Homework 4

November 27, 2020

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1 Problem 1

1.1 Part A

We will show that r is smooth. First, we will let f be an immersion at $a \in M$, and we will also let $(U, \varphi), (V, \psi)$ be the charts centered at a and $f(a)$, respectively. On (V, ψ), we will take the chart adapted to (U, φ) such that $\psi \circ f \circ \varphi^{-1}(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0)$. Now, we will precisely define r with the following, in which we will take $r : V \longrightarrow U$ to be the map for which $r(v) = \varphi^{-1}(x_1, \cdots, x_m, 0, \cdots, 0)$. In this equality for $r(v)$, we took $v \in V$ with $\psi(v) = (x_1, \cdots, x_n)$. From this definition and properties of r, we know that r is smooth because

$$\varphi \circ r \circ \psi^{-1}(x_1, \cdots, x_m, \cdots x_n) = (x_1, \cdots, x_m) \ .$$

Above, observe that this expression is a projection onto the first m coordinates which is smooth. We also have that r is a left inverse of f because φ is locally a homeomorphism, in which

$$\begin{aligned} \varphi \circ r \circ f \circ \varphi^{-1}(x_1, \cdots, x_m) &= \varphi \circ r \circ \psi^{-1} \psi \circ f \circ \varphi^{-1}(x_1, \cdots, x_m) \\ &= \varphi \circ r \circ \psi^{-1}(x_1, \cdots, x_m, 0, \cdots, 0) \\ &= (x_1, \cdots, x_m) \ . \end{aligned}$$

Because r is a left inverse of f, we have that there exists open neighborhoods U of a, and also V of $f(a)$, with $f(U) \subseteq V$. Beyond merely claiming that $f(U) \subseteq V$, it is important to note that we have a retraction $r : V \longrightarrow U$ of f, which we have shown is smooth above.

To show that f is an immersion, at this point it is sufficient to show that $T_a f$ is injective. We know that this claim holds because if we assume that these open neighborhoods and a smooth retraction exists, we can look at the identity mapping $rf : U \longrightarrow U$. By definition, $r \circ f = \mathbf{1}_U$. By the chain rule, differentiating this equality gives that $\mathrm{id} = T_{f(a)} r \circ T_a f$. Finally, this implies that $T_a f$ is a left invertible linear transformation. Then by definition, we know that $T_a f$ is injective and also that f is an immersion because $\mathrm{rank}(\mathrm{id}) \leq \mathrm{rank}(T_a f) \leq m$ which gives that $\mathrm{rank}(T_a f) = m$.

1.2 Part B

We will recall some of our arguments from **A**. That is, if we have an immersion f at $a \in M$, there exists open neighborhood of a, U, and an open neighborhood of $f(a)$, V, such that $f(U) \subseteq V$ and f has a smooth retraction $r : V \longrightarrow U$. r being smooth means that there exists a smooth map $r : V \longrightarrow U$ with $r \circ f : U \longrightarrow U$ being the identity. Now, we know that f is injective on U because if we take some $u_1, u_2 \in U$, $f(u_1) = f(u_2) \Rightarrow rf(u_1) = rf(u_2) \Rightarrow u_1 = u_2$. Therefore, f being injective on U allows us to conclude that f is locally injective at a because there is some neighborhood U of a on which $f : U \longrightarrow V$ is injective. This establishes that any immersion f is locally injective.

1.3 Part C

If f is a submersion, by the Submersion Theorem we know that there exisgs adapted charts (U, ϕ) and (V, ψ) centered at a and $f(a)$ respectively, so that

$$\psi \circ f \circ \varphi^{-1}(x_!, \cdots, x_m) = (x_1, \cdots, x_n) \ .$$

In the above equality, we have that $m \leq n$. Moreover, we know that $\varphi(U)$ contains a rectangular prism, which is useful for us because if $\varphi(U)$ is a rectangular prism, then changing any of the coordinates of $(x_1, \cdots, x_m) \in \varphi(U)$ to 0 stays in $\varphi(U)$. So $\varphi(U)$ containing this rectangular prism enables us to restrict U and $\varphi(U)$. As a result, we will analyze $\phi \circ f \circ \varphi^{-1}$. In particular, $\phi \circ f \circ \varphi^{-1}$ is a projection onto the first n coordinates, which implies that $\phi \circ f \circ \varphi^{-1}(\phi(U))$ is open because $\phi \circ f \circ \varphi^{-1}$ is an open map. In turn, it more conveniently follows that $\psi \circ f(U)$ is open, and because ψ is a homeomorphism, $\psi^{-1}(\psi(f(U))) = f(U)$ is then open. So if we take $v \in f(U)$, with $\psi(v) = (x_1, \cdots, x_n)$, we can define a map $s : f(U) \longrightarrow U$ with $s(v) = \varphi^{-1}(x_1, \cdots, x_n, 0, \cdots, 0)$. In the last expression for $s(v)$, note that $(x_1, \cdots, x_n, 0, \cdots, 0)$ is in $\phi(U)$ because from our previous discussion of the rectangular prism, setting one or more coordinates of $(x_1, \cdots, x_m)$ equal to 0 implies that the new expression remains in $\phi(U)$. Furthermore, s can be expressed in the coordinates of ψ and φ as

