On a complex-analytic approach to classifying stationary measures on S^1 with respect to the countably supported measures on $PSU(1,1)$

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Abstract

We provide a complex-analytic approach to the classification of stationary probability measures on S^1 with respect to complex finite Borel measures on $PSU(1,1)$ satisfying the finite first moment condition by studying their Poisson-like and Cauchy-Szegő transforms from the perspective of generalized analytic continuation.

1 Introduction

1.1 Background

One of the most important notions in dynamical systems is of an **invariant measure**: given a topological space X and a self-map $T : X \to X$, one can study Borel measures $\mu \in Bor(X)$ on X which satisfy

$$(T_*\mu)(A) = \mu(T^{-1}(A)) = \mu(A). \tag{1}$$

This notion works quite well when provided with a single map $X \to X$. However, one often encounters nice spaces equipped with group actions $\Gamma \curvearrowright X$, and it is entirely possible that there are no measures invariant with respect to every element $\gamma \in \Gamma$.

Nevertheless, there is a natural weakening of the above definition, requiring a measure to be invariant "on average".

Definition 1.1. *Consider a group action $\Gamma \curvearrowright X$. Let ν be a Borel probability measure on X. Given a Borel measure μ on Γ, we say that ν is μ-**stationary** (with respect to the action) if*

$$\nu = \int_\gamma \gamma_* \nu d\mu(\gamma). \tag{2}$$

It is easy to see that any invariant measure is stationary with respect to any probability measure on Γ, but the inverse is, of course, not true. Being a μ-stationary measure is, evidently, a much weaker condition. Stationary measures exist in very general settings, unlike invariant ones, but they are no less important, as they are closely related to the Poisson boundaries and long-term dynamics of random walks on groups.

Given an admissible random walk (X_n) on a non-elementary discrete subgroup $\Gamma \subset PSU(1,1)$ with a finite first moment, we know (due to Furstenberg ([Fur71]) and Kaimanovich ([Kai00])) that (X_n) converges to the Poisson boundary almost surely. The respective pushforward of the resulting measure to S^1 via the identification $\partial\Gamma \simeq S^1$ with the Gromov boundary yields a (unique) μ-stationary measure ν_μ with respect to the action of Γ, called the **hitting measure** of the random walk. A big open problem in measured group theory is to understand when hitting measures are singular or absolutely continuous with respect to the Lebesgue measure on S^1. In particular, recall the Kaimanovich-le Prince's singularity conjecture:

Conjecture 1.1 ([KL11]). *For every finite-range admissible random walk (X_n) generated by a probability measure μ on a cocompact Fuchsian group Γ, the hitting measure ν_μ is singular with respect to the Lebesgue measure on $S^1 \simeq \partial\Gamma$.*

This conjecture is known to hold for non-cocompact lattices due to [GL90], and the author's thesis [Kos23] provides affirmative results for nearest-neighbour random walks on cocompact Fuchsian groups, but the conjecture is still widely open, as we did show that even the recently developed geometric ideas are insufficient to completely settle the conjecture.

In our paper we will study the actions of subgroups $\Gamma \subset G = PSU(1,1)$ induced by the action of G on S^1 via hyperbolic isometries, or, more concretely, Möbius transformations. We aim to present a complex-analytic framework which, as we believe, can unify the majority existing results about stationary measures on S^1 with respect to the action of Γ and probability measures satisfying a finite first moment condition. Keep in mind that the analysis will be different depending on several factors:

- Whether μ has finite support or not,
- If μ is infinitely supported, the moment conditions on μ will matter (first moment, exponential moment, superexponential moment, and so on...),
- Whether the subgroup of G generated by the support of μ is discrete or not,
- If the generated subgroup is discrete, whether it is of first or second type,
- And, finally, if it is of first kind, whether it is cocompact or not.

First results about stationary measures and Poisson boundaries for discrete subgroups of $SL_n(\mathbb{R})$ were established by Furstenberg in [Fur63]. In particular, the question of when the Lebesgue measure is μ-stationary was first studied by Furstenberg as well in [Fur71]. Pure Fourier-like approaches were independently demonstrated by [Bou12] and [BPS12], which allow us to study μ-stationary measures for **dense** subgroups of $PSU(1,1)$. However, their methods do not apply for discrete groups and are quite delicate with respect to the initial data, requiring complicated number-theoretic and analytic methods to properly apply. There have been multiple independently developed improvements to Bourgain's approach, see [Leq22] and [Kog22] for latest examples, but they still do not apply to the discrete case and non-finite supports. We also want to mention [Kit23], which provides an entirely different analytic framework to study stationary measures, allowing us to consider stationary measures with continuous but not necessarily differentiable densities. Finally, we want to mention recent attempts to understand the structure of harmonic and Patterson-Sullivan measures using thermodynamic formalism, for example, [GL23] and [CT22]. Once again, these approaches are not universal, as Garcìa-Lessa's paper does not generalize to first-kind Fuchsian groups, and the thermodynamic approach of Cantrell-Tanaka provides considerably more information for Patterson-Sullivan measures than harmonic measures.

As one can see, up until now there was no single method which unified all above settings, and until very recently, the general consensus was that no such method should have exist, in light of the incredible variety of techniques used to study different settings.

1.2 Main results

Inspired by the standard techniques used to study affine self-similar measures on $\mathbb{R}^n$, and their respective Parseval frames, papers of R.S.Strichartz (see [Str90] and sequels), together with [JP98] and more recent papers of E.Weber and J.Herr [Web17] and [HW17], we have developed a promising approach which, in theory, could unify many standard results about stationary measures on S^1 with respect to the action of $PSU(1,1)$. The idea is to consider an appropriate integral transform on S^1 which respects the action of $PSU(1,1)$ and "preserves" (2). The Fourier transform is known to not respect this action, and the resulting exponential terms $e^{(az+b)/(cz+d)}$ are difficult to control. The Helgason-Fourier transform seems to be a better candidate, but integrating the powers of the Poisson kernel $(z,\xi)\mapsto\left(\frac{1-|z|^2}{|z-\xi|^2}\right)^{\lambda i+1}$ against a stationary measure does not actually preserve (2) in a way we want. Essentially, given a μ-stationary measure ν on S^1, one can easily check that the resulting smooth eigenvector of the hyperbolic Laplacian

$$\psi(z) := \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-|z|^2}{|z-e^{it}|^2}\right)^{\lambda i+1} d\nu(t)$$

does not exhibit any nice properties with respect to the action of $PSU(1,1)$, unlike what we see for Patterson-Sullivan measures. However, replacing the Poisson kernel with its logarithm, which is closely related to the Busemann cocycle, does the trick, turning a multiplicative relation into an additive one. The resulting functional equation (4) serves as a proper holomorphic version of (2), and, in a way, it allows us to change the perspective, as we shift from the measurable setting to a holomorphic one on $\mathbb{D}$, granting access to the powerful complex-analytic machinery.

