A Discrete Calculus

Peter Moore[1]
Birmingham University

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[1]Tel.: +44 (0)7866 725686 Email: pjm@worc0260.freeserve.co.uk

Abstract

This paper introduces the reader to a *'Discrete Calculus'* – a reformulation of present-day Calculus which allows the requirement to limit what is typically denoted by h, to be relaxed. Rather, this 'Discrete Calculus' is an environment where such an h is allowed to be a non-zero real.

Throughout this paper, Calculus, as it is traditionally defined, is referred to as *'Continuous Calculus'*, to save confusion.

The significance of *Discrete Derivatives* and *Discrete Integrals* is discussed.

A new notation is developed to be consistent with Continuous Calculus. It is shown that this new Calculus encompasses all that is described by Continuous Calculus, moreover, Continuous Derivatives and Continuous Integrals are shown to be a special case of this new type. In this sense, Discrete Calculus logically extends Continuous Calculus. Discrete Calculus describes a more general Calculus than Continuous Calculus, and so in this paper, standard results from Calculus are adapted to hold generally under the new definitions.

A more general and entirely original form of the Euler-Maclaurin sum is derived and studied.

Finally, there is an summary of further results, stated without proof.

It was intended to have proofs for all propositions and results, but sadly I was unable to complete this in time. However, many results are quite straightforward and do not really require justification.

This paper is dedicated to Alun Williams, the greatest of teachers, who inspired this work.

Contents

Chapter 1

Introduction

There are many problems in mathematics which are concerned with both evaluating and understanding discrete sums. It is my intention to introduce some mathematical tools to help investigate such problems, to save us from approaching each problem from first principles.

In the past, people have considered the application of Discrete Calculus to be limited to problems in computation, numerical methods and the study of sequences. I fervently believe that this is a serious underestimation. I hope to introduce the reader to some surprising results.

But first, let's start with an example.

1.1 The Chessboard problem

Consider a regular chessboard, with eight by eight squares. The question is, how many squares are there in total?

The answer is there are 204. Why is this?

Let's start by considering the biggest square present, the perimeter of the board. Clearly there is only one square present of this size.

Logically, the next size square to consider is the next largest, 7 by 7. By just considering the possible places where the bottom left corner of the square can go, we see that there are four such squares on the board.

With each iteration of square size, we see that the possible positions to put the bottom left corner of the considered square itself forms a square grid of points, the size of this square being $8 - r$ by $8 - r$.

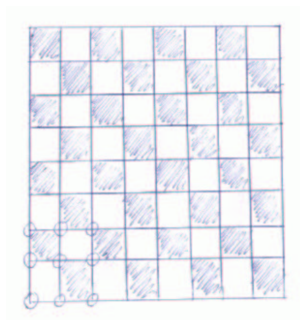

Figure 1.1: Where the bottom left corner can go for 6×6 squares.

We finish the iteration when we reach the fact that there are 64 one by one squares on the board, since no smaller squares exist.

Let's denote the total number of squares by n. We now know that

$$n = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 = \sum_{i=1}^{8} i^2 = 204 \tag{1.1}$$

And this is when we reach the crunch point! This quantity can be interpreted geometrically as the area represented in Figure 1.2.

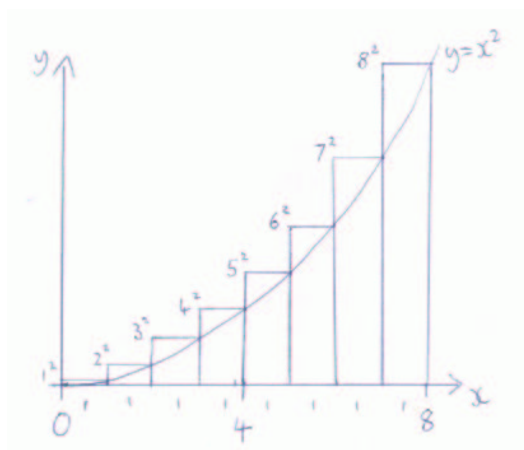

Figure 1.2: $y = x^2$ with right-sided strips

Since the width of each strip is 1, the total area is numerically equal to the sum of the heights of the strips, i.e. $\sum_{i=1}^{8} i^2 = 204$.

As can be seen, the considered area looks somewhat similar to the integral of x^2 evaluated between 0 and 8 (which is $170\frac{2}{3}$). However, the area of the Discrete integral is larger than the area of the *continuous* integral.

Take a look at Figure 1.3.

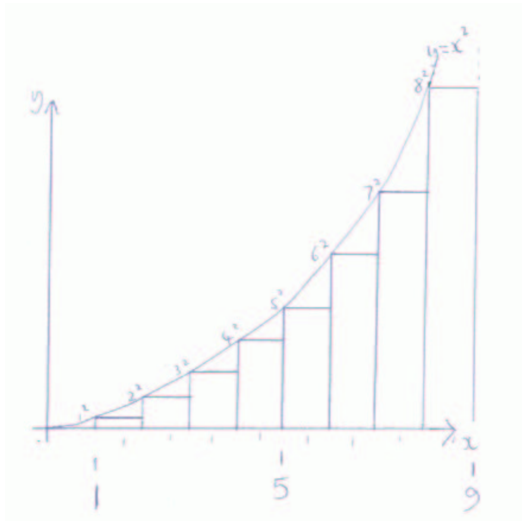

Figure 1.3: $y = x^2$ with left-sided strips

Here we again see that the area of the strips is equal to $\sum_{i=1}^{8} i^2 = 204$. However, this time we have an alternative construction. The curve is the same ($y = x^2$), however this time we have used left-sided strips, and our Discrete integral is now somewhat similar to the integral of x^2 evaluated between 1 and 9. Also note that this time the Discrete integral is numerically less than its continuous analogue (which is 243).

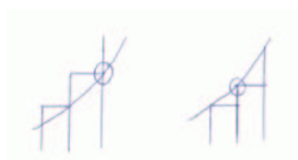

Figure 1.4: How left-sided and right-sided strips compare

Common sense says that we should now make a decision. We must surely choose which type of construction we wish to be concerned with, and ignore the other. But which do we choose?

Well perhaps it would make sense to choose neither, but instead consider a third type of integral, a centered one, as represented in Figure 1.5.

This time we have a more 'independent' construction, one which favors neither left nor right. This time the Discrete Integral is much closer in value to the Continuous Integral evaluated over the same range. The Continuous Integral of x^2 between $\frac{1}{2}$ and $8\frac{1}{2}$ is $204\frac{17}{24}$. So it seems that this is a much better 'unbiased' choice of construction.

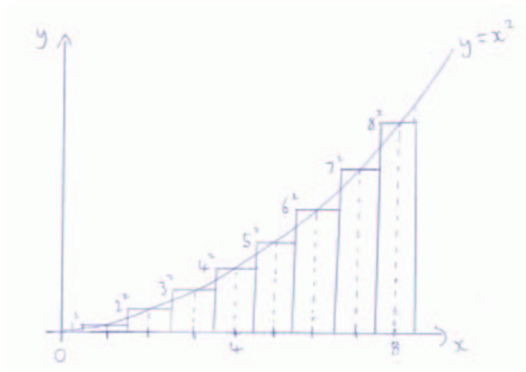

Figure 1.5: $y = x^2$ with centered strips

Well I am still not completely happy with this choice. In some scenarios it might be convenient to choose one of the former two constructions in preference to this 'centered' type. Well there is a solution which turns out to be very powerful. This is to have a *variable* type, one which allows for an arbitrary offset, including the three pre-mentioned types.

After many choices of notation, the most sensible choice seemed to be to slightly modify the notation of Continuous Calculus to contain two additional parameters (the width of the strip and the offset described above). In this notation, a left-sided integral is one where $u = -\frac{1}{2}$, centered when $u = 0$ and right-sided when $u = \frac{1}{2}$. This is because u represents the offset from the center of the strip in multiples of h, the width of the strips.

So the quantity we are calculating turns out to be precisely

$$\int_0^8 x^2 \, \mathrm{d}^1_{\frac{1}{2}} x$$

The $\mathrm{d}^1_{\frac{1}{2}} x$ tells us that we are integrating x with $u = \frac{1}{2}$ and $h = 1$.

It can be shown that

$$\int_a^b x^2 \, \mathrm{d}^h_u x = \left[\frac{1}{3}x^3 + uhx^2 + (u^2 - \frac{1}{12})h^2 x\right]_{x=a}^{x=b} \tag{1.2}$$

Here are a few of the trivial results that one should expect from this formula. This is by no means an exhaustive list of such results, merely a demonstration of a few.

$$\lim_{h\to 0}\left[\int_a^b x^2 \,\mathrm{d}_u^h x\right] = \int_a^b x^2 \,\mathrm{d}x \tag{1.3}$$

$$\int_0^8 x^2 \,\mathrm{d}_{\frac{1}{2}}^1 x = 204 \tag{1.4}$$

$$\int_0^8 x^2 \,\mathrm{d}_{\frac{1}{2}}^8 x = 8 \times 8^2 = 512 \tag{1.5}$$

As the width of the strips decreases, the area represented by the strips closer approximates the value of the integral. Hence (1.3). (1.4) follows from (1.1) and finally, (1.5) makes sense when you consider that this integral represents there being one strip of width 8 and with height $8^2 = 64$ since this is the value of the function at $x = 8$.

