x > 0$.

This is not a linear second-order ODE, but there is a strong connection between Euler's and linear ODEs. How to see this connection?

One way to do this is to find a particular solution of the ODE, then use the Abel's theorem or variation parameters. To guess a solution, we will try $y(x) = x^k$ for some $k \in \mathbb{R}$. We get

$$x^2(k-1)kx^{k-2} + pxkx^{k-1} + qx^k = 0 \Leftrightarrow x^k((k-1)k + pk + q) = 0.$$

The equation

$$(k-1)k + pk + q = 0$$

is often referred to as **indical equation**, and it is very much similar to the characteristic equation of a linear ODE: finding the roots of the indical equation gives the solutions to the Euler's equation.

Once again, we get three cases: we can get two real roots, one repeated roots or two complex conjugate roots. As an exercise, you can try applying the Abel's formula to deal with repeated roots and Euler's formula to deal with the complex case.

However, there is a relatively nice way of reducing Euler's equations to linear ones: we can try a change of variables! Consider $t = \ln(x)$. Let us denote $\dot{y} = \frac{dy}{dt}$. Then the chain rule yields

$$y'x = \dot{y}, \quad \ddot{y} = \frac{d(y'x)}{dt} = x\frac{dy'}{dt} + y'\frac{dx}{dt} = x^2y'' + y'x.$$

This can be substituted in the original ODE to get

$$x^2y'' + pxy' + qy = \ddot{y} + (p-1)\dot{y} + qy = 0.$$

As we can see, we get a linear equation whose characteristic equation coincides with the indicial equation.

So, in terms of t our solutions look like this:

1. (simple roots) If $k_1 \neq k_2$ are two distinct real roots of the indicial equation, then

$$y(t) = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

2. (repeated root) If k is a repeated root of the indicial equation, then

$$y(t) = C_1 e^{kt} + C_2 t e^{kt}$$

3. (complex roots) If $k \pm il$ are two complex conjugate roots of the indicial equation, then

$$y(t) = e^{kt}(C_1 \cos(lt) + C_2 \sin(lt)).$$

Applying the change of variables, we get the solutions in terms of x:

1. (simple roots) If $k_1 \neq k_2$ are two distinct real roots of the indicial equation, then

$$y(t) = C_1 x^{k_1} + C_2 x^{k_2}$$

2. (repeated root) If k is a repeated root of the indicial equation, then

$$y(t) = C_1 x^k + C_2 \ln(x) x^k$$

3. (complex roots) If $k \pm il$ are two complex conjugate roots of the indicial equation, then

$$y(t) = x^k(C_1 \cos(l \ln(x)) + C_2 \sin(l \ln(x))).$$

Definition 10.2. A linear ODE of order n is an equation of form

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_0(x)y = G(x). \tag{31}$$

Equivalently, we can represent every linear ODE as follows:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = g(x), \tag{32}$$

where

$$p_i(x) = \frac{P_i(x)}{P_n(x)}, \quad g(x) = \frac{G(x)}{P_n(x)}$$

for $0 \leq i \leq n-1$.

As always, if $g \equiv 0$, then we call the resepctive ODE **homogeneous**, and if all functional coefficients are constant, then we call the respective ODE **constant-coefficient**.

Notation. It will be convenient to denote the left-hand side by $L[y]$:

$$L[y] = \sum_{i=0}^{n} P_i(x)y^{(i)} = P_i(x)\frac{dy}{dx^i}(x).$$

Think of L as an operator $C^n[a,b] \to C[a,b]$, taking n-times continuously differentiable functions to continuous functions on $[a,b]$. For example, if $L[y] = y'''' + xy'$, then

$$L[e^x] = e^x + xe^x, \quad L[x^4] = 24 + 4x^4.$$

Theorem 10.1 (superposition principle)**.** If y_1 and y_2 solve

$$L[y] = P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_0(x)y = 0,$$

then for every $a, b \in \mathbb{R}$ the function $ay_1(x) + by_2(x)$ is also a solution.

Corollary 10.1. All solutions of a linear homogeneous ODE from a vector space over $\mathbb{R}$.

Theorem 10.2 (existence and uniqueness)**.** If p_i and g are continuous on some interval $I = (a, b) \subseteq \mathbb{R}$, then any initial value problem

$$\begin{cases} y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_0(x)y = g(x) \\ y(x_0) = y_0, \\ y'(x_0) = y_0' \\ \vdots \\ y^{(n-1)}(x_0) = y_0^{(n-1)}. \end{cases}$$

for $x_0 \in (a, b)$ and $y_0^{(i)} \in \mathbb{R}$ admits a unique solution on (a, b).

Corollary 10.2. The dimension of the solution space of a (reduced) linear homogeneous ODE order n equals n.

Proof. Let S denote the space of solutions to

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_0(x)y = 0.$$

Then we consider the following map:

$$\mathrm{ev}_{x_0} : S \to \mathbb{R}^n, \quad \mathrm{ev}_{x_0}(f) = \begin{pmatrix} f(x_0) \\ f'(x_0) \\ \vdots \\ f^{(n-1)}(x_0). \end{pmatrix}$$

This map satisfies the following properties:

1. $$\mathrm{ev}_{x_0}(y_1 + y_2) = \mathrm{ev}_{x_0}(y_1 + y_2),$$ $$\mathrm{ev}_{x_0}(ay) = a\mathrm{ev}_{x_0}(y).$$

2. This map is an injection because of the uniqueness of a solution.

3. This map is a surjection because the solution with given initial conditions always exists.

Therefore, ev_{x_0} is an isomorphism of vector spaces. □

Corollary 10.3. To find the general soltuion to a linear homogeneous ODE, find a basis $\{y_1, \dots, y_n\}$ in S (**fundamental set**), then any solution $y(x)$ can be written as

$$y(x) = \sum_{i=1}^{n} C_i y_i(x).$$

10.1 Constant-coefficient ODEs

Consider

$$L[y] = a_n y^{(n)} + \dots a_0 y = 0,$$

where $a_i \in \mathbb{R}$. How to approach such equations?

As in the case $n = 2$, we try $y(x) = e^{kx}$. When this function is a solution to this ODE?

$$L[e^{kx}] = e^{kx}\left(\sum_{i=0}^{n} a_i k^i\right) = 0 \Leftrightarrow \sum_{i=0}^{n} a_i k^i = 0.$$

This might be considered as an abuse of notation, but us denote $L(x) := \sum_{i=0}^{n} a_i x^i$, and call it the **characteristic polynomial**.

Thus we have proven the following simple theorem:

Theorem 10.3. The function $y(x) = e^{kx}$ is a solution if an only if k is a root of $L(x)$.

However, in general, a polynomial can have multiple roots and complex roots, as the following example shows:

Example 10.1. Consider
$$y''' - 3y' + 2y = 0.$$
The characteristic polynomial is
$$L(x) = x^3 - 3x + 2,$$
and we can quickly guess the root $x = 1$. Therefore, we obtain
$$x^3 - 3x + 2 = (x - 1)P(x).$$
Polynomial division yields $P(x) = x^2 + x - 2$, and $P(x) = (x - 1)(x + 2)$, so
$$L(x) = (x - 1)^2(x + 2).$$
This implies that $y_1(x) = e^x$ and $y_2(x) = e^{-2x}$ are solutions, but the subspace they generate is of dimension two – we are lacking another solution in our fundamental set! Indeed, 1 is a double root, so we suspect (using our second-order intuition) that $y_3 = xe^x$ might be a solution, and it is, as
$$L[xe^x] = (3x + 1)e^x - 3(x + 1)e^x + 2xe^x = 0.$$
So, $\{e^x, xe^x, e^{-2x}\}$ forms a fundamental set of solutions, and the general solution can be written down as follows:
$$y(x) = C_1e^x + C_2xe^x + C_3e^{-2x}.$$

However, how do we deal with tricker characteristic polynomials?

Theorem 10.4 (fundamental theorem of algebra)**.** Every complex polynomial has a root. As a corollary, if $P(x) = \sum_{i=0}^{n} a_i z^i \in \mathbb{C}[x]$, then
$$P(x) = a_0 \prod_{j=1}^{k} (z - k_j)^{m_j},$$
where k_j are the **roots**of P, and $m_j > 0$ are the **multiplicities** of k_j.

Remark. Multiplicities are defined in such a way that $m_1 + \cdots + m_j = n$.

Apart from polynomial division, one might find the Vieta's theorem useful for low-degree polynomials.

Definition 10.3. Let $k, n \in \mathbb{N}$. The **elementary symmetric polynomials** $e_k \in \mathbb{Z}[x_1, \ldots x_n]$ is defined as follows:
$$e_k(x_1, \ldots x_n) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} x_{j_1} x_{j_2} \ldots x_{j_k}.$$
Finally, we define $e_0 = 1$.

For example, if $n = 3$, then
$$e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3,$$
$$e_2(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_1x_3,$$
$$e_3(x_1, x_2, x_3) = x_1x_2x_3.$$

Theorem 10.5 (Vieta's theorem). Let

$$P(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{C}[x].$$

If $k_1, \dots, k_n$ are the roots of P (with repeats according to multiplicities), then

$$\begin{cases} e_1(k_1, \dots, k_n) = -\frac{a_{n-1}}{a_n} \\ e_2(k_1, \dots, k_n) = \frac{a_{n-2}}{a_n} \\ \vdots \\ e_{n-1}(k_1, \dots, k_n) = (-1)^{n-1}\frac{a_1}{a_n} \\ e_n(k_1, \dots, k_n) = (-1)^n \frac{a_0}{a_n}. \end{cases}$$

Proof. The fundamental theorem of algebra implies that

$$P(x) = a_n(x-k_1)(x-k_2)\dots(x-k_n).$$

However,

$$(x-k_1)(x-k_2)\dots(x-k_n) = x^n - e_1(k_1, \dots, k_n)x^{n-1} + e_2(k_i)x^{n-2} - \dots$$

Then we finish the argument by observing that two polynomials are equal if and only if the coefficients are equal. □

Now let us formulate the main theorem of this week, that you will be able to apply directly, and you don't need to know the proof by heart, but it is useful to think about how this theorem relates to the two-dimensional case.