Before stating our results, we would like to point out that it is sufficient to study pure μ-stationary measures due to the fact that the action of $PSU(1,1)$ respects the Lebesgue decomposition, this is a standard reduction.

In order to formulate the main result, we need the definition of the Cauchy-Szegö transform:

$$f_\nu(z) := \frac{1}{2\pi}\int_0^{2\pi}\frac{d\nu(t)}{e^{it}-z}.$$

Theorem 1.1. *Let μ be a probability measure on $G = PSU(1,1)$ satisfying the following moment condition:*

$$\int_G \log\left(\frac{1+|\gamma.z|}{1-|\gamma.z|}\right) d|\mu|(\gamma) < \infty \tag{3}$$

for any $z\in\mathbb{D}$.

- *If a probability measure ν on S^1 is μ-stationary, then*

$$\int_G f_\nu(\gamma^{-1}.z)(\gamma^{-1})'(z)d\mu(\gamma) - f_\nu(z) = \int_G \frac{d\mu(\gamma)}{z-\gamma.\infty}, \tag{4}$$

  for every $z\in\mathbb{D}$.

- *If, in addition, μ is countably supported, then the above equation holds for all $z\in\overline{\mathbb{C}}\setminus\{\mathbb{T}\cup\{\gamma.\infty\}_{\gamma\in supp\mu}\}$.*

- *Finally, if the support of μ generates a countable discrete subgroup of $PSU(1,1)$, then a probability measure ν on S^1 is μ-stationary* ***if and only if*** *(4) holds for all $z\in\overline{\mathbb{C}}\setminus\{\mathbb{T}\cup\{\gamma.\infty\}_{\gamma\in supp\mu}\}$.*

Remark. The necessary condition is universal with respect to μ, we only require the moment condition to hold

The power of this theorem lies in the fact that we managed to successfully transform a measurable functional equation on the circle into a holomorphic condition on the unit disk, which allows us to make use of powerful complex-analytic techniques.

Evidently, we are able to extract the most amount of information from (4) for countably supported probability measures μ. Before stating the corollary, let us recall the **Blaschke condition** for a sequence $\{z_n\}\subset\mathbb{D}$:

$$\sum_{\gamma\in\operatorname{supp}\mu}(1-|\gamma.0|)<\infty. \tag{5}$$

We will say that μ satisfies the Blaschke condition if and only if $\{\gamma.0\}_{\gamma\in\operatorname{supp}\mu}$ satisfies (5).

Corollary 1.1. *Let μ be a countably supported probability measure on $PSU(1,1)$ with a finite first moment.*

1. *Assume that μ satisfies the Blaschke condition, and there is an element $\gamma \in supp\,\mu$ with $\gamma.0 \neq 0$. Then there are no entire solutions to (4). In particular, there are no μ-stationary measures with the Fourier series $\nu \sim \sum_{k\in\mathbb{Z}} a_k e^{ikt}$ with $\limsup_{n\to\infty} |a_k|^{1/k} = 0$.*
2. *Assume that $\limsup\limits_{n\to\infty} \left\| \int_G \frac{d\mu^{*n}(\gamma)}{z-\gamma.\infty} \right\|_1 = \infty$, where $\|\cdot\|_1$ stands for the norm in $H^1(\mathbb{D})$. Then there are no μ-stationary measures with $L^{1+\varepsilon}(S^1, m)$-density for any $\varepsilon > 0$.*

Equation (4) gives us quite a lot of insight into the measures μ for which the (normalized) Lebesgue measure is μ-stationary.

Corollary 1.2. *Let μ be a finite Borel measure supported on a discrete subgroup $\Gamma \subset PSU(1,1)$ with a finite first moment. Let's call a measure μ on $PSU(1,1)$ the* ***Furstenberg measure*** *if the normalized Lebesgue measure on S^1 is μ-stationary.*

1. *The measure μ is a Furstenberg measure if and only if*
$$\int_G \frac{d\mu(\gamma)}{z-\gamma.\infty} = 0, \quad |z| < 1. \tag{6}$$
2. *If μ is a Furstenberg measure, then*
$$\limsup_{n\to\infty} |\mu(\gamma_n)|^{1/n} = 1.$$
3. *(Brown-Shields-Zeller) Suppose μ is a Furstenberg measure. Then $\{\gamma.0\}_{\gamma\in supp\,\mu}$ is non-tangentially dense in $\mathbb{T}$, which means that m-**almost every** point $\xi \in \mathbb{T}$ can be approached by a subsequence $\gamma_n.0$ inside a Stolz angle $\{z \in \mathbb{D} : \frac{|z-\xi|}{1-|z|} < \alpha\}$ for some $\alpha > 1$. As a corollary from [BSZ60, Remark 2], we get*
$$\sum_{\gamma\in supp\,\mu} (1 - |\gamma.0|) = \infty.$$

Remark. One fascinating detail of this theorem lies in the fact that the Brown-Shields-Zeller theorem detects the divergence of the Poincaré series for first-kind groups at the critical exponent $\delta = 1$, as
$$\sum_\gamma e^{-d_{\mathbb{H}^2}(0,\gamma.0)} = \sum_\gamma e^{-\ln\left(\frac{1+|\gamma.0|}{1-|\gamma.0|}\right)} \sim \sum_\gamma (1 - |\gamma.0|).$$

Finally, as a corollary from Fatou's theorem, we get a functional-analytic necessary condition for existence of μ-stationary measures with L^p-density for $1 < p < \infty$.

Corollary 1.3. *Let the support of μ satisfy the Blaschke condition. Then for any μ-stationary measure μ with L^p-density for $1 < p < \infty$, we have*
$$f_\nu(z) \in (\overline{T^*_\mu(B_\mu H^q)})^\perp \subset H^p,$$
where

- *$T_\mu(f) := \sum_\gamma \mu(\gamma)(f\circ\gamma^{-1})(\gamma^{-1})' - f$ is considered as a bounded linear operator $T_\mu : H^p(\mathbb{D}) \to H^p(\mathbb{D})$, and $\frac{1}{p} + \frac{1}{q} = 1$,*
- *$B_\mu H^p$ denotes the subspace of functions in H^q which vanish on the support of μ. This is a non-trivial closed subspace due to the Blaschke condition.*

In particular, if $T_\mu^*(B_\mu H^q)$ is dense in H^q, then there are no μ-stationary measures with L^p-density.

Corollary 1.1.1 strictly strengthens the very last remark in [Bou12], where it was proven that the Lebesgue measure is never stationary with respect to finitely supported measures on $PSU(1,1)$. Corollary 1.1.2, in theory, provides a purely computational heuristic to showing singularty of stationary measures, as for lattices in $PSU(1,1)$, one expects the poles to converge to $\mathbb{T}$, whereas for dense subgroups one would expect the poles to accumulate inside $\mathbb{D}$, thus forcing the H^1-norms to stay bounded.