It is possible to rewrite (1.2) in a simpler form:

$$\begin{aligned}\int_a^b x^2 \,\mathrm{d}_u^h x &= \int_a^b (x+uh)^2 \,\mathrm{d}_0^h x \\ &= \left[\frac{1}{3}(x+uh)^3 - \frac{1}{12}h^2(x+uh)\right]_{x=a}^{x=b} \end{aligned} \tag{1.6}$$

This makes sense when you consider that if a function to be integrated is to be offset by uh, it does not have to be expressed by the $\mathrm{d}_u^h x$, but it could be offset in the function itself, and a centered integral ($\mathrm{d}_0^h x$) could be used instead.

Chapter 2

The Discrete Derivative

The Discrete Derivative is fundamental in the theory of Discrete Calculus. But before we introduce it, it helps to first introduce the *Shift Operator*. The Shift Operator is present in the definition of the Discrete Derivative, and will play a central part in simplifying many proofs involving the Discrete Derivative.

2.1 The Shift Operator

Definition 2.1 (The Shift Operator). *Consider a function $f : \mathbb{C} \to \mathbb{C}$, and a constant $\delta \in \mathbb{C}$. Then we define the* Shift Operator $\mathbf{E}^\delta$ *to act on $f(x)$ by*

$$\mathbf{E}^\delta f(x) \stackrel{\text{def}}{=} f(x+\delta) \quad ^\dagger$$

So, in simple terms, all that the $\mathbf{E}^\delta$ operator does is offset the input variable of the function that it operates on.

2.1.1 Properties of the Shifting Operator

Proposition 2.1. $\mathbf{E}^\delta$ *satisfies the following properties:*

(i) $\mathbf{E}^\delta \mathbf{E}^\varepsilon = \mathbf{E}^\varepsilon \mathbf{E}^\delta = \mathbf{E}^{\delta+\varepsilon}$

(ii) $\mathbf{E}^\delta(\mathbf{E}^\alpha + \mathbf{E}^\beta) = \mathbf{E}^\delta \mathbf{E}^\alpha + \mathbf{E}^\delta \mathbf{E}^\beta$

(iii) $\mathbf{E}^\delta\{\lambda f(x) + \mu g(x)\} = \lambda \mathbf{E}^\delta f(x) + \mu \mathbf{E}^\delta g(x)$

(iv) $(\mathbf{E}^\delta)^n = \mathbf{E}^{n\delta} \qquad (n \in \mathbb{N})$

†Here I use $\stackrel{\text{def}}{=}$ to mean *'equal to by definition'*.

Proof. All results follow directly from the definition. For example, for property (iii):

$$\begin{aligned}\mathbf{E}^{\delta}\{\lambda f(x)+\mu g(x)\} &= \lambda f(x+\delta)+\mu g(x+\delta)\\ &= \lambda\mathbf{E}^{\delta}f(x)+\mu\mathbf{E}^{\delta}g(x)\end{aligned}$$

□

The following property (Property 2.2) will simplify later proofs.

Proposition 2.2. *If f is differentiable, then*

$$\mathbf{E}^{\delta}\frac{\mathrm{d}}{\mathrm{d}x}f(x)=\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{E}^{\delta}f(x).$$

Proof.

$$\begin{aligned}\mathbf{E}^{\delta}\frac{\mathrm{d}}{\mathrm{d}x}f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}f(x+\delta)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{E}^{\delta}f(x)\end{aligned}$$

□

Now that we have proved all of the necessary properties of the Shifting Operator, we can move on and define the Discrete Derivative. We will use the information that we have gathered about the Shifting Operator to prove further properties of the Discrete Derivative.

2.2 The Discrete Derivative

Certain conditions need to be placed on any function that is to be discretely differentiated or integrated. For the sake of simplicity, I will only consider defining the Discrete Derivative on *analytic* functions. This does not mean that it is of little worth to consider other types of functions.

In the same way that the derivative in *Continuous* Calculus can be considered as an operator, so too can the Discrete Derivative.

Let us remind ourselves how the Continuous Derivative is defined on a function $f:\mathbb{C}\to\mathbb{C}$.

Consider a function $f:\mathbb{C}\to\mathbb{C}$ together with a constant $h\in\mathbb{C}$, $h\neq 0$. Define $g:\mathbb{C}^2\to\mathbb{C}$ by

$$\begin{aligned} g(x,h) &= \frac{f(x+h)-f(x)}{h} \\ &= \left(\frac{\mathbf{E}^h - \mathbf{E}^0}{h}\right) f(x). \end{aligned} \tag{2.1}$$

If $\lim_{h\to 0} g(x,h)$ exists (and is the same for all limiting sequences of h) then we call it the derivative of $f(x)$ with respect to x and write

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \stackrel{\text{def}}{=} \lim_{h\to 0} g(x,h). \tag{2.2}$$

To formulate Discrete Calculus we suspend this last stage of taking a limiting value of $g(x,h)$. Instead we leave h as an additional variable in the system. Consider defining the Discrete Derivative of $f(x)$ as $g(x,h)$.

We shall see that defining g by

$$\begin{aligned} g(x,h) &= \frac{f(x)-f(x-h)}{h} \\ &= \left(\frac{\mathbf{E}^0 - \mathbf{E}^{-h}}{h}\right) f(x) \end{aligned} \tag{2.3}$$

gives us exactly the same *continuous* derivative when the limit of $h \to 0$ is taken.

In fact it is generally true, as we shall see later, that the *continuous* derivative is unaffected if we define g as

$$\begin{aligned} g(x,r,h) &= \left(\frac{\mathbf{E}^{h-rh} - \mathbf{E}^{-rh}}{h}\right) f(x) \\ &= \left(\frac{\mathbf{E}^{(1-r)h} - \mathbf{E}^{-rh}}{h}\right) f(x) \end{aligned} \tag{2.4}$$

for any $r \in \mathbb{R}$.

So, for example, (2.1) is achieved by letting $r = 0$ and (2.3) is achieved by letting $r = 1$.

Using this more general definition, we have a whole family of Discrete Derivatives of $f(x)$ for all possible combinations of values for r and h. And as required, when a limiting value of $h \to 0$ is taken, $g(x,r,h)$ is equal to the continuous derivative at x.

There is one final adjustment to make. It makes sense to try to remove any bias that might exist in the offset. When $r = 0$ we see that we are taking a right-sided derivative, and when $r = 1$ we have a left-sided derivative. By using $u = r - \frac{1}{2}$ instead of r we will now have an unbiased variable u which when equal to 0 corresponds to a 'centered' derivative. Additionally, when $u = -\frac{1}{2}$ and $u = \frac{1}{2}$ the corresponding derivative is 'right-sided' and 'left-sided' respectively.

Definition 2.2 (The Discrete Derivative). *Consider an analytic function f, and two constants $u, h \in \mathbb{C}$. Then we define the* Discrete Derivative of $f(x)$ *by*

(i) $h \neq 0$

$$\frac{\mathrm{d}}{\mathrm{d}_u^h x} f(x) \stackrel{\text{def}}{=} \frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right) f(x)$$

(ii) $h = 0$. *If it exists,*

$$\frac{\mathrm{d}}{\mathrm{d}_u^0 x} f(x) \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{\mathrm{d}}{\mathrm{d}_u^h x} f(x)$$

There are some very important things to notice from the above definition:

(i) The Discrete Derivative of a function of one variable becomes a function of three variables (x, u, h).

(ii) For a given function f, the Discrete Derivative can easily be formulated. This is the same as in Continuous Calculus. We will later see that this is not generally the case for Discrete Integrals.

