Homework 3

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1 Problem 1

1.1 Part A

Let $(M, \mathcal{A})$ be a manifold with boundary. We will show that both directions of the statement hold. First, suppose that $\phi(x) \in \partial H^n$ for all charts $(U, \phi) \in \mathcal{A}$ at x. From this assumption, if $\mathcal{A}$ covers M, then a chart (U, ϕ) exists. It follows that x is a boundary point because from this chart (U, ϕ), we have that $\phi(x) \in \partial H^n$. This completes the **first** direction. Conversely, suppose that x is a boundary point. We then know that there exists a chart $(U, \phi) \in \mathcal{A}$ at x with $\phi(x) \in \partial H^n$. With this, we can take another chart (V, ψ) in $\mathcal{A}$ centered at x. From basic facts, we know that the map $\psi \circ \phi^{-1} : \phi(U \cap V) \longrightarrow \psi(U \cap V)$ is a diffeomorphism. We can show that this statement holds with a proof by contradiction. If $\psi(x) \notin \partial H^n$, then there exists some $r_0 \in \mathbf{R}_{>0}$ so that an open ball with radius r_0, $B_{r_0}(x)$, is contained in $H^n \setminus \partial H^n$. Also, we have that $B_{r_0}(x) \cap \psi(U \cap V)$ would be open in $H^n \setminus \partial H^n$. More specifically, this means that there exists some $r \in \mathbf{R}_{>0}$, with $r < r_0$, so that the open ball $B_r(x)$ centered at $\psi(x)$ is contained in $\psi(U \cap V)$. Moreover, $\psi \circ \phi^{-1}$ being a diffeomorphism implies that the image of $B_r(x)$ under $\psi \circ \phi^{-1}$ is open; this observation follows from the fact that diffeomorphisms taking $H^n \longrightarrow H^n$ preserve open sets in $\mathbf{R}^n$.

With the observations that have been laid out, we will now reach the contradiction. It comes from the fact that if we have some $x \in B_r(x)$, then $x \in ((\psi \circ \phi^{-1})^{-1}(B_r(x)) = \phi \circ \psi^{-1}(B_r(x)) \subseteq \phi(U \cap V)$. From this expression, the contradiction is clear because the last coordinate of $\phi(x)$ is 0. There is no open set in $\mathbf{R}^n$ containing $\phi(x)$ that is *only* in the upper half plane. If we could have an open set that contains x with a ball of radius ϵ around x, then the coordinates of this ball would be negative. Quite clearly, this would contradict the fact that the last coordinate of this expression is 0. To conclude, $\phi(\psi^{-1}(B_r(x)))$ cannot be this type of set. Finally, $\psi(x) \in \partial H^n$, which implies that $\psi(x) \in \partial H^n$ holds for charts $(V, \psi) \in \mathcal{A}$ at x. This establishes the contradiction, which proves the remaining direction.

1.2 Part B

To show that the boundary is an $n-1$ manifold, we begin by taking an atlas $\mathcal{A}_0$ on ∂M. If we endow this boundary with the subspace topology of M, we can take (U, ϕ) as any chart in $\mathcal{A}$ and $U \cap \partial M \neq \emptyset$. If we take $U_0 = U \cap \partial M$, it is clear that U_0 is open in ∂M because $U \subseteq M$ is open. We also would have that $\phi_0(U_0)$ is open in $\partial H^n \cong \mathbf{R}^{n-1}$. With (U, ϕ), we can also define $\phi_0 : U_0 \longrightarrow \mathbf{R}^{n-1}$. We define $\phi(x) = (x_1, \cdots, x_{n-1}, 0)$ and from **Part A**, we set $\phi_0(x) = (x_1, \cdots, x_{n-1})$. Besides this short background, we know that $\phi_0 : U_0 \longrightarrow H^n$, and also that $\phi_0(x) = \phi(x)$ for all $x \in U_0$. With the maps ϕ, ϕ_0, we know that ϕ_0 inherits several characteristics of ϕ. ϕ_0 is an injective and continuous map because ϕ is. Similarly, ϕ_0 would inherit the subspace topology from ϕ, as well as being continuous under this subspace topology. Therefore, we have that (U_0, ϕ_0) is a chart for all (U, ϕ) in $\mathcal{A}$ when $U \cap \partial M \neq \emptyset$. Now, we can set $\mathcal{A}_0 = \{(U_0, \phi_0) | (U, \phi) \in \mathcal{A}, U \cap \partial M \neq \emptyset\}$. This definition of $\mathcal{A}_0$ is important because it covers ∂M because we can take some $x \in \partial M$ that is contained in some chart $(U, \phi) \in \mathcal{A}$ and therefore would also be contained in (U_0, ϕ_0). This takes care of describing how ϕ_0 and ϕ relate to each other.