Corollary 1.2 provides several new insights into Furstenberg measures on $PSU(1,1)$. In particular, as our approach deals with signed and complex measures, we are able to talk about complex Furstenberg measures, which is not possible with any geometric approaches. In particular, we obtain Borel sums with poles in the orbits of a non-cocompact lattice which vanishes in $\mathbb{D}$, despite the fact that such counterexamples should be impossible due to Guivarch'-le Jan ([GL90]). The catch is that our condition is only a necessary one; and the Brown-Shields-Zeller theorem does not control the moments of the resulting coefficients. We also exhibit the first known result restricting the moment conditions of a Furstenberg measure, once again, improving on [Bou12]. The notion of a non-tangential limit seems to be key in this approach. Finally, we remark that studying positive Furstenberg measures should be possible using techniques in [BW89] and [HL90], as they deal with Borel-like series having strictly positive coefficients.

Corollary 1.3 provides a pretty significant restriction on the stationary measures in the L^p-class for $1 < p < \infty$, and, in theory, the images with respect to the adjoint operator T_μ^* can be computed explicitly for any measure μ satisfying the Blaschke condition.

The structure of the paper is as follows. In Section 2 we recall all necessary facts about transformations $PSU(1,1)$ and provide a brief recap of complex-analytic tools we are going to use. In Section **??** we introduce an appropriate integral transform which fully respects the action of $PSU(1,1)$ to obtain a holomorphic necessary condition for μ-stationarity, thus proving Theorem 1.1. In Section 4 we extract the most we can from the resulting equation, using state-of-the-art techniques related to generalized analytic continuations.

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2 Preliminaries

2.1 Everything you need to know about isometries of the disk model of $\mathbb{H}^2$

In this subsection we will recall basic facts about $PSU(1,1)$ considered as a isometry group of the disk model $\mathbb{D} = \{|z| < 1\}$ of the hyperbolic plane.

Definition 2.1.

$$PSU(1,1) = \left\{ z \mapsto \frac{az+b}{\overline{b}z+\overline{a}} : a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}.$$

From the definition it is evident that every transformation in $PSU(1,1)$ can be represented by a matrix $\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$ (mod scalar matrices). In particular, if $\gamma(z) = \frac{az+b}{\overline{b}z+\overline{a}}$ then $\gamma^{-1}(z) = \frac{\overline{a}z-b}{-\overline{b}z+a}$.

Also, it will turn out that sometimes working with ∞ as a basepoint is more conventient than choosing $0 \in \mathbb{H}^2$, we will use

$$\overline{\gamma\left(\overline{z}^{-1}\right)} = \frac{1}{\gamma(z)}, \quad z \in \overline{\mathbb{C}}, \tag{7}$$

and, as a simple corollary,

$$\gamma.\infty = \frac{a}{\overline{b}} = \left(\frac{\overline{b}}{a}\right)^{-1} = (\overline{\gamma.0})^{-1}. \tag{8}$$

Lemma 2.1. *Let $\gamma(z) = \frac{az+b}{\overline{b}z+\overline{a}}$. then*

$$\gamma'(z) = \frac{1}{(\overline{b}z+\overline{a})^2}.$$

Proof.

$$\gamma'(z) = \frac{a(\overline{b}z+\overline{a}) - (az+b)\overline{b}}{(\overline{b}z+\overline{a})^2} = \frac{|a|^2-|b|^2}{(\overline{b}z+\overline{a})^2} = \frac{1}{(\overline{b}z+\overline{a})^2}.$$

□

Finally, recall that for any $\gamma(z) = \frac{az+b}{\overline{b}z+\overline{a}}$ with $|a|^2 - |b|^2 = 1$ we have

$$\frac{1}{2}\frac{\gamma''(z)}{\gamma'(z)} = \frac{1}{2}\frac{-2\overline{b}}{(\overline{b}z+\overline{a})^3}\left(\frac{1}{(\overline{b}z+\overline{a})^2}\right)^{-1} = \frac{-\overline{b}}{\overline{b}z+\overline{a}} = -\frac{1}{z+\frac{\overline{a}}{\overline{b}}} = -\frac{1}{z-\gamma^{-1}.\infty}. \tag{9}$$

2.2 Complex-analytic prerequisites: Hardy spaces

We will heavily rely on standard complex-analytic techniques related to Cauchy transforms and generalized analytic continuation, we refer to standard textbooks on these topics: [RS02], [Cim00], [CMR06].

Let us denote $\mathbb{D} = \{|z| < 1\}$ and $\mathbb{D}_e := \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Given a domain $U \subset \overline{\mathbb{C}}$, we will denote the space of holomorphic functions on U by $\mathfrak{H}(U)$ and the space of meromorphic functions on U by $\mathfrak{M}(U)$.

Definition 2.2. *Let $0 < p < \infty$. The* ***Hardy space*** $(H^p(\mathbb{D}), ||\cdot||_p)$ *is a space of holomorphic functions on $\mathbb{D}$ defined as follows.*

$$H^p(\mathbb{D}) = \left\{ f \in \mathfrak{H}(\mathbb{D}) \mid ||f||_p := \sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p} < \infty \right\}.$$

If $p = \infty$, then we define $(H^\infty(\mathbb{D}), ||f||_\infty)$ as the space of bounded holomorphic functions on $\mathbb{D}$ equipped with the sup-norm.

Finally, we define $H^p(\mathbb{D}_e) := \{z \mapsto f(1/z) : f \in H^p(\mathbb{D})\}$, with $H^p_0(\mathbb{D}) \subset H^p(\mathbb{D}_e)$ denoting functions vanishing at infinity.

It is well-known that for $1 \le p \le \infty$ the function $||\cdot||_p : H^p(\mathbb{D}) \to \mathbb{R}_{\ge 0}$ defines a norm, so the respective Hardy spaces $H^p(\mathbb{D})$ are Banach spaces for $1 \le p \le \infty$. For $0 < p < 1$ the Hardy spaces $H^p(\mathbb{D})$ admit a complete translation-invariant metric defined by $d(f,g) := ||f-g||_p^p$, but the topology it defines is not non-locally convex.

Definition 2.3. *Let f be a meromorphic function on $\mathbb{D}$ ($\mathbb{D}_e$, resp.). If the limit $\lim\limits_{r\to 1^-} f(re^{it})$ ($\lim\limits_{r\to 1^+} f(re^{it})$ resp.) exists Leb-almost everywhere on $\mathbb{T}$, then we say that f admits a* ***non-tangential limit*** *on the boundary.*

Definition 2.4. *A sequence of points $\{z_n\} \subset \mathbb{D}$ is said to* ***non-tangentially*** *converge to $\xi \in \partial\mathbb{D}$ if there exists a* ***Stolz angle*** $A = \{\frac{|\xi - z|}{1-|z|} \le M\}$ *and $N > 0$ such that $z_n \to \xi$ and $z_n \subset A$ for $n > N$.*

We will frequently use the following classical theorems.

