On a complex-analytic approach to classifying stationary measures on S^1 with respect to the countably supported measures on $PSU(1,1)$

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May 28, 2024

Abstract

We provide a complex-analytic approach to the classification of positive stationary measures on S^1 with respect to complex finite Borel measures on $PSU(1,1)$ satisfying the finite first moment condition by studying their Poisson-like and Cauchy-like transforms from the perspective of generalized analytic continuation. This approach allows us to establish a functional-analytic criterion for singularity of a hitting measure for random walks on lattices in $PSU(1,1)$. In addition, this approach suggests alternative proofs to the several well-known related results by Furstenberg, Kaimanovich, Guivarc'h-le Jan and Bourgain, among several others.

1 Introduction

1.1 Background

One of the most important notions in dynamical systems is of an **invariant measure**: given a topological space X and a self-map $T : X \to X$, one can study Borel measures $\mu \in Bor(X)$ on X which satisfy

$$(T_*\mu)(A) = \mu(T^{-1}(A)) = \mu(A). \tag{1}$$

This notion works quite well when provided with a single map $X \to X$. However, one often encounters nice spaces equipped with group actions $\Gamma \curvearrowright X$, and it is entirely possible that there are no measures invariant with respect to every element $\gamma \in \Gamma$.

Nevertheless, there is a natural weakening of the above definition, requiring a measure to be invariant "on average".

Definition 1.1. *Consider a group action $\Gamma \curvearrowright X$. Let ν be a Borel probability measure on X. Given a Borel measure μ on Γ, we say that ν is **μ-stationary** (with respect to the action) if*

$$\nu = \int_\gamma \gamma_* \nu d\mu(\gamma). \tag{2}$$

It is easy to see that any invariant measure is stationary with respect to any probability measure on Γ, but the inverse is, of course, not true. Being a μ-stationary measure is, evidently, a much weaker condition. Stationary measures exist in very general settings, unlike invariant ones, but they are no less important, as they are closely related to the Poisson boundaries and long-term behaviour of random walks on groups.

Our primary motivation stems from studying admissible random walks (X_n) on lattices $\Gamma \subset PSU(1,1)$ with a finite first moment. We know (due to Furstenberg ([Fur71]) and Kaimanovich ([Kai00])) that almost sure (X_n) converges to the Gromov boundary. The respective pushforward of the resulting measure to S^1 via the identification $\partial\Gamma \simeq S^1$ yields a (unique) μ-stationary measure ν_μ with respect to the action of Γ, called the **hitting measure** of the random walk. A big open problem in measured group theory is to understand when hitting measures are singular or absolutely continuous with respect to the hitting measure. In particular, recall the Kaimanovich-le Prince's singularity conjecture:

Conjecture 1.1 ([KL11]). *For every finite-range admissible random walk (X_n) generated by a probability measure μ on a cocompact Fuchsian group Γ, the hitting measure ν_μ is singular with respect to the Lebesgue measure on $S^1 \simeq \partial\Gamma$.*

This conjecture is known to hold for non-cocompact lattices due to [GL90], and the author's thesis [Kos23] provides affirmative results for nearest-neighbour random walks on cocompact Fuchsian groups, but the conjecture is still widely open, as we did show that even the recently developed geometric ideas are insufficient to completely settle the conjecture.

In our paper we will study the actions of subgroups $\Gamma \subset G = PSU(1,1)$ induced by the action of G on S^1 via hyperbolic isometries, or, more concretely, Möbius transformations. We aim to present a complex-analytic framework which, as we believe, can unify the majority existing results about stationary measures on S^1 with respect to the action of Γ and probability measures satisfying a finite first moment condition. Keep in mind that the analysis will be different depending on several factors:

- Whether μ has finite support or not,
- If μ is infinitely supported, the moment conditions on μ will matter (first moment, exponential moment, superexponential moment, and so on...),
- Whether the subgroup of G generated by the support of μ is discrete or not,
- If the generated subgroup is discrete, whether it is of first or second type,
- And, finally, if it is of first kind, whether it is cocompact or not.

First results about stationary measures and Poisson boundaries for discrete subgroups of $SL_n(\mathbb{R})$ were established by Furstenberg in [Fur63]. In particular, the question of when the Lebesgue measure is μ-stationary was first studied by Furstenberg as well in [Fur71]. Pure Fourier-like approaches were independently demonstrated by [Bou12] and [BPS12], which allow us to study μ-stationary measures for **dense** subgroups of $PSU(1,1)$. However, their methods do not apply for discrete groups and are quite delicate with respect to the initial data, requiring complicated number-theoretic and analytic methods to properly apply. There have been multiple improvements to Bourgain's approach, see [Leq22] and [Kit23] for latest examples, but they still do not apply to the discrete case and non-finite supports. Finally, we want to mention recent attempts to understand the structure of harmonic and Patterson-Sullivan measures using thermodynamic formalism, for example, [GL23] and [CT22]. Once again, these approaches are not universal, as Garcìa-Lessa's paper does not generalize to first-kind Fuchsian groups, and the thermodynamic approach of Cantrell-Tanaka provides considerably more information for Patterson-Sullivan measures than harmonic measures.

As one can see, up until now there was no single method which unified all above settings, and the general consensus is that no such method should have exist, in light of the incredible variety of techniques used to study different settings.

1.2 Main results

Inspired by the standard techniques used to study affine self-similar measures and their Parseval frames on $\mathbb{R}^n$, papers of R.S.Strichartz (see [Str90] and sequels), together with [JP98] and more recent papers of E.Weber [Web17] and [HW17], we have developed a promising approach which, in theory, could unify many standard results about stationary measures on S^1. The idea is to consider an appropriate integral transform on S^1 which respects the action of $PSU(1,1)$ and "preserves" (2). The Fourier transform is known to not respect this action, and the resulting exponential terms $e^{(az+b)/(cz+d)}$ are difficult to control. The Helgason-Fourier transform seems to be a better candidate, but integrating the powers of the Poisson kernel $(z,\xi) \mapsto \left(\frac{1-|z|^2}{|z-\xi|^2}\right)^{\lambda i+1}$ against a stationary measure does not actually preserve (2) in a way we want.

Moreover, given a μ-stationary measure ν on S^1, one can easily check that the resulting eigenvector of the hyperbolic Laplacian

$$\psi(z) := \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-|z|^2}{|z-e^{it}|^2}\right)^{\lambda i+1} d\nu(t)$$

does not exhibit any nice properties with respect to the action of $PSU(1,1)$, unlike Patterson-Sullivan measures. However, replacing the Poisson kernel with its logarithm, which is closely related to the Busemann cocycle, does the trick, turning a multiplicative relation into an additive one. The resulting functional equation (4) serves as a proper holomorphic version of (13), and, in a way, it allows us to change the perspective, as we shift from studying measurable functions on unit circle to holomorphic functions on $\mathbb{D}$, allowing us to make use of the vast complex-analytic machinery.

Before stating our results, we would like to point out that it is sufficient to study pure μ-stationary measures due to the fact that the action of $PSU(1,1)$ respects the Lebesgue decomposition, this is a standard reduction.

Our main result is a following necessary condition for stationarity.

Theorem 1.1. *Let μ be a complex finite Borel measure on $G = PSU(1,1)$ satisfying the following moment condition:*

$$\int_G \log\left(\frac{1+|\gamma.z|}{1-|\gamma.z|}\right) d|\mu|(\gamma) < \infty \tag{3}$$

for any $z \in \mathbb{D}$. Then a probability measure ν on S^1 is μ-stationary implies that the Cauchy transform

$$f_\nu(z) := \frac{1}{2\pi}\int_0^{2\pi}\frac{d\nu(t)}{e^{it}-z}$$

satisfies

$$\int_G f_\nu(\gamma^{-1}.z)(\gamma^{-1})'(z)d\mu(\gamma) - f_\nu(z) = \int_G \frac{d\mu(\gamma)}{z-\gamma.\infty}. \tag{4}$$

for every $z \in \mathbb{D}$. Moreover, if μ is countably supported then the above holds for all $z \in \overline{\mathbb{C}} \setminus \{\mathbb{T} \cup \{\gamma.\infty\}_{\gamma \in supp\mu}\}$.

Remark. We don't claim that this is a sufficient condition for any μ, as we, predictably, expect to lose some information by "cutting" ν in half and applying μ-stationarity.

The power of this theorem lies in the fact that we managed to successfully transform a measurable functional equation on the circle into a holomorphic condition on the unit disk, which allows us to make use of powerful complex-analytic techniques. Moreover, the theory of Cauchy transforms is quite mature, and, in theory, one should be able to use Tumarkin's and Alexandrov's theorems about Cauchy transforms to control the subtle properties of the resulting stationary measure as well.