Theorem 10.6. Consider the ODE

$$L[y] = \sum_{i=0}^{n} a_i y^{(i)} = 0.$$

If $L(x) = a_0 \prod_{j=1}^{k} (z-k_j)^{m_j}$, then the functions

$$y_{j,p} = x^p e^{k_j x}, \quad j = 1, \dots, k, p = 0, \dots, m_j - 1$$

form the fundamental set of solutions.

Lemma 10.1. For any functions $f(x), g(x)$ and $n > 0$ we have

$$\frac{d}{dx^n}(fg)(x) = \sum_{i=0}^{n} \binom{n}{i} \frac{df}{dx^i}(x) \frac{dg}{dx^{n-i}}(x).$$

Lemma 10.2. If k_i is a root of $L(x)$ with the multiplicity m_i, then

$$L(x) = T(x)(x-k_i)^{m_i}$$

for some polynomial $T(x)$, and $T(k_i) = \dfrac{L^{(m_i)}(k_i)}{m!}$.

Proof. The fact that T is a well-defined polynomial follows from the polynomial remainder theorem.

Previous lemma implies that

$$L^{(m_i)}(x) = \sum_{j=0}^{m_i} C_j \binom{n}{j} T^{(j)}(x)(x-k_i)^{m_i - j}$$

for some constants C_j. However, we can observe that $T^{(j)}(x)(x-k_i)^{m_i-j}$ vanishes for $x = k_i$ for all j except $j = m_j$, which immediately implies the formula for $T(k_i)$. □

Proof of the Theorem. Let us introduce some notation first. Let us treat the expression $\left(\frac{d}{dx}-a\right)$ as an operator. For example,

$$\left(\frac{d}{dx}-a\right)[f(x)] := \frac{d}{dx}f(x) - af(x).$$

But then

$$\left(\frac{d}{dx}-a\right)[e^{ax}] = ae^{ax} - ae^{ax} = 0,$$

and

$$\left(\frac{d}{dx}-a\right)[xe^{ax}] = e^{ax} + axe^{ax} - axe^{ax} = e^{ax}.$$

So, it is easily proven via induction that

$$\left(\frac{d}{dx}-a\right)^k [x^k e^{ax}] := \left(\frac{d}{dx}-a\right)\dots\left(\frac{d}{dx}-a\right)[x^k e^{ax}] = 0.$$

Finally, we notice that

$$L[y] = \left(T \circ \left(\frac{d}{dx}-k_i\right)^{m_i}\right)[y]$$

for every root k_i with multiplicity m_i, so

$$L[x^p e^{k_j x}] = T\left[\left(\frac{d}{dx}-k_i\right)^{m_i}[x^p e^{k_j x}]\right] = 0.$$

Linear independence is harder to show, but the shortest (and really handwavy) way is to consider any linear combination

$$P_1(x)e^{a_1 x} + \dots P_n(x)e^{a_n x}$$

for $a_1 < a_2 < \dots < a_n$. (won't work for complex a_i...) We can just divide both sides by $e^{a_n x}$, and then argue that

$$\lim_{x\to\infty} P_j(x)e^{(a_j-a_n)x} = 0$$

for all $j \le n-1$, and then our expression asymptotically behaves like $P_n(x)$, which was assumed to be a non-zero polynomial. $\square$

Remark. The best way to prove linear independence is to show that the **Laplace transform** of any linear combination is not zero.

Remark. This theorem is stated using the complex exponentials, in examples you are required to convert them into sines and cosines via Euler's formula.

11 Method of undetermined coefficients for higher-order linear ODEs

Suppose that we want to solve inhomogeneous ODE of the following forms:

$$L[y] := \sum_{i=0}^{n} a_i y^{(i)} = Ce^{kx}, \tag{33}$$

$$L[y] := \sum_{i=0}^{n} a_i y^{(i)} = Cx^l e^{kx}, \tag{34}$$

$$L[y] := \sum_{i=0}^{n} a_i y^{(i)} = Cx^l \cos(kx), \tag{35}$$

First of all, recall that, given a polynomial $L(x) = \sum\limits_{i=0}^{n} a_i x^i \in \mathbb{R}[x]$, we consider the respective **polynomial differential operator** $L[y]$ as follows:

$$L[y] := \sum_{i=0}^{n} a_i y^{(i)},$$

or, equivalently,

$$L(D) = L\left(\frac{d}{dx}\right) = \sum_{i=0}^{n} a_i \frac{d^i}{dx^i}.$$

Lemma 11.1. Let

$$L[y] = \sum_{i=0}^{n} a_i y^{(i)}.$$

Then for every $k \in \mathbb{R}, l \in \mathbb{N}$ we have

$$L[x^l e^{kx}] = \sum_{p=0}^{n} \frac{\binom{l}{p}}{p!} L^{(p)}(k) x^{l-p} e^{kx}. \tag{36}$$

In particular,

$$L[e^{kx}] = L(k) e^{kx}. \tag{37}$$

Proof. It is slightly tricky to see, but (36) is actually, linear with respect to L. First of all, we notice that

$$\sum_{p=0}^{n} \frac{\binom{l}{p}}{p!} L^{(p)}(k) x^{l-p} e^{kx} = \sum_{p=0}^{\infty} \frac{\binom{l}{p}}{p!} L^{(p)}(k) x^{l-p} e^{kx}.$$

Now we can consider another operator $\tilde{L}$, and we observe that

$$\sum_{p=0}^{\infty} \frac{\binom{l}{p}}{p!} (aL + L')^{(p)}(k) x^{l-p} e^{kx} = \sum_{p=0}^{\infty} \frac{\binom{l}{p}}{p!} (aL^{(p)}(k) + \tilde{L}^{(p)}(k)) x^{l-p} e^{kx}.$$

Therefore, it suffices to prove the statement for $L = \frac{d^n}{dx^n}$ (we just take the polynomial x^n), but this follows from the product rule almost directly:

$$\frac{d^n}{dx^n}(x^l e^{kx}) = \sum_{p=0}^{n} \binom{n}{p} \frac{d^p}{dx^p}(x^l) k^{n-p} e^{kx} = \sum_{p=0}^{n} \binom{n}{p} \binom{l}{p} p! x^{l-p} k^{n-p} e^{kx}.$$

Now we finish the argument by noticing that $\binom{n}{p} p! k^{n-p} = L^{(p)}(k)$. □

Theorem 11.1. There exists a solution to (33) in the form of

$$y(x) = A_m x^m e^{kx},$$

where m is the multiplicity of k as a root of the characteristic polynomial. Moreover, $A_m = C/L^{(m)}(k)$.

Proof. Recall that $L[y] = \left(T \circ \left(\frac{d}{dx} - k\right)^m\right)[y]$, so we have to find such $A_m \neq 0$ that

$$\left(T \circ \left(\frac{d}{dx} - k\right)^m\right)(A_m x^m e^{kx}) = Ce^{kx}.$$

However,

$$\left(\frac{d}{dx} - k\right)(A_m x^m e^{kx}) = mA_m x^{m-1} e^{kx},$$

so

$$\left(T\circ\left(\frac{d}{dx}-k\right)^m\right)(A_m x^m e^{kx}) = T[A_m m! e^{kx}].$$

We don't have a polynomial term in RHS, so this operator can be immediately computed:

$$T[A_m m! e^{kx}] = A_m m! T[e^{kx}] \overset{(37)}{=} A_m m! T(k) e^{kx} = Ce^{kx} \Rightarrow A_m L^{(m)}(k) = C.$$

□

Theorem 11.2. There exists a solution to (34) in the form of

$$y(x) = \sum_{j=0}^{l} B_j x^{m+j} e^{kx},$$

where m is the multiplicity of k as a root of the characteristic polynomial. Moreover, $B_{m+l} \neq 0$.

Proof. The idea is the same, let us compute $\left(\frac{d}{dx}-k\right)[y]$.

$$\left(\frac{d}{dx}-k\right)[y] = \sum_{j=0}^{l} B_j \left(\frac{d}{dx}-k\right)[x^{m+j}e^{kx}] = \sum_{j=0}^{l} B_j (m+j) x^{m+j-1} e^{kx}.$$

Therefore,

$$L[y] = \left(T\circ\left(\frac{d}{dx}-k\right)^m\right)[y] = T\left[\sum_{j=0}^{l} B_j \binom{m+j}{m} m! x^j e^{kx}\right] = \sum_{j=0}^{l} B_j \binom{m+j}{m} m! T[x^j e^{kx}] = Cx^l e^{kx}.$$

The formula (36) implies

$$T[x^j e^{kx}] = \sum_{p=0}^{\infty} \frac{\binom{j}{p}}{p!} T^{(p)}(k) x^{j-p} e^{kx},$$

We can rearrange the terms to get

$$B_l \binom{m+l}{m} T(k) x^l + \left(B_l \binom{m+l}{m}(l-1)T'(k) + B_{l-1}\binom{m+l-1}{m} T(k)\right) x^{l-1} + \cdots = \frac{C}{m!} x^l.$$

This is a triangular system, which admits a unique solution, moreover, $B_l \neq 0$. □

Theorem 11.3. There exists a solution to (35) in the form of

$$y(x) = \sum_{j=0}^{l} x^{m+j}(C_j \cos(kx) + D_j \sin(kx)),$$

where m is the multiplicity of ik as a root of the characteristic polynomial.

This theorem immediately follows from Theorem 1.2 via passing to complex exponentials.

Example 11.1. Let us find a particular solution to $y''' - 3y' + 2y = 58\cos(x)$.

First of all, we need to find the roots of the characteristic equations:

$$x^3 - 3x + 2 = (x-1)(x^2+x-2) = (x-1)^2(x+2).$$

Because $\pm i$ are not the roots, we can try $y(x) = A\cos(x) + B\sin(x)$. Computing the derivatives, we get

$$y' = -A\sin(x) + B\cos(x)$$
$$y'' = -y,$$
$$y''' = -y'.$$

So,

$$y''' - 3y' + 2y = -4y' + 2y = (4A + 2B)\sin(x) + (2A - 4B)\cos(x) = 58\cos(x).$$

Therefore,

$$\begin{cases} 4A + 2B = 0, \\ 2A - 4B = 58. \end{cases}$$

And we end up with $A = \frac{29}{5}$, $B = -\frac{58}{5}$.