Theorem 2.1 (Fatou's theorem)**.** *Every holomorphic function $f \in H^p(\mathbb{D})$ for $0 < p \leq \infty$ admits a non-tangential limit $f(\zeta)$ for Leb-almost every $\zeta \in S^1$ which belongs to $L^p(S^1, Leb)$. Moreover, for $0 < p < \infty$ we have*

$$||f||_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^p dt\right)^{1/p}.$$

Theorem 2.2 (F. Riesz, M. Riesz)**.** *For $p \geq 1$ we have a complete realization of the Hardy spaces $H^p(\mathbb{D})$ as subspaces $L^p(S^1)$: these are exactly the functions with vanishing negative Fourier coefficients.*

Let us briefly list some examples of holomorphic functions in Hardy spaces.

Example 2.1.

- *Analytic polynomials $p(z) = a_0 + \cdots + a_n z^n$ are dense in $H^p(\mathbb{D})$ for all $0 < p < \infty$, and are wk*-dense in $H^\infty(\mathbb{D})$. ([CMR06, Theorem 1.9.4])*
- *If $0 < p < q \leq \infty$, then $H^q(\mathbb{D}) \subsetneq H^p(\mathbb{D})$.*
- *For any $z_0 \in \mathbb{D}_e$ and $k > 0$, we have $\frac{1}{(z-z_0)^k} \in H^\infty(\mathbb{D})$, hence $\frac{1}{(z-z_0)^k} \in H^p(\mathbb{D})$ for any $0 < p \leq \infty$. Keep in mind that this will not hold for any $|z_0| = 1$, see later examples.*

2.3 Complex-analytic prerequisites: pseudocontinuations

Definition 2.5. *Let f be a meromorphic function on $\mathbb{D}$. If there exists a function T_f which is meromorphic on $\mathbb{D}_e$ such that the non-tangential limits of f and $\tilde{f}$ coincide Leb-almost everywhere, then we say that f is **pseudocontinuable**, and $\tilde{f}$ is a **pseudocontinuation** of f, and vice versa.*

In our paper we will use several important results about non-tangential limits and pseudocontinuations.

Theorem 2.3 (Lusin-Privalov, [Pri56])**.** *If f is pseudocontinuable, then its pseudocontinuation is unique.*

As a corollary, we get that pseudocontinuations are compatible with analytic continuations.

Definition 2.6. *Let ν be a complex finite Borel measure on S^1. Then its **Cauchy transform** is the integral*

$$C_\nu(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{d\nu(t)}{1 - e^{-it}z}. \tag{10}$$

*Its **Cauchy-Szegő transform** is the integral*

$$f_\nu(z) := \frac{1}{2\pi}\int_0^{2\pi} \frac{d\nu(t)}{e^{it} - z}. \tag{11}$$

It is easy to see that $C_\nu(z) = f_{\nu'}(z)$ for $d\nu'(t) = e^{it}d\nu(t)$, but we will still use both transforms when convenient.

The properties of $C_\nu(z)$ as a holomorphic function on $\mathbb{D}$ strongly depend on ν itself, but the following theorem of Smirnov ensures that we at least end up in $H^p(\overline{\mathbb{C}} \setminus \mathbb{T})$ for $p < 1$.

Theorem 2.4 (Smirnov)**.** *Let $f(z) = C_\nu(z)$ for some complex finite Borel measure ν. Then $f \in H^p(\mathbb{D})$ and $f \in H^p(\mathbb{D}_e)$ for all $0 < p < 1$.*

In particular, we get that Cauchy transforms cannot grow too fast:

Theorem 2.5. *Let $f(z) = C_\nu(z)$ for some complex Borel measure ν on S^1. Then*

$$|f(z)| = O\left(\frac{1}{1-|z|}\right)$$

for $z \in \mathbb{D}$, and

$$|f(z)| = O\left(\frac{1}{1-|z|^{-1}}\right)$$

for $z \in \mathbb{D}_e$.

We can do better if we know that $\nu \ll Leb$ due to a theorem of M. Riesz.

Theorem 2.6 (Riesz)**.** *Let ν be an absolutely continuous measure on S^1 with the L^p-density for $1 < p < \infty$. Then $C_\nu(z) \in H^p(\mathbb{D})$.*

Remark. This theorem cannot not hold for $p = 1$ or $p = \infty$, as it is well-known that there are no continuous projections $L^1 \to H^1$ and $L^\infty \to H^\infty$.

Finally, the Cauchy transform of a positive Borel measure on S^1 is unique in the following sense:

Theorem 2.7 ([CMR06], Proposition 4.1.4)**.** *Let ν_1, ν_2 be two probability measures on S^1. Then $C_{\nu_1} = C_{\nu_2}$ if and only if $f_{\nu_1} = f_{\nu_2}$ if and only if $\nu_1 = \nu_2$.*

Example 2.2.

- *Consider $f(z) = (1-z)^{-1}$. This is the Cauchy transform of the Dirac delta measure δ_1, so $f(z) \in H^p(\mathbb{D})$ for all $0 < p < 1$, but $f(z) \notin H^1(\mathbb{D})$. The idea is that the integral*

$$\int_0^1 \frac{dx}{x^p}$$

  converges for $p < 1$ and diverges for $p = 1$.

- *This suggests that for $0 < p < 1$ it makes sense to talk about the closure of all simple poles $z \to \frac{1}{1-e^{it}z}$ on $\mathbb{T}$, and it turns out that this closure admits a very nice description:*

$$\overline{span}^{H^p}\left\{\frac{1}{1-\xi z} : |\xi| = 1\right\} := H^p \cap \overline{H^p_0}, \tag{12}$$

  where by $H^p \cap \overline{H^p_0}$ we denote the subspace of all functions $f(z) \in \mathcal{H}(\overline{\mathbb{C}} \setminus \mathbb{T})$, such that both inner and outer components lie in H^p, $f(\infty) = 0$ and inner + outer non-tangential limits a.e. exist and coincide. See [Ale79] for a proof.

The above space $H^p \cap \overline{H^p_0}$ is very important because of the following theorem.

Theorem 2.8 (Fatou)**.** *A probability measure ν on S^1 is singular if and only if $f_\nu(z) \in H^p \cap \overline{H^p_0}$ if and only if $C_\nu(z) \in H^p \cap \overline{H^p_0}$.*

A direct computation yields the following proposition.

Proposition 2.1. *Let $\gamma \in PSU(1,1)$ and consider the linear operator*

$$V_\gamma(f)(z) := f(\gamma^{-1}(z))(\gamma^{-1})'(z).$$

Then for every $w \in \mathbb{C}$ we have

$$V_\gamma\left(\frac{1}{w-z}\right) = \frac{1}{\gamma.w - z} - \frac{1}{\gamma.\infty - z}.$$

3 Holomorphic stationarity condition

In this section we will provide proofs of the main results.