We are able to extract the most amount of information from (4) for **countably supported probability** measures μ. Before stating the corollary, let us recall the **Blaschke condition** for a sequence $\{z_n\} \subset \mathbb{D}$:

$$\sum_{\gamma \in \operatorname{supp}\mu}(1-|\gamma.0|) < \infty. \tag{5}$$

We will say that μ satisfies the Blaschke condition if and only if $\{\gamma.0\}_{\gamma \in \operatorname{supp}\mu}$ satisfies (5).

Corollary 1.1. *Let μ be a countably supported probability measure on $PSU(1,1)$ with a finite first moment.*

1. *Assume that μ satisfies the Blaschke condition, and there is an element $\gamma \in supp\,\mu$ with $\gamma.0 \neq 0$. Then there are no entire solutions to (4). In particular, there are no μ-stationary measures with the Fourier series $\nu \sim \sum_{k\in\mathbb{Z}} a_k e^{ikt}$ with $\limsup_{n\to\infty}|a_k|^{1/k} = 0$.*

2. *Assume that* $\limsup_{n\to\infty} \left\| \int_G \frac{d\mu^{*n}(\gamma)}{z-\gamma.\infty} \right\|_1 = \infty$, *where* $||\cdot||_1$ *stands for the norm in* $H^1(\mathbb{D})$*. Then there are no* μ*-stationary measures with* $L^{1+\varepsilon}(S^1, m)$*-density for any* $\varepsilon > 0$*.*

Equation (4) gives us quite a lot of insight into the measures μ for which m is μ-stationary.

Corollary 1.2. *Let* μ *be a finite Borel measure supported on a discrete subgroup* $\Gamma \subset PSU(1,1)$ *with a finite first moment. Let's call measures* μ *on* $PSU(1,1)$ *such that the Lebesgue measure* m *on* S^1 *is* μ*-stationary the* ***Furstenberg measures****.*

1. *Denote the Lebesgue measure on* S^1 *by* m*. Then* m *is* μ*-stationary implies*

$$\int_G \frac{d\mu(\gamma)}{z-\gamma.\infty} = 0, \quad |z| < 1. \tag{6}$$

2. *Suppose that* m *is* μ*-stationary. Then*

$$\limsup_{n\to\infty} |\mu(\gamma_n)|^{1/n} = 1.$$

3. *Suppose that* m *is* μ*-stationary. Then* $\{\gamma.0\}_{\gamma\in supp\,\mu}$ *is non-tangentially dense in* $\mathbb{T}$*, which means that* m*-****almost every*** *point* $\xi \in \mathbb{T}$ *can be approached by a subsequence* $\gamma_n.0$ *inside a Stolz angle* $\{z \in \mathbb{D} : \frac{|z-\xi|}{1-|z|} < \alpha\}$ *for some* $\alpha > 1$*. As a corollary from [BSZ60, Remark 2], we get*

$$\sum_{\gamma\in supp\,\mu} (1 - |\gamma.0|) = \infty.$$

Finally, as a corollary from Fatou's theorem, we get a functional-analytic necessary condition for existence of μ-stationary measures with L^p-density for $1 < p < \infty$.

Corollary 1.3. *Let the support of* μ *satisfy the Blaschke condition. Then for any* μ*-stationary measure* μ *with* L^p*-density for* $1 < p < \infty$*, we have*

$$f_\nu(z) \in (\overline{T_\mu^*(B_\mu H^q)})^\perp \subset H^p,$$

where

- $T_\mu(f) := \sum_\gamma \mu(\gamma)(f\circ\gamma^{-1})(\gamma^{-1})' - f$ *is considered as a bounded linear operator* $T_\mu : H^p(\mathbb{D}) \to H^p(\mathbb{D})$*, and* $\frac{1}{p} + \frac{1}{q} = 1$*,*
- $B_\mu(z)$ *is the Blaschke function corresponding to the support of* μ*:*

$$B_\mu(z) := \prod_{\gamma\in supp(\mu)} \gamma^{-1}(z).$$

In particular, if $T_\mu^*(B_\mu H^q)$ *is dense in* H^q*, then there are no* μ*-stationary measures with* L^p*-density.*

Corollary 1.1.1 strictly strengthens the very last remark in [Bou12], where it was proven that the Lebesgue measure is never stationary with respect to finitely supported measures on $PSU(1,1)$. Corollary 1.1.2, in theory, provides a purely computational heuristic to showing singularty of stationary measures, as for lattices in $PSU(1,1)$, one expects the poles to converge to $\mathbb{T}$, whereas for dense subgroups one would expect the poles to accumulate inside $\mathbb{D}$, thus forcing the H^1-norms to stay bounded.

Corollary 1.2 provides several new insights into Furstenberg measures on $PSU(1,1)$. In particular, as our approach deals with signed and complex measures, we are able to talk about complex Furstenberg measures, which is not possible with any geometric approaches. In particular, we obtain Borel sums with poles in the orbits of a non-cocompact lattice which vanishes in $\mathbb{D}$, despite the fact that such counterexamples should be impossible due to Guivarch'-le Jan ([GL90]). The catch is that our condition is only a necessary one; and the Brown-Shields-Zeller theorem does not control the moments of the resulting coefficients. We also exhibit the first known result restricting the moment conditions of a Furstenberg measure, once again, improving on [Bou12]. The notion of a non-tangential limit seems to be key in this approach. Finally, we remark that studying positive Furstenberg measures should be possible using techniques in [BW89] and [HL90], as they deal with Borel-like series having strictly positive coefficients.

Finally, Corollary 1.3 provides a pretty significant restriction on the stationary measures in the L^p-class, and, in theory, the images with respect to the adjoint operator T_μ^* can be computed explicitly for any measure μ satisfying the Blaschke condition.

The structure of the paper is as follows. In Section 2 we recall all necessary facts about transformations $PSU(1,1)$ and provide a brief recap of complex-analytic tools we are going to use. In Section 3 we introduce an appropriate integral transform which fully respects the action of $PSU(1,1)$ to obtain a holomorphic necessary condition for μ-stationarity, thus proving Theorem 1.1. In Section 4 we extract the most we can from the resulting equation, using state-of-the-art techniques related to generalized analytic continuations.

Remark. This is very much work in progress – we believe that the established connection between the structure of stationary measures and closed invariant subspaces with respect to the backward shift goes much, much deeper. We will outline some of the open questions in the concluding section of this preprint.

1.3 Acknowledgements

I would like to thank Ilyas Bayramov, Ivan Nikitin, Kunal Chawla, Mathav Murugan, Pablo Shmerkin, Lior Silberman and Omer Angel for useful discussions. Also I am tremendously grateful to Alexander Kalmynin, Giulio Tiozzo and Tianyi Zheng for reading the preliminary version of this preprint and for providing continued support and encouragement throughout the past year. Finally, I would like to thank Giulio Tiozzo for organizing a short visit to the Fields institute on Feb 19–22, 2024.

2 Preliminaries

2.1 Everything you need to know about isometries of the disk model of $\mathbb{H}^2$

In this subsection we will recall basic facts about $PSU(1,1)$ considered as a isometry group of the disk model $\mathbb{D} = \{|z| < 1\}$ of the hyperbolic plane.

Definition 2.1.

$$PSU(1,1) = \left\{ z \mapsto \frac{az+b}{\overline{b}z+\overline{a}} : a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}.$$

From the definition it is evident that every transformation in $PSU(1,1)$ can be represented by a matrix $\begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$ (mod scalar matrices). In particular, if $\gamma(z) = \frac{az+b}{\overline{b}z+\overline{a}}$ then $\gamma^{-1}(z) = \frac{\overline{a}z-b}{-\overline{b}z+a}$.

Also, it will turn out that sometimes working with ∞ as a basepoint is more conventient than choosing $0 \in \mathbb{H}^2$, we will use

$$\overline{\gamma\left(\overline{z}^{-1}\right)} = \frac{1}{\gamma(z)}, \quad z \in \overline{\mathbb{C}}, \tag{7}$$

and, as a simple corollary,

$$\gamma.\infty = \frac{a}{\overline{b}} = \left(\frac{\overline{b}}{a}\right)^{-1} = (\overline{\gamma.0})^{-1}. \tag{8}$$

Lemma 2.1. *Let* $\gamma(z) = \frac{az+b}{\bar{b}z+\bar{a}}$*. then*

$$\gamma'(z) = \frac{1}{(\bar{b}z+\bar{a})^2}.$$

Proof.

$$\gamma'(z) = \frac{a(\bar{b}z+\bar{a}) - (az+b)\bar{b}}{(\bar{b}z+\bar{a})^2} = \frac{|a|^2-|b|^2}{(\bar{b}z+\bar{a})^2} = \frac{1}{(\bar{b}z+\bar{a})^2}.$$

$\square$

Definition 2.2.

$$\hat{c}_B(z,\xi) := \log\left(\frac{1-|z|^2}{|z-\xi|^2}\right), \quad |z|<1, |\xi|=1. \tag{9}$$

Remark. This notation is motivated by the explicit form of the **Busemann cocycle** for discrete subgroups of $PSU(1,1)$ acting geometrically on $\mathbb{H}^2$.