2.2.1 Properties of the Discrete Derivative

Proposition 2.3. *Discrete derivatives extend the traditional concept of a derivative — if a discrete derivative exists, then the 'continuous' derivative is equal to the discrete derivative with $h = 0$.*

Proof. Consider an analytic function f and a two constants $u, h \in \mathbb{C}, h \neq 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}_u^h x} f(x) = \frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right) f(x).$$

Since f is analytic, we can expand $f(x)$ using a *Taylor Expansion*. i.e. $\exists R \in \mathbb{R}, R > 0$ such that $\forall |h| < R$,

$$\begin{aligned} f(x - uh + \tfrac{1}{2}h) &= f(x - uh - \tfrac{1}{2}h) + hf'(x - uh - \tfrac{1}{2}h) \\ &\quad + \frac{h^2}{2!} f''(x - uh - \tfrac{1}{2}h) + \ldots \end{aligned}$$

Rearranging gives:

$$\frac{f(x-uh+\frac{1}{2}h)-f(x-uh-\frac{1}{2}h)}{h} = f'(x-uh-\tfrac{1}{2}h)+\frac{h}{2!}f''(x-uh-\tfrac{1}{2}h)+\frac{h^2}{3!}f'''(x-uh-\tfrac{1}{2}h)+\dots$$

Giving

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_u^0 x}f(x) &= \lim_{h\to 0}\left\{\frac{\mathrm{d}}{\mathrm{d}_u^h x}f(x)\right\}\\
&= \lim_{h\to 0}\left\{\frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h}-\mathbf{E}^{(-\frac{1}{2}-u)h}\right)f(x)\right\}\\
&= \lim_{h\to 0}\left\{\frac{f(x-uh+\frac{1}{2}h)-f(x-uh-\frac{1}{2}h)}{h}\right\}\\
&= \lim_{h\to 0}\left\{f'(x-uh-\tfrac{1}{2}h)+\frac{h}{2!}f''(x-uh-\tfrac{1}{2}h)\right.\\
&\qquad\left.+\frac{h^2}{3!}f'''(x-uh-\tfrac{1}{2}h)+\dots\right\}\\
&= f'(x)
\end{aligned}$$

□

Proposition 2.4.
$$\frac{\mathrm{d}}{\mathrm{d}_u^h x}f(x) = \mathbf{E}^{-uh}\frac{\mathrm{d}}{\mathrm{d}_0^h x}f(x)$$

This proposition tells us that so long as we know what the Discrete Derivative is when $u = 0$, we can calculate the Discrete Derivative for all other values of u. For example, with this freedom in mind, tables of derivatives only need to include the case $u = 0$.

Proof. Case $h = 0$ is trivial since the Discrete Derivative is the same for all values for u.

For $h \neq 0$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_u^h x}f(x) &= \frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h}-\mathbf{E}^{(-\frac{1}{2}-u)h}\right)f(x)\\
&= \mathbf{E}^{-uh}\frac{1}{h}\left(\mathbf{E}^{\frac{1}{2}h}-\mathbf{E}^{-\frac{1}{2}h}\right)f(x)\\
&= \mathbf{E}^{-uh}\frac{\mathrm{d}}{\mathrm{d}_0^h x}f(x)
\end{aligned}$$

□

Proposition 2.5 (Linearity of the Discrete Derivative). *Consider an analytic function $f : \mathbb{C} \to \mathbb{C}$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}_u^h x}\Big[\lambda f(x) + \mu g(x)\Big] = \lambda \frac{\mathrm{d}}{\mathrm{d}_u^h x} f(x) + \mu \frac{\mathrm{d}}{\mathrm{d}_u^h x} g(x).$$

Proof. There are two cases to consider:

(i) $h \neq 0$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_u^h x}\Big[\lambda f(x) + \mu g(x)\Big] &= \frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right)\Big[\lambda f(x) + \mu g(x)\Big] \\
&= \lambda \frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right) f(x) \\
&\quad + \mu \frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right) g(x) \\
&= \lambda \frac{\mathrm{d}}{\mathrm{d}_u^h x} f(x) + \mu \frac{\mathrm{d}}{\mathrm{d}_u^h x} g(x)
\end{aligned}$$

(ii) $h = 0$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_u^h x}\Big[\lambda f(x) + \mu g(x)\Big] &= \frac{\mathrm{d}}{\mathrm{d}x}\Big[\lambda f(x) + \mu g(x)\Big] \\
&= \lambda \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \mu \frac{\mathrm{d}}{\mathrm{d}x} g(x) \\
&= \lambda \frac{\mathrm{d}}{\mathrm{d}_u^h x} f(x) + \mu \frac{\mathrm{d}}{\mathrm{d}_u^h x} g(x)
\end{aligned}$$

$\square$

Proposition 2.6.

$$\frac{\mathrm{d}}{\mathrm{d}_u^h x} f(x) = \frac{\mathrm{d}}{\mathrm{d}_{-u}^{-h} x} f(x)$$

Proof. Consider an analytic function $f : \mathbb{C} \to \mathbb{C}$. There are two cases to consider:

(i) $h \neq 0$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_{-u}^{-h}x} f(x) &= \frac{1}{-h}\left(\mathbf{E}^{(\frac{1}{2}-(-u))(-h)} - \mathbf{E}^{(-\frac{1}{2}-(-u))(-h)}\right) f(x) \\
&= \frac{1}{-h}\left(\mathbf{E}^{(\frac{1}{2}+u)(-h)} - \mathbf{E}^{(-\frac{1}{2}+u)(-h)}\right) f(x) \\
&= \frac{1}{-h}\left(\mathbf{E}^{(-\frac{1}{2}-u)h} - \mathbf{E}^{(\frac{1}{2}-u)h}\right) f(x) \\
&= \frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right) f(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}_{u}^{h}x} f(x)
\end{aligned}
$$

(ii) $h = 0$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_{-u}^{0}x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x} f(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}_{u}^{0}x} f(x)
\end{aligned}
$$

□

Proposition 2.7. *Discrete Derivatives commute with each other.*

$$
\frac{\mathrm{d}}{\mathrm{d}_{u}^{h}x}\frac{\mathrm{d}}{\mathrm{d}_{v}^{j}x} f(x) = \frac{\mathrm{d}}{\mathrm{d}_{v}^{j}x}\frac{\mathrm{d}}{\mathrm{d}_{u}^{h}x} f(x)
$$

Proof. We can say without loss of generalization that there are three cases to consider:

(i) $h \neq 0, j \neq 0$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_{u}^{h}x}\frac{\mathrm{d}}{\mathrm{d}_{v}^{j}x} f(x) &= \frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right)\frac{1}{j}\left(\mathbf{E}^{(\frac{1}{2}-v)j} - \mathbf{E}^{(-\frac{1}{2}-v)j}\right) f(x) \\
&= \frac{1}{j}\left(\mathbf{E}^{(\frac{1}{2}-v)j} - \mathbf{E}^{(-\frac{1}{2}-v)j}\right)\frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right) f(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}_{v}^{j}x}\frac{\mathrm{d}}{\mathrm{d}_{u}^{h}x} f(x)
\end{aligned}
$$

(ii) $h \neq 0, j = 0$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_u^h x}\frac{\mathrm{d}}{\mathrm{d}_v^j x}f(x) &= \frac{\mathrm{d}}{\mathrm{d}_u^h x}\frac{\mathrm{d}}{\mathrm{d}x}f(x)\\
&= \frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right)\frac{\mathrm{d}}{\mathrm{d}x}f(x)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right)f(x)\right\}\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}_u^h x}f(x)\\
&= \frac{\mathrm{d}}{\mathrm{d}_v^j x}\frac{\mathrm{d}}{\mathrm{d}_u^h x}f(x)
\end{aligned}$$

(iii) $h = 0, j = 0$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_u^0 x}\frac{\mathrm{d}}{\mathrm{d}_v^0 x}f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}f(x)\\
&= \frac{\mathrm{d}}{\mathrm{d}_v^0 x}\frac{\mathrm{d}}{\mathrm{d}_u^0 x}f(x)
\end{aligned}$$

□

Proposition 2.8. *The Centered (i.e. $u = 0$) Discrete Derivative of an odd function is an even function, and vice versa.*

Proof. Follows directly from definition. □

2.3 Higher Order Derivatives

Just as in Continuous Calculus, it is of interest to study higher order derivatives. By this I mean functions which are produced from the repeated act of some differential operator.

Definition 2.3 (Higher Order Derivatives). *We define higher order derivatives inductively.*

$$\begin{aligned}
\frac{\mathrm{d}^1}{\mathrm{d}_u^h x^1}f(x) &\stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}_u^h x}f(x)\\
\textit{and}\quad \frac{\mathrm{d}^{i+1}}{\mathrm{d}_u^h x^{i+1}}f(x) &\stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}_u^h x}\left(\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i}\right) \quad \forall i \in \mathbb{N}
\end{aligned}$$

2.3.1 Properties of Higher Order Derivatives

Proposition 2.9 (Linearity of Higher Order Discrete Derivatives). *Consider two analytic functions $f, g : \mathbb{C} \to \mathbb{C}$, $\lambda, \mu \in \mathbb{C}$. Then $\forall i \in \mathbb{N}$*

$$\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i}\Big\{\lambda f(x) + \mu g(x)\Big\} = \lambda \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x) + \mu \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} g(x)$$

Proof. By induction on i.

Let $p(i)$ be the statement above about i.

Certainly $p(1)$ holds by Proposition 2.5.