Additionally, we can show that the usual maps that we have analyzed in previous homeworks are diffeomorphisms. That is, we can take 2 charts (U_0, ϕ_0) and (V_0, ψ_0) in $\mathcal{A}_0$. Immediately, we can form the maps $\psi_0 \circ \phi_0^{-1} : \phi_0(U_0 \cap V_0) \longrightarrow \psi_0(U_0 \cap V_0)$. Quite clearly, $\psi_0 \circ \phi_0^{-1}$ is associated with the map $\psi \circ \phi^{-1} : \phi(U \cap V \cap \partial M) \longrightarrow$

$\psi(U \cap V \cap \partial M)$. We know that $\psi \circ \phi^{-1}$ is a diffeomorphism because (U, ϕ) and (V, ψ) are charts. Therefore, we have shown that any 2 charts in $\mathcal{A}_0$ are compatible by following the definitions. To conclude, we have that $\mathcal{A}_0$ is an atlas and also that ∂M is a manifold. This completes the **second** part.

1.3 Part C

We will show that these notions are well-defined by taking two different representatives of the tangent vector v and showing that the expressions that we will obtain for each representative are the same. To begin, take $x \in \partial M$ and $(U, \phi, x, h), (U', \phi', x, h')$ the 2 representatives of v. We will set $f(y) = \phi' \circ \phi^{-1}(y)$, and for $1 \leq i \leq n$, set h_i equal to the unit vector that only has an entry 1 in the i coordinate. For $i < n$, the expression $\phi(x) + h_i$ is in ∂H^n. So we know that the image of $\phi(x) + h_i$ under f is also contained in ∂H^n and has n coordinate that is 0. Directly, this implies that $\frac{\partial f_n}{\partial y_i(\phi(x))}$ vanishes for all $1 \leq i < n$. Then, for all $t > 0$, $f(\phi(x)_t h_n)$ must have a positive n th coordinate. We know this because $\phi(x) + th_n \notin \partial H^n$ and $f(\phi(x) + th_n) \in H^n \setminus \partial H^n$; we have that $\frac{\partial f_n}{\partial_{y_i}(\phi(x))} = r$ where $r > 0$. With all of this information, we know that

$$h' = D(\phi' \circ \phi^{-1})(\phi(x))h \ .$$

For h'_n we would then have that

$$h'_n = \left[\frac{\partial f_n(x)}{\partial y_1}, \cdots, \frac{\partial f_n(x)}{\partial y_n}\right] h \ .$$

Rearranging these terms a bit more implies that

$$\begin{aligned} &\Rightarrow [0 \cdots 0c]h \\ &= ch_n \ . \end{aligned}$$

From the simple rearrangements above, it is clear that because $c > 0$, $ch_n = h'_n \Leftrightarrow h_n \equiv 0$. Furthermore, we also have that the expression would be negative if and only if h_n is negative. Altogether, this displays the fact that the notions of being tangent to the boundary, in which the tangent vector could point inwards or outwards, are notions that are indeed well-defined. These notions are exactly independent of different representatives that we have specified above.