Let us prove some properties of $\hat{c}_B(z,\xi)$ first.

Lemma 2.2. *For every* $\gamma \in PSU(1,1)$ *and* $|z|<1$ *we have*

$$\log(1-|\gamma.z|^2) - \log(1-|z|^2) = \log(|\gamma'(z)|), \quad z \in \overline{\mathbb{C}}, \tag{10}$$

and

$$\hat{c}_B(\gamma.z, \gamma.\xi) = \hat{c}_B(z,\xi) - \hat{c}_B(\gamma^{-1}.0, \xi), \quad |z|<1, |\xi|=1. \tag{11}$$

Proof. Let $\gamma(z) := \frac{az+b}{\bar{b}z+\bar{a}}$. Then

$$\begin{aligned}\frac{1-|\gamma.z|^2}{1-|z|^2} &= \frac{1-\left|\frac{az+b}{\bar{b}z+\bar{a}}\right|^2}{1-|z|^2} = \frac{|\bar{b}z+\bar{a}|^2-|az+b|^2}{(1-|z|^2)|\bar{b}z+\bar{a}|^2} = \\ &= \frac{(\bar{b}z+\bar{a})(b\bar{z}+a)-(az+b)(\bar{a}\bar{z}+\bar{b})}{(1-|z|^2)|\bar{b}z+\bar{a}|^2} = \\ &= \frac{1-|z|^2}{(1-|z|^2)|\bar{b}z+\bar{a}|^2} = \frac{1}{|\bar{b}z+\bar{a}|^2} = |\gamma'(z)|.\end{aligned}$$

This implies (10). As for (11), observe that for every $\gamma(z) = \frac{az+b}{\bar{b}z+\bar{a}}$ we have

$$\begin{aligned}|\gamma.z-\gamma.\xi|^2 &= \left(\frac{az+b}{\bar{b}z+\bar{a}} - \frac{a\xi+b}{\bar{b}\xi+\bar{a}}\right)\left(\frac{\overline{az}+\bar{b}}{b\bar{z}+a} - \frac{\bar{a}\bar{\xi}+\bar{b}}{b\bar{\xi}+a}\right) = \\ &= |\gamma'(z)||\gamma'(\xi)||z-\xi|^2,\end{aligned}$$

and

$$\hat{c}_B(\gamma^{-1}.0,\xi) = \log\left(\frac{1-|\frac{b}{a}|^2}{|-\frac{b}{a}-\xi|^2}\right) = \log\left(\frac{|a|^2-|b|^2}{|-b-a\xi|^2}\right) = \log(|\gamma'(\xi)|).$$

Therefore,

$$\begin{aligned}\hat{c}_B(\gamma.z,\gamma.\xi) &= \log\left(\frac{1-|\gamma.z|^2}{|\gamma.z-\gamma.\xi|^2}\right) = \log\left(\frac{(1-|z|^2)|\gamma'(z)|}{|\gamma'(z)||\gamma'(\xi)||z-\xi|^2}\right) = \\ &= \log\left(\frac{1-|z|^2}{|\gamma'(\xi)||z-\xi|^2}\right) = \hat{c}_B(z,\xi) - \log(|\gamma'(\xi)|) = \\ &= \hat{c}_B(z,\xi) - \hat{c}_B(\gamma^{-1}.0,\xi)\end{aligned}$$

$\square$

Remark. Compare with the property of being a 2-cocycle:

$$c(gh, x) = c(g, hx) + c(h, x).$$

Finally, recall that for any $\gamma(z) = \frac{az+b}{\bar{b}z+\bar{a}}$ with $|a|^2 - |b|^2 = 1$ we have

$$\frac{1}{2}\frac{\gamma''(z)}{\gamma'(z)} = \frac{1}{2}\frac{-2\bar{b}}{(\bar{b}z+\bar{a})^3}\left(\frac{1}{(\bar{b}z+\bar{a})^2}\right)^{-1} = \frac{-\bar{b}}{\bar{b}z+\bar{a}} = -\frac{1}{z+\frac{\bar{a}}{\bar{b}}} = -\frac{1}{z-\gamma^{-1}.\infty}. \tag{12}$$

2.2 Complex-analytic prerequisites

We will heavily rely on standard complex-analytic techniques related to Cauchy transforms and generalized analytic continuation, we refer to standard textbooks on these topics: [RS02], [Ros06].

Let us denote $\mathbb{D} = \{|z| < 1\}$ and $\mathbb{D}_e := \overline{\mathbb{C}} \setminus \mathbb{D}$. Given a domain $U \subset \overline{\mathbb{C}}$, we will denote the space of holomorphic functions on U by $\mathfrak{H}(U)$ and the space of meromorphic functions on U by $\mathfrak{M}(U)$.

Definition 2.3. *Let $0 < p < \infty$. The **Hardy space** $(H^p(\mathbb{D}), ||\cdot||_p)$ is a space of holomorphic functions on $\mathbb{D}$ defined as follows.*

$$H^p(\mathbb{D}) = \left\{ f \in \mathfrak{H}(\mathbb{D}) \mid ||f||_p := \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p} < \infty \right\}.$$

If $p = \infty$, then we define $(H^\infty(\mathbb{D}), ||f||_\infty)$ as the space of bounded holomorphic functions on $\mathbb{D}$ equipped with the sup-norm.

Finally, we define $H^p(\mathbb{D}_e) := \{z \mapsto f(1/z) : f \in H^p(\mathbb{D})\}$, with $H^p_0(\mathbb{D}) \subset H^p(\mathbb{D}_e)$ denoting functions vanishing at infinity.

It is well-known that for $1 \le p \le \infty$ the respective Hardy spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{D}_e)$ are Banach spaces, whereas for $0 < p < 1$ they are topological non-locally convex vector spaces.

Definition 2.4. *Let f be a meromorphic function on $\mathbb{D}$ ($\mathbb{D}_e$, resp.). If the limit $\lim\limits_{r\to 1^-} f(re^{it})$ ($\lim\limits_{r\to 1^+} f(re^{it})$ resp.) exists Leb-almost everywhere on $\mathbb{T}$, then we say that f admits a **non-tangential limit** on the boundary.*

Definition 2.5. *Let f be a meromorphic function on $\mathbb{D}$. If there exists a function T_f which is meromorphic on $\mathbb{D}_e$ such that the non-tangential limits of f and $\tilde{f}$ coincide Leb-almost everywhere, then we say that f is **pseudocontinuable**, and T_f is a **pseudocontinuation** of f, and vice versa.*

In our paper we will use several important results about non-tangential limits and pseudocontinuations.

Theorem 2.1 (Lusin-Privalov, [Pri56])**.** *If f is pseudocontinuable, then its pseudocontinuation is unique.*

As a corollary, we get that pseudocontinuations are compatible with analytic continuations. The next theorem is a well-known fact about non-tangential limits of functions in the Hardy spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{D}_e)$.

Theorem 2.2 (Fatou's theorem + Riesz-Riesz)**.** *Every holomorphic function in $H^p(\mathbb{D})$ for $0 < p \le \infty$ admits a non-tangential limit which belongs to $L^p(S^1, Leb)$. Moreover, for $p \ge 1$ we have a complete characterization of such functions in $L^p(S^1)$: these are exactly the functions with vanishing negative Fourier coefficients.*

Definition 2.6. *Let ν be a complex finite Borel measure on S^1. Then its **Cauchy transform** is the integral*

$$C_\nu(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{d\nu(t)}{1 - e^{-it}z}.$$

The properties of $C_\nu(z)$ as a holomorphic function on $\mathbb{D}$ strongly depend on ν itself, but the following theorem of Smirnov ensures that we at least end up in H^p for $p < 1$.

Theorem 2.3 (Smirnov)**.** *Let $f(z) = C_\nu(z)$ for some complex finite Borel measure ν. Then $f \in H^p(\overline{C}\setminus\mathbb{T})$ for all $0 < p < 1$.*

We can do better if we know that $\nu \ll Leb$ due to a theorem of M. Riesz.

Theorem 2.4 (Riesz)**.** *Let ν be an absolutely continuous measure on S^1 with the L^p-density for $1 < p < \infty$. Then $C_\nu(z) \in H^p(\mathbb{D})$.*

Remark. This theorem cannot not hold for $p = 1, \infty$, as it is well-known that there are no continuous projections $L^1 \to H^1$ and $L^\infty \to H^\infty$.

As it turns out, there is an elegant criterion for the singularity of ν in terms of its Cauchy transform. Recall theorems of Tumarkin and Aleksandrov.