Example 11.2. Let us find a particular solution to $y''' - 3y' + 2y = 8\cosh(2x)$.

Since $8\cosh(2x) = 4e^{2x} + 4e^{-2x}$, we can use the superposition principle to consider two separate ODEs. As 2 is not a root, we try $y(x) = Ae^{2x}$ to solve

$$y''' - 3y' + 2y = 4e^{2x}.$$

We quickly get $A = \frac{4}{L(2)} = e^{2t}$.

However, as -2 is a simple root, we have to plug in $y(x) = Bxe^{-2x}$ to solve

$$y''' - 3y' + 2y = 4e^{-2x}.$$

We get

$$B = \frac{4}{L'(-2)} = \frac{4}{12 - 3} = \frac{4}{9}.$$

12 Variation of parameters for higher-order ODEs

As for second-order ODEs, there is a universal way to solve inhomogeneous ODEs if we already know how to solve the respective homogeneous ODE.

Let us consider an equation

$$L[y] = \sum_{i=0}^{n} a_i(x) y^{(i)} = g(x),$$

where $a_i(x)$ and $g(x)$ are continuous functions on some interval I.

Remark. You can divide by a_n right away or while applying the method itself, as you will see later.

Suppose that we are given a fundamental set of solutions $\{y_1, \ldots, y_n\}$ which solves the homogeneous ODE. Then we want to find the solution to the inhomogeneous ODE in the form of

$$y(x) = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x) = \sum_{j=1}^{n} u_j(x)y_j(x).$$

Consider $y'(x)$:

$$y'(x) = \sum_{j=1}^{n} u_j'(x)y_j(x) + \sum_{j=1}^{n} u_j(x)y_j'(x).$$

Let us impose an extra condition:

$$\sum_{j=1}^{n} u_j'(x)y_j(x) = 0,$$

so that

$$y'(x) = \sum_{j=1}^{n} u_j(x)y_j'(x).$$

Then

$$y''(x) = \sum_{j=1}^{n} u_j'(x)y_j'(x) + \sum_{j=1}^{n} u_j(x)y_j''(x),$$

and let;s impose another condition:

$$\sum_{j=1}^{n} u_j'(x) y_j'(x) = 0,$$

so

$$y''(x) = \sum_{j=1}^{n} u_j(x) y_j''(x).$$

Continuing in the same fashion, we get the following system of equations:

$$\begin{cases} \sum_{j=1}^{n} u_j'(x) y_j(x) = 0 \\ \sum_{j=1}^{n} u_j'(x) y_j'(x) = 0 \\ \vdots \sum_{j=1}^{n} u_j'(x) y_j^{(n-2)}(x) = 0, \end{cases}$$

and $y^{(k)}(x) = \sum_{j=1}^{n} u_j(x) y_j^{(k)}(x)$ for all $0 \le k < n$.

However, we never computed $y^{(n)}$:

$$y^{(n)} = \sum_{j=1}^{n} u_j'(x) y_j^{(n-1)}(x) + \sum_{j=1}^{n} u_j(x) y_j^{(n)}(x).$$

Therefore,

$$L[y] = \sum_{i=0}^{n} a_i(x) \left(\sum_{j=1}^{n} u_j(x) y_j^{(i)}(x) \right) + a_n(x) \sum_{j=1}^{n} u_j'(x) y_j^{(n-1)}(x) = g(x).$$

However,

$$\sum_{i=0}^{n} a_i(x) \left(\sum_{j=1}^{n} u_j(x) y_j^{(i)}(x) \right) = \sum_{j=1}^{n} u_j(x) \left(\sum_{i=0}^{n} a_i y_j^{(i)}(x) \right) = 0,$$

so

$$\sum_{i=0}^{n} a_i(x) \left(\sum_{j=1}^{n} u_j(x) y_j^{(i)}(x) \right) + a_n(x) \sum_{j=1}^{n} u_j'(x) y_j^{(n-1)}(x) = a_n(x) \sum_{j=1}^{n} u_j'(x) y_j^{(n-1)}(x) = g(x)$$

recovers the last equation we need. So, our complete system of equations is

$$\begin{cases} \sum_{j=1}^{n} u_j'(x) y_j(x) = 0 \\ \sum_{j=1}^{n} u_j'(x) y_j'(x) = 0 \\ \vdots \\ \sum_{j=1}^{n} u_j'(x) y_j^{(n-2)}(x) = 0, \\ \sum_{j=1}^{n} u_j'(x) y_j^{(n-1)}(x) = \frac{g(x)}{a_n(x)}. \end{cases}$$

The determinant of this system is $W[y_1, \ldots, y_n](x)$. It does not vanish (why?), so we can apply the Cramer's rule to solve the system.

Proposition 12.1. For all $1 \le j \le n$ we have

$$u_j'(x) = (-1)^{n+j} \frac{W[y_1, \ldots y_{j-1}, y_{j+1}, \ldots, y_n](x)}{W[y_1, \ldots y_{j-1}, y_j, y_{j+1}, \ldots, y_n](x)} \frac{g(x)}{a_n(x)}.$$

Now we integrate this ratio and multiply by y_j to recover $y(x)$.

Theorem 12.1. The general solution to

$$L[y] := \sum_{i=0}^{n} a_i(x) y^{(i)} = g(x)$$

on an interval I is given by

$$y(x) = \int_{x_0}^{x} \left(\sum_{k=1}^{n} (-1)^{n+k} \frac{W[y_1, \dots y_{j-1}, y_{j+1}, \dots, y_n](s)}{W[y_1, \dots, y_n](s)} y_k(x) \right) \frac{g(s)}{a_n(s)} ds$$

for some $x_0 \in I$.

Hint: You are heavily encouraged to make use of **Abel's formula** for higher order ODEs (**without proof!**):

$$\frac{W'[y_1, \dots, y_n]}{W[y_1, \dots, y_n]} = -\frac{a_{n-1}(x)}{a_n(x)}.$$

However, I recommend proving the formula at least for $n = 3$.

Example 12.1. Find the general solution of $y''' - 7y' + 6y = \frac{40}{e^{2x}+1}$.

The characteristic equation is $L(k) = k^3 - 7k + 6 = (k-1)(k-2)(k+3)$. So, the roots are $k_{123} = 1, 2, -3$, and

$$y_h(x) = C_1 e^x + C_2 e^{2x} + C_3 e^{-3x}$$

is the solution for the homogeneous ODE.

Now we are looking for such $u_i(x)$, which satisfy the following system of equations:

$$\begin{cases} u_1' e^x + u_2' e^{2x} + u_3' e^{-3x} = 0 \\ u_1' e^x + 2u_2' e^{2x} - 3u_3' e^{-3x} = 0 \\ u_1' e^x + 4u_2' e^{2x} + 9u_3' e^{-3x} = \frac{40}{e^{2x}+1} \end{cases}$$

The simplest way to approach this system to subtract the first equation from the second and third equations. We get

$$\begin{cases} u_2' e^{2x} - 4u_3' e^{-3x} = 0 \\ 3u_2' e^{2x} + 8u_3' e^{-3x} = \frac{40}{e^{2x}+1}. \end{cases}$$

We can simplify further by multiplying the first equation by 3 and subtracting from the second equation.

$$20u_3' e^{-3x} = \frac{40}{e^{2x}+1}, \quad 5u_2' e^{2x} = \frac{40}{e^{2x}+1}$$

Now we will have to integrate:

$$u_2 = \int \frac{8}{e^{4x} + e^{2x}} dx \overset{t=e^{2x}}{=} \int \frac{8}{t^2+t} \frac{dt}{2t} = 4\left(-e^{-2x} - 2x + \ln(1+e^{2x})\right) + C_2.$$

$$u_3 = \int \frac{2e^{3x}}{e^{2x}+1} \overset{t=e^x}{=} \int \frac{2t^2 dt}{t^2+1} = 2(e^x - \arctan(e^x)) + C_3.$$

Finally, we need to find u_1.

$$u_1' = -u_2' e^x - u_3' e^{-4x} = -\frac{8}{e^{3x}+e^x} - \frac{2}{e^{3x}+e^x} = -\frac{10}{e^{3x}+e^x}.$$

Integrating, we get $u_1(x) = 10(e^{-x} - \arctan(e^{-x})) + C_1$.

13 Systems of first-order linear ODEs

13.1 Two-dimensional case

Definition 13.1. A 2×2 system of first-order linear ODEs is a system

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 \\ x_2' = a_{21}x_1 + a_{22}x_2 \end{cases}$$

for some constants a_{ij}. A **solution** of such a system is a C^1-function $X : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, such that

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + a_{12}x_2(t) \\ x_2'(t) = a_{21}x_1(t) + a_{22}x_2(t) \end{cases}$$

holds for $t \in (b_0, b_1)$.

Example 13.1. The simplest possible system is

$$\begin{cases} x_1' = 0x_1 + 0x_2 = 0 \\ x_2' = 0x_1 + 0x_2 = 0. \end{cases}$$

Solving such a system is easy: $t \mapsto \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$ works for any two constants C_1, C_2.

Example 13.2. A bit more difficult example is

$$\begin{cases} x_1' = 1x_1 + 0x_2 = x_1 \\ x_2' = 0x_1 + 1x_2 = x_2. \end{cases}$$

We can solve both ODEs separately, thus getting $X(t) = e^t \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$.

But keep in mind that more difficult systems cannot be solved like that... we need to develop some tools to solve such systems

13.2 General facts

First of all, let us rewrite the system using the language of linear algebra. If we denote $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $A = (a_{ij})_{1 \le i,j \le 2}$, then the system can be rewritten like this:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = X' = AX.$$

This representation immediately implies

Theorem 13.1 (superposition principle)**.** If $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$ are two solutions, then any linear combination $aX + Y$ is also a solution.