Theorem 3.1. *Let μ be a probability measure on $G = PSU(1,1)$ satisfying the finite first moment condition:*

$$\int_G \log\left(\frac{1+|\gamma.z|}{1-|\gamma.z|}\right) d\mu(\gamma) < \infty \tag{13}$$

for every $z \in \mathbb{D}$. Then a probability measure ν on S^1 is μ-stationary implies

$$\int_G f_\nu(\gamma^{-1}.z)(\gamma^{-1})'(z)d\mu(\gamma) - f_\nu(z) = \int_G \frac{d\mu(\gamma)}{z-\gamma.\infty}. \tag{14}$$

for every $z \in \mathbb{D}$.

Proof. Let $z \in \mathbb{D}$. Due to Theorem 2.1 we have

$$\begin{aligned} T_\gamma(f_\nu)(z) = T_\gamma\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\nu(t)}{e^{it}-z}\right) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{\gamma.e^{it}-z} - \frac{1}{\gamma.\infty - z} - \frac{1}{e^{it}-z}\right)d\nu(t) = \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{d\gamma_*\nu(t)}{e^{it}-z} - \frac{1}{2\pi}\int_0^{2\pi}\frac{d\nu(t)}{e^{it}-z} - \frac{1}{\gamma.\infty - z}. \end{aligned}$$

The moment condition allows us to use the DCT, so we use the fact that ν is μ-stationary to argue as follows:

$$\begin{aligned} &\int_G f_\nu(\gamma^{-1}.z)(\gamma^{-1})'(z)d\mu(\gamma) - f_\nu(z) = \int_G T_\mu(f_\nu)(z)d\mu(\gamma) = \int_G\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d(\gamma_*\nu-\nu)(t)}{e^{it}-z} - \frac{1}{\gamma.\infty-z}\right)d\mu(\gamma) = \\ &= \frac{1}{2\pi}\int_G\int_0^{2\pi}\frac{d(\gamma_*\nu-\nu)(t)}{e^{it}-z}d\mu(\gamma) - \int_G\frac{1}{\gamma.\infty-z}d\mu(\gamma) = \frac{1}{2\pi}\int_0^{2\pi}\int_G\frac{d(\gamma_*\nu-\nu)(t)}{e^{it}-z}d\mu(\gamma) - \int_G\frac{1}{\gamma.\infty-z}d\mu(\gamma) = \\ &= \int_G\frac{1}{z-\gamma.\infty}d\mu(\gamma). \end{aligned}$$

which is precisely (14). ∎

Remark. In the above proof we didn't assume anything about μ except the moment condition. If we focus on countable supported μ, then (14) holds for all $z \in \overline{\mathbb{C}}$ outside of the poles of $\int_G \frac{d\mu(\gamma)}{z-\gamma.\infty}$.

Theorem 3.2. *Let μ be a probability measure on $G = PSU(1,1)$ satisfying the following moment condition:*

$$\int_G \log\left(\frac{1+|\gamma.z|}{1-|\gamma.z|}\right) d|\mu|(\gamma) < \infty \tag{15}$$

for any $z \in \mathbb{D}$. Also, assume that (S^1,ν) is the Poisson boundary of (G,μ). If $f_{\nu'}(z)$ satisfies (14) for all $z \in \mathbb{D}$, then $\nu' = \nu$. In other words, ν is the μ-stationary measure if and only if $f_\nu(z)$ solves (14).

Proof. Observe that (14) is equivalent to saying that

$$\begin{aligned} \lambda_{z_0} : G \to \mathbb{C}, \quad \gamma &\mapsto f_\nu(\gamma^{-1}.z_0)(\gamma^{-1})'(z_0) - \frac{1}{z_0-\gamma.\infty}, \\ \lambda'_{z_0} : G \to \mathbb{C}, \quad \gamma &\mapsto f_{\nu'}(\gamma^{-1}.z_0)(\gamma^{-1})'(z_0) - \frac{1}{z_0-\gamma.\infty} \end{aligned}$$

are both μ-harmonic functions on G. The Poisson representation allows us to rewrite both λ_{z_0} and λ'_{z_0} as follows:

$$\begin{aligned} \lambda_{z_0}(id) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\lim_{\gamma\to e^{it}} f_\nu(\gamma^{-1}.z_0)(\gamma^{-1})'(z_0) - \frac{1}{z_0-\gamma.\infty}\right)d\nu(t) = \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\lim_{\gamma\to e^{it}} f_\nu(\gamma^{-1}.z_0)(\gamma^{-1})'(z_0)\right) - \frac{1}{z_0-e^{it}}d\nu(t). \end{aligned}$$

Similarly, we have

$$\lambda'_{z_0}(id) = \frac{1}{2\pi}\int_0^{2\pi}\left(\lim_{\gamma\to e^{it}} f_{\nu'}(\gamma^{-1}.z_0)(\gamma^{-1})'(z_0)\right) - \frac{1}{z_0 - e^{it}}d\nu(t)$$

Because $f_{\nu'}$ satisfies the growth condition, the limit $\lim\limits_{\gamma\to e^{it}} f_{\nu'}(\gamma^{-1}.z_0)(\gamma^{-1})'(z_0)$ vanishes for almost all t due to [CMR06, Equation 2.1.8]:

$$f_\nu(\gamma^{-1}.z_0)(\gamma^{-1})'(z_0) = f_\nu(\gamma^{-1}.z_0)(\gamma^{-1})'(z_0)\frac{|(\gamma^{-1})'(z_0)|}{1-|\gamma^{-1}.z_0|} = \frac{C}{|\bar{b}z_0+\bar{a}|(|\bar{b}z_0+\bar{a}|-|az_0+b|)} =$$
$$= \frac{C(|\bar{b}z_0+\bar{a}|+|az_0+b|)}{|\bar{b}z_0+\bar{a}|(|\bar{b}z_0+\bar{a}|^2-|az_0+b|^2)}.$$

(compare [CMR06, eq 2.1.8]), leaving us with

$$\lambda_{z_0}(id) = f_{\nu'}(z_0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{e^{it}-z_0}d\nu(t) = f_\nu(z_0).$$

□

4 Squeezing water from a stone: a deep dive into (14)

In this section we will explore the functional equation (14) in much more detail. From now on, we will restrict ourselves to countably supported probability measures μ, denoting by $\Gamma \le G = PSU(1,1)$ the subgroup generated by the support of μ.