By hypothesis,

$$\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i}\Big\{\lambda f(x) + \mu g(x)\Big\} = \lambda \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x) + \mu \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} g(x),$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}_u^h x}\Big\{\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i}\Big\{\lambda f(x) + \mu g(x)\Big\}\Big\} = \frac{\mathrm{d}}{\mathrm{d}_u^h x}\Big\{\lambda \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x) + \mu \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} g(x)\Big\}$$

giving

$$\frac{\mathrm{d}^{i+1}}{\mathrm{d}_u^h x^{i+1}}\Big\{\lambda f(x) + \mu g(x)\Big\} = \lambda \frac{\mathrm{d}^{i+1}}{\mathrm{d}_u^h x^{i+1}} f(x) + \mu \frac{\mathrm{d}^{i+1}}{\mathrm{d}_u^h x^{i+1}} g(x)$$

which completes the induction. $\square$

Proposition 2.10.

$$\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} = \frac{1}{h^i}\sum_{j=0}^{i}(-1)^j \binom{i}{j} \mathbf{E}^{(i(\frac{1}{2}-u)-j)h}$$

Proof. By induction on i.

Let $p(i)$ be the statement above about i.

Certainly $p(1)$ holds by Definition 2.3.

By hypothesis,

$$\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} = \frac{1}{h^i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\mathbf{E}^{(i(\frac{1}{2}-u)-j)h},$$

and so

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_u^h x}\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} &= \frac{\mathrm{d}}{\mathrm{d}_u^h x}\frac{1}{h^i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\mathbf{E}^{(i(\frac{1}{2}-u)-j)h} \\
&= \frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right)\frac{1}{h^i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\mathbf{E}^{(i(\frac{1}{2}-u)-j)h} \\
&= \frac{1}{h^{i+1}}\Bigg\{\Big(\sum_{j=0}^{i}(-1)^j\binom{i}{j}\mathbf{E}^{(i(\frac{1}{2}-u)-j)h+(\frac{1}{2}-u)h}\Big) \\
&\quad -\Big(\sum_{j=0}^{i}(-1)^j\binom{i}{j}\mathbf{E}^{(i(\frac{1}{2}-u)-j)h+(-\frac{1}{2}-u)h}\Big)\Bigg\} \\
&= \frac{1}{h^{i+1}}\Bigg\{\Big(\sum_{j=0}^{i}(-1)^j\binom{i}{j}\mathbf{E}^{-jh+(i+1)(\frac{1}{2}-u)h}\Big) \\
&\quad -\Big(\sum_{j=1}^{i+1}(-1)^{j-1}\binom{i}{j-1}\mathbf{E}^{-jh+(i+1)(\frac{1}{2}-u)h}\Big)\Bigg\} \\
&= \frac{1}{h^{i+1}}\Bigg\{\mathbf{E}^{(i+1)(\frac{1}{2}-u)h} + (-1)^{i+1}\mathbf{E}^{(i+1)(-\frac{1}{2}-u)h} \\
&\quad +\Big(\sum_{j=1}^{i}(-1)^j\left\{\binom{i}{j}+\binom{i}{j-1}\right\}\mathbf{E}^{((i+1)(\frac{1}{2}-u)-j)h}\Big)\Bigg\} \\
&= \frac{1}{h^{i+1}}\Bigg\{\mathbf{E}^{(i+1)(\frac{1}{2}-u)h} + (-1)^{i+1}\mathbf{E}^{(i+1)(-\frac{1}{2}-u)h} \\
&\quad +\Big(\sum_{j=1}^{i}(-1)^j\binom{i+1}{j}\mathbf{E}^{((i+1)(\frac{1}{2}-u)-j)h}\Big)\Bigg\}
\end{aligned}$$

giving

$$\frac{\mathrm{d}^{i+1}}{\mathrm{d}_u^h x^{i+1}} = \frac{1}{h^{i+1}}\sum_{j=0}^{i+1}(-1)^j\binom{i+1}{j}\mathbf{E}^{((i+1)(\frac{1}{2}-u)-j)h}$$

which completes the induction. □

Proposition 2.11. *Discrete Higher Order Derivatives logically extend Continuous Higher Order Derivatives, when they exist. Consider an analytic function $f : \mathbb{C} \to \mathbb{C}$. Then*

$$\forall i \in \mathbb{N}, \quad \frac{\mathrm{d}^i}{\mathrm{d}_u^0 x^i} f(x) = \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x).$$

Proof. By induction on i.

Let $p(i)$ be the statement above about i.

Certainly $p(1)$ holds by Proposition 2.3.

By hypothesis,

$$\frac{\mathrm{d}^i}{\mathrm{d}_u^0 x^i} f(x) = \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x)$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}_u^0 x}\left\{\frac{\mathrm{d}^i}{\mathrm{d}_u^0 x^i} f(x)\right\} = \frac{\mathrm{d}}{\mathrm{d}_u^0 x}\left\{\frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x)\right\}$$

giving

$$\frac{\mathrm{d}^{i+1}}{\mathrm{d}_u^0 x^{i+1}} f(x) = \frac{\mathrm{d}^{i+1}}{\mathrm{d}x^{i+1}} f(x)$$

which completes the induction. □

Proposition 2.12.

$$\forall i \in \mathbb{N}, \quad \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x) = \frac{\mathrm{d}^i}{\mathrm{d}_{-u}^{-h} x^i} f(x)$$

Proof. By induction on i.

Let $p(i)$ be the statement above about i.

Certainly $p(1)$ holds by Proposition 2.6.

By hypothesis,

$$\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x) = \frac{\mathrm{d}^i}{\mathrm{d}_{-u}^{-h} x^i} f(x)$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}_u^h x}\left\{\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x)\right\} = \frac{\mathrm{d}}{\mathrm{d}_u^h x}\left\{\frac{\mathrm{d}^i}{\mathrm{d}_{-u}^{-h} x^i} f(x)\right\}$$

giving

$$\begin{aligned}\frac{\mathrm{d}^{i+1}}{\mathrm{d}_u^h x^{i+1}} f(x) &= \frac{\mathrm{d}}{\mathrm{d}_{-u}^{-h} x}\left\{\frac{\mathrm{d}^i}{\mathrm{d}_{-u}^{-h} x^i} f(x)\right\} \\ &= \frac{\mathrm{d}^{i+1}}{\mathrm{d}_{-u}^{-h} x^{i+1}} f(x)\end{aligned}$$

which completes the induction. □

Proposition 2.13. *First order Discrete Derivatives commute with arbitrary order Discrete Derivatives.*

$$\forall i \in \mathbb{N}, \quad \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} \frac{\mathrm{d}}{\mathrm{d}_v^j x} f(x) = \frac{\mathrm{d}}{\mathrm{d}_v^j x} \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x)$$

Proof. By induction on i.

Let $p(i)$ be the statement above about i.

Certainly $p(1)$ holds by Proposition 2.7.

By hypothesis,

$$\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} \frac{\mathrm{d}}{\mathrm{d}_v^j x} f(x) = \frac{\mathrm{d}}{\mathrm{d}_v^j x} \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x)$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}_u^h x}\left\{\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} \frac{\mathrm{d}}{\mathrm{d}_v^j x} f(x)\right\} = \frac{\mathrm{d}}{\mathrm{d}_u^h x}\left\{\frac{\mathrm{d}}{\mathrm{d}_v^j x} \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x)\right\}$$

giving

$$\begin{aligned}\frac{\mathrm{d}^{i+1}}{\mathrm{d}_u^h x^{i+1}} \frac{\mathrm{d}}{\mathrm{d}_v^j x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}_v^j x} \frac{\mathrm{d}}{\mathrm{d}_u^h x} \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x) \\ &= \frac{\mathrm{d}}{\mathrm{d}_v^j x} \frac{\mathrm{d}^{i+1}}{\mathrm{d}_u^h x^{i+1}} f(x)\end{aligned}$$

which completes the induction. □

We can supersede this proposition now with a more general one.

Proposition 2.14. *Arbitrary order Discrete Derivatives commute with other arbitrary order Discrete Derivatives.*

$$\forall i, k \in \mathbb{N}, \quad \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i}\frac{\mathrm{d}^k}{\mathrm{d}_v^j x^k} f(x) = \frac{\mathrm{d}^k}{\mathrm{d}_v^j x^k}\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x)$$

Proof. By induction on i.

Let $p(i)$ be the statement above about i.

Certainly $p(1)$ holds by Proposition 2.13.