Theorem 13.2 (existence and uniqueness for linear systems)**.** Consider the following IVP:

$$\begin{cases} X' = AX, \\ X(t_0) = \begin{pmatrix} a \\ b \end{pmatrix}. \end{cases}$$

where $a, b \in \mathbb{R}$. Then there exists a unique globally defined solution $X(t)$.

Corollary 13.1. The space of soluitons for $X' = AX$ is two-dimensional. The explicit isomorphism can be defined like this:

$$X(t) \mapsto X(t_0) \in \mathbb{R}^2$$

In particular, if we find two solutions $X_1(t)$ and $X_2(t)$, then the general solution can be written down like this:

$$X(t) = C_1X_1(t) + C_2X_2(t).$$

13.3 Let's try to find some solutions!

As we are still dealing with linear ODEs, we want to try solutions of form $X(t) = e^{kt}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = e^{kt}v$. Let us plug this function into our ODE:

$$X' = kX = ke^{kt}v, \quad AX = e^{kt}Av.$$

Therefore, $Av = kv$. How do we call such vectors v?

Definition 13.2. Let $A \in \mathrm{Mat}_n(\mathbb{R})$. A vector $v \in \mathbb{R}^n$ is called an **eigenvector** of A if there exists a number k (called **eigenvalue** of v) such that $Av = kv$.

Definition 13.3. A matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ is called **diagonalizable** if $\mathbb{R}^n$ admits a basis, consisting of eigenvectors of A.

So, if A is diagonalizable, then there exist two eigenvectors v_1, v_2 with eigenvalues k_1, k_2. Therefore, the general solution looks like this:

$$X(t) = C_1e^{k_1t}v_1 + C_2e^{k_2t}v_2.$$

Example 13.3. Let $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Its characteristic polynomial is $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = \lambda^2 - 5\lambda + 6$. The roots are $2, 3$, so there are two distinct eigenvalues.

Now it remains to find the eigenvectors. The first eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ can be guessed immediately, and the other vector can be found as follows:

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2a_1 + a_2 \\ 3a_2 \end{pmatrix} = 3\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

We get $a_1 = a_2$, so $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is another eigenvector. So, theeneral solution is

$$X(t) = C_1e^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Remark. Before we move on to non-diagonalizable cases, we want to talk about the relation between second-order ODEs and systems. As you have noticed, there are several striking similarities between two solution schemes. This is not a coincidence!

Example 13.4. Consider $y'' + py' + qy = 0$. Let us denote $y = x_1$, $y' = x_2$. Then

$$y'' + py' + qy = 0 \Leftrightarrow y'' = x_2' = -px_2 - qx_1 \Leftrightarrow \begin{cases} x_1' = x_2 \\ x_2' = -px_2 - qx_1 \end{cases}.$$

Therefore, every second-order ODE is equivalent to solving the system

$$X' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}X.$$

Let us check this by solving $y'' + 4y' + 3y = 0$. The respective system is given by

$$X' = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} X.$$

The characteristic equation is $\lambda^2 + 4\lambda + 3 = 0$, so the eigenvalues are $-1, -3$. Let us find the eigenvectors:

$$\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ -3a - 4b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}.$$

Therefore, $b = -a$, and $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ will work.

$$\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ -3a - 4b \end{pmatrix} = \begin{pmatrix} -3a \\ -3b \end{pmatrix}.$$

Therefore, $b = -3a$, and $v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ will work. So, the general solution looks like this:

$$X(t) = C_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

In particular,

$$x_1(t) = y(t) = -C_1 e^{-t} + C_2 e^{-3t},$$

which is the general solution for the second-order ODE.

13.4 Non-diagonalizable cases

Unfortunately, not every real-valued matrix is diagonalizable. It is well-known that every **symmetric** matrix admits an orthogonal eigenbasis, but there are two cases that might happen:

1. The eigenvalues are complex
2. The eigenvalue is repeated and there is only one eigenvector.

So, suppose that A admits two complex conjugate complex eigenvalues $\lambda_{12} = k \pm il$. Then it is easy to see that $v_1 = \overline{v_2}$. So,

$$X(t) = C_1 e^{kt}(\cos(lt) + i\sin(lt))v_1 + C_2 e^{kt}(\cos(lt) - i\sin(lt))\overline{v_1}$$

is the complex general solution to the system.

Recall that

$$e^{kt}(\cos(lt) + i\sin(lt))v_1 + e^{kt}(\cos(lt) - i\sin(lt))\overline{v_1} = 2\mathrm{Re}(e^{kt}(\cos(lt) + i\sin(lt))v_1)$$
$$e^{kt}(\cos(lt) + i\sin(lt))v_1 - e^{kt}(\cos(lt) - i\sin(lt))\overline{v_1} = 2i\mathrm{Im}(e^{kt}(\cos(lt) + i\sin(lt))v_1),$$

and these from the **real** fundamental set of solutions.

Example 13.5. Consider $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The eigenvalues are $\pm i$, with the eigenvectors $\begin{pmatrix} 1 \\ \pm i \end{pmatrix}$. Therefore, the fundamental set is

$$t \mapsto \mathrm{Re}\left((\cos(t) + i\sin(t))\begin{pmatrix} 1 \\ i \end{pmatrix}\right) = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix},$$

$$t \mapsto \mathrm{Im}\left((\cos(t) + i\sin(t))\begin{pmatrix} 1 \\ i \end{pmatrix}\right) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}.$$

And, well, that makes sense, as this system **does** arrive from a second-order ODE with periodic solutions.

Now, suppose that A is not diagonalizable and has only one real eigenvalue k and one eigenvector v. So, $X_1(t) = e^{kt}v$ is a solution, but how to find another linearly independent solution?

Proposition 13.1. If a 2-by-2 matrix A is not diagonalizable, and it admits a single eigenvector with the eigenvalue k, then there exists a matrix C such that

$$C^{-1}AC = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}.$$

Such matrices are called (two-dimensional) **Jordan blocks**.

Proof. The idea is that $A - k\mathrm{Id}$ is a nilpotent operator, so there exists a vector w, such that v and w form a basis, and $(A - k\mathrm{Id})w = v$. Then we can choose $C = \begin{pmatrix} v & w \end{pmatrix}$. ☐

So, suppose that $B = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$. Then we can still solve the system naively:

$$\begin{cases} x_1' = kx_1 + x_2 \\ x_2' = kx_2 \end{cases} \Rightarrow \begin{cases} x_1' = kx_1 + C_2e^{kt} \\ x_2(t) = C_2e^{kt} \end{cases} \Rightarrow \begin{cases} x_1(t) = (C_2t + C_1)e^{kt} \\ x_2(t) = C_2e^{kt} \end{cases}.$$

In this case $v = e_1, w = e_2$. Therefore,

$$X(t) = C_1e^{kt}v + C_2e^{kt}w + C_2te^{kt}v,$$

and, in particular, we claim that $X_2(t) = e^{kt}w + te^{kt}v$ is also a solution:

$$X_2'(t) = ke^{kt}w + e^{kt}v + kte^{kt}v, \quad AX_2(t) = e^{kt}(kw + v) + tke^{kt}v.$$

As we can see, both expressions have the same terms, so $X_2' = AX_2$

14 Dynamics of solutions

In this section we will discuss how solutions behave when $t \to \infty$. We recommend looking at the respective phase portraits in other lecture notes provided in this course. This will be a pretty basic overview of how the solutions behave for different types of systems.

Definition 14.1. For systems $X' = AX$, the origin is referred to as an **equilibrium point**. It is called

- **asymptotically stable**, if all solutions tend to the origin for $t \to \infty$, regardless of the initial condition,
- **stable**, if all solutions stay in some bounded region, which depends only on the initial condition,
- otherwise, we call it an **unstable** equilibrium.

Remark. This definition differs from the standard one, as, in general, we only need to consider initial values close the equilibrium.

14.1 The diagonalizable case

In this case the solution always has the form

$$X(t) = C_1e^{k_1t}v_1 + C_2e^{k_2t}v_2.$$

- We can have $0 < k_1, k_2$. Then both exponential terms tend to infinity when $t \to \infty$ and to zero when $t \to -\infty$ for **any** non-zero pair (C_1, C_2). Therefore,

$$\lim_{t\to\infty} ||X(t)|| = \infty, \quad \lim_{t\to-\infty} ||X(t)|| = 0.$$

- We can have $0 > k_1, k_2$. Then both exponential terms tend to infinity when $t \to -\infty$ and to zero when $t \to \infty$ for **any** non-zero pair (C_1, C_2). Therefore,

$$\lim_{t\to\infty} ||X(t)|| = 0, \quad \lim_{t\to-\infty} ||X(t)|| = \infty.$$

  This case can be reduced to the first one by setting $t = -t$.

- We can have $k_1 < 0 < k_2$. Then for $C_1 = 0$ we get the solution $t \mapsto C_2 e^{k_2 t} v_2$ which tends to infinity for $t \to \infty$, and for $C_2 = 0$ we get a solution with the opposite behavior. Therefore, in this case the equilibrium is unstable. Sometimes, when we have one stable direction and one unstable direction, we call the respective equilibrium **semistable**.

- When one of the eigenvalues is zero (for example, k_1), we get

$$X(t) = C_1 v_1 + C_2 e^{k_2 t} v_2,$$

  and the behaviour only depends on the absolute value of k_2. If $k_2 < 0$ then the equilibrium is stable, if $k_2 > 0$ then the solutions tend to infinity.

- For $k_1 = k_2$ we get a relatively simple description: all solutions just flow along straight lines. And the direction only depends on the sign of $k_1 = k_2$.

14.2 Complex eigenvalues

In this case the solution always has the form

$$X(t) = C_1 \mathrm{Re}(e^{(k+il)t} v) + C_2 \mathrm{Im}(e^{(k+il)t} v)$$

where $k + il$ is the complex eigenvalue and v is the complex eigenvector.