Theorem 2.5 (Tumarkin)**.** *Let f be a holomorphic function in $\overline{\mathbb{C}} \setminus \mathbb{T}$ vanishing at infinity. Then $f(z) = C_\nu(z)$ for some complex Borel finite measure μ if and only if*

$$\sup_{0<r<1} \int_{S^1} |(f(re^{it}) - f(e^{it}/r))| dt < \infty$$

Theorem 2.6 (Aleksandrov)**.** *Let $f(z) = C_\nu(z)$ be a Cauchy transform of a complex finite Borel measure ν on S^1. Then the following statements are equivalent.*

1. *The measure ν is singular with respect to the Lebesgue measure.*
2. *The function $\xi \mapsto (f(r\xi) - f(\xi/r))$ vanishes Leb-a.e. on S^1.*
3. *We have*
$$0 < \liminf_{p\to 1^-} (||f||_p(1-p)) < \infty.$$

We will also need some facts about weighted composition operators with respect to Möbius transformations.

Theorem 2.7. *Let $\gamma \in PSU(1,1)$ and consider the operator*

$$V_\gamma(f)(z) := f(\gamma^{-1}(z))(\gamma^{-1})'(z).$$

- *For every $w \in \mathbb{C}$ we have*
$$V_\gamma\left(\frac{1}{w-z}\right) = \frac{1}{\gamma.w - z} - \frac{1}{\gamma.\infty - z}.$$
- *$V_\gamma : H^p(\mathbb{D}) \to H^p(\mathbb{D})$ is a bounded linear operator for $1 \le p \le \infty$, being an isometry for $p = 1$.*

The first identity quickly follows from a residue computation via (complex) change of variables. The proof of the second statement is a classical application of the Littlewood subordination theorem (see [Car95] for details).

3 The log-Poisson transform of a stationary measure

Definition 3.1. *Let μ be a complex finite Borel measure on $G = PSU(1,1)$. Then a probability measure ν on S^1 is μ-stationary if*

$$\int_G \gamma_*(\nu) d\mu(\gamma) = \nu. \tag{13}$$

Remark. We put minimal restrictions on μ, for example, we allow measures μ which are absolutely continuous with respect to the Haar measure on G.

Let ν be a Borel probability measure on S^1. Consider the function (see (9) for notation)

$$p_\nu(z) := \int_{S^1} \hat{c}_B(z,\xi) d\nu(\xi). \tag{14}$$

It is easily seen that for any $|z| < 1$ this is a well-defined function as $\hat{c}_B(z,\xi)$ is bounded with respect to ξ on S^1. However, it is not always non-negative, as the Poisson kernel itself might take values less than 1.

Lemma 3.1. *Let μ be a complex finite Borel measure satisfying the **finite first moment** condition:*

$$\int_G \log\left(\frac{1+|\gamma.z|}{1-|\gamma.z|}\right) d|\mu|(\gamma) < \infty \tag{15}$$

for any $|z| < 1$. If ν is a μ-stationary probability measure, then

$$\int_G p_\nu(\gamma^{-1}.z) d\mu(\gamma) - p_\nu(z) = const \tag{16}$$

for all $|z| < 1$.

Proof. First of all, we need to resolve any convergence issues. Finite first moment ensures that the cocycle is absolutely integrable for all $|z| < 1$:

$$\begin{aligned}\int_G \int_{S^1} |\hat{c}_B(\gamma^{-1}.z,\xi)| d\nu(\xi) d|\mu|(\gamma) &= \int_G \int_{S^1} \left|\log\left(\frac{1-|\gamma^{-1}.z|^2}{|\gamma^{-1}.z-\xi|^2}\right)\right| d\nu(\xi) d|\mu|(\gamma) \leq \\ &\leq \int_G \log\left(\frac{1-|\gamma.z|^2}{(1-|\gamma.z|)^2}\right) d|\mu|(\gamma) = \\ &= \int_G \log\left(\frac{1+|\gamma.z|}{1-|\gamma.z|}\right) d|\mu|(\gamma) < \infty.\end{aligned} \tag{17}$$

Therefore, we are able to make use of the Fubini's theorem and DCT. If ν were to be μ-stationary, then

$$\begin{aligned} p_\nu(z) &\overset{(13)}{=} \int_G \left(\int_{S^1} \hat{c}_B(z,\gamma.\xi) d\nu(\xi)\right) d\mu(\gamma) \overset{(11)}{=} \\ &\overset{(11)}{=} \int_G \left(\int_{S^1} \hat{c}_B(\gamma^{-1}.z,\xi) - \hat{c}_B(\gamma^{-1}.0,\xi) d\nu(\xi)\right) d\mu(\gamma) \overset{(17)}{=} \\ &\overset{(17)}{=} \int_G p_\nu(\gamma^{-1}.z) d\mu(\gamma) - \int_G p_\nu(\gamma^{-1}.0) d\mu(\gamma),\end{aligned}$$

and both integrals $\int_G p_\nu(\gamma^{-1}.0) d\mu(\gamma)$ and $\int_G p_\nu(\gamma^{-1}.z) d\mu(\gamma)$ are well-defined due to (17). □

Remark. Keep in mind that the value $l_{\mu,\nu} := \int_G p_\nu(\gamma^{-1}.0) d\mu(\gamma)$ is a constant which only depends on μ and ν. Observe that if the subgroup Γ generated by the support of μ is discrete, we obtain the Furstenberg's formula for the drift (see [Fur63]), which does not depend on the choice of ν as well.

Now, let us recall the Fourier expansion of the logarithm of the Poisson kernel:

$$\log(1 - 2x\cos(t) + x^2) = -2\sum_{k=1}^{\infty}\frac{x^k}{k}\cos(kt) \tag{18}$$

where the right series converges uniformly for $|x| < 1$. From this we can deduce

$$\begin{aligned}\hat{c}_B(|z|e^{i\theta}, e^{it}) &= \log(1-|z|^2) - \log(1 - 2|z|\cos(\theta - t) + |z|^2) \overset{(18)}{=} \\ &\overset{(18)}{=} \log(1-|z|^2) + 2\sum_{k=1}^{\infty}\frac{|z|^k}{k}\cos(k(\theta - t))\end{aligned}$$

for all $|z| < 1$. In particular, denoting $\xi = e^{it}$, we can switch from the trigonometric basis to the exponential one to obtain

$$\begin{aligned}\hat{c}_B(z,\xi) &= \log(1-|z|^2) + \sum_{k=1}^{\infty}\frac{z^k}{k}e^{-ikt} + \frac{\overline{z}^k}{k}e^{ikt} = \\ &= \log(1-|z|^2) + \sum_{k=1}^{\infty}\frac{z^k}{k}\xi^{-k} + \frac{\overline{z}^k}{k}\xi^k\end{aligned} \tag{19}$$

Now, assume that $\nu \sim \sum_{k\in\mathbb{Z}} a_k e^{ikt}$, where

$$a_k := \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}d\nu(t). \tag{20}$$

Combining (19) and (20), we obtain

$$p_\nu(z) = \log(1-|z|^2) + \sum_{k=1}^{\infty}\frac{z^k}{k}a_k + \frac{\overline{z}^k}{k}a_{-k}. \tag{21}$$

Remark. The equation (21) proves that $p_\nu(z)$ uniquely defines ν, as we can recover the coefficients by considering

$$\begin{aligned}a_k &= \frac{1}{(k-1)!}\left.\frac{\partial^k}{\partial z^k}\right|_{z=0}(p_\nu(z) - \log(1-|z|^2)), \quad k > 0, \\ a_{-k} &= \frac{1}{(k-1)!}\left.\frac{\partial^k}{\partial \overline{z}^k}\right|_{z=0}(p_\nu(z) - \log(1-|z|^2)), \quad k > 0.\end{aligned}$$

Combining with (16), we get

$$\begin{aligned}&\left(\int_G \log(1-|\gamma^{-1}.z|^2)d\mu(\gamma) - \log(1-|z|^2)\right) + \\ &+ \left(\int_G f_+(\gamma^{-1}.z)d\mu(\gamma) - f_+(z)\right) + \\ &+ \left(\int_G f_-(\overline{\gamma^{-1}.z})d\mu(\gamma) - f_-(\overline{z})\right) = l_{\mu,\nu},\end{aligned}$$

where

$$\begin{aligned}f_+(z) &:= \sum_{k=1}^{\infty}\frac{a_k}{k}z^k, \\ f_-(z) &:= \sum_{k=1}^{\infty}\frac{a_{-k}}{k}z^k.\end{aligned}$$

Keep in mind that the integrability condition allows us to split the integral like this. Moreover, we can immediately observe that both f_+ and f_- are holomorphic inside $\mathbb{D}$ as ν is a probability measure, so $|a_k| \leq 1$ for all $k \in \mathbb{Z}$.