Besides this, we must also verify the basic axioms of the vector space. If we let $T_x M$ be a vector space over $\mathbf{R}$, then we can take the same representatives $v_1 = (U, \phi, x, h), v_2 = (U', \phi', x, h')$. Naturally, we can take the summation of v_1, v_2. We know that

$$v_1 + v_2 = [U, \phi, x, h + D(\phi \circ \phi^{-1})(\phi'(x))h'] \ , \alpha v_1 = [U, \phi, x, rh] \ ,$$

where $\alpha \in \mathbf{R}$. We then know that the vector space $T_x M$ satisfies the two properties, and this implies, by the linearity of the derivative operator, that we indeed have a vector space. Besides this, we also have that $T_x M \cong \mathbf{R}^n$ because $[U, \phi, x, h] \mapsto h$ for fixed U, ϕ.

Additionally, we can show that the subspace consisting of all vectors tangent to the boundary. First, let $A \subseteq T_x M$ be a subspace with all vectors tangent to the boundary. Then, we can define a map $\gamma : A \longrightarrow T_x \partial M$, from $\gamma([U, \phi, x, (h_1, \cdots, h_{n-1}, 0)]$. Elementwise, γ is a projection of all of the nonzero coordinates for h. Besides taking the projection of all nonzero coordinates of h, we also restrict the chart appropriately. With this, we then know that γ is a bijective and smooth map. γ^{-1} is also a smooth map, which implies that $T_x\partial M$ is in fact the collection of all vectors that are tangent to the boundary. From γ, it is clear that $T_x\partial M$ can be viewed as the subspace A. This completes the **second** part.

2 Problem 2

2.1 Part A

We will show that $\mathfrak{M}_x$ is a maximal ideal with the following. To begin, we know that $\mathfrak{M}_x$ is an ideal; we can take $[f], [g] \in \mathfrak{M}_x$. It is clear that $f(x) = g(x) = 0$, which automatically implies that $f(x)+g(x) = [f]+[g] = [f+g] \in \mathfrak{M}_x$. We can also check that if we have another $[h] \in C^\infty_{M,x}$, then $(fh)(x) = f(x)h(x) = 0 \times h(x) = 0 \in \mathfrak{M}_x$. This affirms that the multiplication of two arbitrary elements of $\mathfrak{M}$ belongs to the ideal, in which $[f][g] \in \mathfrak{M}_x \Rightarrow [fg] \in \mathfrak{M}_x$. Finally, we can take some $r \in \mathbf{R}$, in which $rf(x) = r \times 0 = 0 \in \mathfrak{M}_x \Rightarrow [rf] \in \mathfrak{M}_x$. So the ideal $\mathfrak{M}_x$ is closed under addition within itself, as well as being closed under multiplication from arbitrary elements that we choose from $\mathbf{R}$. This **completes** showing that $\mathfrak{M}_x$ is an ideal.

However, besides showing that $\mathfrak{M}_x$ is an ideal, we must now show that it is maximal. First, let that $[f] \in C^\infty_{M,x}$ be a germ that is not in $\mathfrak{M}_x$. If we want to show that the ideal is maximal, we would have to take generators of it to be $\mathfrak{M}_x \cup \{f\}$. If we have these generators, then we presumably would be able to obtain $C^\infty_{M,x}$. It is also clear that if we take some ideal that strictly contains $\mathfrak{M}_x$ would have to be $C^\infty_{M,x}$. By hypothesis, if we take $[f] \notin \mathfrak{M}_x$, this would imply that $f(x) \neq 0$ because $f(x) \equiv r$. Therefore, we can take some $s \in \mathbf{R}$ and rescale f. This rescaling would merely consist of dividing f by s and checking whether another germ $[g] \in C^\infty_{M,x}$ could be generated by the germ $f(x)$ when it is rescaled by s.

