STSCI 5080 Homework 1

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1 Problem: Rice 2.5.44

1.1 Part A

To find the pmf of X, we observe that from the definition of X, taking the ceiling of the exponential random variable T gives a pmf of the form,

$$\begin{aligned}\mathbf{P}(T = n) &= \mathbf{P}(T \in [an, a(n+1)] \\ &= \int_{an}^{a(n+1)} \lambda \exp(-\lambda x) \mathrm{d}x \\ &= (1-p)^n p \ .\end{aligned}$$

This shows that the pmf of X is geometric.

1.2 Part B

To show that the given inequality holds for the exponential random variable, we will calculate the probability

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which by definition is the indefinite integral of exponential random variables,

$$\mathbf{P}(X \geq k) = \int_t^\infty \lambda e^{-\lambda x} \mathrm{d}x = e^{-\lambda t} \ .$$

Next, from the other direction we can compute the conditional probability directly by applying the formula,

$$\mathbf{P}(X \geq n+k | X \geq n) = \frac{\mathbf{P}(\{X \geq n+k\} \cap \{X \geq n\})}{\mathbf{P}(X \geq n)} \ .$$

Because the intersection of the two events is equivalent to $\{X \geq n+k\}$, this gives

$$\begin{aligned}\frac{\mathbf{P}(X \geq n+k}{\mathbf{P}(X \geq k} &= \frac{\exp(-\lambda(n+k))}{\exp(-\lambda k)} \\ &= \exp(-\lambda k) \ .\end{aligned}$$

Hence the inequality holds.

2 Problem: Rice 2.5.60 Modified

To find the pdf corresponding to the lognormal density, we begin by observing that the cdf of the lognormal density is of the form

$$F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(e^x \leq y) ,$$

in turn implying that the expression above vanishes when $y \leq 0$ because the pdf is not even defined for this interval of y values, and furthermore, that

$$\mathbf{P}(e^X \leq y) = \mathbf{P}(X \leq \log(y)) ,$$

for $y \geq 0$. Applying the straightforward definition of the probability measure above gives the rearrangements

$$\int_{-\infty}^{\log(y)} f_X(x)\mathrm{d}x = F_X(\log(y)) .$$

From this expression, we can now apply the 'General Strategy' that was discussed in class. That is, given that we have expressed the cdf of Y in terms of the cdf of X, we can now differentiate the expression above to obtain the pdf of Y,

$$f_Y(y) = \bigg(F_X(\log(y))\bigg)' = F_X'(\log(y))\frac{1}{y} = f(\log(y))\frac{1}{y} = \frac{1}{y\sqrt{2\pi}}\exp\bigg(\frac{-\log(y)^2}{2}\bigg)\bigg) .$$

In the final step, observe that the exponential term that we have listed is equivalent to the expression from the marginal density that we obtained after differentiating.

3 Problem: The Weibull CDF, Rice 2.5.67

3.1 Part A

To find the Weibull pdf, we begin by calculating the following derivative with respect to x,

$$\begin{aligned}(F_X(x))' &= \bigg(1 - \exp\bigg(-\Big(\frac{x}{\alpha}\Big)^{\beta}\bigg)\bigg)' \\ &= -\exp\bigg(-\Big(\frac{x}{\alpha}\Big)^{\beta}\bigg)\frac{\partial}{\partial x}\bigg(-\Big(\frac{x}{a}\Big)^{\beta}\bigg) .\end{aligned}$$

Finally, the derivative works out to be of the form,

$$\frac{\beta}{\alpha^{\beta}}x^{\beta-1}\exp(-\Big(\frac{x}{\alpha}\Big)^{\beta}) .$$

3.2 Part B

To show that W also follows an exponential distribution if it follows the Weibull distribution, we first set $g(x) = (\frac{x}{\alpha})^{\beta}$, and calculate,

$$\begin{aligned} f_X(x) &= f_W(g^{-1}(x))|\frac{\partial}{\partial x}g^{-1}(x)| \\ &= \left(\frac{\beta}{\alpha^{\beta}}\right)(\alpha x^{\frac{1}{\beta}})^{\beta-1}\exp(-x)\frac{\alpha}{\beta}x^{\frac{1-\beta}{\beta}} \\ &= \beta^0 \times \alpha^{-\beta+\beta-1+1}x^{\frac{\beta-1}{\beta}+\frac{1-\beta}{\beta}} \times e^{-x} \\ &= e^{-x} \ , \end{aligned}$$

because $g^{-1}(x) = \alpha x^{\frac{1}{\beta}}$. Therefore, the distribution also follows an exponential distribution.

3.3 Part C

To explain how Weibull random variables could be generated by a random number generator, we know that if we have some uniform $(0, 1)$ random variable, from this random variable we can generate random numbers by recalling that $-\log(U)$ follows an exponential distribution. As a result, this quantity following an exponential distribution implies that, from our computation in **B**, that the random variable $Y = \alpha(-\log(U))^{\frac{1}{\beta}}$ will follow the Weibull distribution.

4 Problem: Rice 3.8.9

4.1 Part A

To find the marginal densities of X and Y, we observe that given that (X, Y) is uniformly distributed over the defined region, this region is precisely the portion of the parabola $y = 1 - x^2$ that is above the $y - axis$, which occurs for $-1 \leq x \leq 1$. Next, the marginal densities of X and Y are respectively calculated with,

$$f_X(x) = \int_0^{-x^2+1} \frac{3}{4}\mathrm{d}y = \frac{3}{4}(1 - x^2) \ ,$$

and

$$f_Y(y) = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{3}{4}\mathrm{d}x = 2(\frac{3}{4}\sqrt{1-y}) = \frac{3}{2}\sqrt{1-y} \ .$$

4.2 Part B

From the marginal densities calculated in **A**, the corresponding conditional densities of X and Y, are, respectively,

$$f_{X|Y}(x|y) = \frac{\frac{3}{4}}{\frac{3}{2}\sqrt{1-y}} = \frac{1}{2\sqrt{1-y}} \ ,$$

for $-1 \leq x \leq 1$, and,

$$f_{Y|X}(y|x) = \frac{\frac{3}{4}}{-\frac{3}{4}x^2 + \frac{3}{4}} = \frac{-1}{x^2+1} \ ,$$

for $0 \leq y \leq 1$.

5 Problem: CDF and PDF of R

First, we will determined the cdf of the given region R by making use of properties of the uniformly distributed random vector. That is, given some region R, the uniformly distributed random vector has distribution with magnitude $\frac{1}{|R|}$, where $|R|$ is the region of the area R. Therefore, the probability

$$\mathbf{P}((X,Y) \in R) = \frac{|R|}{\pi} \ ,$$

which in the case of this particular R implies that the random variable $R = \sqrt{X^2 + Y^2}$, for $0 \leq r \leq 1$, is equivalent to the event that the random vector (X,Y) belongs to the semicircular region R. As a result,

$$\mathbf{P}(R \leq r) = \frac{\pi r^2}{\pi} = r \ ,$$

because πr^2 is the area of R, demonstrates that the equality holds as long as $0 \leq r \leq 1$. From this information, we can then find the pdf of R by evaluating the integral in the given hint. That is, we know that the pdf of R is of the given form with the double integral because the integral exactly represents, with a polar coordinates transformation, the joint pdf,

$$\mathbf{P}(R \leq r) = \frac{1}{\pi} \int_R \mathrm{d}x\mathrm{d}y \ .$$

From this integral taken over the semicircle region R, simplifying terms yields

$$\Rightarrow \frac{1}{\pi} \int_0^{2\pi} \int_0^r \rho \mathrm{d}\rho \mathrm{d}\theta = \int_0^r 2\rho \mathrm{d}\rho = r^2 \ .$$

Altogether, the steps above demonstrate that $\mathbf{P}(R^2 \leq r) = \mathbf{P}(R \leq \sqrt{r}) = r$, given that $0 \leq r \leq 1$. Observe that we derived the PDF and CDF for the given region R with r in general, which in turn absolutely works for $r \equiv 1$ because each of the inequalities that we listed above are satisfied when $r \equiv 1$.

6 Problem: Rice 3.8.23 with Poisson Random Variable

To show that the unconditioned distribution would be a Poisson distribution, we will follow the hint by first computing the joint pmf. Because $X|N \sim \text{Bin}(N,p)$, and furthermore that $N \sim \text{Poisson}(\lambda)$, it is clear that the probability

$$\mathbf{P}(X = x|N = n) \ ,$$

can be precisely expressed as,

$$\binom{n}{x} p^x (1-p)^{n-x} \ ,$$

for $n \geq x$, and 0 otherwise. Next, we observe that the pmdf of the Poisson random variable is of the form,

$$\exp(-\lambda)\frac{\lambda^n}{n!} \ ,$$

for $n - 0, 1, \cdots, \infty$, $x = 0, 1., \cdots n$, with $x < n$, and again 0 otherwise. Because the joint pmdf $\mathbf{P}(X = x, N = n) = 0$ if $x > n$, it is possible to express the conditioned distribution of a binomial random variable, conditioned against a Poisson random variable, with

$$\begin{aligned} \mathbf{P}(X = x) = \sum_n \binom{n}{x} p^x (1-p)^{n-x} \exp(-\lambda)\frac{\lambda^n}{n!} \\ p^x \exp(-\lambda) \sum_{n=x}^{\infty} \frac{n!(1-p)^{n-x}}{x!(n-x)!}\frac{\lambda^n}{n!} \ , \end{aligned}$$

from which combining, and rearranging, terms around the summation above gives

$$\begin{aligned} \frac{p^x}{\exp(-\lambda)\lambda^x} n! \sum_{n=x}^{\infty} \frac{(\lambda(1-p))^{n-x}}{(n-x)!} \\ = \exp(-\lambda)\frac{(\lambda p)^x}{x!} \sum_{y=0}^{\infty} \frac{\lambda(1-p)^y}{y!} \\ \frac{\exp(-\lambda(\lambda p)^x}{x!} \exp(\lambda(1-p)) \ . \end{aligned}$$

In the penultimate step, observe that we introduce a change of variables in the summation, with $y = 0$ as the beginning term instead of $n = x$. Finally, because the exponential function, for $a \in \mathbf{R}$ satisfies,

$$\exp(a) = \sum_{k=0}^{\infty} \frac{a^k}{k!} \ ,$$

which for $a \equiv \lambda p$ is equivalent to the series expansion

$$\begin{aligned} \exp(-\lambda + \lambda - \lambda p)\frac{(\lambda p)^x}{x!} \\ = \exp(-\lambda p)\frac{(\lambda p)^x}{x!} \ . \end{aligned}$$

Observe that the equality holds for $x = 0, 1, \cdots, \infty$. Moreover, this shows that the unconditional pdf of the given distribution is Poisson because the unconditional pmf is exactly of the form given above, which is indeed a Poisson random variable with parameter λ. From these rearrangements, we know that the pmf of a Poisson random variable is nonzero, and equal to the expression above, when $x = -0, 1, \cdots, \infty$, and 0 otherwise.

7 Problem: Rice 3.8.33 Modified

For $X, Y \sim U[-1/2, 1/2]$, we know that

$$f(x) = 1 \ , \ f(y) = 1$$

for $-1/2 < x < 1/2$ and $-12 < y < 1/2$, respectively. This implies that

$$f(x, y) = f(x)f(y) \ ,$$

because the random variables are assumed to have been independent. Therefore, $f(x, y) = 1$, and if we take $Z = X + Y$, the Jacobian for Z, which will be used to compute the pdf, is of the form

$$\mathcal{J} = \frac{\partial(x, y)}{\partial(z, w)} = 1 \ ,$$

because evaluating the determinant of the resulting 2 bvy 2 matrrix that we form for the Jacobian has terms, across that first row, that are equal to 1 and -1, and terms along the second row that are equal to 0 and 1. Evaluating the determinant gives 1. Therefore, the jonit pdf $f(z, w) = |\mathcal{J}| f(x, y)$, which implies that $f(z, w) = 1$.

Now that we have determined the pdf of Z. we can sketch the feasible region below. In particular, from this graph, we know that the pdf of Z has the marginal pdfs,

$$f(z) = \int_{w=-\frac{1}{2}}^{z+\frac{1}{2}} f(z, w) \mathrm{d}w \ ,$$

for $-1 < z < 0$, and

$$\int_{w=z-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}w \ ,$$

for $0 < z < 1$. Observe that this definition for $f(z)$ is equivalent to

$$z + 1 \ ,$$

and

$$1 - z \ ,$$

for $-1 < z < 0$ and $0 < z < 1$, respectively.

From the sketch above, we know that the pdf of Z is the triangular distribution, as shown below

8 Problem: On Handout

From the hint, it is clear from the moment generating functions for independent Poisson random variables that, for $X, Y \sim \text{Poisson}(\lambda)$,

$$\mathcal{M}_Y(s) = \prod_{i=1}^{n} \mathcal{M}_{Y_i}(s) \ ,$$

implying that

$$\begin{aligned} \mathcal{M}_{X+Y}(s) &= \mathcal{M}_X(s)\mathcal{M}_Y(s) \\ &= \exp(\lambda(s-1))\exp(\mu(s-1)) \\ &= \exp((\lambda+\mu)(s-1)) \ , \end{aligned}$$

so that $X + Y \sim \text{Poisson}(\lambda + \mu)$. From the second hint, we also know that the events stated on the left and right of the second hint are equivalent because if the 2 events on the left are to occur simultaneously, then it is equivalent to stipulating that the other Poisson random variable Y will equal $n - x$. Furthermore, we can look towards rearranging terms so that the product of exponential distributions from the Poisson distribution of $Z = X + Y$ is of the form

$$\exp((\lambda+\mu)(s-1)) = \exp(\lambda+\mu)\exp(s-1) \ ,$$

from which the second hint implies that, because the events on the left and right are equivalent, we can form the product of exponentials above by changing the parameters of the individual Poisson random variables. More specifically, we have that one of the Poisson random variables represents $1 - p$ raised to the power $n - x$, while the remaining term in the multiplication can be obtained by keeping the condition that $X = x$, which is maintained on both the left and right hand sides of the equality in the second Hint.

With this in mind, to justify that the remaining combinatorial coefficient is appropriate, we recall that, again from the second hint, that the product of the Poisson random variables is equivalent to the product of the random variables, with $Z = X + Y$, can be expressed as

$$p_Z(z) = \sum_{j=0}^{z} \mathbf{P}(X = x \text{and} Y = n - x) = \sum_{x=0}^{z} \exp(-\lambda)\frac{\lambda^x}{x!}\exp(-\mu)\frac{\mu^{n-x}}{(n-x)!} \ ,$$

therefore implying that, with appropriate choice of dividing and multiplying by the individual factors corresponding to the power series expansions of the exponentials, we recover the correct combinatorial coefficient, which can be interpreted as the number of n objects that can be selected from the ratio π (which represents the first Poisson distribution with parameter λ),

$$\frac{n!}{(\frac{\lambda}{\lambda+\mu})!\left(n - \frac{\lambda}{\lambda+\mu}\right)!}p^n(1-p)^{n-x} = \binom{n}{\frac{\lambda}{\lambda+\mu}}p^n(1-p)^{n-x} \sim \text{Bin}(n, \pi) \ .$$