We can simplify to get

$$\int_G \log(|(\gamma^{-1})'(z)|)d\mu(\gamma) + \int_G \left(f_+(\gamma^{-1}.z) - f_+(z)\right) d\mu(\gamma) + \int_G \left(f_-(\overline{\gamma^{-1}.z}) - f_-(\overline{z})\right) d\mu(\gamma) = l_{\mu,\nu}. \quad (22)$$

Now we observe that (22) features a sum of a harmonic, holomorphic and an antiholomrphic function being constant. Moreover, recall that

$$\log(|(\gamma^{-1})'(z)|) = Re(\log((\gamma^{-1})'(z))) = \frac{1}{2}(\log((\gamma^{-1})'(z)) + \overline{\log((\gamma^{-1})'(z))})$$

is precisely the harmonic decomposition of $\log(|(\gamma^{-1})'(z)|)$. We fix the principal branches of the complex logarithm to make this equality unambiguous which is possible due to $\gamma'(z) \neq 0$ for any $\gamma \in PSU(1,1)$ and $|z| < 1$.

Now observe that a sum of a holomorphic function and an antiholomorphic function is constant iff both parts are constant as well, so we get

$$\frac{1}{2}\int_G \log((\gamma^{-1})'(z))d\mu(\gamma) + \int_G f_+(\gamma^{-1}.z)d\mu(\gamma) - f_+(z) = l_{\mu,\nu} \quad (23)$$

$$\frac{1}{2}\int_G \log(\overline{(\gamma^{-1})'(z)})d\mu(\gamma) + \int_G f_-(\overline{\gamma^{-1}.z})d\mu(\gamma) - f_-(\overline{z}) = l_{\mu,\nu} \quad (24)$$

These equations are, essentially, the same due to the symmetry $a_k = \overline{a_{-k}}$, as we assume that ν is positive. Let us redenote $f_+(z) = F_\nu(z)$, so that we are trying to solve

$$\frac{1}{2}\int_G \log((\gamma^{-1})'(z))d\mu(\gamma) + \int_G F_\nu(\gamma^{-1}.z)d\mu(\gamma) - F_\nu(z) = l_{\mu,\nu}. \quad (25)$$

Differentiating both sides and making use of the moment condition again to interchange the derivative and the integrals, we can get rid of the logarithm and reduce the equation to

$$\frac{1}{2}\int_G \frac{(\gamma^{-1})''(z)}{(\gamma^{-1})'(z)}d\mu(\gamma) + \int_G f_\nu(\gamma^{-1}.z)(\gamma^{-1})'(z)d\mu(\gamma) - f_\nu(z) = 0, \quad (26)$$

where $f_\nu(z) = \sum_{k=0}^{\infty} a_{k+1}z^k$. Using (12), we can rewrite (26) as follows:

$$\int_G f_\nu(\gamma^{-1}.z)(\gamma^{-1})'(z)d\mu(\gamma) - f_\nu(z) = \int_G \frac{d\mu(\gamma)}{z - \gamma.\infty}, \quad |z| < 1. \quad (27)$$

Thus, we have proven the first part of Theorem 1.1.

Theorem 3.1. *Let μ be a complex finite Borel measure with the finite first moment with respect to the hyperbolic metric. Consider a probability measure ν on S^1 admitting the Fourier series $\nu \sim \sum_{k\in\mathbb{Z}} a_k z^k$. Then ν is μ-stationary if and only if the function $f_\nu(z) = \sum_{k=0}^{\infty} a_{k+1}z^k$ satisfies the functional equation* (27) *for $|z| < 1$.*

4 Squeezing water from a stone: a deep dive into (27)

In this section we will explore the functional equation (27) in much more detail. From now on, we will restrict ourselves to countably supported probability measures μ, denoting by $\Gamma \leq G = PSU(1,1)$ the subgroup generated by the support of μ.

Theorem 4.1 (Corollary 1.1.1)**.** *Let μ be a probability measure with finite support. Then there are no entire solutions to* (27)*.*

Proof. Choose an element $\tau \in \operatorname{supp}(\mu)$ which does not fix the origin. In particular, $\tau.\infty = (\overline{\tau.0})^{-1} \neq \infty$. Fix small enough contour C_τ around $\tau.\infty$. Integrating both sides over this contour, we get

$$\int_{C_\tau}\left(\int_G \frac{d\mu(\gamma)}{z-\gamma.\infty} + \int_G f(\gamma^{-1}.z)(\gamma^{-1})'(z)d\mu(\gamma) - f(z)\right)dz =$$
$$= \sum_{\gamma.\infty=\tau.\infty}\mu(\gamma) + \int_\Gamma\int_{\gamma^{-1}(C_\tau)} f(z)dz - \int_{C_\tau} f(z) = 0$$

by applying the change of variables. As $f(z)$ is entire, the contour integrals vanish, leaving us with $\mu(\gamma) = 0$ for all γ with the same pole as τ, which leads to a contradiction. □

Corollary 4.1. *Let μ be a probability measure with finite support. Then $\limsup_{k\to\infty}|a_k|^{1/k} > 0$ for every μ-stationary measure with the Fourier series $\nu \sim \sum_{k\in\mathbb{Z}} a_k e^{ikt}$.*

Proof. Consider a μ-stationary measure ν with $\limsup_{k\to\infty}|a_k|^{1/k} = 0$. Then $f_\nu(z)$ is an entire function which solves (27) for $|z| < 1$. The LHS of (27) can be analytically continued to a meromorphic function on $\mathbb{C}$, therefore, $f_\nu(z)$ solves (27) for all $\mathbb{C}$. This allows us to apply Theorem 4.1, yielding a contradiction. □

Lemma 4.1. *Let ν be a probability measure on S^1. Then we have*

$$f_\nu(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\nu(t)}{e^{it}-z} = C_{e^{-it}\nu}(z). \tag{28}$$

for all $|z| < 1$.

Proof. It is easy to see that $f_\nu(z)$ is indeed a Cauchy transform, but not of ν itself: we have to consider a measure $\nu' = e^{-it}\nu$, and then it is true that

$$f_\nu(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-it}d\nu(t)}{1-e^{-it}z} = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\nu(t)}{e^{it}-z}.$$

To show that this equality holds for $|z| < 1$, we can simply expand the geometric series as follows:

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\nu(t)}{e^{it}-z} = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=0}^\infty e^{-i(k+1)t}z^k\right)d\nu(t) = \sum_{k=0}^\infty \frac{1}{2\pi}\int_0^{2\pi} e^{-i(k+1)t}d\nu(t)z^k = f_\nu(z).$$

□

Due to Theorem 2.3 we know that $f_\nu(z)$ is holomorphic on $\overline{\mathbb{C}}\setminus\mathbb{T}$. Moreover, (27) makes sense for all $z \in \overline{\mathbb{C}}$ outside of the poles of RHS (namely, $\gamma.\infty$ for all $\gamma \in \operatorname{supp}\mu$).

Theorem 4.2 (Theorem 1.1)**.** *A probability measure ν is μ-stationary only if $f_\nu(z)$ solves* (27) *for all $z \in \overline{\mathbb{C}}\setminus(\mathbb{T}\cup\{\gamma.\infty\}_{\gamma\in supp\mu})$.*

Proof. Let $z \in \overline{\mathbb{C}}\setminus(\mathbb{T}\cup\{\gamma.\infty\}_{\gamma\in\operatorname{supp}\mu})$. Due to Theorem 2.7 we have

$$T_\mu\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\nu(t)}{e^{it}-z}\right) = \frac{1}{2\pi i}\int_{\mathbb{T}}\left(\frac{1}{\gamma.e^{it}-z} - \frac{1}{\gamma.\infty - z} - \frac{1}{e^{it}-z}\right)d\nu(t) =$$
$$= \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{d\gamma_*\nu(w)}{e^{it}-z} - \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{d\nu(w)}{e^{it}-z} - \frac{1}{\gamma.\infty-z}.$$

However, as ν is μ-stationary, we can see that

$$\left(\sum_\Gamma \mu(\gamma)T_{\gamma^{-1}}\right)\left(\int_{\mathbb{T}}\frac{d\nu(w)}{w(w-z)}\right) = \int_{\mathbb{T}}\frac{d(\mu*\nu-\nu)(w)}{w(w-z)} - \int_\Gamma\frac{d\mu(\gamma)}{\gamma.\infty-z} = \int_\Gamma\frac{d\mu(\gamma)}{z-\gamma.\infty},$$

which is precisely (27). □

Theorem 4.3 (Corollary 1.1.2)**.** *Let μ be a countably supported probability measure. If* $\limsup_{n\to\infty} \left\| \int_G \frac{d\mu^* n(\gamma)}{z-\gamma.\infty} \right\|_1 = \infty$*, then there are no μ-stationary measures with $L^{1+\varepsilon}(S^1, Leb)$-density for any $\varepsilon > 0$.*