$$\varphi \circ \psi^{-1}(x_1, \cdots, x_n) = (x_1, \cdots, x_n, 0, \cdots 0) \ .$$

From this expression, observe that the final term that I have given demonstrates that s is injecting into a large space which in turn makes it smooth. Also, we know that s is a smooth section because $f \circ s(\psi^{-1}(x_1, \cdots, x_n)) = f(\phi^{-1}(x_1, \cdots, x_n) = \psi^{-1}(x_1, \cdots, x_n)$. Finally, we have that $s(f(a)) = a$ because $\psi(f(a)) = (0, \cdots, 0)$, which implies that $\phi(s(f(a)) = (0, \cdots, 0)$.

The other direction is much shorter. If we assume that an open neighborhood U of a and such a smooth section exist, then we can look at the identity mapping $fs : f(U) \longrightarrow f(U)$ and observe that $\mathrm{Id} = T_{f(a)}(fs) = T_a f \circ T_{f(a)} s$. This equality implies that, as a linear transformation, $T_a f$ has a right inverse which therefore makes $T_a f$ surjective. Finally, we have that f is a submersion which is what we originally wanted to show from this direction.

1.4 Part D

We will show that submersions are open maps. If f is a submersion and $U \subseteq M$ is an open set, then we can proceed by taking some $x \in U$, and from **C**, we know that there exists an open neighborhood $U_x \subseteq M$ of x with $f(U_x)$ open. Now, if we let $V_x = U \cap U_x$, we know that V is open and also that $V \subseteq U$. As stated previously, we then know that $s : f(U_x) \longrightarrow U_x$ is a smooth section and because s is by definition a left inverse, $s \circ (f(v)) = v$, and this equality implies that s takes $f(V_x)$ into V_x. As a result, if we have some $y \in f(U_x) \backslash f(V_x)$ then $y = f(z)$ for some $z \notin V_x$, which finally implies that $x(y) = z \notin V_x$. Altogether, we now have that $f(V_x)$ is open because $s^{-1}(V_x) = f(V_x)$ because s is smooth, in addition to the preimage of V_x under s being open.

Furthermore, we can take some open neighborhood $f(V_x)$ of $f(x)$, and with this neighorhood, we know that $f(V_x) \subseteq f(U)$ because $V_x \subseteq U$. From this, we know that $f(V_x)$ is open. Extending this observation for every $x \in U$ implies that $\cup_{x \in U} f(V_x) = f(U)$ and because $f(U)$ is a union of open sets, $f(U)$ is therefore open. This establishes that all submersions are open maps, which is what we wanted to show.

2 Problem 2

2.1 Part A

We will show that f is an injective homomorphism with the following. We will take $t, x \in (-1, \infty)$ and $f(t) = f(s)$. If we suppose that $t-s = 0$, we can then show that f is surjective by rearranging terms between the first coordinates because they are equal

$$\begin{aligned}\frac{3at}{1+t^3} &= \frac{3as}{1+s^2} \\ t(1+s^2) &= s(1+t^3) \\ t-s &= ts(t-s)(t+s) \\ 1 &= ts(t+s)\end{aligned}$$

Because all of the other coordinates are equal, we can manipulate the following expressions in a similar way

$$\begin{aligned}\frac{3at^2}{1+t^3} &= \frac{3as^2}{1+s^3} \\ t^2+s^3t^2 &= s^2+t^3s^2 \\ (t-s)(t+s) &= t^2s^2(t-s) \\ t+s &= t^2s^2 \\ 1 &= t^3s^3 \\ 1 &= ts = t+s \\ 1 &= t+\frac{1}{t} \\ t &= t^2+1\end{aligned}$$