Continuing in this direction, we know that $[\frac{s}{r}f] \in I$, and also that $\frac{s}{r}f(x) = s$. If we take another germ $[g] \in C^\infty_{M,x}$, with $g(x) = s$, we then have that

$$\left(g - \frac{s}{r}f\right)(x) = s - s \equiv 0 \ .$$

So far, this establishes that $[g]$ is actually an element of I. Furthermore, it also establishes that $C^\infty_{M,x} \subseteq I$. So the ideal is maximal as well.

We will now show that the given quotient defines a morphism of $\mathbf{R}$ algebras. To do this part, we will define such a morphism. Specifically, a morphism of $\mathbf{R}$ algebras is a map

$$\alpha : \frac{C^\infty_{M,x}}{\mathfrak{M}_x} \longrightarrow \mathbf{R} \ ,$$

with $\alpha : [f]+\mathfrak{M}_x \mapsto f(x)$. We can easily show that α is well defined by following the theme of the first problem, in which we will take two arbitrary representatives. We would like to show that applying these representatives to α does **not** affect their image under α. So if $[f]+\mathfrak{M}_x = [g]+\mathfrak{M}_x$, there must exist $[h] \in \mathfrak{M}_x$ so that $[f]+[g] = [h]$. More clearly, this means that $f(x) + h(x) = g(x) \Rightarrow f(x) + 0 = g(x)$. These simple rearrangements imply that $f(x) = g(x)$ whenever $[f] + \mathfrak{M}_x = [g] + \mathfrak{M}_x$, which shows that α is well-defined.

Besides these previous arguments, we must verify more properties to be sure that α is an $\mathbf{R}$ algebra morphism. Immediately, we know that taking $r \in \mathbf{R}$ implies that $r\alpha([f] + \mathfrak{M}_x) = rf(x) = \alpha([rf] + \mathfrak{M}_x)$. Also, we know that taking $[f], [g] \in C^\infty_{M,x}$ similarly implies that

$$\alpha([f] + [g] + \mathfrak{M}_x) = (f + g)(x) = f(x) + g(x) = \alpha([f] + \mathfrak{M}_x) + \alpha([g] + \mathfrak{M}_x) \ ,$$

which confirms that α satisfies one of the usual vector space properties. The second one follows naturally, in which we have that

$$\alpha([f] + [g] + \mathfrak{M}_x) = (fg)(x) = f(x)g(x) = \alpha([f] + \mathfrak{M}_x)\alpha([g] + \mathfrak{M}_x) \ .$$

Checking of all of these properties directly implies that α is the **R** algebra morphism that we want. To show that α is an isomorphism, we must check that the map is surjective and injective. From α, we can take some $[f] \in \ker\alpha$. From basic definitions, $f(x) = 0 \Rightarrow [f] \in \mathfrak{M}_x \Rightarrow [f] + \mathfrak{M}_x = \mathfrak{M}_x$. In short, we have shown that the kernel is trivial, which takes care of injectivity.

As for surjectivity, we will take some $r \in \mathbf{R}$. We will define $f : U \longrightarrow \mathbf{R}$ with $f(u) = r$ for all $u \in U$, with u being a chart that contains x. It is clear that α is surjective because if we take $[f] \in C^{\infty}_{M,x}$, we then know that $\alpha([f] + \mathfrak{M}_x) = f(x) = r$. These equalities imply that α is surjective because for each element in the codomain, there exists a unique element in the domain that is mapped to it through α. To conclude, this completes all of the properties that we had to show in the **first** part.

2.2 Part B

We can show that a unique maximal ideal exists with the following. If we take another maximal ideal $\mathfrak{n} \neq \mathfrak{M}_x$ in $C^{\infty}_{M,x}$, then we would want to show that the ideal $\mathfrak{n}$ is not actually a maximal ideal through a contradiction. We know that $\mathfrak{n}$ contains some germ $[f]$ with $f(x) \neq 0$. f is continuous, and from basic characteristics of continuous functions, we know that there exists some open neighborhood U of x in which $f(y)$ does not vanish for any $y \in U$.