By hypothesis,

$$\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i}\frac{\mathrm{d}^k}{\mathrm{d}_v^j x^k} f(x) = \frac{\mathrm{d}^k}{\mathrm{d}_v^j x^k}\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x)$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}_u^h x}\left\{\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i}\frac{\mathrm{d}^k}{\mathrm{d}_v^j x^k} f(x)\right\} = \frac{\mathrm{d}}{\mathrm{d}_u^h x}\left\{\frac{\mathrm{d}^k}{\mathrm{d}_v^j x^k}\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x)\right\}$$

giving

$$\begin{aligned} \frac{\mathrm{d}^{i+1}}{\mathrm{d}_u^h x^{i+1}}\frac{\mathrm{d}^k}{\mathrm{d}_v^j x^k} f(x) &= \frac{\mathrm{d}^k}{\mathrm{d}_v^j x^k}\frac{\mathrm{d}}{\mathrm{d}_u^h x}\frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x) \\ &= \frac{\mathrm{d}^k}{\mathrm{d}_v^j x^k}\frac{\mathrm{d}^{i+1}}{\mathrm{d}_u^h x^{i+1}} f(x) \end{aligned}$$

which completes the induction. □

2.4 Theorems involving the Discrete Derivative

There are a few theorems that can now be introduced.

Theorem 2.1 (Expanding a function in terms of its Discrete Derivatives). *Consider an analytic function f and a constant $h \in \mathbb{R}$. Then if a is a multiple of h,*

(i)

$$\begin{aligned} f(x+a) &= f(x) + a\frac{\mathrm{d}}{\mathrm{d}^h_{-\frac{1}{2}}x}f(x) + \frac{a(a-h)}{2!}\frac{\mathrm{d}^2}{\mathrm{d}^h_{-\frac{1}{2}}x^2}f(x) \\ &\quad + \frac{a(a-h)(a-2h)}{3!}\frac{\mathrm{d}^3}{\mathrm{d}^h_{-\frac{1}{2}}x^3}f(x) + \dots \\ &= f(x) + \sum_{i=0}^{\infty}\left(\prod_{j=0}^{i}\frac{a-jh}{j+1}\right)\frac{\mathrm{d}^{i+1}}{\mathrm{d}^h_{-\frac{1}{2}}x^{i+1}}f(x) \end{aligned}$$

(ii)

$$\begin{aligned} f(x+a) &= f(x) + a\frac{\mathrm{d}}{\mathrm{d}^h_{\frac{1}{2}}x}f(x) + \frac{a(a+h)}{2!}\frac{\mathrm{d}^2}{\mathrm{d}^h_{\frac{1}{2}}x^2}f(x) \\ &\quad + \frac{a(a+h)(a+2h)}{3!}\frac{\mathrm{d}^3}{\mathrm{d}^h_{\frac{1}{2}}x^3}f(x) + \dots \\ &= f(x) + \sum_{i=0}^{\infty}\left(\prod_{j=0}^{i}\frac{a+jh}{j+1}\right)\frac{\mathrm{d}^{i+1}}{\mathrm{d}^h_{\frac{1}{2}}x^{i+1}}f(x) \end{aligned}$$

Proof. For $n \in \mathbb{N}, h \in \mathbb{C}, h \neq 0$,

$$\begin{aligned} \mathbf{E}^{nh}f(x) &= (1 + \{\mathbf{E}^h - 1\})^n f(x) \\ &= (1 + h\frac{\mathrm{d}}{\mathrm{d}^h_{-\frac{1}{2}}x})^n f(x) \\ &= \Big\{1 + nh\frac{\mathrm{d}}{\mathrm{d}^h_{-\frac{1}{2}}x} + \frac{n(n-1)}{2!}h^2\frac{\mathrm{d}^2}{\mathrm{d}^h_{-\frac{1}{2}}x^2} \\ &\quad + \frac{n(n-1)(n-2)}{3!}h^3\frac{\mathrm{d}^3}{\mathrm{d}^h_{-\frac{1}{2}}x^3} + \dots\Big\}f(x) \end{aligned}$$

introducing a change of variable $a = nh$,

$$\begin{aligned} \mathbf{E}^{a}f(x) &= \Big\{1 + a\frac{\mathrm{d}}{\mathrm{d}^h_{-\frac{1}{2}}x} + \frac{a(a-h)}{2!}\frac{\mathrm{d}^2}{\mathrm{d}^h_{-\frac{1}{2}}x^2} \\ &\quad + \frac{a(a-h)(a-2h)}{3!}\frac{\mathrm{d}^3}{\mathrm{d}^h_{-\frac{1}{2}}x^3} + \dots\Big\}f(x) \end{aligned}$$

which gives (i). To get (ii), send h to $-h$, and make use of Proposition 2.12. Then send a to $-a$ to get the final result. $\square$

Conjecture 2.1. *Theorem 2.1 holds for all $a \in \mathbb{C}$.*

It seems reasonable to imagine that this should be the case, especially as f is analytic, which imposes many constraints, but sadly I have not managed to prove this.

There are some interesting things to note from Theorem 2.1:

(i) The Taylor Expansion expands a function about a point in terms of its Continuous Derivatives, whereas the above expansions expand a function in terms of its *more general* Discrete Derivatives. The equations additionally hold for $h = 0$, since they then simply express Taylor's Theorem.

(ii) We see the special values of $u = -\frac{1}{2}$ and $u = \frac{1}{2}$ in the derivatives above. This corresponds to 'right-sided' and 'left-sided' derivatives respectively. This is in contrast to the Discrete Integral whereby the values refer to 'left-sided' and 'right-sided' integrals respectively.

Theorem 2.2 (Expressing the Discrete Derivative of a function in terms of the function's continuous derivatives). *Any Discrete Derivative of an analytic function can be expanded in terms of the Continuous Derivatives of the same function.*

Proof. Consider an analytic function f.

$$\begin{aligned}\mathbf{E}^{(\frac{1}{2}-u)h} f(x) &= f(x + (\tfrac{1}{2} - u)h) \\ &= f(x) + (\tfrac{1}{2} - u)h \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{(\frac{1}{2} - u)^2 h^2}{2!} \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) \\ &\quad + \frac{(\frac{1}{2} - u)^3 h^3}{3!} \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) + \ldots\end{aligned}$$

by Taylor Expansion. Similarly

$$\begin{aligned}\mathbf{E}^{(-\frac{1}{2}-u)h} f(x) &= f(x + (-\tfrac{1}{2} - u)h) \\ &= f(x) + (-\tfrac{1}{2} - u)h \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{(-\frac{1}{2} - u)^2 h^2}{2!} \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) \\ &\quad + \frac{(-\frac{1}{2} - u)^3 h^3}{3!} \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) + \ldots\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_u^h x} f(x) &= \frac{1}{h}\left(\mathbf{E}^{(\frac{1}{2}-u)h} - \mathbf{E}^{(-\frac{1}{2}-u)h}\right) f(x) \\
&= \left[(\tfrac{1}{2}-u) - (-\tfrac{1}{2}-u)\right] \frac{\mathrm{d}}{\mathrm{d}x} f(x) \\
&\quad + \frac{h}{2!}\left[(\tfrac{1}{2}-u)^2 - (-\tfrac{1}{2}-u)^2\right] \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) \\
&\quad + \frac{h^2}{3!}\left[(\tfrac{1}{2}-u)^3 - (-\tfrac{1}{2}-u)^3\right] \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) + \ldots \\
&= \frac{\mathrm{d}}{\mathrm{d}_0^1 u}(-u) \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{h}{2!} \frac{\mathrm{d}}{\mathrm{d}_0^1 u}(-u)^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) \\
&\quad + \frac{h^2}{3!} \frac{\mathrm{d}}{\mathrm{d}_0^1 u}(-u)^3 \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) + \ldots \\
&= \sum_{i=1}^{\infty} \frac{h^{i-1}}{i!} \frac{\mathrm{d}}{\mathrm{d}_0^1 u}(-u)^i \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x)
\end{aligned}$$

□

This expression simplifies considerably for the three special cases of $u \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$. Here are the expansions for these cases:

(i) $u = -\frac{1}{2}$

$$\frac{\mathrm{d}}{\mathrm{d}_{-\frac{1}{2}}^h x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{h}{2!} \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) + \frac{h^2}{3!} \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) + \frac{h^3}{4!} \frac{\mathrm{d}^4}{\mathrm{d}x^4} f(x) + \ldots$$

(ii) $u = 0$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}_0^h x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x} f(x) + \frac{1}{3!}\left(\frac{h}{2}\right)^2 \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) + \frac{1}{5!}\left(\frac{h}{2}\right)^4 \frac{\mathrm{d}^5}{\mathrm{d}x^5} f(x) \\
&\quad + \frac{1}{7!}\left(\frac{h}{2}\right)^6 \frac{\mathrm{d}^7}{\mathrm{d}x^7} f(x) + \ldots
\end{aligned}$$

(iii) $u = \frac{1}{2}$

$$\frac{\mathrm{d}}{\mathrm{d}_{\frac{1}{2}}^h x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) - \frac{h}{2!} \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) + \frac{h^2}{3!} \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) - \frac{h^3}{4!} \frac{\mathrm{d}^4}{\mathrm{d}x^4} f(x) + \ldots$$

That just about rounds up all we need to say about the Discrete Derivative for now. To finish up, we shall take a look at some common Discrete Derivatives.