First of all, let us compute the real parts and the imaginary parts as follows:

$$\mathrm{Re}(e^{(k+il)t} v) = e^{kt} \cos(lt) \mathrm{Re}(v) - e^{kt} \sin(lt) \mathrm{Im}(v),$$

$$\mathrm{Im}(e^{(k+il)t} v) = e^{kt} \sin(lt) \mathrm{Re}(v) + e^{kt} \cos(lt) \mathrm{Im}(v).$$

As $\cos(lt)\mathrm{Re(v)} - \sin(lt)\mathrm{Im(v)}$ and $\sin(lt)\mathrm{Re(v)} + \cos(lt)\mathrm{Im(v)}$ are two periodic functions, the stability only depends on k. In other words, we only need to look at the real part of the complex eigenvalue.

- If $k > 0$, then the solutions tend to infinity.
- If $k < 0$, then the solutions converge to zero.
- If $k = 0$, then the solutions are periodic.

14.3 The case of a Jordan block (non-diagonalizable case)

In this case we only have one eigenvalue, so the dynamics only depend on the sign of the eigenvalue.

- If $k > 0$, then the solutions tend to infinity.
- If $k < 0$, then the equlibrium is asymptotically stable, and they converge to zero along the lone eigenvector.
- If $k = 0$, then we get a weird behaviour: the solutions along the eigenvector stay in place, and for other initial conditions the solutions flow in parallel to the eigenvector, but in the opposite sides to the right and to the left of the eigenvector.

Remark. As you can observe, stability in **all** cases only depends on the sign of the real part of the eigenvalue(s).

15 Non-homogeneous systems of first-order linear ODEs

Consider the following systems of ODEs:

$$X'(t) = AX(t) + G(t), \tag{38}$$

where $G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$. Such systems are called **non-homogeneous systems of first-order linear ODEs**, and in this section we will develop several methods of solving them.

As always, the solutions exist and are unique provided we specify an initial condition. Also, the superposition holds:

Theorem 15.1 (superposition principle)**.** Difference of any two solutions of (38) is a solution to $X' = AX$. In particular, any solution to (38) is a sum of a solution $X_h(t)$ to $X' = AX$ and some particular solution $X_p(t)$.

But how can we find a particular solution?

15.1 Diagonalization

Suppose that D is a diagonal matrix, let $D = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$, and we want to solve

$$Y' = DY + H, \tag{39}$$

where $H(t) = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$This system can be rewritten as follows:

$$\begin{cases} y_1'(t) = k_1 y_1(t) + h_1(t), \\ y_2'(t) = k_2 y_2(t) + h_2(t). \end{cases}$$

As you can see, these ODEs can be solved separately. So, we get

$$y_i(t) = e^{k_i t} \int_{t_0}^{t} e^{-k_i s} h_i(s) ds + C_i e^{k_i t}.$$

Now, suppose that A is a diagonalizable matrix, and we want to solve

$$X' = AX + G.$$

First of all, we replace $X = TY$, where T is the transition matrix corresponding to the eigenbasis. In other words, $T^{-1}AT = D$. So, we can rewrite the system as follows:

$$X' = AX + G \Leftrightarrow (TY)' = ATY + G \Leftrightarrow T(Y') = ATY + G.$$

Multiplying both parts by T^{-1} from the left, we get

$$T(Y') = ATY + G \Leftrightarrow Y' = (T^{-1}AT)Y + T^{-1}G = DY + T^{-1}G.$$

Denoting $T^{-1}G = H$, we arrive at (39), which we already know how to solve.

16 Fundamental matrices

Definition 16.1. Consider a linear system of ODEs $X' = AX$, and fix a fundamental set of solutions $S = \{X_1(t), X_2(t)\}$. Then the **fundamental matrix** $\Psi(t)$ associated to S is a matrix, columns of which are X_i. In other words, if we denote

$$X_1(t) = \begin{pmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{pmatrix}, \quad X_2(t) = \begin{pmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{pmatrix},$$

then

$$\Psi(t) = \begin{pmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) \end{pmatrix} := (X_1|X_2).$$

Remark. Of course, the matrix $\Psi(t)$ depends on the choice of the set. We will explore the relation between the fundamental matrices for different fundamental sets later in the notes.

Example 16.1. Consider the system $X' = AX$ for $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then the general solution to the system is

$$X(t) = C_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can choose the fundamental system in several ways, for example, define

$$S_1 = \left\{ e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad S_2 = \left\{ e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Then

$$\Psi_{S_1} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}, \quad \Psi_{S_2} = \begin{pmatrix} e^t & 0 \\ e^{2t} & e^{2t} \end{pmatrix}.$$

Example 16.2. Consider the system $X' = AX$ for $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Then we know that there are two eigenvectors: $v_1 = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ corresponding to the eigenvalue 2, and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ corresponding to the eigenvalue 3. So, the general solution is

$$X(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We can choose the fundamental system in several ways, for example, define

$$S_1 = \left\{ e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \quad \left\{ e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Then

$$\Psi_{S_1}(t) = \begin{pmatrix} e^{2t} & e^{3t} \\ 0 & e^{3t} \end{pmatrix}, \quad \Psi_{S_2}(t) = \begin{pmatrix} e^{2t} + e^{3t} & e^{3t} \\ e^{3t} & e^{3t} \end{pmatrix}$$

Can you guess how these pairs of matrices are related?

16.1 Properties of fundamental matrices

Proposition 16.1. Let $X' = AX$ be a linear system of ODEs equipped with a fundamental set of solutions (X_1, X_2). Then, for every vector $v \in \mathbb{R}^2$ the function $X(t) := \Psi(t)v$ is a solution of this system.

Proof. For each $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ we have

$$\Psi(t)v = v_1 X_1(t) + v_2 X_2(t),$$

but the superposition principle implies that this is always a solution of the provided system. □

Corollary 16.1. Let $X' = AX$ be a linear system of ODEs equipped with a fundamental set of solutions (X_1, X_2). Consider the following IVP:

$$\begin{cases} X' = AX, \\ X(t_0) = v. \end{cases}$$

Then the function $X(t) := \Psi(t)\Psi^{-1}(t_0)v$ solves the IVP.

Proof. The previous proposition already implies that the following function solves the system, now we only need to verify the initial condition.

$$X(t_0) = \Psi(t_0)\Psi^{-1}(t_0)v = v.$$

□

16.2 Matrix exponential

Definition 16.2. Let A be a matrix. The **matrix exponential** of A is the value of a series

$$\exp(A) := I + A + \frac{A^2}{2} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

It is not trivial to show that this series converges in $\mathrm{Mat}_2(\mathbb{R})$. A slick way to prove the convergence makes use of a **submultiplicative norm** on $\mathrm{Mat}_2(\mathbb{R})$. In other words, we want to find a non-negative function $||\cdot|| : \mathrm{Mat}_2(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ which satisfies the following properties:

1. $||A|| = 0 \iff A = 0$,
2. $||aA|| = |a|||A||$,
3. $||A+B|| \leq ||A|| + ||B||$,
4. $||AB|| \leq ||A||||B||$.

Such norms exist: a natural choice is the **operator norm**: $||A||_{op} := \sup\limits_{|v|=1} |Av|$.

Any norm defines a topology on $\mathrm{Mat}_2(\mathbb{R})$ which is equivalent to the Eucledian one (non-trivial fact!). Therefore, we only need to show that the series converges absolutely, and it does:

$$||\exp(A)||_{op} = \left\|\sum_{k=0}^{\infty} \frac{A^k}{k!}\right\|_{op} \leq \sum_{k=0}^{\infty} \left\|\frac{A^k}{k!}\right\|_{op} \leq \sum_{k=0}^{\infty} \frac{\|A\|_{op}^k}{k!} = e^{||A||_{op}} < \infty.$$

Remark. You are not required to understand the details of this proof, let's just take the fact that the matrix exponential is well-defined for granted.

Proposition 16.2. Let A, B, C be matrices. Then the following properties hold:

1. if $AB = BA$ then $\exp(A+B) = \exp(A)\exp(B)$.
2. $\exp(-A) = \exp(A^{-1})$. In particular, every matrix exponential is invertible.
3. $\exp(CAC^{-1}) = C\exp(A)C^{-1}$.

Proof.

1. Because $AB = BA$, for every $k > 0$ we have

$$(A+B)^k = \sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i}.$$

Therefore,

$$\exp(A+B) = \sum_{k=0}^{\infty} \frac{\sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i}}{k!} = \sum_{\substack{0\leq k<\infty \\ 0\leq i\leq k}} \binom{k}{i} \frac{A^i B^{k-i}}{k!}$$

Now we set $k - i = j$. Then we can rewrite the sum in terms of k, j:

$$\binom{k}{i} \frac{A^i B^{k-i}}{k!} = \binom{k}{k-j} \frac{A^{k-j} B^j}{k!} = \frac{A^{k-j} B^j}{j!(k-j)!}.$$

However, now both k and j run from 0 to ∞, so

$$\sum_{\substack{0\leq k\leq\infty \\ 0\leq i\leq k}} \binom{k}{i} \frac{A^i B^{k-i}}{k!} = \sum_{\substack{0\leq k<\infty \\ 0\leq j<\infty}} \frac{A^{k-j} B^j}{j!(k-j)!} = \left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right)\left(\sum_{j=0}^{\infty} \frac{B^j}{j!}\right) = \exp(A)\exp(B).$$

2. We know that A commutes with $-A$, so, using the first property, we get
$$\exp(A)\exp(-A) = \exp(-A)\exp(A) = \exp(A - A) = \exp(0) = I.$$
3. Here we just use the matrix associativity:
$$\exp(CAC^{-1}) = \sum_{k=0}^{\infty} \frac{(CAC^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{CA^kC^{-1}}{k!} = C\left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right)C^{-1} = C\exp(A)C^{-1}.$$

□

Example 16.3. Now we will discuss how to compute matrix exponentials. First of all, we need to compute the exponentials of the following matrices:

- Consider $D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Then
$$D^k = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix},$$
so
$$\exp(D) = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}.$$

- Consider $J = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$. We observe that the matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ do commute, so
$$\exp(J) = \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix}\exp(N).$$
However, N is a **nilpotent matrix**, in other words, $N^k = 0$ for some $k > 0$. It is easily checked that $N^2 = 0$, so $\exp(N) = I + N$. Therefore,
$$\exp(J) = \begin{pmatrix} e^a & e^a \\ 0 & e^a \end{pmatrix}$$

- Now consider $C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. A quick way to compute this exponential is to note that this matrix corresponds to a complex number $a + ib$, and its exponential will correspond to e^{a+ib}, as the correspondence is, in fact, a ring isomorphism. Thus, we can use the Euler's formula:
$$\exp(C) = \begin{pmatrix} e^a\cos(b) & -e^a\sin(b) \\ e^a\sin(b) & e^a\cos(b) \end{pmatrix}.$$

However, every 2 by 2 matrix is conjugate to one of these matrices, so we use the third property to conclude the argument.