From this last equality, we can observe the following. The quadratic equation $t^2 - t + 1$ has no real solutions unless there exists t, s with $t - s = 0$. The steps above establish that $f(t) = f(s) \longrightarrow t - s = 0 \longrightarrow t = s$. This equality directly implies that f is surjective, which takes care of one of the things that we wanted to show. Now, we will move onto showing that f is an immersion by taking $x \in I$. With this x, we know that

$$T_x f = \left(\frac{3a(1+x^3) - 3ax(3x^2)}{(1+x^3)^2}, \frac{6ax(1+x^3) - 3ax^2(3x^2)}{(1+t^3)^2} \right) = \left(\frac{3a-6ax^3}{(1+x^3)^2}, \frac{6ax-3ax^4}{(1+x^3)^2} \right).$$

By definition, we know that f would be an immersion if T_xM has constant rank which is 1. To show that this claim holds, we observe that the expression above would have rank 1 if the point that we are choosing to evaluate T_xf at is $(0, 0)$. For the sake of contradiction, assume that $T_xf = (0, 0)$ for some $x \in I$. Then, we would have that each of the components of T_xf vanish, namely that $3a - 6ax^3 = 0$ and $6ax - 3ax^4 = 0$. From each of these equations,we will concentrate on the first one. Solving for x in this expression gives that

$$1 = 2x^3 \Rightarrow x = \frac{1}{2^{\frac{1}{3}}},$$

which is helpful. Substituting this value of x into the second equation implies that

$$\begin{aligned}\frac{6a}{2^{\frac{1}{3}}} &= 3a\left(\frac{1}{2^{\frac{1}{3}}}\right)^4\\ \frac{6}{2^{\frac{1}{3}}} &= \frac{3}{2^{\frac{4}{3}}}\\ 6 &\neq \frac{3}{2}\ .\end{aligned}$$

This contradiction demonstrates that the expression for T_xf that we have been looking at has constant rank $= 1$. By the Immersion Theorem, it follows that f is a immersion, and, again from our previous arguments, we have also shown that f is surjective.

2.2 Part B

In short, we know that f is not an embedding because it has a discontinuous inverse. More specifically, we can see that this claim holds by taking an open neighborhood $\mathcal{U}$ of 0 that is in I. We know that $f^{-1}(\mathcal{U})$ would not be open in $\mathbf{R}^2$ because $f^{-1}(\mathcal{U})$ contains the origin but not any ball around the origin. From these observations, we also know that $f(I)$ would not be a topological manifold because topological manifolds have the property that they are path connected. In this case, $f(I)$ is not locally path connected because it does not admit a basis of open coordinates, specifically because $f(I) \cap B_0$ is not path connected for any ball B_0 around the origin.

From my plot of the expression, we see that $f(I)$ is not path connected because any sufficiently small ball around the origin illustrates that $f(I) \cap B_0$ is not path connected. See my picture for such a ball centered around the origin. This takes care of the last part of **B**.

2.3 Part C

We will show that an injective immersion onto a Hausdorff Manifold N is not an embedding by showing that the map $f^{-1} : fM \longrightarrow M$ is continuous. In this claim, f^{-1} is a map that takes each point $y \in fM$ to its unique inverse $x \in M$. We know that the inverse of y under f^{-1} is well-defined because f is surjective. So we will begin by taking an open set $U \subseteq M$, as well as taking some $x \in U$. With this, we will take open sets $\mathcal{V}'_{xy}, \mathcal{V}'_{yx} \subseteq N$, with $x \in \mathcal{V}_{xy}, y \in \mathcal{V}'_{yx}, x \notin \mathcal{V}'_{yx}$ and $y \notin \mathcal{V}'_{xy}$. From this, we will also set $\mathcal{V}_{xy} = \mathcal{V}'_{xy} \cap fM, \mathcal{V}_{yx} = \mathcal{V}'_{yx} \cap fM$. We have that the sets V_{ij} are each open in fM.