2.5 Some Standard Discrete Derivatives

We have talked about Discrete Derivatives in some depth now, and so here I shall introduce a few common ones. Please note that in the following tables

I have only tabulated the Discrete Derivative with $u = 0$. This is all that is needed since the more general derivative can be found by substituting x with $x - uh$ (see Proposition 2.4).

$f(x)$	$\frac{\mathrm{d}}{\mathrm{d}_0^h x} f(x)$
$c \in \mathbb{C}$	0
x	1
x^2	$2x$
x^3	$3x^2 + \frac{1}{4}h^2$
x^4	$4x^3 + xh^2$
x^5	$5x^4 + \frac{5}{2}x^2h^2 + \frac{1}{16}h^4$
x^6	$6x^5 + 5x^3h^2 + \frac{3}{8}xh^4$
a^x	$\frac{\sinh\{(h/2)\ln a\}}{(h/2)}\, a^x$

$f(x)$	$\frac{\mathrm{d}}{\mathrm{d}_0^h x} f(x)$
e^x	$\frac{\sinh(h/2)}{(h/2)}\, \mathrm{e}^x$
e^{ix}	$\frac{i\sin(h/2)}{(h/2)}\, \mathrm{e}^{ix}$
$\cos x$	$\frac{-\sin(h/2)}{(h/2)}\, \sin x$
$\sin x$	$\frac{\sin(h/2)}{(h/2)}\, \cos x$
$\tan x$	$\frac{\tan(h/2)}{\frac{h}{2}\left(1 - \frac{\sin^2 x}{\cos^2(h/2)}\right)}$
$\ln x$	$\frac{1}{h}\ln\left(1 + \frac{2h}{2x - h}\right)$

Chapter 3

The Discrete Integral

3.1 The Partial Discrete Derivative

Before introducing the Discrete Integral, we must first define a certain type of *Partial Discrete Derivative* which will play a fundamental role in defining the Discrete Integral.

Definition 3.1. *Consider a function of three complex variables $f(x, u, h)$ which for any fixed pair u, h is analytic on x. Then the* Partial Discrete Derivative *of $f(x)$ is defined as*

$$\frac{\partial}{\partial_u^h x} f(x, u, h) \stackrel{\text{def}}{=} \frac{f(x + (\frac{1}{2} - u)h, u, h) - f(x + (-\frac{1}{2} - u)h, u, h)}{h}.$$

This does not define the *Partial Discrete Derivative* in general.

3.2 The Indefinite Discrete Integral

Definition 3.2. *If we have a function $g(x)$ such that*

$$g(x) = \frac{\partial}{\partial_u^h x} f(x, u, h)$$

for some function $f(x, u, h)$ then we call $f(x, u, h)$ an Indefinite Discrete Integral *of $g(x)$ and write*

$$\int g(x)\,\mathrm{d}_u^h x \stackrel{\text{def}}{=} f(x,u,h).$$

3.2.1 Properties of the Indefinite Discrete Integral

Proposition 3.1. *If f_1 and f_2 are both Indefinite Discrete Integrals of a function g, then $f_1 - f_2$ is a function of u and h only.*

Proposition 3.2. *All analytic functions have Indefinite Discrete Integrals.*

Proposition 3.3.

$$\int g(x)\,\mathrm{d}_u^0 x = \int g(x)\,\mathrm{d}x$$

Proof. For the Discrete Integral to exist, there must be some function f which the Discrete Integral is equal to, such that

$$g(x) = \frac{\partial}{\partial_u^0 x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x).$$

Therefore

$$f(x) = \int g(x)\,\mathrm{d}_u^0 x = \int g(x)\,\mathrm{d}x$$

□

Proposition 3.4.

$$\int f(x)\,\mathrm{d}_u^h x = \int f(x+uh)\,\mathrm{d}_0^h x$$

Proposition 3.5.

$$\int f(x)\,\mathrm{d}_u^h x = \int f(x)\,\mathrm{d}_{-u}^{-h} x$$

Proposition 3.6.

$$\frac{\mathrm{d}}{\mathrm{d}_u^h x} \int f(x)\,\mathrm{d}_u^h x = f(x)$$

Proposition 3.7.

$$\int \lambda f(x) + \mu g(x)\,\mathrm{d}_u^h x = \lambda \int f(x)\,\mathrm{d}_u^h x + \mu \int g(x)\,\mathrm{d}_u^h x$$

Proposition 3.8.

$$\int \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} f(x)\,\mathrm{d}_u^h x = \frac{\mathrm{d}^i}{\mathrm{d}_u^h x^i} \int f(x)\,\mathrm{d}_u^h x + A(u,h)$$

Proposition 3.9. *The Centered (i.e. with $u = 0$) Discrete Integral of an odd function is an even function, and vice versa.*

Proof. Follows from Proposition 2.8. $\square$

Proposition 3.10.

$$\int f(\alpha x)\,\mathrm{d}_u^h x \quad = \quad \frac{1}{\alpha}\int f(x)\,\mathrm{d}_u^{\alpha h}(\alpha x)$$

Proposition 3.11. *Given that (x_1, u_1, h_1) are independent of each other, and (x_2, u_2, h_2) are independent of each other, then*

$$\int\int f(x_2)\,\mathrm{d}_{u_2}^{h_2}x_2\,\mathrm{d}_{u_1}^{h_1}x_1 = \int\int f(x_1)\,\mathrm{d}_{u_1}^{h_1}x_1\,\mathrm{d}_{u_2}^{h_2}x_2$$

even if $(x_1, u_1, h_1, x_2, u_2, h_2)$ are not independent of each other in general.

3.3 The Definite Discrete Integral

Definition 3.3. *If a function has an Indefinite Discrete Integral, then the Definite Discrete Integral of the function between a and b is defined as*

$$\int_a^b g(x)\,\mathrm{d}_u^h x \stackrel{\mathrm{def}}{=} f(b,u,h) - f(a,u,h).$$

3.3.1 Properties of the Definite Discrete Integral

Proposition 3.12. *The Definite Discrete Integral of a function is* unique.

Proposition 3.13. *Consider a complex function g which has a Discrete Integral, together with constants $a, b, u, h \in \mathbb{C}, h \neq 0$, such that $b - a = nh$ for some $n \in \mathbb{Z}$. Then*

$$\int_a^b g(x)\,\mathrm{d}_u^h x = \begin{cases} 0 & \frac{b-a}{h} = 0 \\ h\sum_{i=1}^{\frac{b-a}{h}} f(a + (u + i - \frac{1}{2})h) & \frac{b-a}{h} > 0 \\ -h\sum_{i=1}^{\frac{a-b}{h}} f(b + (u + i - \frac{1}{2})h) & \frac{b-a}{h} < 0 \end{cases}$$

Proposition 3.14.

$$\int_a^b f(x)\,\mathrm{d}_u^h x = -\int_b^a f(x)\,\mathrm{d}_u^h x$$

Proposition 3.15.

$$\int_a^b f(x)\,\mathrm{d}_u^h x + \int_b^c f(x)\,\mathrm{d}_u^h x = \int_a^c f(x)\,\mathrm{d}_u^h x$$

Proposition 3.16.

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}_u^h x} f(x)\,\mathrm{d}_u^h x = f(b) - f(a)$$

3.4 Theorems involving the Discrete Integral

Theorem 3.1.

$$\int f(x)\,\mathrm{d}_{un}^{h/n} x = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{E}^{ih} \int f(x)\,\mathrm{d}_u^h x\,\mathrm{d}_0^{1/n} i$$

For example, if we consider $n = 3, u = 0$ we get

$$\begin{aligned}
\int f(x)\,\mathrm{d}_0^{h/3} x &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{E}^{ih} \int f(x)\,\mathrm{d}_0^h x\,\mathrm{d}_0^{1/3} i \\
&= \mathbf{E}^{-\frac{1}{3}h} \int f(x)\,\mathrm{d}_0^h x + \mathbf{E}^0 \int f(x)\,\mathrm{d}_0^h x + \mathbf{E}^{\frac{1}{3}h} \int f(x)\,\mathrm{d}_0^h x \\
&= (\mathbf{E}^{-\frac{1}{3}h} + \mathbf{E}^0 + \mathbf{E}^{\frac{1}{3}h}) \int f(x)\,\mathrm{d}_0^h x
\end{aligned}$$

3.5 Some Standard Discrete Integrals

It is only necessary to tabulate Discrete Integrals with $u = 0$. The more general derivative can be found by substituting x with $x + uh$ (see Proposition 3.4).