On U, we can also define another continuous function $g : U \longrightarrow \mathbf{R}$. This function g is given by $g(y) = \frac{1}{f(y)}$. From our previous arguments, we have that $f(y) \neq 0$, which ensures that g doesn't blow up. It also ensures that g is continuous and that this map is well-defined. Because $\mathfrak{n}$ is supposed to be a maximal ideal, we would then have that $[f] \in \mathfrak{n} \Rightarrow [f][g] \in \mathfrak{n} \Rightarrow [1] \in \mathfrak{n}$. However, we have recovered a contradiction in the maximality of the ideal $\mathfrak{n}$ because if $\mathfrak{n}$ *were* maximal, then $[h][1] = [h] \in \mathfrak{n} \Rightarrow C^{\infty}_{M,x} \subseteq \mathfrak{n}$ would not hold. It is not possible that $C^{\infty}_{M,x} \subseteq \mathfrak{n}$. This contradiction completes the **second** part.

2.3 Part C

We will show that the covector map is well-defined. For this part, we will take $[f] = [g]$. If f is defined on some open neighborhood U, then we can take $x \in U$ and if g is defined on another open neighborhood V, then we can take $x \in V$. From these open sets, we can form charts (U, ϕ) and (V, ψ). By definition, there must then exist an open neighborhood $W \subseteq U \cup V$ on which restricting f and g to this open neighborhood makes them equal, in which $f|_W = g|_W$. Now, we can take another chart for W, in which $(W, \varphi = \phi|_W)$. With this third chart, we know that on W $f^{\sim} = f \circ \phi^{-1} = g \circ \phi^{-1} = g^{\sim}$. This capture the fact that the compositions with either f or g with ϕ^{-1} on the open set W, on which f and g are equal, are the same. So for $v = [c, x, h]$, $d_x f(v) = Df^{\sim}(\phi(x))h = Dg^{\sim}(\phi(x))h = d_x g(v)$. This captures the fact that from the 2 charts that we started out with, the derivatives of f and g are equal on W, in which $d_x f = d_x g$. This implies that the map ðis in fact well-defined.

We will now show that ð: $\mathfrak{n} \longrightarrow T^*_x M$ is injective. One possible case that we could have entails that $M \cong \mathbf{R}^k$ for some k, and also that $\beta : M \longrightarrow \mathbf{R}^n$. If we have this case, we know that $T^*_x M$ consists of linear functions $\gamma : \mathbf{R}^k \longrightarrow \mathbf{R}$. With this background, we can then take $f : M \longrightarrow \mathbf{R}$ with $f = \gamma \circ \beta$. From f, we can calculate the derivative directly by the chain rule, in which we will have to look at the composition $f \circ \beta^{-1}$. We have that

$$d_x f = D(f \circ \beta^{-1})(\beta(x)) = D(\gamma)(\beta(x)) \ .$$

Observe that these terms are constant because γ is linear, and also because $d_x f h = \gamma(h)$. So we have that ðis surjective because ð$([f]) = \gamma$. This takes care of one case.

The remaining possibilities are more general, in which we will take M to be a manifold. We will also take $\gamma \in T_x^* M$ be some linear function taking $\mathbf{R}^k \to \mathbf{R}$. In these cases, we want M to be k-dimensional, with a chart (U, ϕ) centered at x. Now, we can define $f : \phi(U) \longrightarrow \mathbf{R}$ with $d_{\phi(x)} f = \gamma$. We can then restrict this f to M, in which $f|_M = f \circ \phi$. We can extend f_M to all of M, and in doing this, we will be able to show that ðis surjective. It follows directly by observing that

$$d_x f_M = D\widetilde{f_M}(\phi(x)) = D(f \circ \phi \circ \phi^{-1})(\phi(x)) = Df(\phi(x)) \ .$$

Finally, we have that

$$\Rightarrow Df(\phi(x)) = d_{\phi(x)} f = \gamma \ ,$$

which guarantees surjectivity because ð$([f_M]) = \gamma$.