Proof. Let ν be μ-stationary with density in $L^{1+\varepsilon}(S^1)$. Due to Fatou's theorem we know that $f_\nu(z) \in H^{1+\varepsilon}(\mathbb{D})$. In particular, $f_\nu(z) \in H^1(\mathbb{D})$. As all composition operators in LHS of (27) are exactly the operators treated in Theorem 2.7, they are isometries, in particular, the H^1-norm of LHS is at most $2||f||_1$. Make note of the fact that this application of the triangle inequality does not depend on μ at all. Applying H^1-norm to both sides, we get

$$2\,\|f_\nu\|_1 \geq \left\| \int_G \frac{d\mu(\gamma)}{z-\gamma.\infty} \right\|_1 .$$

However, keep in mind that any μ-stationary measure is μ^{*n}-stationary, therefore, WLOG one can replace μ with μ^{*n} in the above inequality without changing LHS. This would imply

$$2\,\|f_\nu\|_1 \geq \limsup_{n\to\infty} \left\| \int_G \frac{d\mu^{*n}(\gamma)}{z-\gamma.\infty} \right\|_1 = \infty,$$

which leads to a contradiction. $\square$

Example 4.1. *Consider $\mu = \delta_\gamma$ for a non-elliptic $\gamma \in PSU(1,1)$. Then the H^1-norm of $\frac{1}{z-\gamma^n.\infty}$ goes to infinity as $n \to \infty$, so there are no absolutely continuous measures with densities in $L^{1+\varepsilon}(S^1)$, as we expected.*

However, as simple as this criterion seems, given a measure μ supported on a lattice in $PSU(1,1)$, it is not at all easy to estimate $\left\| \int_G \frac{d\mu^* n(\gamma)}{z-\gamma.\infty} \right\|_1$, therefore, a potential argument should rely on how non-uniformly the poles will be distributed in small neighbourhoods of $\mathbb{T}$.

4.1 Functional-analytic necessary condition for existence of absolutely continuous stationary measures

In this subsection we treat LHS of (27) as a bounded operator: define

$$T_\mu : H^p(\mathbb{D}) \to H^p(\mathbb{D}), \quad T_\mu(f)(z) := \sum_\gamma \mu(\gamma) f(\gamma^{-1}.z)(\gamma^{-1})'(z) - f(z).$$

It is well-known that T_μ is a bounded operator for all $0 < p \leq \infty$, and in such generality, not much else is known about T_μ. If $p > 1$, we can at least explicitly compute its adjoint $T_\mu^* : H^q \to H^q$.

Proposition 4.1. *Consider $V_\gamma(f)(z) = f(\gamma^{-1})(\gamma^{-1})'(z)$ as a bounded operator $H^p \to H^p$. Then*

$$V_\gamma^*(f)(z) = S^*(f(\gamma.z)\gamma.z), \quad f \in H^q(\mathbb{D}),$$

where S^* stands for the backwards shift $S^*(g)(z) = \frac{g(z)-g(0)}{z}$.

Proof. As in many similar computations (see [Cow88, Theorem 2] for an example), we use the reproducing kernel property of $\frac{1}{1-\bar{a}z}$: for any $f \in H^p$

$$\left\langle f(z), \frac{1}{1-\bar{a}z} \right\rangle := \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{it})dt}{1-a\bar{z}} = f(a).$$

A slight modification yields

$$\left\langle f(z), \frac{1}{a-z} \right\rangle = \frac{f(\bar{a}^{-1})}{\bar{a}}.$$

As we know how T_γ acts on $\frac{1}{a-z}$, reflexivity of H^p for all $1 < p < \infty$ allows us to write

$$\frac{V_\gamma^* f(\overline{a}^{-1})}{\overline{a}} = \left\langle (V_\gamma^*) f(z), \frac{1}{a-z} \right\rangle = \left\langle f(z), \frac{1}{\gamma.a - z} - \frac{1}{\gamma.\infty - z} \right\rangle =$$

$$= \frac{f(\overline{\gamma.a}^{-1})}{\overline{\gamma.a}} - \frac{f(\overline{\gamma.\infty}^{-1})}{\overline{\gamma.\infty}} \overset{(7)}{=} \gamma.\overline{a^{-1}} f(\gamma.\overline{a^{-1}}) - \gamma.0 f(\gamma.0).$$

Replacing $\overline{a}^{-1}$ with ω, we get

$$V_\gamma^* f(w) = \frac{\gamma.w f(\gamma.w) - \gamma.0 f(\gamma.0)}{w} = S^*(f(\gamma.w)\gamma.w).$$

□

As a quick corollary, we get that

$$T_\mu^*(f)(z) = S^* \left(\sum_\gamma \mu(\gamma) f(\gamma.z) \gamma.z \right) - f(z). \tag{29}$$

Theorem 4.4. *Let μ satisfy the Blaschke condition. Then for any μ-stationary measure μ with L^p-density for $1 < p < \infty$, we have*

$$f_\nu(z) \in (\overline{T_\mu^*(B_\mu H^q)})^\perp \subset H^p,$$

where $B_\mu(z)$ is the Blaschke function corresponding to the support of μ:

$$B_\mu(z) := \prod_{\gamma \in supp(\mu)} \gamma^{-1}(z).$$

Proof. Let $T_\mu(f) = \sum_\gamma \frac{\mu(\gamma)}{z - \gamma.\infty}$. It is easy to see that $\sum_\gamma \frac{\mu(\gamma)}{z-\gamma.\infty}$ is a linear combination of reproducing kernels $\frac{1}{1-\overline{\gamma.0}z}$. In particular, $\sum_\gamma \frac{\mu(\gamma)}{z-\gamma.\infty} \in (B_\mu H^q)^\perp$. Therefore,

$$0 = \langle T_\mu(f), B_\mu H^q \rangle = \left\langle f, \overline{T_\mu^*(B_\mu H^q)} \right\rangle.$$

□

This proves Corollary 1.3. Recall the Douglas-Shields-Shapiro theorem:

Theorem 4.5 (Douglas-Shields-Shapiro, $(p > 1)$-version)**.** *Let $1 < p < \infty$ and consider $f \in H^p$. Then the following are equivalent:*

1. *f is non-cyclic with respect to S^*, that is, $span\{(S^*)^k(f)\}_{k \geq 0}$ is **not** dense in $H^p(\mathbb{D})$.*
2. *There exists a holomorphic function $\varphi \in H^2$ with $|\varphi(e^{it})| = 1$ a.e. on S^1 (inner function) such that $f \in (\varphi H^q)^\perp$.*
3. *There exists an inner function $\varphi \in H^2$ such that f/φ admits a pseudocontinuation to a function $\tilde{f} \in H_0^p(\mathbb{D}_e)$.*

As a corollary from this theorem, we can deduce the following:

Corollary 4.2. *Let μ satisfy the Blaschke condition, and assume that $\overline{T_\mu^*(B_\mu H^q)} = \varphi H^q$ for some inner $\varphi \in H^2$. Then every solution to (27) is pseudocontinuable.*

Proof. Beurling's theorem implies that $\overline{T_\mu^*(B_\mu H^q)}$ is invariant with respect to the forward shift $f(z) \mapsto zf(z)$. Therefore, the orthogonal complement is S^*-invariant in H^p. Douglas-Shields-Shapiro theorem applies, and we get that f is pseudocontinuable. □

Keep in mind that any Cauchy transform of a singular measure is pseudocontinuable, but the converse it not true, therefore, the above corollary is still a much weaker statement. Nevertheless, we can still make use of this observation: suppose that ν is an absolutely continuous μ-stationary measure. As the operator T_μ takes pseudocontinuations to pseudocontinuations, and RHS itself has both inner and outer non-tanegntial limits which coincide a.e., we would get two distinct (meromorphic!) solutions of (27) on $\mathbb{D}_e$: one comes from the Cauchy transform itself, and another from the previous Corollary. This would cause the operator T_μ to have a meromorphic function on $\mathbb{D}_e$ in its kernel, which seems unlikely in the discrete case. From what we understand, the injectivity of T_μ is still an open question.

To conclude this subsection, we want to remind the reader that the difficulty of this problem lies precisely in the fact that T_μ does **not** commute with the backward shift S^*, which disallows us from easily arguing that T_μ preserves the S^*-invariant subspaces in any way.

4.2 When the Lebesgue measure is stationary?

Earlier we have reproved the well-known fact that the Lebesgue measure cannot be a stationary measure if μ has finite support. To understand this case better, we need to look at (27) and observe that f_ν vanishes, leaving us with vanishing of the following **Borel series**.

$$\sum_\gamma \frac{\mu(\gamma)}{z-\gamma.\infty} = 0, \quad |z| < 1. \tag{30}$$

This immediately proves Corollary 1.1.1. At first glance, it might seem counter-intuitive that the above sum can vanish on the entire disc, but recall the following fundamental fact about Borel series.