$f(x)$	$\int f(x)\,\mathrm{d}_0^h x$
$c \in \mathbb{C}$	cx
x	$\frac{1}{2}x^2$
x^2	$\frac{1}{3}x^3 - \frac{1}{12}h^2 x$
x^3	$\frac{1}{4}x^4 - \frac{1}{8}h^2 x^2$
x^4	$\frac{1}{5}x^5 - \frac{1}{6}h^2 x^3 + \frac{7}{240}h^4 x$
x^5	$\frac{1}{6}x^6 - \frac{5}{24}h^2 x^4 + \frac{7}{96}h^4 x^2$
x^6	$\frac{1}{7}x^7 - \frac{1}{4}h^2 x^5 + \frac{7}{48}h^4 x^3$
	$-\frac{31}{1344}h^6 x$

$f(x)$	$\int f(x)\,\mathrm{d}_0^h x$
a^x	$\dfrac{(h/2)}{\sinh\{(h/2)\ln a\}}\ a^x$
e^x	$\dfrac{(h/2)}{\sinh(h/2)}\ \mathrm{e}^x$
e^{ix}	$\dfrac{(h/2)}{i\sin(h/2)}\ \mathrm{e}^{ix}$
$\cos x$	$\dfrac{(h/2)}{\sin(h/2)}\ \sin x$
$\sin x$	$-\dfrac{(h/2)}{\sin(h/2)}\ \cos x$
$\cos^2 x$	$\dfrac{x}{2} + \dfrac{h}{\sin h}\cdot\dfrac{\sin 2x}{4}$
$\sin^2 x$	$\dfrac{x}{2} - \dfrac{h}{\sin h}\cdot\dfrac{\sin 2x}{4}$

Chapter 4

The Discrete Integral Expansion Formula

The *Discrete Integral Expansion Formula* is a more general form of the *Euler-Maclaurin Sum Formula*. The Euler-Maclaurin sum formula is essentially a formula for summing a function over the domain $\{1, 2, \ldots, n\}$.

The Discrete Integral Expansion Formula gives an expansion for an arbitrary Discrete Integral, i.e. where the sum domain is of the form

$$\{a + uh + \tfrac{h}{2}, a + uh + \tfrac{3h}{2}, a + uh + \tfrac{5h}{2}, \ldots, b + uh - \tfrac{3h}{2}, b + uh - \tfrac{h}{2}\}$$

which gives rise to significant differences between the Discrete Integral Expansion Formula and the Euler-Maclaurin Formula.

4.1 The Euler-Maclaurin Sum Formula

Consider an analytic function f. Then the *Euler-Maclaurin Sum Formula* states that

$$\begin{aligned}\sum_{i=1}^{n} f(i) &= \left[\int f(x)\,\mathrm{d}x + \tfrac{1}{2} f(x) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x)\right]_{x=0}^{x=n} \\ &\quad + n\frac{B_{2m+2} f^{(2m+2)}(\xi)}{(2m+2)!} \qquad (4.1)\end{aligned}$$

for some $\xi \in (0, n)$, where B_n is the n^{th} Bernoulli number (see Appendix B).

4.2 The Discrete Integral Expansion Formula

Consider a function $f(x,u,h) f : \mathbb{C}^3 \to \mathbb{C}$ which is analytic in all three variables. Then

$$\begin{aligned}
\frac{\partial}{\partial u} f(x+uh) &= hf'(x+uh) = \frac{\partial}{\partial x}(hf(x+uh)) \\
&= \frac{\partial}{\partial x}\left\{\frac{\mathrm{d}}{\mathrm{d}_u^h x}\int hf(x+uh)\,\mathrm{d}_u^h x\right\} \\
&= \frac{\partial}{\partial x}\left\{\frac{1}{h}\left[\int hf(x+\tfrac{1}{2}h)\,\mathrm{d}_u^h x - \int hf(x-\tfrac{1}{2}h)\,\mathrm{d}_u^h x\right]\right\} \\
&= \frac{\partial}{\partial x}\left\{\int f(x)\,\mathrm{d}_{u+\frac{1}{2}}^h x - \int f(x)\,\mathrm{d}_{u-\frac{1}{2}}^h x\right\} \\
&= \frac{\partial}{\partial_0^1 u}\left\{\frac{\partial}{\partial x}\left\{\int f(x)\,\mathrm{d}_u^h x\right\}\right\}
\end{aligned}$$

Discretely integrating gives

$$\frac{\partial}{\partial x}\left\{\int f(x)\,\mathrm{d}_u^h x\right\} \equiv \frac{\partial}{\partial u}\left\{\int f(x+uh)\,\mathrm{d}_0^1 u\right\} + C \tag{4.2}$$

for some constant C.

Normally we would have an arbitrary function of two variables instead of a constant. However, since we are integrating with pre-decided values of 0 and 1 $(\mathrm{d}_0^1 u)$, it is only a constant, and as such is independent of x, u, h.

Since this constant is independent of h, let us consider (4.2) for the case $h = 0$.

We get

$$\frac{\partial}{\partial x}\left\{\int f(x)\,\mathrm{d}x\right\} = \frac{\partial}{\partial u}\left\{\int f(x)\,\mathrm{d}_0^1 u\right\} + C,$$

giving

$$\begin{aligned}
f(x) &= f(x)\frac{\partial}{\partial u}(u+c_1) + C \\
&= f(x) + C.
\end{aligned}$$

Thus $C \equiv 0$. Hence we can rewrite (4.2) as

$$\frac{\partial}{\partial x}\left\{\int f(x)\,\mathrm{d}_u^h x\right\} \equiv \frac{\partial}{\partial u}\left\{\int f(x+uh)\,\mathrm{d}_0^1 u\right\} \tag{4.3}$$

Although this does not look like the Euler-Maclaurin Expansion, it can be manipulated to be of the same form. However, this version of the formula is the simplest and perhaps the more elegant.

Before continuing, let us consider other virtues of this identity.

Essentially we have something of the form

$$\frac{\partial}{\partial x} f_1(x,u) \equiv \frac{\partial}{\partial u} f_2(x,u). \tag{4.4}$$

Had it also been the case that

$$\frac{\partial}{\partial u} f_1(x,u) \equiv -\frac{\partial}{\partial x} f_2(x,u), \tag{4.5}$$

then we would have had a spectacular result. We would have been able to engineer the function

$$f_1(x,u) + i f_2(x,u)$$

which would have been analytic on $x + iu$. Sadly though, this identity does not hold. Maybe a variation does. It is certainly something worth investigating!

Let's now go back to considering (4.3).

Integrating both sides with respect to x we get

$$
\begin{aligned}
\int f(x)\,\mathrm{d}_u^h x &\equiv \int \frac{\partial}{\partial u}\left\{\int f(x+uh)\,\mathrm{d}_0^1 u\right\}\mathrm{d}x \\
&\equiv \int \frac{\partial}{\partial u}\left\{\int\left\{f(x)+(uh)f'(x)+\frac{(uh)^2}{2!}f''(x)+\ldots\right\}\mathrm{d}_0^1 u\right\}\mathrm{d}x \\
&\equiv \int \frac{\partial}{\partial u}\left\{\int\left\{f(x)+(uh)f'(x)+\frac{(uh)^2}{2!}f''(x)+\ldots\right\}\mathrm{d}x\right\}\mathrm{d}_0^1 u \\
&\equiv \frac{\partial}{\partial u}\left\{\left(\int 1\,\mathrm{d}_0^1 u\right)\int f(x)\,\mathrm{d}x+\left(\int u\,\mathrm{d}_0^1 u\right)hf(x)\right. \\
&\quad \left.+\left(\int u^2\,\mathrm{d}_0^1 u\right)\frac{h^2}{2!}f'(x)+\left(\int u^3\,\mathrm{d}_0^1 u\right)\frac{h^3}{3!}f''(x)+\ldots\right\} \\
&\equiv \int f(x)\,\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}u}\left(\int u\,\mathrm{d}_0^1 u\right)hf(x)+\frac{\mathrm{d}}{\mathrm{d}u}\left(\int u^2\,\mathrm{d}_0^1 u\right)\frac{h^2}{2!}f'(x) \\
&\quad +\frac{\mathrm{d}}{\mathrm{d}u}\left(\int u^3\,\mathrm{d}_0^1 u\right)\frac{h^3}{3!}f''(x)+\frac{\mathrm{d}}{\mathrm{d}u}\left(\int u^4\,\mathrm{d}_0^1 u\right)\frac{h^4}{4!}f'''(x)
\end{aligned}
$$

We can write this more elegantly as

$$
\begin{aligned}
\int f(x)\,\mathrm{d}_u^h x &\equiv \int f(x)\,\mathrm{d}x+e_1(u)hf(x)+e_2(u)h^2f'(x) \\
&\quad +e_3(u)h^3f''(x)+\ldots, \qquad (4.6)
\end{aligned}
$$

$$
\text{where } e_n(u) = \frac{1}{n!}\frac{\mathrm{d}}{\mathrm{d}u}\left(\int u^n\,\mathrm{d}_0^1 u\right). \qquad (4.7)
$$

The Euler-Maclaurin Formula can be deduced from (4.6) by letting $u=\frac{1}{2}, h=1$ and evaluating the whole expression between values 0 and n.