2.4 Part D

Let (U, ϕ) be a chart at x. We will define $d : U \longrightarrow \mathbf{R}$ by $d(y) = ||\phi(y) - \phi(x)||$, namely the Euclidean distance in $\phi(U)$. Then, because $d(x) \equiv 0$, $[d] \in \mathfrak{n}$. Then, if we take some $[f] \in$ kerð, we can compute

$$d_x f = D(f \circ \phi^{-1})(\phi(x)) = 0 \ .$$

This helps us because it confirms that every partial derivative in the image of f vanishes. Similarly, we also know that the function g that we have defined in previous parts has a limit as $y \longrightarrow x$. In particular, this limit is smooth and continuous, because $g(y) = \frac{f(y)}{d(y)}$ is smooth and continuous. g is also defined at 0. So far, we then have that $[g] \in \mathfrak{m}_x$, and also that $[f] = [d][g] \Rightarrow [f] \in \mathfrak{m}_x^2$. These steps imply that kerð$\subseteq \mathfrak{m}_x^2$.

If we have that $[f] = [gh] = [g][h]$, for $[f], [g], [h] \in \mathfrak{m}_x$, then we can show that the **other** direction of containment holds. With this goal in mind, we can look at ð$([f])$. We have that ð$([f])$ =ð$([g][h]) = d_x(gh) - d_x g h(x) + g(x) d_x h = d_x g \times 0 + 0 \times d_x h = 0$. These rearrangements confirm that $[f] \in$ kerð. In more stark terms, this communicates the fact that any sum of the products is in the kernel. So $[f] \in$ kerð. We then have that the other direction of containment holds because the sum of products of $[f], [g]$ illustrates that taking this sum is equivalent to taking the sums of their multiples. So we have that $\mathfrak{m}_x^2 \subseteq$ kerð. With these two directions of containment, it is clear that we have equality.

Also, we can confirm that the map is linear by checking that the two equalities from linear algebra hold. In particular, we can take $a, b \in \mathbf{R}$, and with these constants, we know that ð$(a[f] + b[g]) = d_x(af + bg) = a d_x f + b d_x g = a$ð$([f]) + b$ð$([g])$. From our previous work, we have shown that the map is surjective and that the kernel is in fact the maximal ideal. By the first isomorphism theorem, we have that $\frac{\mathfrak{m}_x}{\ker \partial} \cong \mathrm{Im}\partial$. We conclude that

$$\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2} \cong T_x^* M \ ,$$

which finishes the last part of the second problem.

3 Problem 3

3.1 Part A

We will show that such a manifold would have codimension n. To begin, we will take $x \in V^{\sim}$. If we want to form the ordered basis of V that we want, we denote it with $\mathcal{B} = (e_0, \cdots, e_n)$. We will also observe that we have the chart $(U \times V, \phi \times \text{id}_{n+1})$ on $\mathbf{P}V \times V$ because $V \cong \mathbf{R}^{n+1}$. As a result, we can take the coordinates $(u_1, \cdots, u_n, v_0, \cdots v_n)$ on $U \times V$. Because $U \times V \cong \mathbf{R}^n \times \mathbf{R}^{n+1}$, we have that $v = (v_0, \cdots, v_n)$ is on the line $[1, u_1, \cdots, u_n] \Leftrightarrow v_0$ scales each of the factor the same for *each* coordinate. Precisely, for all $1 \leq i \leq n$, this means that $v_i = v_0 u_i$. As we took the coordinates for v, we can also take the coordinates $(x_1, \cdots, x_n, y_0, \cdots, y_n)$ for x. We also have to define $f_i \in C^\infty(U)$ though, which we denote by $f_i(v) = v_i - v_o u_i$. Directly from our previous work, we know that $v \in V^{\sim} \Leftrightarrow f_i(v) = 0$ for all i. As we have several times previously, we can compute $d_x f_i$. Qualitatively, it has $2n+1$ components, with the ith component being $-y_0$, the $n+1$ component being $-x_i$, and finally the $n+1$ component being 1. Besides these components that we have specified, the components on all remaining indices are 0. We then have that the submanifold is of codimension n because with different i, one of the f_i entries will have a nonzero $n+i+1$ st component. This makes the f_i's linearly independent.