Theorem 4.1 (Corollary 1.1.1)**.** *Let μ be a probability measure with finite support. Then there are no entire solutions to* (14)*.*

Proof. Choose an element $\tau \in \mathrm{supp}(\mu)$ which does not fix the origin. In particular, $\tau.\infty = (\overline{\tau.0})^{-1} \neq \infty$. Fix small enough contour C_τ around $\tau.\infty$. Integrating both sides over this contour, we get

$$\int_{C_\tau}\left(\int_G \frac{d\mu(\gamma)}{z-\gamma.\infty} + \int_G f(\gamma^{-1}.z)(\gamma^{-1})'(z)d\mu(\gamma) - f(z)\right)dz =$$
$$= \sum_{\gamma.\infty=\tau.\infty}\mu(\gamma) + \int_\Gamma\int_{\gamma^{-1}(C_\tau)} f(z)dz - \int_{C_\tau} f(z) = 0$$

by applying the change of variables. As $f(z)$ is entire, the contour integrals vanish, leaving us with $\mu(\gamma) = 0$ for all γ with the same pole as τ, which leads to a contradiction. □

Corollary 4.1. *Let μ be a probability measure with finite support. Then $\limsup\limits_{k\to\infty}|a_k|^{1/k} > 0$ for every μ-stationary measure with the Fourier series $\nu \sim \sum_{k\in\mathbb{Z}} a_k e^{ikt}$.*

Proof. Consider a μ-stationary measure ν with $\limsup_{k\to\infty}|a_k|^{1/k} = 0$. Then $f_\nu(z)$ is an entire function which solves (14) for $|z| < 1$. The LHS of (14) can be analytically continued to a meromorphic function on $\mathbb{C}$, therefore, $f_\nu(z)$ solves (14) for all $\mathbb{C}$. This allows us to apply Theorem 4.1, yielding a contradiction. □

Due to Theorem 2.4 we know that $f_\nu(z)$ is holomorphic on $\overline{\mathbb{C}} \setminus \mathbb{T}$. Moreover, (14) makes sense for all $z \in \overline{\mathbb{C}}$ outside of the poles of RHS (namely, $\gamma.\infty$ for all $\gamma \in \mathrm{supp}\mu$).

Theorem 4.2 (Corollary 1.1.2)**.** *Let μ be a countably supported probability measure. If $\limsup\limits_{n\to\infty}\left\|\int_G \frac{d\mu^{*n}(\gamma)}{z-\gamma.\infty}\right\|_1 = \infty$, then there are no μ-stationary measures with $L^{1+\varepsilon}(S^1, Leb)$-density for any $\varepsilon > 0$.*

Proof. Let ν be μ-stationary with density in $L^{1+\varepsilon}(S^1)$. Due to Fatou's theorem we know that $f_\nu(z) \in H^{1+\varepsilon}(\mathbb{D})$. In particular, $f_\nu(z) \in H^1(\mathbb{D})$. As all composition operators in LHS of (14) are exactly the operators treated in Theorem 2.1, they are isometries, in particular, the H^1-norm of LHS is at most $2||f||_1$. Make note of the fact that this application of the triangle inequality does not depend on μ at all. Applying H^1-norm to both sides, we get

$$2\,\|f_\nu\|_1 \geq \left\| \int_G \frac{d\mu(\gamma)}{z - \gamma.\infty} \right\|_1 .$$

However, keep in mind that any μ-stationary measure is μ^{*n}-stationary, therefore, WLOG one can replace μ with μ^{*n} in the above inequality without changing LHS. This would imply

$$2\,\|f_\nu\|_1 \geq \limsup_{n\to\infty} \left\| \int_G \frac{d\mu^{*n}(\gamma)}{z - \gamma.\infty} \right\|_1 = \infty,$$

which leads to a contradiction. $\square$

Example 4.1. *Consider $\mu = \delta_\gamma$ for a non-elliptic $\gamma \in PSU(1,1)$. Then the H^1-norm of $\frac{1}{z-\gamma^n.\infty}$ goes to infinity as $n \to \infty$, so there are no absolutely continuous measures with densities in $L^{1+\varepsilon}(S^1)$, as we expected.*

However, as simple as this criterion seems, given a measure μ supported on a lattice in $PSU(1,1)$, it is not at all easy to estimate $\left\| \int_G \frac{d\mu^{*n}(\gamma)}{z-\gamma.\infty} \right\|_1$, therefore, a potential argument should rely on a very precise analysis of how non-uniformly the poles will be distributed in small neighbourhoods of $\mathbb{T}$.

Finally, recall that in the proof of Theorem 3.2 we introduced harmonic functions

$$\lambda_{z_0} : G \to \mathbb{C}, \quad \gamma \mapsto f_\nu(\gamma^{-1}.z_0)(\gamma^{-1})'(z_0) - \frac{1}{z_0 - \gamma.\infty}$$

for every $|z_0| < 1$. In particular, we can take $z_0 = 0$, and rearrange the formula to get

$$f(\gamma^{-1}.0) = \frac{\lambda_0(\gamma) - \overline{\gamma.0}}{(\gamma^{-1})'(0)}.$$

In particular, this formula shows that all information about the harmonic measure ν is encoded in the behaviour of a single harmonic function on G.

4.1 Functional-analytic necessary condition for existence of absolutely continuous stationary measures

In this subsection we treat LHS of (14) as a bounded operator: define

$$T_\mu : H^p(\mathbb{D}) \to H^p(\mathbb{D}), \quad T_\mu(f)(z) := \sum_\gamma \mu(\gamma) f(\gamma^{-1}.z)(\gamma^{-1})'(z) - f(z).$$

It is well-known that T_μ is a bounded operator for all $0 < p \leq \infty$, and in such generality, not much else is known about T_μ. If $p > 1$, we can at least explicitly compute its adjoint $T_\mu^* : H^q \to H^q$.

Proposition 4.1. *Consider $V_\gamma(f)(z) = f(\gamma^{-1})(\gamma^{-1})'(z)$ as a bounded operator $H^p \to H^p$. Then*

$$V_\gamma^*(f)(z) = S^*(f(\gamma.z)\gamma.z), \quad f \in H^q(\mathbb{D}),$$

where S^* stands for the backwards shift $S^*(g)(z) = \frac{g(z)-g(0)}{z}$.

Proof. As in many similar computations (see [Cow88, Theorem 2] for an example), we use the reproducing kernel property of $\frac{1}{1-\overline{a}z}$: for any $f \in H^p$

$$\left\langle f(z), \frac{1}{1-\overline{a}z} \right\rangle := \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})dt}{1-a\overline{z}} = f(a).$$

A slight modification yields

$$\left\langle f(z), \frac{1}{a-z} \right\rangle = \frac{f(\overline{a}^{-1})}{\overline{a}}.$$

As we know how T_γ acts on $\frac{1}{a-z}$, reflexivity of H^p for all $1 < p < \infty$ allows us to write

$$\begin{aligned}\frac{V_\gamma^* f(\overline{a}^{-1})}{\overline{a}} &= \left\langle (V_\gamma^*) f(z), \frac{1}{a-z} \right\rangle = \left\langle f(z), \frac{1}{\gamma.a - z} - \frac{1}{\gamma.\infty - z} \right\rangle = \\ &= \frac{f(\overline{\gamma.a}^{-1})}{\overline{\gamma.a}} - \frac{f(\overline{\gamma.\infty}^{-1})}{\overline{\gamma.\infty}} \overset{(7)}{=} \gamma.\overline{a^{-1}} f(\gamma.\overline{a^{-1}}) - \gamma.0 f(\gamma.0).\end{aligned}$$