Now, we will set $\mathcal{A} = \{U\} \cup \{f^{-1}(\mathcal{V}_{yx})|y \in fM\backslash f(U)\}$. Because the map f is continuous, the inverse image $f^{-1}(\mathcal{V}_{yx})$ is open. Furthermore, we know that $\mathcal{A}$ covers U because for all $z \in M$, it is possible that $z \in U$ or $f(z) \in fM\backslash f(U)$, which implies that $z \in f^{-1}(\mathcal{V}_{f(z)x})$. By definition, M being compact implies that $\mathcal{A}$ has a finite subcover. Additionally, because U is the only set that contains x, U would be in the finite subcover of $\mathcal{A}$. The existence of this finite subcover guarantees that we have $y_1, \cdots, y_r$ so that $f^{-1}(\mathcal{V}_{y_1x}), \cdots, f^{-1}(\mathcal{V}_{y_nx})$ together cover $M\backslash U$. Then, $\mathcal{V}_{y_1x}, \cdots, \mathcal{V}_{n_nx}$ cover $fM\backslash f(U)$. Altogether, we have that $\mathcal{V}_{x_1y} \cap \cdots \cap \mathcal{V}_{x_ry}$ is a finite intersection of open sets. This implies that $\mathcal{V}_{x_1y} \cap \cdots \cap \mathcal{V}_{x_ny}$ is an open set which does not intersect any $\mathcal{V}_{x_iy}$, and that this open set must be contained in $f(U)$. Similarly, we also know that $f(x) \in \mathcal{V}_{xy_1} \cap \cdots \cap \mathcal{V}_{xy_r}$ because $f(x) \in \mathcal{V}_{xy_i}$ for all i. With all of these arguments, we have that $f(x) \in f(U)$, in which there exists an open set $\mathcal{V}_{xy_1} \cap \cdots \cap \mathcal{V}_{xy_r} \subseteq f(U)$. This establishes that $f(U)$ is open in fM. To conclude, we have shown that f^{-1} is continuous because for all $f^{-1}(f(U)) \subseteq M$ that are open, $f(U)$ is open in fM. We also have that f is an embedding, which finishes **C**.

3 Problem 3

We will show that $M^{\sim}$ has the desired structure with the following. We begin by taking $x \in M$, as well as an open neighborhood U at x with $\pi(u)$ open and $\pi : U \longrightarrow \pi(U)$ a homeomorphism which exists because π is a local homeomorphism. Also, we will take another chart (V, ψ) at $\pi(x)$, and $V_0 = \pi(U) \cap V$, with $\psi_0 = \psi|_{V_0}$. From the definitions that we are given, (V_0, ψ_0) is a chart at x with $V_0 \subset \pi(U)$ open. Next, observe that $U_0 = \pi^{-1}(V_0)$ is an open set because π is a local homeomorphism. Furthermore, we know htat U_0 is still an open neighborhood of x.

From here, we will look at the map $\phi : 0 \longrightarrow \mathbf{R}^n$. In this map, the n in the image space denotes the dimension of M, with $\phi(y) = \psi_0(\pi(y))$, and $y \in U_0$. By construction, we know that ϕ is a homeomorphism from U_0 to $\mathbf{R}^n$ because π is a homeomorphism on U_0 and ψ_0 is a homeomorphism on $\pi(U_0) = V_0$. Altogether, this implies that (U_0, ϕ) is a chart at x. As we have done in past homeworks, we can then look at the following composition because from the coordinates of the adapted charts ϕ and ψ_0, we know that

$$\psi_0 \circ \pi \circ \phi^{-1}(x_1, \cdots, x_n) = \psi_0 \circ \pi \circ \pi^{-1}(x_1, \cdots, x_n) = (x_1, \cdots x_n) \ .$$

From this equality, we know that π is smooth at x with respect to $(U_0 m, \phi)$. In a similar way to the definitions that we are given, we know that M is locally a homeomorphism for any smooth structure on $M^{\sim}$. In addition, we also know that the smooth structure on $M^{\sim}$ has to contain the chart (U_1, ϕ_1) at x, and that the coordinates of π are given by the composition that I have expressed above. Then, it follows that the smooth structure on $M^{\sim}$ on which the map π is smooth must contain the chart (U_0, ϕ)m . To show that this claim holds, we will consider the transition map for the charts (U_0, ϕ_0) at $x \in M^{\sim}$ and the second chart (U_1, ϕ_1) at $y \in M^{\sim}$. With these charts, we know that the charts have corresponding adapted charts (V_0, ψ_0) and (V_1, ψ_1) which are respectively at $\pi(x)$ and $\pi(y)$. With these charts, we have that

$$\begin{aligned} \phi_0 \circ \phi_1^{-1}(x_1, \cdots, x_n) &= \psi_0 \circ \pi \circ \pi^{-1} \circ \psi^{-1}(x_1, \cdots, x_n) \\ &= \psi_0 \circ \psi_1^{-1}(x_1, \cdots, x_n) \ . \end{aligned}$$

We know that the last expression above is smooth because (V_0, ψ_0) and (V_1, ψ_1) are compatible charts. This statement implies that every (U_0, ϕ_0) and (U_1, ϕ_1) are compatible charts. Finally, we have the desired smooth structure on $M^{\sim}$ because the compatible charts (U_0, ϕ_0) and (U_1, ϕ_1) covering $M^{\sim}$ determine an atlas and the smooth structure on $M^{\sim}$ because π is smooth with constant rank n. From the arguments in previous problems, we know that π is a submersion.