3.2 Part B

We will show that the inverse image consists only of a single point by restating properties of the projection map b. We know that b is the projection map onto the second component, and also that it is continuous. From this map, we can take the restriction β of b to $V^{\sim}$. This would be helpful because we know that β is continuous. Also β has the property that the preimage of any open set $\mathcal{S} \subseteq V$ would be open in $\mathbf{P}V \times V$. The intersection of this open set with $V^{\sim}$ would clearly be open in $V^{\sim}$ as well. From these arguments so far, it is also clear that b being smooth in all of its coordinates also implies that β is smooth in all of its coordinates. This property follows from the fact that b is a projection on the last $n+1$ coordinates.

Now, we can show that the preimage of v under *beta* only consists of a single point. We know that this holds because if we take some nonzero $v \in V$, then we can specify a unique line that determines v by $[v]$. That is, we can make use of the equivalence relation, in which we know that $\mathbf{P}V$ is the quotient of V. The equivalence relation tells us that any other point u lying on the same line that determines v would make u and v equivalent. More precisely, this means that $u \equiv v \Rightarrow u = \lambda v$ for some $\lambda \in \mathbf{R}$ and of course $u \in [v]$. The conclusion now immediately follows because if we have the pair $([v], v)$, then the inverse image of v under β is the set $\{[v], v\}$. This is **only** a single point and finishes the second part of the third problem.

3.3 Part C

We will show that the restriction is a diffeomorphism by checking that all of the necessary properties hold. From our **previous** work, we know that β restricted to $V^{\sim} \backslash E$ is injective. From **Part B**, we have also shown that this restriction map is surjective because the preimage of some $v \in V$ is *only* a single point. As we have already mentioned too, β is smooth because it is a projection onto the last $n+1$ coordinates. Now that we have established that β is injective, surjective and smooth, all that remains is to check that the inverse of β is smooth. Explicitly, the map is

$$\beta^{-1} : V \backslash \{0\} \longrightarrow V^{\sim} \backslash E \ .$$

We can analyze whether β^{-1} is smooth by taking $v \neq 0$. In this case, the image of v under β^{-1} would be $\beta^{-1}(v) = ([v], v)$. This image demonstrates that β^{-1} is smooth because the image $\beta^{-1}(v)$ embeds $V \backslash \{0\}$ onto the last $n+1$ coordinates of $V \backslash E$. This shows that β^{-1} is smooth, and also establishes that the map we are looking at is a diffeomorphism.

3.4 Part D

We will show that the given statements hold with the following. To begin, we let $\zeta : V^{\sim} \longrightarrow \mathbf{R}$, with

$$\zeta(u_1, \cdots, u_n, v_0, u_1, v_0, \cdots, u_n v_0) = v_0 \ .$$

We know that ζ is a projection onto one coordinate, so it is smooth. It is also trivial that ζ is linearly independent because we only have ζ itself. But we know that E is a submanifold of codimension 1 because if we take some $x \in V^{\sim}$, $x \in E \Leftrightarrow v_0 = u_1 v_0 = \cdots = u_n v_0 = 0 \Leftrightarrow \zeta(x) = 0$. From the submanifold, we know that it has coordinates $(u_1, \cdots, u_n, v_0, v_0 u_1, \cdots, v_0 u_n) \in V^{\sim}$, which have the form $(u_1, \cdots, u_n, 0, \cdots, 0)$. It follows that E is *diffeomorphic* to $\mathbf{P}V$ by the mapping

$$(u_1, \cdots, u_n, 0, \cdots 0) \Leftrightarrow [1, u_1, \cdots, u_n] \ .$$

The statement above implies that the mapping has a coordinate map that is a projection onto all of the nonzero coordinates. We have checked of **all** properties to show that ζ is a diffeomorphism. Besides this, we can also show that the expression for the tangent space holds by making use of the surjectivity. From previous arguments, β is surjective. This directly implies that it has rank $n+1$, which makes it a subimmersion; we know from the Fibre theorem that the tangent bundle of the submanifold $\beta^{-1}(0)$ is $TE = \ker(T\beta)$. Finally, we have that $T_{([v],0)}E = \ker(T_{([v],0)}\beta)$.