Replacing $\overline{a}^{-1}$ with ω, we get

$$V_\gamma^* f(w) = \frac{\gamma.w f(\gamma.w) - \gamma.0 f(\gamma.0)}{w} = S^*(f(\gamma.w)\gamma.w).$$

□

As a quick corollary, we get that

$$T_\mu^*(f)(z) = S^* \left(\sum_\gamma \mu(\gamma) f(\gamma.z)\gamma.z \right) - f(z). \tag{16}$$

Theorem 4.3. *Let μ satisfy the Blaschke condition. Then for any μ-stationary measure ν with L^p-density for $1 < p < \infty$, we have*

$$f_\nu(z) \in (\overline{T_\mu^*(B_\mu H^q)})^\perp \subset H^p,$$

where $B_\mu(z)$ is the Blaschke function corresponding to the support of μ:

$$B_\mu(z) := \prod_{\gamma \in supp(\mu)} \gamma^{-1}(z).$$

Proof. Let $T_\mu(f) = \sum_\gamma \frac{\mu(\gamma)}{z-\gamma.\infty}$. It is easy to see that $\sum_\gamma \frac{\mu(\gamma)}{z-\gamma.\infty}$ is a linear combination of reproducing kernels $\frac{1}{1-\overline{\gamma.0}z}$. In particular, $\sum_\gamma \frac{\mu(\gamma)}{z-\gamma.\infty} \in (B_\mu H^q)^\perp$. Therefore,

$$0 = \langle T_\mu(f), B_\mu H^q \rangle = \left\langle f, \overline{T_\mu^*(B_\mu H^q)} \right\rangle.$$

□

This proves Corollary 1.3.

4.2 When the Lebesgue measure is stationary?

Earlier we have reproved the well-known theorem of J. Bourgain that the Lebesgue measure cannot be a stationary measure if μ has finite support. To understand this case better, we need to look at (14) and observe that f_ν vanishes, leaving us with vanishing of the following **Borel series**.

$$\sum_{\gamma} \frac{\mu(\gamma)}{z - \gamma.\infty} = 0, \quad |z| < 1. \tag{17}$$

This immediately proves Corollary 1.1.1. At first glance, it might seem counter-intuitive that the above sum can vanish on the entire disc, but recall the following fundamental fact about Borel series.

Theorem 4.4 ([BSZ60], Theorem 3)**.** *Let $A = \{z_n\} \subset \mathbb{D}$ be a sequence of points **inside** the unit disk without interior limit points. Then there exists a sequence $\{c_n\} \in l^1$ such that*

$$\sum_n \frac{c_n}{z - z_n} = 0, \quad |z| > 1$$

if and only if almost every point in S^1 is a non-tangential limit of a subsequence in $\{z_n\}$.

This theorem almost gives what we want, however, the above theorem gives series with poles **inside** the disk which vanishes **outside** of it, whereas we need the opposite – Borel series with poles **outside** the disk and vanishing **inside** the disk.

One can easily mitigate this by considering the change of variables $z \mapsto 1/z$:

$$\sum_\gamma \frac{\mu(\gamma)}{z^{-1} - \gamma.\infty} = \sum_\gamma \frac{\mu(\gamma) z}{1 - (\gamma.\infty) z} = \sum_\gamma \frac{(\gamma.\infty)^{-1}\mu(\gamma) z}{(\gamma.\infty)^{-1} - z} =$$
$$= \sum_\gamma -\frac{\mu(\gamma)}{\gamma.\infty} + \frac{\mu(\gamma)}{(\gamma.\infty)^2}\frac{1}{(\gamma.\infty)^{-1} - z}.$$

However, as we can plug in $z = 0$ in (17), we get that

$$\sum_\gamma \frac{\mu(\gamma)}{z^{-1} - \gamma.\infty} = \sum_\gamma \frac{\mu(\gamma)}{(\gamma.\infty)^2}\frac{1}{(\gamma.\infty)^{-1} - z} = 0$$

for all $|z| > 1$. Applying the Brown-Shields-Zeller theorem, we obtain Corollary 1.2.3.

Remark. Recall that the orbit of a point with respect to an action of a discrete subgroup of $PSU(1,1)$ is non-tangentially dense if and only if the subgroup is of the first type. Therefore, Theorem 4.4 confirms that $\Gamma \subset PSU(1,1)$ being a first-kind Fuchsian group should be a necessary condition for a Furstenberg measure on Γ to exist.

Moreover, due to another theorem of Beurling, referring to [RS02, Corollary 4.2.24]:

Theorem 4.5 ([Beu34], [BC89])**.** *Let $\{z_n\}$ be a sequence of points **outside** of the unit disk with $|z_n| \downarrow 1$. If $\limsup_{n\to\infty} |c_n|^{1/n} < 1$ and*

$$\sum_n \frac{c_n}{z - z_n} = 0, \quad |z| < 1,$$

then all $c_n = 0$.

Applying this theorem to $z_n = \gamma_n.\infty$ (relative to a suitable enumeration of Γ), we get that a Fuchsian group of first kind $\Gamma \subset PSU(1,1)$ admits a Furstenberg measure only if

$$\limsup_{n\to\infty} |\mu(\gamma_n)|^{1/n} = 1,$$

thus proving Corollary 1.2.2. Combined with the exponential growth of Fuchsian groups, this condition implies that a Furstenberg measure μ cannot have a double-exponential moment with respect to the hyperbolic distance: if we let $c > 0$, then

$$\sum_n \mu(\gamma_n) e^{e^{cd(0,\gamma_n.0)}} < \infty \iff \sum_n \mu(\gamma_n) e^{cn} < \infty \Rightarrow \limsup_{n\to\infty} |\mu(\gamma_n)|^{1/n} < e^{-c} < 1.$$

As for the strongest known moment conditions: it is known that J.Li's counterexample, given in the Appendix of [LNP21], provides a Furstenberg measure with an exponential moment, our approach shows that a Furstenberg measure cannot have a double-exponential moment. It is widely believed that the approach developed in [CM07] should yield an example of a Furstenberg measure with a superexponential moment, but we are not aware of a complete and self-contained argument being published.

Finally, we would like to remark that the proof of [BSZ60, Theorem 3] is, essentially, non-constructive. In context of our problem, the idea is as follows.

1. We start by considering an operator $H^\infty(\mathbb{D}) \to l^\infty(\Gamma)$,

$$T(f)_\gamma = f((\gamma.\infty)^{-1}).$$

2. Its image is proven to be wk^*-closed in $l^\infty(\Gamma)$.

3. Consider $p = \delta_e \in l^\infty(\Gamma)$. A clever argument shows that $\mathrm{dist}(p, T(H^\infty(\mathbb{D}))) = \frac{1}{2}$, so the image cannot coincide with the whole $l^\infty(\Gamma)$. and applying the Hahn-Banach theorem, we prove the existence of an element $a \in l^1(\Gamma)$, such that:

   - for any $y \in T(H^\infty(\mathbb{D})) \subset l^\infty(\Gamma)$ we have $y(a) = 0$
   - $p(a) = 1$
   - $||a|| < 2 + \varepsilon$ for a small enough $\varepsilon > 0$.