Example 16.4. Consider $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$. Then its eigenvalues are $2, 3$, and the corresponding eigenvectors are $v_1 = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So,
$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$
Now we take matrix exponentials:
$$\begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\exp\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \exp\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}.$$
Computing the inverse and multiplying the results, we get
$$\exp\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} e^2 & e^3 - e^2 \\ 0 & e^3 \end{pmatrix}.$$

Example 16.5. Consider $B = \begin{pmatrix} 4 & -1 \\ 1 & 6 \end{pmatrix}$. Let us write the characteristic equation:

$$p(x) = x^2 - 10x + 25 = (x-5)^2.$$

Therefore, 5 is a repeated eigenvalue. Let us find the eigenvector:

$$B - 5Id = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix},$$

so

$$(B - 5Id)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff -x = y.$$

So, $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the eigenvector. Now we have to find the generalized eigenvector w:

$$(B - 5Id)\begin{pmatrix} x \\ y \end{pmatrix} = v = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \iff -x - y = 1,$$

so we can choose $w = \begin{pmatrix} -1 \\ 0. \end{pmatrix}$ Therefore, we consider the transition matrix

$$C = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix},$$

and

$$C^{-1}BC = \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix} \Rightarrow C^{-1}\exp(B)C = \begin{pmatrix} e^5 & e^5 \\ 0 & e^5 \end{pmatrix} \Rightarrow \exp(B) = C\begin{pmatrix} e^5 & e^5 \\ 0 & e^5 \end{pmatrix}C^{-1}.$$

Now we just compute the product of these three matrices:

$$\begin{pmatrix} e^5 & e^5 \\ 0 & e^5 \end{pmatrix}C^{-1} = \begin{pmatrix} -e^5 & -2e^5 \\ -e^5 & -e^5 \end{pmatrix}, \quad C\begin{pmatrix} -e^5 & -2e^5 \\ -e^5 & -e^5 \end{pmatrix} = \begin{pmatrix} 0 & -e^5 \\ e^5 & 2e^5 \end{pmatrix}.$$

Example 16.6. Consider $D = \begin{pmatrix} 3 & 1 \\ -8 & -1 \end{pmatrix}$. Its characteristic polynomial is

$$p(x) = x^2 - 2x + 5,$$

the roots are $x_{12} = 1 \pm 2i$. Let us find the complex eigenvector:

$$(D - (1+2i)Id)\begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff \begin{pmatrix} 2-2i & 1 \\ -8 & -2-2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow (2-2i)x + y = 0.$$

Therefore, the complex eigenvector is $v = \begin{pmatrix} 1 \\ -2+2i \end{pmatrix}$. Therefore, we want to find the real and the imaginary part of the vector:

$$\mathrm{Re}(v) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathrm{Im}(v) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Now consider $C = (\mathrm{Re}(v)|\mathrm{Im}(v)) = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$. We can check that $C^{-1}DC = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Therefore,

$$\exp(D) = C\begin{pmatrix} e\cos(2t) & e\sin(2t) \\ -e\sin(2t) & e\cos(2t) \end{pmatrix}C^{-1} = \begin{pmatrix} e\cos(2) + e\sin(2) & \frac{1}{2}e\sin(2) \\ -4e\sin(2) & e\cos(2) - e\sin(2) \end{pmatrix}.$$

Now, let us discuss how can we use matrix exponentials to our advantage.

Proposition 16.3. Consider a system $X' = AX$. Then for any vector $v \in \mathbb{R}^2$ the function $t \mapsto \exp(At)v$ is a solution of the system.

Proof. MAT23x teaches that we can safely differentiate series by differentiating each term separately. In this case, we get

$$\frac{d}{dt}\exp(At)v = \left(\sum_{k=0}^{\infty} \frac{A^k v t^k}{k!}\right)' = \sum_{k=0}^{\infty} \frac{kA^k v t^{k-1}}{k!} = A\exp(At)v.$$

So, matrix exponentials solve systems of ODEs. $\square$

But we already know that the solution of an IVP is unique. So, the following corollary hints at the connection between fundamental matrices and matrix exponentials:

Corollary 16.2. Let $X' = AX$ be a system equipped with a fundamental set (X_1, X_2). Then

$$\exp(At) = \Psi(t)\Psi^{-1}(0).$$

This corollary suggests that we can find the general solution of a system by finding $\exp(At)$.

17 Variation of parameters

Now, that we know about fundamental matrices, we can discuss another, more general way to solve non-homogeneous linear 2 by 2 systems.

Consider a system $X' = AX + G$, where $G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$, and g_i are continuous. Then, in the spirit of the one-dimensional variation pf parameters, we solve the homogeneous system $X' = AX$ first. So, the resulting general solution looks like $X_h(t) = \Psi(t)v$. Now we replace v with $U(t)$, and we try to look for a particular solution in the form

$$X_p(t) = \Psi(t)U(t).$$

Let us plug this function into the non-homogeneous system:

$$X_p'(t) = \Psi'(t)U(t) + \Psi(t)U'(t) = A\Psi(t)U(t) + G(t).$$

As $\Psi'(t)U(t) = A\Psi(t)U(t)$, this simplifies to

$$\Psi(t)U'(t) = G(t) \iff U'(t) = \Psi^{-1}(t)G(t) \Rightarrow U(t) = \int \Psi^{-1}(t)G(t)dt + C,$$

where $C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \in \mathbb{R}^2$ Therefore, we get the following formula for the general solution:

$$X(t) = \Psi(t)\int \Psi^{-1}(t)G(t)dt + \Psi(t)C.$$

As we expected, $\Psi(t)C$ is the general solution to the respective homogeneous system.

Example 17.1 (Exercise 1, ch. 7.9)**.** Consider

$$X' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} X + \begin{pmatrix} e^t \\ t \end{pmatrix}.$$

First of all, we meed to compute **a** fundamental matrix of this system. Let's do it the old-fashioned way by just solving the homogeneous system.

Characteristic equation:

$$p(x) = x^2 - 1, x = -1, 1.$$

The respective eigenvectors are $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So, the general solution is

$$X(t) = C_1 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Choosing two solutions $e^{-t}v_1$ and $e^t v_2$, we get

$$\Psi(t) = \begin{pmatrix} e^{-t} & e^t \\ 3e^{-t} & e^t \end{pmatrix}.$$

Now we have to compute $\Psi^{-1}(t)G(t)$:

$$\Psi^{-1}(t)G(t) = \frac{1}{2}\begin{pmatrix} -e^{2t} + te^t \\ 3 - te^{-t} \end{pmatrix}.$$

Integrating this function we get

$$U(t) = \begin{pmatrix} \frac{1}{2}\left(e^t(t-1) - \frac{e^{2t}}{2}\right) + C_1 \\ \frac{3t}{2} + \frac{1}{2}e^{-t}(t+1) + C_2 \end{pmatrix}.$$

18 General theory of linear systems of first-order ODEs

In this section we will outline the basic properties and approaches to $n \times n$ systems of first-order ODEs.

Definition 18.1. An $n \times n$ **system of linear equations** is a system

$$\begin{cases} x_1'(t) = a_{11}x_1 + \dots a_{1n}x_n(t) \\ x_2'(t) = a_{21}x_1 + \dots a_{2n}x_n(t) \\ \vdots \\ x_n'(t) = a_{n1}x_1 + \dots a_{nn}x_n(t). \end{cases}$$

As always, the first step we need to take is to rewrite this system in matrix form by defining $A = (a_{ij})$. The system in the definition will be equivalent to

$$X' = AX,$$

where $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

As always, the superposition principle holds, and the solutions are uniquely defined by setting the initial condition $X(t_0) = v$ for any $t_0 \in \mathbb{R}$ and $v \in \mathbb{R}^n$.

In other words, we can always find a fundamental set of solutions $(X_i(t))$, such that the general solution can be written as follows:

$$X(t) = C_1X_1(t) + \dots + C_nX_n(t).$$

But we run into the same trouble as for 2×2 systems: not every matrix is diagonalizable. However, each matrix admits the **Jordan normal form**.

Theorem 18.1 (complex-valued Jordan normal form)**.** Let $A \in \mathrm{Mat}_n(\mathbb{C})$. Then A is similar to a block-diagonal matrix of the following form:

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \dots & J_m \end{pmatrix},$$

where $J_i = \begin{pmatrix} k_i & 1 & 0 & \dots & 0 & 0 \\ 0 & k_i & 1 & \dots & 0 & 0 \\ 0 & 0 & k_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & k_i & 1 \\ 0 & 0 & 0 & \dots & 0 & k_i \end{pmatrix}$, and all $k_i \in \mathbb{C}$.

Keep in mind that we will be mostly dealing with real matrices and we want to obtain real-valued solutions. So, we need to formulate the real-valued version as well.