3.5 Part E

We will show that all of the statements hold. To begin, we will take a chart (U_a, ϕ_a) centered at a on M. On this chart, we want the first coordinate to be 0, as well as rotating the chart so that $\zeta \in T_a M$ is given by $\zeta = (1, 0, \cdots, 0)$. Now, we will take a basis $\mathcal{B} = (e_0, \cdots, e_n)$ of V. Taking $e_0 = T_a f(\zeta)$ is advantageous, and we will take $\mathcal{U}_{\mathcal{B}}$ to be the set which has coordinates $(u_1, \cdots, u_n)$, which correspond to the coordinates $(v_0, \cdots, v_n)$ on V. With our basis of V, we are also able to define $f(x) \in V$ with $\mathcal{B}$ that we have chosen. If $f(x) \in V$, it can be expressed componentwise with $f(x) = (f_0, \cdots, f_n)$.

If we take some x outside of A, we can similarly define $f^{\sim}(x) = \left(\frac{f_1(x)}{f_0(x)}, \cdots, \frac{f_n(x)}{f_0(x)}, f_0(x), f_1(x), \cdots, f_n(x)\right)$. For $x \notin A$, the denominator $f_0(x)$ that appears in some of the components of $f^{\sim}(x)$ never vanishes, which guarantees that each of the components in $f^{\sim}(x)$ behave well. Above all, $f^{\sim}(x)$ is also smooth. Finally, taking the composition $\beta \circ f^{\sim}(x) = (f_0(x), f_1(x), \cdots, f_n(x)) = f(x)$.

We will now make use of *L' Hopital's Rule*. As demonstrated in the past HW, we want to extend the function uniquely to another one that is defined on all of $\mathbf{R}^{k+1}$. Quite surely, we can do this by observing that because $A = f^{-1}(0)$, extending $f^{\sim}(x)$ to get $f^{\sim}(a)$ would give us that $f_i(a) = f_0(a) \equiv 0$. So if we take the chart (U_a, ϕ_a), we know by *L' Hopital's Rule* that the function $h_i(x) = \frac{f_i(x)}{f_0(x)}$ is smooth because we can differentiate terms in the numerator and denominator without running into difficulties of terms being undefined. Moreover, we also that have $h_i(x)$ is defined for $x \notin f_0^{-1}(0)$. Therefore, we can extend the function uniquely to obtain $f^{\sim}(a)$ as we have been talking about. Explicitly, we would then have that $f^{\sim}(a) = (h_1^{\sim}(a), \cdots, h_n^{\sim}(a), f_0(a), f_1(a), \cdots, f_n(a))$. It is clear by the usual *L' Hopital's Rule* that $f^{\sim}(a)$ is smooth. As we stated previously when looking at $f^{\sim}(x)$, observe that the composition $\beta \circ f^{\sim}(a) = (f_0(a), \cdots, f_n(a))$. So far, so good.

We will now analyze each of the coordinates of $f^{\sim}(a)$. Within these coordinates, we will apply *L' Hopital's Rule* to analyze the behavior of the terms that are quotients of $f_i(x)$ and $f_0(x)$. By *L' Hopital's Rule*, we have that $h_i^{\sim}(a) = \frac{\frac{\partial f_i(a)}{\partial x_0}}{\frac{\partial f_0(a)}{\partial x_0}}$. This expression for $h_i^{\sim}(a)$ is the ith coordinate of $T_a f(\zeta)$ because ζ is a unit vector that lies along x_0. To conclude, we have that $f^{\sim}(a) = ([T_a f(\zeta)], 0)$. This completes the last part of **Problem 3**.