Definition 4.1. *A sequence of points $\{z_n\} \subset \mathbb{D}$ is said to* ***non-tangentially*** *converge to $\xi \in \partial\mathbb{D}$ if there exists a* ***Stolz angle*** *$\{\frac{|\xi-z|}{1-|z|} \leq M\}$ and $N > 0$ such that $z_n \to \xi$ and $z_n \subset A$ for $n > N$.*

Theorem 4.6 ([BSZ60], Theorem 3)**.** *Let $A = \{z_n\} \subset \mathbb{D}$ be a sequence of points* ***in*** *the unit disk without interior limit points. Then there exists a sequence $\{c_n\} \in l^1$ such that*

$$\sum_n \frac{c_n}{z-z_n} = 0, \quad |z| > 1$$

if and only if almost every point in S^1 is a non-tangential limit of a subsequence in $\{z_n\}$.

This theorem almost gives what we want, however, the above theorem gives series with poles **inside** the disk which vanishes **outside** of it, whereas we need the opposite – series with poles **outside** the disk and vanishing **inside** the disk.

One can easily mitigate this by considering the change of variables $z \mapsto 1/z$:

$$\sum_\gamma \frac{\mu(\gamma)}{z^{-1}-\gamma.\infty} = \sum_\gamma \frac{\mu(\gamma)z}{1-(\gamma.\infty)z} = \sum_\gamma \frac{(\gamma.\infty)^{-1}\mu(\gamma)z}{(\gamma.\infty)^{-1}-z} =$$
$$= \sum_\gamma -\frac{\mu(\gamma)}{\gamma.\infty} + \frac{\mu(\gamma)}{(\gamma.\infty)^2}\frac{1}{(\gamma.\infty)^{-1}-z}.$$

However, as we can plug in $z = 0$ in (30), we get that

$$\sum_\gamma \frac{\mu(\gamma)}{z^{-1}-\gamma.\infty} = \sum_\gamma \frac{\mu(\gamma)}{(\gamma.\infty)^2}\frac{1}{(\gamma.\infty)^{-1}-z} = 0$$

for all $|z| > 1$. Applying the Brown-Shields-Zeller theorem, we obtain Corollary 1.2.3.

Remark. Recall that the orbit of a point with respect to an action of a discrete subgroup of $PSU(1,1)$ is non-tangentially dense if and only if the subgroup is of the first type. Therefore, Theorem 4.6 confirms that $\Gamma \subset PSU(1,1)$ being a first-kind Fuchsian group should be a necessary condition for a Furstenberg measure on Γ to exist.

Moreover, due to another theorem of Beurling, referring to [RS02, Corollary 4.2.24]:

Theorem 4.7 ([Beu34], [BC89])**.** *Let $\{z_n\}$ be a sequence of points **outside** of the unit disk with $|z_n| \downarrow 1$. If $\limsup_{n\to\infty} |c_n|^{1/n} < 1$ and*

$$\sum_n \frac{c_n}{z - z_n} = 0, \quad |z| < 1,$$

then all $c_n = 0$.

Applying this theorem to $z_n = \gamma_n.\infty$ (relative to a suitable enumeration of Γ), we get that a Fuchsian group of first kind $\Gamma \subset PSU(1,1)$ admits a Furstenberg measure only if

$$\limsup_{n\to\infty} |\mu(\gamma_n)|^{1/n} = 1,$$

this proving Corollary 1.2.2. Combined with the exponential growth of Fuchsian groups, this condition implies that a Furstenberg measure μ cannot have a double-exponential moment with respect to the hyperbolic distance: if we let $c > 0$, then

$$\sum_n \mu(\gamma_n) e^{e^{cd(0,\gamma_n.0)}} < \infty \iff \sum_n \mu(\gamma_n) e^{cn} < \infty \Rightarrow \limsup_{n\to\infty} |\mu(\gamma_n)|^{1/n} < e^{-c} < 1.$$

As for the strongest known moment conditions: it is known that J.Li's counterexample, given in the Appendix of [LNP21], provides a Furstenberg measure with an exponential moment, our approach shows that a Furstenberg measure cannot have a double-exponential moment. However, it is not even known whether there exists an example of a Furstenberg measure with a superexponential moment.

Finally, we would like to remark that the proof of [BSZ60, Theorem 3] is, essentially, non-constructive. In context of our problem, the idea is as follows.

1. We start by considering an operator $H^\infty(\mathbb{D}) \to l^\infty(\Gamma)$,

$$T(f)_\gamma = f((\gamma.\infty)^{-1}).$$

2. Its image is proven to be wk^*-closed in $l^\infty(\Gamma)$.

3. Consider $p = \delta_e \in l^\infty(\Gamma)$. Identifying $l^\infty(\Gamma)$ with $l^1(\Gamma)$. A clever argument shows that $\mathrm{dist}(p, T(H^\infty(\mathbb{D}))) = \frac{1}{2}$, so the image cannot coincide with the whole $l^\infty(\Gamma)$. and applying the Hahn-Banach theorem, we prove the existence of an element $a \in l^1(\Gamma)$, such that:

   - for any $y \in T(H^\infty(\mathbb{D})) \subset l^\infty(\Gamma)$ we have $y(a) = 0$
   - $p(a) = 1$
   - $||a|| < 2 + \varepsilon$ for a small enough $\varepsilon > 0$.

4. The sequence a solves our problem, that is,

$$\sum_\gamma \frac{a_\gamma}{z - (\gamma.\infty)^{-1}} = 0.$$

5 The proof of the singularity conjecture

Let μ be a finitely supported probability measure on a non-elementary discrete subgroup $\Gamma \subset PSU(1,1)$ generating a random walk which we will denote by (X_n). Let us define two sequences of functions

$$g_n(z) := \mathbb{E}\left[\frac{1}{X_n.\infty - z}\right], \tag{31}$$

$$h_n(z) := \mathbb{E}\left[\frac{1}{\frac{X_n.\infty}{|X_n.\infty|} - z}\right]. \tag{32}$$

It is easy to see that

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{X_n.\infty - z}\right] = \mathbb{E}\left[\frac{1}{X_\infty.\infty - z}\right] = f_\nu(z)$$

for any $|z| < 1$, where by $X_\infty.\infty$ we, via an abuse of notation, denote the a.s. limit of $X_m.\infty$. The convergence can be trivially upgraded to convergence on compact subsets, and this fact precisely corresponds to the wk^*-convergence of μ^{*n} to the hitting measure ν. Of course, this does not show singularity by itself, but we aim to show that the rational approximation induced by g_n converges in H^p for all $p \in (0,1)$, which corresponds to a much stronger condition, "forcing" singularity by pinning down $f_\nu(z)$ into an exotic subspace $H^p \cap \overline{H_0^p}$, which has trivial intersections with $H^{1+\varepsilon}$ for any $\varepsilon > 0$ and $p \in (0,1)$.

Theorem 5.1. *Let (X_n) be a finitely supported admissible random walk on a non-elementary discrete subgroup $\Gamma \leq PSU(1,1)$. Then for every $0 < p < 1$ the following statements hold:*

1. $g_n(z) \xrightarrow{H^p} f_\nu(z)$.
2. $\lim\limits_{n\to\infty} d_{H^p}(g_n, h_n) = 0$.

Before we prove the theorem, we need the following standard lemma that is commonly used to work with H^p-spaces for $p \in (0,1)$.

Lemma 5.1. *Let $f(z) \in \mathcal{H}(\mathbb{D})$ with $\Re(f(z)) > 0$ for all $z \in \mathbb{D}$. Then for all $p \in (0,1)$ we have*

$$\|f\|_p \leq A_p |f(0)|^p,$$

where $A_p = 1/\cos(p\pi/2)$.

Proof of Theorem 5.1.

- Let $\delta > 0, \theta \in \mathbb{R}$. Define an event $E_{\theta,\delta} := \{\omega \in \Omega : |X_\infty - e^{i\theta}| < \delta\}$.

  Consider two integrals:

$$\left|\int_{\Omega\setminus E_{\theta,\delta}} \frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega)\right| \tag{33}$$

$$\left|\int_{E_{\theta,\delta}} \frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega)\right|. \tag{34}$$

  We will proceed by estimating these two integrals in two different ways. As for the first integral, the random walk almost surely misses $e^{i\theta}$:

$$\left|\int_{\Omega\setminus E_{\theta,\delta}} \frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega)\right| \leq \int_{\Omega\setminus E_{\theta,\delta}} \left|\frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}}\right| d\mathbb{P}(\omega) =$$
$$= \int_{\Omega\setminus E_{\theta,\delta}} \frac{|X_\infty - X_n|}{|X_n - e^{i\theta}||X_\infty - e^{i\theta}|} d\mathbb{P}(\omega) \leq \int_{\Omega\setminus E_{\theta,\delta}} \frac{|X_\infty - X_n|}{\delta^2} d\mathbb{P}(\omega).$$

Now we observe that as $n \to \infty$, the expectation $\int_{\Omega \setminus E_{\theta,\delta}} |X_\infty - X_n| d\mathbb{P}(\omega)$ goes to zero as $X_n \to X_\infty$ almost surely.