Theorem 18.2 (real-valued Jordan normal form)**.** Let $A \in \mathrm{Mat}_n(\mathbb{R})$. Then A is similar to a block-diagonal matrix of the following form:

$$\begin{pmatrix} J^r & 0 \\ 0 & J^c \end{pmatrix},$$

where

$$J^r = \begin{pmatrix} J_1^r & 0 & \dots & 0 \\ 0 & J_2^r & \dots & 0 \\ \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \dots & J_m^r \end{pmatrix},$$

where $J_i^r = \begin{pmatrix} k_i & 1 & 0 & \dots & 0 & 0 \\ 0 & k_i & 1 & \dots & 0 & 0 \\ 0 & 0 & k_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & k_i & 1 \\ 0 & 0 & 0 & \dots & 0 & k_i \end{pmatrix}$, and all $k_i \in \mathbb{R}$, and

$$J^c = \begin{pmatrix} J_1^c & 0 & \dots & 0 \\ 0 & J_2^c & \dots & 0 \\ \vdots & \vdots & \ddots & \dots \\ 0 & 0 & \dots & J_l^c \end{pmatrix},$$

$$J_j^c = \begin{pmatrix} \begin{matrix} a_j & b_j \\ -b_j & a_j \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 & \dots & 0 & 0 \\ 0 & \begin{matrix} a_j & b_j \\ -b_j & a_j \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \dots & 0 & 0 \\ 0 & 0 & \begin{matrix} a_j & b_j \\ -b_j & a_j \end{matrix} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \begin{matrix} a_j & b_j \\ -b_j & a_j \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ 0 & 0 & 0 & \dots & 0 & \begin{matrix} a_j & b_j \\ -b_j & a_j \end{matrix} \end{pmatrix}.$$

19 Laplace transform: basic examples and properties

Definition 19.1. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ be a function. Then we define its **Laplace transform** $\mathcal{L}(f)$ as follows:

$$\mathcal{L}(f)(s) = \int_{0^-}^{\infty} e^{-st} f(t) dt$$

for $s > 0$.

Remark. We never specified the functional domain where is transform is well-defined, as this integral might diverge. We will do that later. Also, if you are comfortable with complex integration, you can also define this integral for any $s \in \mathbb{C}$. Again, we won't use these values of s in our course.
Remark. Also, as you have noticed, we wrote 0^- instead of just 0. As we are going to deal with pretty nasty "functions" (yes, quotations are intentional), we will want to consider

$$\int_{0^-}^{\infty} e^{-st} f(t) dt := \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\infty} e^{-st} f(t) dt.$$

If f is continuous at 0 then we can just write 0. For example,

Example 19.1. Let $f(t) \equiv 1$. Then

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st}dt = -\frac{e^{-st}}{s}\bigg|_{t=0}^{t=\infty} = \frac{1}{s}$$

for all $s > 0$.

Example 19.2. Let $f(t) = e^{at}$. Then

$$\mathcal{L}(f)(s) = \int_0^\infty e^{(a-s)t}dt = \frac{e^{(a-s)t}}{a-s}\bigg|_{t=0}^{t=\infty},$$

however, this difference is only well-defined for $s > a$. For such s we have

$$\mathcal{L}(f)(s) = \frac{1}{s-a}.$$

19.1 Properties of Laplace transform

In this subsection we will formulate several important properties of the Laplace transform.

Definition 19.2. A function $f : [\alpha, \beta] \to \mathbb{C}$ is called **piecewise continious** if it has finitely many jump discontinuities.

Remark. In this definition, "finitely many" is the same as saying "isolated".

Definition 19.3. A function $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$ is of **exponential order** if there exist constants $a \in \mathbb{R}$ and $M, K > 0$ such that for any $t \geq M$ we have

$$|f(t)| \leq Ke^{at}.$$

Theorem 19.1 (Theorem 6.1.2)**.** Let f be a piecewise continuous function of exponential order. Then $\mathcal{L}(f)(s)$ is well-defined for $s > a$, where a is a constant chosen in Definition 19.3.

Proof. First of all, we notice that

$$\int_{0^-}^\infty e^{-st}f(t)dt = \int_{0^-}^M e^{-st}f(t)dt + \int_M^\infty e^{-st}f(t)dt.$$

Riemann integrability criterion ensures that we can separate integrals like this, and $\int_{0^-}^M e^{-st}f(t)dt$ converges. Therefore, we need to deal with the right integral. Let us show that it converges absolutely:

$$\left|\int_M^\infty e^{-st}f(t)dt\right| \leq \int_M^\infty |e^{-st}||f(t)|dt \leq \int_M^\infty e^{-st}e^{at}dt.$$

But the previous example show that this integral converges precisely when $s > a$. □

Remark. Almost every function we will consider will be piecewise continuous and of exponential order, this will ensure that their Laplace transforms are well defined for $s \to \infty$.

Example 19.3 (Dealing with discontinuous functions)**.** Let f be a piecewise continuous function on $[\alpha, \beta]$. For every $t_0 \in [\alpha, \beta]$ let us define

$$f(t_0^+) := \lim_{t \to t_0^+} f(t), \quad f(t_0^-) := \lim_{t \to t_0^-} f(t).$$

In the first limit (with $+$) we approach t_0 from the right, and in the second one (with $-$) we approach t_0 from the left.

Consider a **step function**

$$f_k(t) := \begin{cases} 1, & 0 \le t < 1, \\ k, & t = 1, \\ 0, & t > 1. \end{cases}$$

In particular, notice that

$$f(1^+) = 0, \quad f(1^-) = 1.$$

Let us compute its Laplace transform:

$$\int_0^\infty e^{-st} f_k(t) dt = \int_0^1 e^{-st} dt = \frac{1 - e^{-s}}{s}.$$

Proposition 19.1. Let f, g be functions satisfying the conditions of Theorem 19.1.

1. For every constants $a, b \in \mathbb{R}$ we have $\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f) + b\mathcal{L}(g)$.
2. For any $a > 0$ we have
$$\mathcal{L}(f(at)) = \frac{1}{a}\mathcal{L}(f)\left(\frac{s}{a}\right)$$
3. For any $a \in \mathbb{R}$ we have
$$\mathcal{L}(e^{at} f(t)) = \mathcal{L}(f)(s - a).$$
4. Assume that f' satisfies the conditions of Theorem 19.1. Denoting $\mathcal{L}(f)(s) = F(s)$, we get
$$\mathcal{L}(f')(s) = sF(s) - f(0^-).$$
5. Assume that $f^{(n)}$ satisfies the conditions of Theorem 19.1. Then
$$\mathcal{L}(f^{(n)})(s) = s^n F(s) - \sum_{i=0}^{n-1} s^{n-1-i} f^{(i)}(0^-).$$
6. For any $n \in \mathbb{N}$ we have
$$\mathcal{L}(t^n f)(s) = (-1)^n F^{(n)}(s).$$
7. Define the **Heaviside function**
$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \ge c \end{cases}.$$
Then
$$\mathcal{L}(f(t - c)u_c(t))(s) = e^{-cs} F(s).$$

Proof.

1. Follows from the linearity of the integral:
$$\int_{0^-}^\infty e^{-st}(af(t) + bg(t))dt = a\int_{0^-}^\infty e^{-st} f(t)dt + b\int_{0^-}^\infty e^{-st} g(t)dt = a\mathcal{L}(f)(s) + b\mathcal{L}(g)(s).$$
2. Let $a > 0$.
$$\int_{0^-}^\infty e^{-st} f(at)dt \overset{at=u}{=} \frac{1}{a}\int_{0^-}^\infty e^{-\frac{su}{a}} f(u)du = \frac{1}{a}\mathcal{L}(f)\left(\frac{s}{a}\right)$$
3. Let $a \in \mathbb{R}$.
$$\int_{0^-}^\infty e^{-st} e^{at} f(t)dt = \int_{0^-}^\infty e^{(a-s)t} f(t)dt = \mathcal{L}(f)(s - a).$$

4. As with all natural integral transforms, here we integrate by parts:
$$\int_{0^-}^{\infty} e^{-st} f'(t)dt + \int_{0^-}^{\infty} -se^{-st} f(t)dt = \lim_{\varepsilon\to 0} \lim_{A\to\infty} \left(e^{-sA} f(A) - e^{s\varepsilon} f(-\varepsilon)\right).$$
As f is of exponential order, $\lim_{A\to\infty} e^{-sA} f(A) = 0$, and $\lim_{\varepsilon\to 0} e^{s\varepsilon} f(-\varepsilon) = f(0^-)$. Therefore,
$$\int_{0^-}^{\infty} e^{-st} f'(t)dt = sF(s) - f(0^-).$$
5. This is proven via induction by n.
6. Let us prove this for $n = 1$, then we use induction.
$$\int_{0^-}^{\infty} e^{-st} t f(t)dt = \int_{0^-}^{\infty} -\frac{d}{ds}(e^{-st} f(t))dt \overset{\text{Leibnitz}}{=} -\frac{d}{ds}\left(\int_{0^-}^{\infty} e^{-st} f(t)dt\right) = -F'(s).$$
7.
$$\int_{0^-}^{\infty} e^{-st} u_c(t) f(t-c)dt = \int_{c^-}^{\infty} e^{-st} f(t-c)dt \overset{t-c=u}{=} \int_{0^-}^{\infty} e^{-su-sc} f(u)du = e^{-cs} F(s).$$

□

Example 19.4. Let us compute $\mathcal{L}(\sin(at))$. The textbook gives us a standard method to compute this transform, but let's try a different approach. We know that
$$\sin(at) = \mathrm{Im}(e^{iat}),$$
so
$$\mathcal{L}(\mathrm{Im}(e^{iat})) = \mathrm{Im}\left(\mathcal{L}(e^{iat})\right) = \mathrm{Im}\left(\frac{1}{s-ia}\right).$$
As
$$(s-ia)^{-1} = \frac{s+ia}{s^2+a^2},$$
we get
$$\mathcal{L}(\mathrm{Im}(e^{iat})) = \mathrm{Im}\left(\frac{1}{s-ia}\right) = \frac{a}{a^2+s^2}.$$
Yes, it is still **illegal** to consider complex values of s, but this is still a neat way to see what the Laplace transform of $\sin(at)$ and $\cos(at)$ should be.

Also, we know that $\cos(at) = \frac{d}{dt}\left(\frac{1}{a}\sin(at)\right)$. So,
$$\mathcal{L}(\cos(at))(s) = s\mathcal{L}\left(\frac{1}{a}\sin(at)\right)(s) = \frac{s}{s^2+a^2}.$$
Pretty cool, right?