Please see Appendix A to see the first few terms of this expression. Appendix B lists the first few $e_i(u)$ polynomials.

4.3 Early Results

Proposition 4.1.

$$
e_i(u) = \frac{B_n(x-\frac{1}{2})}{n!}
$$

where B_n is the n^{th} Bernoulli Polynomial.

Proposition 4.2.

$$
\zeta(x) = \int_0^\infty w^{-x}\,\mathrm{d}_{\frac{1}{2}}^1 w
$$

where $\zeta(x)$ *is the* Riemann Zeta Function.

Proposition 4.3. *If* p *is rational, then so too is* $e_i(p)$.

And here is a list of further results ...

$$e_i(0) = 0 \quad \forall i \in \{1,3,5,7,\ldots\} \tag{4.8}$$

$$e_i(\tfrac{1}{2}) = 0 \quad \forall i \in \{3,5,7,9,\ldots\} \tag{4.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}u} e_n(u) = e_{n-1}(u) \tag{4.10}$$

$$e_i(u) = (-1)^i e_i(-u) \tag{4.11}$$

$$e_i(\tfrac{1}{2}) = \left(\frac{-2^{i-1}}{2^{i-1}-1}\right) e_i(0), \quad i \neq 1 \tag{4.12}$$

$$e_i(\tfrac{1}{3}) = -\frac{1}{2}\left(\frac{3^{i-1}-1}{3^{i-1}}\right) e_i(0), \quad i \in \{2,4,6,8,\ldots\} \tag{4.13}$$

$$e_n(\tfrac{1}{2}) = \frac{B_n}{n!} \tag{4.14}$$

$$\begin{aligned} e_n(u) &= e_n(a) + (u-a)e_{n-1}(a) + \frac{(u-a)^2}{2!} e_{n-2}(a) \\ &\quad + \ldots + \frac{(u-a)^n}{n!} e_0(a) \end{aligned} \tag{4.15}$$

$$\tfrac{1}{2} e_n(\tfrac{1}{2}) = \mathbf{Re}\left\{\sum_{r=1}^{\infty} \frac{-1}{(2\pi i r)^n}\right\} \tag{4.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}_u^h x} f(x) = \frac{\mathrm{d}}{\mathrm{d}_0^1 u}\left[u . \frac{\mathrm{d}}{\mathrm{d}_{\frac{1}{2}}^{uh} x} f(x)\right] \tag{4.17}$$

$$\frac{e_r(un)}{n^r} = \int_{-\frac{1}{2}}^{\frac{1}{2}} e_r(x)\, \mathrm{d}_u^{1/n} x \tag{4.18}$$

$$\frac{e_r(0)}{n^r} = \frac{1}{n}\sum_{i=1}^{n} e_r\left(-\tfrac{1}{2} + (i - \tfrac{1}{2})/n\right) \tag{4.19}$$

Appendix A

The Discrete Integral Expansion Formula

Below are the first few terms of the Discrete Integral Expansion Formula.

$$
\begin{aligned}
\int f(x)\,\mathrm{d}_u^h x \quad = \quad & \int f(x)\,\mathrm{d}x \\
& + uhf(x) \\
& + (u^2 - \tfrac{1}{12})\frac{h^2}{2!} f'(x) \\
& + (u^3 - \tfrac{1}{4}u)\frac{h^3}{3!} f''(x) \\
& + (u^4 - \tfrac{1}{2}u^2 + \tfrac{7}{240})\frac{h^4}{4!} f^{(3)}(x) \\
& + (u^5 - \tfrac{5}{6}u^3 + \tfrac{7}{48}u)\frac{h^5}{5!} f^{(4)}(x) \\
& + (u^6 - \tfrac{5}{4}u^4 + \tfrac{7}{16}u^2 - \tfrac{31}{1344})\frac{h^6}{6!} f^{(5)}(x) \\
& + (u^7 - \tfrac{7}{4}u^5 + \tfrac{49}{48}u^3 - \tfrac{31}{192}u)\frac{h^7}{7!} f^{(6)}(x) \\
& + (u^8 - \tfrac{7}{3}u^6 + \tfrac{49}{24}u^4 - \tfrac{31}{48}u^2 + \tfrac{127}{3840})\frac{h^8}{8!} f^{(7)}(x) \\
& + (u^9 - 3u^7 + \tfrac{147}{40}u^5 - \tfrac{31}{16}u^3 + \tfrac{381}{1280}u)\frac{h^9}{9!} f^{(8)}(x) \\
& + \quad \ldots
\end{aligned}
$$

Appendix B

Bernoulli Numbers and e(u) Polynomials

B.1 The e(u) Polynomials

$$
\begin{aligned}
e_0(u) &= 1 \\
e_1(u) &= u \\
e_2(u) &= \frac{1}{2!}(u^2 - \tfrac{1}{12}) \\
e_3(u) &= \frac{1}{3!}(u^3 - \tfrac{1}{4}u) \\
e_4(u) &= \frac{1}{4!}(u^4 - \tfrac{1}{2}u^2 + \tfrac{7}{240}) \\
e_5(u) &= \frac{1}{5!}(u^5 - \tfrac{5}{6}u^3 + \tfrac{7}{48}u) \\
e_6(u) &= \frac{1}{6!}(u^6 - \tfrac{5}{4}u^4 + \tfrac{7}{16}u^2 - \tfrac{31}{1344}) \\
e_7(u) &= \frac{1}{7!}(u^7 - \tfrac{7}{4}u^5 + \tfrac{49}{48}u^3 - \tfrac{31}{192}u) \\
e_8(u) &= \frac{1}{8!}(u^8 - \tfrac{7}{3}u^6 + \tfrac{49}{24}u^4 - \tfrac{31}{48}u^2 + \tfrac{127}{3840}) \\
e_9(u) &= \frac{1}{9!}(u^9 - 3u^7 + \tfrac{147}{40}u^5 - \tfrac{31}{16}u^3 + \tfrac{381}{1280}u)
\end{aligned}
$$

$e_0(-\frac{1}{2}) = 1$	$e_0(0) = 1$	$e_0(\frac{1}{2}) = 1$
$e_1(-\frac{1}{2}) = \frac{1}{2}$	$e_1(0) = 0$	$e_1(\frac{1}{2}) = -\frac{1}{2}$
$e_2(-\frac{1}{2}) = \frac{1}{12}$	$e_2(0) = -\frac{1}{24}$	$e_2(\frac{1}{2}) = \frac{1}{12}$
$e_3(-\frac{1}{2}) = 0$	$e_3(0) = 0$	$e_3(\frac{1}{2}) = 0$
$e_4(-\frac{1}{2}) = -\frac{1}{720}$	$e_4(0) = \frac{7}{5,760}$	$e_4(\frac{1}{2}) = -\frac{1}{720}$
$e_5(-\frac{1}{2}) = 0$	$e_5(0) = 0$	$e_5(\frac{1}{2}) = 0$
$e_6(-\frac{1}{2}) = \frac{1}{30,240}$	$e_6(0) = -\frac{31}{967,680}$	$e_6(\frac{1}{2}) = \frac{1}{30,240}$
$e_7(-\frac{1}{2}) = 0$	$e_7(0) = 0$	$e_7(\frac{1}{2}) = 0$
$e_8(-\frac{1}{2}) = -\frac{1}{1,209,600}$	$e_8(0) = \frac{127}{154,828,800}$	$e_8(\frac{1}{2}) = -\frac{1}{1,209,600}$
$e_9(-\frac{1}{2}) = 0$	$e_9(0) = 0$	$e_9(\frac{1}{2}) = 0$
$e_{10}(-\frac{1}{2}) = \frac{1}{47,900,160}$	$e_{10}(0) = -\frac{511}{24,524,881,920}$	$e_{10}(\frac{1}{2}) = \frac{1}{47,900,160}$
$e_{11}(-\frac{1}{2}) = 0$	$e_{11}(0) = 0$	$e_{11}(\frac{1}{2}) = 0$

B.2 The Bernoulli Numbers

$B_0 = 1$	$B_1 = -\frac{1}{2}$	$B_2 = \frac{1}{6}$	$B_3 = 0$
$B_4 = -\frac{1}{30}$	$B_5 = 0$	$B_6 = \frac{1}{42}$	$B_7 = 0$
$B_8 = -\frac{1}{30}$	$B_9 = 0$	$B_{10} = \frac{5}{66}$	$B_{11} = 0$
$B_{12} = -\frac{691}{2730}$	$B_{13} = 0$	$B_{14} = \frac{7}{6}$	$B_{15} = 0$
$B_{16} = -\frac{3,617}{510}$	$B_{17} = 0$	$B_{18} = \frac{43,867}{798}$	$B_{19} = 0$
$B_{20} = -\frac{174,611}{330}$	$B_{21} = 0$	$B_{22} = \frac{854,513}{138}$	$B_{23} = 0$