4. The sequence a solves our problem, that is,

$$\sum_\gamma \frac{a_\gamma}{z - (\gamma.\infty)^{-1}} = 0.$$

4.3 A criterion for singularity

Theorem 4.6. *Let μ be a finitely supported probability measure on a discrete subgroup $\Gamma \leq G$. Then one of the following statements holds.*

- *The harmonic measure ν is singular with respect to the Lebesgue measure.*
- *The subspace $span\{(S^*)^k(f_\nu)(z)\}_{k\geq 0} \subset H^p$ is dense for all $p < 1$. In other words, $f_\nu \in H^p$ is a cyclic function.*

Proof. Assume that ν is not cyclic, that is, the closure of the subspace $\mathrm{span}\{(S^*)^k(f_\nu)(z)\}_{k\geq 0}$ is not the entire H^p for a fixed $p < 1$. Then, due to a theorem of Aleksandrov, $f_\nu(z) \in H^p$ admits a pseudocontinuation $\tilde{f}_\nu(z) \in H^p(\mathbb{D}_e)$. Because μ has finite support, the operator T_μ respects pseudocontinuations. In particular, both the restriction of $f_\nu(z)$ on $\mathbb{D}_e$ and $\tilde{f}_\nu(z)$ are taken to the same function $\sum_\gamma \frac{\mu(\gamma)}{z-\gamma.\infty}$. Therefore, arguing in a similar way as in the proof of Theorem 3.2, we get that the function $\lambda_{z_0} : \gamma \to (f_\nu(\gamma^{-1}.z_0) - \tilde{f}_\nu(z_0))(\gamma^{-1})'(z_0)$ is μ-harmonic, and it converges to zero almost surely when $\gamma \to \xi \in S^1$. Therefore, $f_\nu(z) \equiv \tilde{f}_\nu(z)$ for all $z \in \mathbb{D}_e$, so due to the Fatou's theorem, ν is a singular measure. $\square$

4.3.1 A motivating example

We will start this section with a simple but an instructive example.

Example 4.2. *Consider a random walk (X_n) generated by one single atom supported on a non-elliptic element $\gamma \in PSU(1,1)$. Suppose we were to "prove" that the hitting measure for this random walk is singular without actually knowing in advance that it is atomic. Even in this elementary case we know that X_n converges to S^1 almost surely, so, on the level of Cauchy transforms, we know that*

$$\mathbb{E}\left[\frac{1}{1-(\overline{X_n.0})z}\right] \to \mathbb{E}\left[\frac{1}{1-(\overline{X_\infty})z}\right]$$

on compact subsets of $\mathbb{D}$. However, because the random walk is so simple, we can explicitly write

$$\mathbb{E}\left[\frac{1}{1-(\overline{X_n.0})z}\right] = \frac{1}{1-(\overline{\gamma^n.0})z},$$

and it is not difficult to prove (see [Cim00, Lemma 6.2.23]) that

$$\mathbb{E}\left[\frac{1}{1-(\overline{X_n.0})z}\right] = \frac{1}{1-(\overline{\gamma^n.0})z} \xrightarrow{H^p} \frac{1}{1-(\overline{\gamma_+})z} = \mathbb{E}\left[\frac{1}{1-(\overline{X_\infty})z}\right].$$

Moreover, it is also not difficult to work out that

$$\frac{1}{1-\frac{\overline{\gamma^n.0}}{|\gamma^n.0|}z} \xrightarrow{H^p} \frac{1}{1-(\overline{\gamma_+})z},$$

but $\dfrac{1}{1-\frac{\overline{\gamma^n.0}}{|\gamma^n.0|}z} \in \overline{span}^{H^p}\left\{\frac{1}{1-\xi z} : |\xi| = 1\right\}$*, so the limit has to be in* $\overline{span}^{H^p}\left\{\frac{1}{1-\xi z} : |\xi| = 1\right\}$ *as well.*

Conjecture 4.1. *Let μ be a probability measure on a discrete subgroup $\Gamma \leq PSU(1,1)$ satisfying the finite first moment. Then the sequence $\left(\mathbb{E}\left[\frac{1}{\overline{X_n.0}-z}\right]\right)_{n\geq 1}$ converges in H^p to a function in $H^p \cap \overline{H^p_0}$.*

Corollary 4.2. *Assume that μ satisfies the conclusion of the above conjecture. Then the μ-stationary measure ν either*

- *is the Lebesgue measure*
- *has a density in L^1 but not in L^p*
- *is a singular measure.*

If μ satisfies the Blaschke condition, then the first option is impossible. If, in addition, μ has a superexponential moment, then ν is singular.

5 Open questions

- It is easy to see from the proof of Corollary 1.1.1 that we actually get non-existence of solutions $f(z) = \sum a_{k+1} z^k$ to (14) with $\limsup_{n\to\infty} |a_k|^{1/k} < \varepsilon$ for some small ε, as only one preimage of the chosen contour explodes, so we can bound the radius of the convergence of the solution. Ideally, one would like to show that for finitely supported μ every solution of (14) has radius of convergence exactly 1. Keep in mind that this result would almost close the smoothness gap: it is known that absolutely continuous densities stationary with respect to finitely supported measures can belong to $C^n(S^1)$ for any $n > 1$.

The Douglas-Shields-Shapiro theorem implies that any holomorphic function with the radius of convergence exceeding 1 is either cyclic with respect to the backward shift or rational. It is reasonable to assume that (14) only has rational solutions when μ is supported on a single element, and we conjecture that former never happens.

- The Brown-Shields-Zeller theorem has an unexpected consequence – it requires the poles to be non-tangentially dense **almost** everywhere on S^1. Therefore, even if Γ is a non-cocompact lattice, there will be a sequence $(a_\gamma) \in l^1(\Gamma)$ such that

$$\sum \frac{a_\gamma}{z - \gamma.\infty} = 0, \quad |z| < 1.$$

However, due to [GL90] we know that the Lebesgue measure is not stationary with respect to any μ with finite first moment. Therefore, either Theorem 1.1 is not a criterion, (a_γ) does not have the first finite moment, or there is a complex-valued Furstenberg measure – keep in mind that Guivarch'-le Jan's methods only apply for probability measures μ.

- Cauchy transforms were used to study affine self-similar measures on $\mathbb{C}$ in [LSV98]. However, the paper was focused on studying measures supported on fractals with the Hausdorff dimension $\alpha > 1$, which forces the Cauchy transforms to be bounded and Holder with exponent $\alpha - 1$ (see [LSV98, Theorem 2.1(b)]). It should be possible to characterize the Hausdorff dimension of hitting measures using a similar self-similarity condition for the non-tangential limit $f_\nu(e^{it})$, but we expect the Hausdorff dimensions be strictly smaller than 1, and we are not aware of any existing ideas in this direction.

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