As for the second integral, we observe that the probability of $E_{\theta,\delta}$ goes to zero as $\delta \to 0$, because the hitting measure is non-atomic. So, we can separate the integral like this.

$$\left| \int_{E_{\theta,\delta}} \frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega) \right|^p \le \left| \int_{E_{\theta,\delta}} \frac{1}{X_n - e^{i\theta}} d\mathbb{P}(\omega) \right|^p + \left| \int_{E_{\theta,\delta}} \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega) \right|^p .$$

Now we proceed to estimate the H^p-norms themselves.

$$\left\| \mathbb{E} \left[\frac{1}{X_n.\infty - z} - \frac{1}{X_\infty.\infty - z} \right] \right\|_p^p = \int_0^{2\pi} \left| \int_\Omega \frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega) \right|^p \frac{d\theta}{2\pi} =$$
$$= \int_0^{2\pi} \left| \int_{\Omega \setminus E_{\theta,\delta}} \frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega) + \int_{E_{\theta,\delta}} \frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega) \right|^p \frac{d\theta}{2\pi} \le$$
$$\le \int_0^{2\pi} \left| \int_{\Omega \setminus E_{\theta,\delta}} \frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega) \right|^p + \left| \int_{E_{\theta,\delta}} \frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega) \right|^p \frac{d\theta}{2\pi}.$$

Exploiting the above estimates, we get

$$\int_0^{2\pi} \left| \int_{\Omega \setminus E_{\theta,\delta}} \frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega) \right|^p + \left| \int_{E_{\theta,\delta}} \frac{1}{X_n - e^{i\theta}} - \frac{1}{X_\infty - e^{i\theta}} d\mathbb{P}(\omega) \right|^p \frac{d\theta}{2\pi} \le$$
$$\le \frac{1}{\delta^{2p}} \int_0^{2\pi} \left| \int_{\Omega \setminus E_{\theta,\delta}} |X_\infty - X_n| d\mathbb{P}(\omega) \right|^p + \left\| \int_{E_{\theta,\delta}} \frac{1}{X_n - z} d\mathbb{P}(\omega) \right\|_p^p + \left\| \int_{E_{\theta,\delta}} \frac{1}{X_\infty - z} d\mathbb{P}(\omega) \right\|_p^p \overset{L5.1}{\le}$$
$$\le \frac{1}{\delta^{2p}} \int_0^{2\pi} \left| \int_{\Omega \setminus E_{\theta,\delta}} |X_\infty - X_n| d\mathbb{P}(\omega) \right|^p + A_p \left(\left| \int_{E_{\theta,\delta}} (X_n.0) d\mathbb{P}(\omega) \right|^p + \left| \int_{E_{\theta,\delta}} (X_\infty.0) d\mathbb{P}(\omega) \right|^p \right).$$

All that is left to do is to observe that the first integral always goes to zero as $n \to \infty$, and the second term can be estimated by $2A_p \nu(E_{\theta,\delta})^p$, and by choosing δ to be as small as needed, we can force $2A_p \nu(E_{\theta,\delta})^p \to 0$ as well, establishing the convergence.

- The idea is mostly the same, we can just repeat the same estimates.

$$\left\| \mathbb{E} \left[\frac{1}{X_n.\infty - z} - \frac{1}{\frac{X_n.\infty}{|X_n.\infty|} - z} \right] \right\|_p^p = \int_0^{2\pi} \left| \int_\Omega \frac{1}{X_n - e^{i\theta}} - \frac{1}{\frac{X_n.\infty}{|X_n.\infty|} - e^{i\theta}} d\mathbb{P}(\omega) \right|^p \frac{d\theta}{2\pi} \le$$
$$\le \frac{1}{\delta^{2p}} \int_0^{2\pi} \left| \int_{\Omega \setminus E_{\theta,\delta}} \left| \frac{X_n.\infty}{|X_n.\infty|} - X_n \right| d\mathbb{P}(\omega) \right|^p + A_p \left(\left| \int_{E_{\theta,\delta}} (X_n.0) d\mathbb{P}(\omega) \right|^p + \left| \int_{E_{\theta,\delta}} \frac{X_n.0}{|X_n.0|} d\mathbb{P}(\omega) \right|^p \right).$$

Again, a.s. convergence makes the first integral small, and the second term is controlled by $\nu(E_{\theta,\delta})^p$.

□

Corollary 5.1. *Choose $0 < p < 1$ as in the previous theorem. Then $f_\nu(z) \in H^p \cap \overline{H_0^p}$ and therefore, ν cannot have a $L^{1+\varepsilon}(S^1)$-density.*

Proof. The above theorem implies that $(h_n)_{n>0}$ is also a Cauchy sequence in H^p. However, due to Aleksandrov we know that

$$\overline{span}^{H^p}\left\{\frac{1}{1-e^{i\lambda}z}\right\}_{\lambda\in\mathbb{R}} = H^p \cap \overline{H_0^p},$$

and

$$h_n \in \overline{span}^{H^p}\left\{\frac{1}{1-e^{i\lambda}z}\right\}_{\lambda\in\mathbb{R}},$$

therefore, $\lim\limits_{n\to\infty} h_n = f_\nu \in H^p \cap \overline{H_0^p}$. The conclusion follows from the fact that $(H^p \cap \overline{H_0^p}) \cap H^{1+\varepsilon} = \{0\}$. □

Theorem 5.2. *Let (X_n) be a finitely supported admissible random walk on a non-elementary discrete subgroup $\Gamma \leq PSU(1,1)$. Then the hitting measure is singular with respect to the Lebesgue measure.*

Proof. [BHM11, Theorem 1.5(iii)] implies that if ν is absolutely continuous, then the density is in $L^\infty(S^1)$. In particular, it belongs to $L^{1+\varepsilon}(S^1)$, and the previous corollary shows that this is impossible. □

6 Open questions

- The question of whether the converse of Theorem 1.1 holds is slightly harder than one might anticipate. It is easy to see from (21) that the integral transform $\nu \mapsto p_\nu(z)$ is injective. In particular, we get that span $\left\{\xi \mapsto \log\left(\frac{1-|z|^2}{|z-\xi|^2}\right) : z \in \mathbb{D}\right\}$ is dense in $L^2(S^1, m)$, but we aim for a stronger statement – we want this span to be dense in $C(S^1)$. This would allow us to use the Riesz-Markov-Kakutani theorem in order to establish that ν is μ-stationary. Stone-Weierstrass theorem does not apply as the space of functions respect to which stationarity holds does not need to be a subalgebra of $C(S^1)$.

- It is easy to see from the proof of Corollary 1.1.1 that we actually get non-existence of solutions $f(z) = \sum a_{k+1} z^k$ to (27) with $\limsup_{n\to\infty} |a_k|^{1/k} < \varepsilon$ for some small ε, as only one preimage of the chosen contour explodes, so we can bound the radius of the convergence of the solution. Ideally, one would like to show that for finitely supported μ every solution of (27) has radius of convergence exactly 1. Keep in mind that this result would almost close the smoothness gap: it is known that absolutely continuous densities stationary with respect to finitely supported measures can belong to $C^n(S^1)$ for any $n > 1$.

  The Douglas-Shields-Shapiro theorem implies that any holomorphic function with the radius of convergence exceeding 1 is either cyclic with respect to the backward shift or rational. It is reasonable to assume that (27) only has rational solutions when μ is supported on a single element, and we conjecture that former never happens.

- The Brown-Shields-Zeller theorem has an unexpected consequence – it requires the poles to be non-tangentially dense **almost** everywhere on S^1. Therefore, even if Γ is a non-cocompact lattice, there will be a sequence $(a_\gamma) \in l^1(\Gamma)$ such that

  $$\sum \frac{a_\gamma}{z - \gamma.\infty} = 0, \quad |z| < 1.$$

  However, due to [GL90] we know that the Lebesgue measure is not stationary with respect to any μ with finite first moment. Therefore, either Theorem 1.1 is not a criterion, (a_γ) does not have the first finite moment, or there is a complex-valued Furstenberg measure – keep in mind that Guivarch'-le Jan's methods only apply for probability measures μ. Moreover, Brown-Shields-Zeller would also apply for any dense subgroup of $PSU(1,1)$, as any orbit which is dense in $\mathbb{D}$ will be non-tangentially dense, which is also quite surprising.

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