19.2 Solving ODEs using the Laplace transform

Example 19.5. Find the solution of the following IVP:
$$\begin{cases} y'' - y' - 6y = 0, \\ y(0) = 1, y'(0) = 1. \end{cases}$$
Let us apply the Laplace transform to both sides of the equation.
$$s^2Y(s) - y'(0) - sy(0) - sY(s) + y(0) - 6Y(s) = s^2Y(s) - 1 - s - sY(s) + 1 - 6Y(s) = 0.$$

Therefore,

$$Y(s) = \frac{s}{s^2 - s - 6}.$$

We can apply partial fractions, so that we get

$$\frac{s}{s^2 - s - 6} = \frac{s}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2} = \frac{3}{5s-15} + \frac{2}{5s+10}$$

So, we have to find the inverse Laplace transform of $\frac{3}{5s-15}$ and $\frac{2}{5s+10}$.

So, our problem is reduced to finding the **inverse Laplace transform** $\mathcal{L}^{-1}$ of some rational functions. To do this you apply partial fractions, and then just consult Table 6.2.1 in the textbook.

Example 19.6 (cont.)**.** To find $\mathcal{L}^{-1}\left(\frac{3}{5s-15}\right)$ and $\mathcal{L}^{-1}\left(\frac{2}{5s+10}\right)$, we just consult the table:

$$Y(s) = \frac{3}{5s-15} = \frac{3}{5}\frac{1}{s-3} \mapsto y(s) = \frac{3}{5}e^{3t},$$

$$Y(s) = \frac{2}{5s+10} = \frac{2}{5}\frac{1}{s+2} \mapsto y(s) = \frac{2}{5}e^{-2t}.$$

19.3 Partial fractions

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We won't talk about the algorithm itself, but the basic idea of the method of partial fractions is to decompose any rational function $\frac{P(s)}{Q(s)}$ into a sum of elementary functions:

$$\frac{P(s)}{Q(s)} = \sum_{i,j} \frac{A_{ij}}{(s-a_j)^i} + \sum_{k,l} \frac{B_{kl}^{(1)}s + B_{kl}^{(2)}}{(s^2 + b_l^2)^k}.$$

Now, suppose that one is given an ODE of form

$$a_n y^{(n)} + \cdots + a_0 y = g(t),$$

where $g(t)$ is a linear combination of exponential and trigonometric functions. Then we can take the Laplace transforms of both parts, and we can observe (without doing any major computations) that $Y(s)$ is, indeed, a rational function.

However, you need to keep in mind that the table does not have a formula for

$$\frac{1}{(s^2 + a^2)^k}$$

for $k > 1$. Can you find the inverse Laplace transform of this function?

Hint: use the fact that

$$\left(\frac{1}{(s^2 + a^2)^k}\right)' = -\frac{2ks}{(s^2 + a^2)^{k+1}}.$$

20 Solving ODEs with discontinuous forcing functions

In this section we want to understand how to solve constant-coefficient ODEs of form

$$L[y] := a_n y^{(n)} + \ldots a_0 y = g(t),$$

where $g(t)$ has several jump discontinuities. As Laplace transform is the only tool that allows us to reliably deal with discontinuous functions, we will use this approach for the example given below.

Example 20.1. Solve the following IVP:

$$\begin{cases} y'' + y = 1 - u_{3\pi}(t), \\ y(0) = 0, y'(0) = 1. \end{cases}$$

Let us apply the Laplace transform to both parts of the ODE.

$$s^2 Y(s) - 1 + Y(s) = \frac{1 - e^{-3\pi s}}{s}.$$

Therefore,

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - e^{-3\pi s}\frac{1}{s(s^2+1)}.$$

Then

$$\mathcal{L}^{-1}(Y(s))(t) = y(t) = \sin(t) + h(t) - u_{3\pi}(t)h(t - 3\pi),$$

where $\mathcal{L}(h)(s) = H(s) = \frac{1}{s(s^2+1)}$. Applying the partial fractions method, we get

$$\frac{1}{s(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C}{s} \Rightarrow \begin{cases} A + C = 0 \\ B = 0 \\ C = 1, \end{cases}$$

so

$$H(s) = \frac{1}{s} - \frac{s}{s^2+1} \Rightarrow h(t) = 1 - \cos(t),$$

$$y(t) = \sin(t) - \cos(t) + 1 - u_{3\pi}(t)(1 - \cos(t - 3\pi)).$$

21 Dirac delta functions

Let us start by defining the following family of functions:

$$d_\tau(t) = \frac{1}{2\tau}\chi_{[-\tau,\tau]}(t) = \begin{cases} \frac{1}{2\tau}, & \text{if } -\tau \le t \le \tau, \\ 0, & \text{otherwise.} \end{cases}$$

There are two facts we want to observe about this family of functions:

- As $\tau \to 0$, the **supports** of the functions d_τ go to zero. In other words, the region where the functions are non-zero shrinks and becomes smaller and smaller, until it collapses to a single point.
- The integrals of these functions are always equal to 1.

It is good to think of d_τ as a process that happens over a very short amount of time.

Now, try to imagine the "pointwise limit" of d_τ for $\tau \to 0$. Naively, we get the following:

$$\lim_{\tau\to 0} d_\tau(t) = \begin{cases} 0, & \text{if } t \neq 0, \\ \infty, & \text{if } t = 0. \end{cases}$$

Does this limit exist in the traditional sense? Of course not, a function cannot be equal to infinity at any point. Nevertheless, let us denote this mysterious limit by $\delta(t)$. It is called **the Dirac delta function**, and we can make sense of it by trying to compute the following integral for all $\varepsilon > 0$:

$$\int_{-\varepsilon}^{\infty} f(t)\delta(t)dt$$

for a sufficiently "nice" function $f(t)$. As $\delta(t)$ was defined as a pointwise limit, we can **define** this integral as a limit as well:

$$\int_{-\varepsilon}^{\infty} f(t)\delta(t)dt := \lim_{\tau\to 0}\int_{-\varepsilon}^{\infty} f(t)\chi_{[-\tau,\tau]}(t)dt.$$

Choosing $\tau < \varepsilon$, we get that this limit will be equal to

$$\lim_{\tau \to 0} \frac{1}{2\tau} \int_{-\tau}^{\tau} f(t)dt.$$

But we already know that this limit exists and is equal to $f(0)$. Therefore, we can **define**

$$\int_{0^-}^{\infty} f(t)\delta(t)dt = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\infty} f(t)\delta(t)dt = \lim_{\varepsilon \to 0} f(0) = f(0).$$

Same argument shows

$$\int_{0^-}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

for every $t_0 \geq 0$. Now, notice that if $f(t) = e^{-st}$ then we just computed the Laplace transform of the delta function.

And this argument also suggests that is is best to think of the delta function as a "**generaized function**", whose domain is a nice space of functions, and it acts as the evaluation functional at 0.

Remark. Finally, this argument shows why having 0^- instead of 0 or 0^+ is so important: we need the domain to fully contain a small open neighborhood of 0.

21.1 Dealing with the delta function in concrete examples

The previous argument shows that $\mathcal{L}(\delta(t-t_0))(s) = e^{-st_0}$ for $t_0 \geq 0$. We can use this fact to solve some problems featuring the delta function.

Example 21.1. Consider the following IVP featuring a non-homogeneous second-order ODE:

$$\begin{cases} y'' + 2y' + 2y = \delta(t-\pi), \\ y(0) = 1, y'(0) = 0. \end{cases}$$

Let us take the Laplace transform of both sides:

$$s^2Y(s) - s + 2sY(s) - 2 + 2Y(s) = e^{-\pi s}.$$

Therefore,

$$Y(s) = \frac{s+2+e^{-\pi s}}{s^2+2s+2} = \frac{s+2}{s^2+2s+2} + \frac{e^{-\pi s}}{s^2+2s+2}$$

For convenience, let us denote $H_1(s) := \frac{s+2}{s^2+2s+2}$ and $H_2(s) = \frac{1}{s^2+2s+2}$. If $\mathcal{L}(h_i(t))(s) := H_i(s)$, then

$$y(t) = h_1(t) + u_\pi(t)h_2(t-\pi).$$

Now we observe that

$$H_1(s-1) = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1},$$

and

$$H_2(s-1) = \frac{1}{s^2+1}.$$

So,

$$\mathcal{L}^{-1}(H_1(s-1))(t) = \cos(t) + \sin(t) \Rightarrow \mathcal{L}^{-1}(H_1(s))(t) = h_1(t) = e^{-t}\cos(t) + e^{-t}\sin(t),$$

and

$$h_2(t) = e^{-t}\sin(t).$$

Finally, we get

$$y(t) = e^{-t}(\cos(t) + \sin(t)) - u_\pi(t)e^{-t+\pi}(\sin(t)).$$

In particular,

$$y(t) = \begin{cases} e^{-t}(\cos(t) + \sin(t)), & t < \pi, \\ e^{-t}\cos(t) + (e^{-t} - e^{\pi - t})\sin(t) & t \geq \pi. \end{cases}$$

In particular, the plot of this function looks like this:

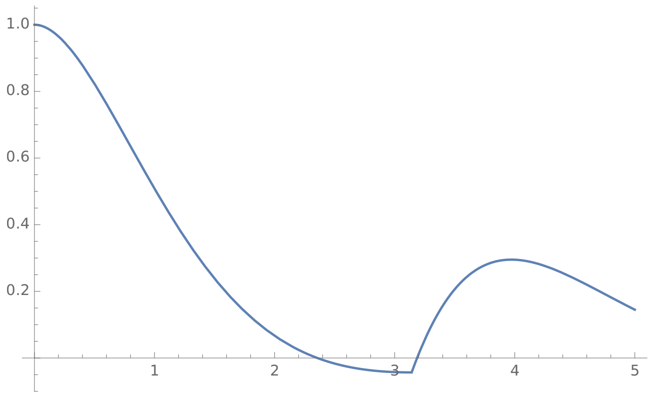

Figure 2: