Homework 9

November 14, 2020

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1 Problem 1

1.1 Part A

With Floquet's Theorem, we observe that the proposed solution y in terms of the given basis functions that we will substitute into the Mathieu Differential Equation has two solutions that are continuously differentiable. Furthermore, Floquet's Theorem states that if we have a solution of the Mahtieu's Equation that is periodic in π, we can then obtain a solution involving the eigenvalue a by looking at the characteristic equation

$$\gamma^2 - (y_1(\pi) + y_2'(\pi))\gamma + 1 = 0 \ .$$

In this characteristic equation, observe that the solutions $y_1(x)$ and $y_2(x)$ have respective eigenvalues $\gamma_1 = e^{i\alpha\pi}$ and $\gamma_2 = e^{-i\alpha\pi}$ with α an arbitrary constant. With the general knowledge of Floquet's Theorem in hand, substituting the proposed solution $y = \sum_{k=0}^{\infty} c_k e^{ikt}$ directly into **1** gives

$$\begin{aligned} \frac{d^2y}{dt^2} - 2q\cos(2t)y &= -ay \\ \Rightarrow \frac{1}{2}\sum_{k=-\infty}^{\infty} c_k(ik)^2 e^{ikt} - 2q\cos(2t)\sum_{k=-\infty}^{\infty} c_k e^{ikt} &= -\frac{a}{2}\sum_{k=-\infty}^{\infty} c_k e^{ikt} \\ \Rightarrow \sum_{k=-\infty}^{\infty} c_k(ik)^2 e^{ikt} - 4q\cos(2t)e^{i\nu t}\sum_{k=-\infty}^{\infty} c_k e^{ikt} &= -a\sum_{k=-\infty}^{\infty} c_k e^{ikt} \ . \end{aligned}$$

Replacing the cosine with a sum of complex exponentials implies that

$$\begin{aligned} \sum_{k=-\infty}^{\infty} c_k(ik)^2 e^{ikt} - 4q\left(\frac{1}{2}(e^{2it} + e^{-2it})\right)\sum_{k=-\infty}^{\infty} c_k e^{ikt} &= -ae^{i\nu t}\sum_{k=-\infty}^{\infty} c_k e^{ikt} \\ \Rightarrow e^{i\nu t}\sum_{k=-\infty}^{\infty} c_k(ik)^2 e^{ikt} - 2q\left(e^{2it} + e^{-2it}\right)\sum_{k=-\infty}^{\infty} c_k e^{ikt} &= -a\sum_{k=-\infty}^{\infty} c_k e^{ikt} \\ \Rightarrow \sum_{k=-\infty}^{\infty} c_k(ik)^2 e^{ikt} - 2q\left(\sum_{k=-\infty}^{\infty} c_k\left(e^{it(2+k)} + e^{it(-2+k)}\right)\right) &= -a\sum_{k=-\infty}^{\infty} c_k e^{ikt} \ . \end{aligned}$$

In particular, it is a well known fact that the entries of a tridiagonal matrix satisfy a reccurence relation. Now that we have substituted the proposed solutions y with the basis functions of our solution the complex exponentials, we can directly read off the tridiagonal matrix from the recurrence relation of the form

$$ac_k = k^2 c_k + qc_{k+2} + qc_{k-2} \ .$$

In particular, the observation of this recurrence relation allows us to conclude that we have the following tridiagonal matrices in one of the following 4 possibilities

1.1.1 Matrix for the Even Solutions, Even k

$$\Rightarrow \begin{bmatrix} 0 & 2q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ q & 4 & q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & q & 16 & q & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & q & 36 & q & \vdots & \vdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & q & 64 & q & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & q & 100 & q & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & (K+2)^2 & q \end{bmatrix} \times \begin{bmatrix} c_0 \\ c_2 \\ c_4 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_K \end{bmatrix} = a \times \begin{bmatrix} c_0 \\ c_2 \\ c_4 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_K \end{bmatrix} .$$

1.1.2 Matrix for the Even Solutions, Odd k

$$\Rightarrow \begin{bmatrix} 1+q & q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ q & 9 & q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & q & 25 & q & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & q & 49 & q & \vdots & \vdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & q & 81 & q & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & q & 121 & q & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & (K+2)^2 & q \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_3 \\ c_5 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_K \end{bmatrix} = a \times \begin{bmatrix} c_1 \\ c_3 \\ c_5 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_K \end{bmatrix} .$$

1.1.3 Matrix for the Odd Solutions, Odd k

$$\Rightarrow \begin{bmatrix} 1-q & q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ q & 9 & q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & q & 25 & q & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & q & 49 & q & \vdots & \vdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & q & 81 & q & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & q & 121 & q & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & (K+2)^2 & q \end{bmatrix} \times \begin{bmatrix} c_1 \\ c_3 \\ c_5 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_K \end{bmatrix} = a \times \begin{bmatrix} c_1 \\ c_3 \\ c_5 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_K \end{bmatrix}.$$

1.1.4 Matrix for the Odd Solutions, Even k

$$\Rightarrow \begin{bmatrix} 4 & q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ q & 16 & q & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & q & 36 & q & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & q & 64 & q & \vdots & \vdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & q & 100 & q & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q & (K+4)^2 & q \end{bmatrix} \times \begin{bmatrix} c_0 \\ c_2 \\ c_4 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_K \end{bmatrix} = a \times \begin{bmatrix} c_0 \\ c_2 \\ c_4 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_K \end{bmatrix}.$$

From the matrix above, observe that we have taken into account, as suggested from the discussion in the HW, the necessary boundary conditions for each case. Specifically, from the equivalence relation that we have listed previously, we know that the Boundary Conditions for the Fourier Coefficients of the odd solutions implies that the first term of the matrix is equal to $1+q$ because from the recurrence relation for our tridiagonal matrix, we directly have that $ac_1 = 1^2c_1 + qc_3 + qc_{-1}$. But from our boundary conditions, we know that $c_{-1} = c_1$, which indeed shows that the first entry is $1+q$ from the $1+q$ factor that we accumulate from combining terms. Repeating similar arguments implies that the first entry of the matrix for the odd solution that we have given is indeed what is shown above. Imposing bondary conditions on our recurrence relation, as well as making observations that relate to the symmetry of the Fourier Coefficients gives the first entries of the matrix for each of the 3 cases, with the even case being the **first** one in which the matrix is not symmetric.

1.2 Part B

We will describe the necessary similarity transformation. We want such a transformation because for the Even Solutions with Even k, the matrix is clearly not symmetric because of the $2q$ term. Therefore, we will introduce an invertible matrix P and another matrix A, so that matrix $\mapsto PAP^{-1}$. In the case of the 2 by 2 block that makes the matrix for our Even Solutions with Even k not symmetric, we observe the following. First of all, we can apply a similarity relation to the 2 by 2 block

$$\begin{bmatrix} 0 & 2q \\ q & 4 \end{bmatrix}$$

by taking

$$P = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$$

Immediately, this gives

$$PAP^{-1} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2q \\ q & 4 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2}q \\ \sqrt{2}q & 4 \end{bmatrix}$$

1.3 Part C

1.3.1 $q = 5$

n	Eigenvalue a_n, Shooting	Spectral
0	$\approx -5.8000460219779635$.	
1	$\approx 1.8581875385611433$.	
2	$\approx 7.4491097333760266$.	
10	$\approx 100.12636911904552$.	
15	$\approx 225.05581226711658$.	

n	Eigenvalue b_n, Shooting	Spectral
0	$\approx -5.790080600003919$.	
1	$\approx 2.0994604400928583$.	
2	$\approx 9.236327696995122$.	
10	$\approx 100.12636875717304$.	
15	$\approx 225.05581226711658$.	

We have included some plots for the $q = 5$ case at the end.

1.3.2 $q = 25$

Repeating the same strategy, we apply our Python function and obtain the eigenvalues a_n and b_n under the specified initial conditions with **q=25** fixed.

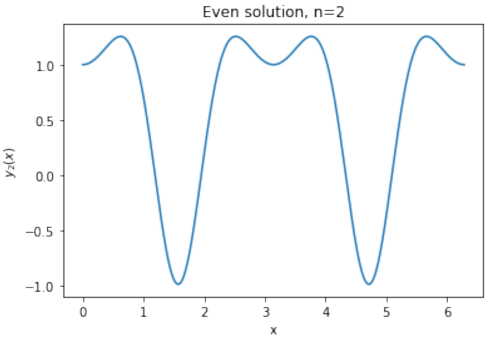
n	Eigenvalue a_n, Shooting	Spectral
0	$\approx -40.256779549657125$.	
1	$\approx -21.314899696391812$.	
2	$\approx -3.5221647357919936$.	
10	$\approx 103.23020470068373$.	
15	$\approx 226.40071983089973$.	

n	Eigenvalue b_n, Shooting	Spectral
0	$\approx -40.25677898794626$.	
1	$\approx -21.314860628151866$.	
2	$\approx -3.5209415361355454$.	
10	$\approx 103.22567964736123$.	
15	$\approx 226.4007189121707$.	

1.4 Part D

The eigenvectors of the matrix correspond to the coefficients of the Fourier Series c_k, no if's and's or but's!

1.5 Part E



1.6 Part F

From our computation and the Python code that we have implemented, we observe that the spectral method is better. In comparison to the shooting method that we implemented last week, we observe that despite this method enabling us to find solutions to the Mathieu Equation, the method was essentially not very precise because we were in effect 'shooting' depending on initial conditions that we fixed for the position and first derivative of the function y. On the other hand, with the Spectral Method, we are able to obtain to directly calculate the eigenvalues that we want from the matrix instead of having to try out various initial conditions to even see which one would work, along with which initial conditions for y and y' would ensure that we can find the desired eigenvalues for $n = 1, 5, 10, 15$ precisely.

2 Code

```
from scipy.linalg import eigh_tridiagonal
from scipy.integrate import solve_ivp
import numpy as np
import matplotlib.pyplot as plt

for q in [25,5]:
  print('q=',q)
  # Even solution, even k
  k1=0; oddk=0
  d00=[(2*k-oddk)**2 for k in range(k1,8)] # diagonal elements
  e00=[q for i in range(k1+1,8)] # off diagonal elements
  e00[0]=np.sqrt(2)*q # similarity transform

  # Even solution, odd k
  k1=1; oddk=1
  d01=[(2*k-oddk)**2 for k in range(k1,8)] # diagonal elements
  e01=[q for i in range(k1+1,8)] # off diagonal elements
  d01[0]=1+q # replace first element of diagonal

  # Odd solution, even k
  k1=1; oddk=0
  d10=[(2*k-oddk)**2 for k in range(k1,8)] # diagonal elements
  e10=[q for i in range(k1+1,8)] # off diagonal elements

  # Odd solution, odd k
```

```
  k1=1; oddk=1
  d11=[(2*k-oddk)**2 for k in range(k1,8)] # diagonal elements
  e11=[q for i in range(k1+1,8)] # off diagonal elements
  d11[0]=1-q # replace first element of diagonal

  w00,v00 = eigh_tridiagonal(d00,e00)
  w01,v01 = eigh_tridiagonal(d01,e01)
  w10,v10 = eigh_tridiagonal(d10,e10)
  w11,v11 = eigh_tridiagonal(d11,e11)

  print(w00); print(w01)
  print(w11); print(w10)

# Plotting even solution for n=1
def p(t,y):
    return(np.array([y[1],-(w00[1]-2*q*np.cos(2*t))*y[0]]))
result = solve_ivp(p,(0,2*np.pi),[1,0],rtol=1e-10,atol=1e-10)
plt.plot(result.t,result.y[0])
plt.title('Even solution, n=2')
plt.xlabel('x')
plt.ylabel('$y_2(x)$')
```

3 References

[1] The Discretised harmonic oscillator: Mathieu functions and a new class of generalised Hermite polynomials, M. Aunola.
[2] Notes on Numerical Methods from http : //depts.washington.edu/bdecon/workshop2012/c2$_s$tability.pdf.