To show that π is a local diffeomorphism, we will take adapted charts (U_0, ϕ_0) at $x \in M^{\sim}$ and (V_0, ψ_0) on M. From the charts that we have defined previously, we know that

$$\psi_0 \circ \pi \circ \phi_0^{-1}(x_1, \cdots, x_n) = \psi_0 \circ \pi \circ \pi^{-1} \circ \psi^{-1}(x_1, \cdots, x_n) = (x_1, \cdots, x_n) \ .$$

From the coordinates above, we have that π is the identity mapping. Therefore, π is a diffeomorphism from U_0 to $\pi(U_0)$. We conclude that π is a local diffeomorphism because for each $x \in M^{\sim}$, there is some open neighborhood U_0, with $x \in U_0$, and π a diffeomorphism from U_0 to the image $\pi(U_0)$. This completes one of the properties that we wanted to show.

We know that π is smooth because it is a local diffeomorphism. If we know that f is smooth, then the composition $\pi \circ f$ is also smooth. Similarly, π is smooth because $g \circ \pi(y) = \pi \circ f(y)$ is smooth at every $y \in M^{\sim}$. So because π is smooth, then $g(x)$ is smooth at every $x = \pi(y) \in M$. By the surjectivity of π, we know that f being smooth implies that g is smooth. In fact, if we assume that g is smooth, then by the smoothness of π, $g \circ \pi(y) = \pi \circ f(y)$ for all $y \in M^{\sim}$. As we stated previously, in this case for $y \in M^{\sim}$ we will take an open neighborhood U, with $f(y) \in U$, on which π is a diffeomorphism. Again by the definitions, π being a local diffeomorphism on U implies that π has a smooth inverse from $\pi(U) \longrightarrow U$. Finally, $\pi \circ f$ being smooth on $f^{-1}(U)$ implies that f is smooth on $f^{-1}(U)$. Therefore, f is smooth at y. Similar to what we have already mentioned, g being smooth implies that f is smooth. Putting everything together, we have that f is smooth iff g is, which completes the last part of **2**.

4 Problem 4

4.1 Part A

We will show that the conditions hold with the following. We begin taking a chart (B, ψ) of the submanifold $M \subseteq \mathbf{R}^n$ at the origin. From M, observe that $d_0\psi_1, \cdots, d_0\psi_m$ are linearly independent, so they form a basis for $T_0M = E$. From this basis, we will set $e_i = d_0\psi_i$ for $1 \leq i \leq m$. If we have an induced chart on M, (A, ϕ), from (B, ψ), and the inclusion map $i : M \longrightarrow \mathbf{R}^n$, we can identify ϕ as a mapping from $A \longrightarrow E$, because $E \cong \mathbf{R}^k$. From this isomorphism, it becomes apparent that we can also choose $d_0\psi_1, \cdots, d_0\psi_m \cup \mathcal{B}$, where $\mathcal{B} = \{f_{m+1}, \cdots, f_n\}$. But to show that the **first** condition holds, we will study the composition $i \circ \phi^{-1} : E \longrightarrow \mathbf{R}^n$. From $i \circ \phi^{-1}$, we know that the first m coordinates is the identity mapping on the m coordinates. For the last $n - m$ coordinates, we define g as

$$i \circ \phi^{-1}(x_1, \cdots, x_n) = (x_1, \cdots, x_m, g(x_1, \cdots, x_m)) \ ,$$

as mentioned in class. So we know that if π is a projection map onto the last $n - m$ coordinates of $\mathbf{R}^n$, then automatically we have that π, i and ϕ^{-1} are all smooth. Because the composition of smooth maps is smooth, we have that $g = \pi \circ i \circ \phi^{-1}$ is also smooth.

We will then denote $U \subseteq E$ as the image of ϕ, and we observe that this image is open because A is. Also, we will denote the image $V \subseteq F$ of $g(U)$, and we observe that this image is open because i and π are. From these claims, we can show that $g(0) = 0$ because with B centered at the origin, A is also centered at the origin. Furthermore, $0 \in U$ and $i \circ \phi^{-1}(0) = 0$. This equality holds because the origin is on M, which allows us to conclude that $g(0) = 0$, and also that $0 \in V$.

We will now show that the rest of the desired properties hold. Suppose that $x \in \mathbf{R}^n$ is in $U \times V$ and on M. We then know that $x = (x_1, \cdots, x_n)$ with $(x_1, \cdots, x_m) \in U$ and that $i \circ \phi^{-1}(x_1, \cdots, x_n) = (x_1, \cdots, x_n)$. Therefore, we know that x is in the graph of G because $g(x_1, \cdots, x_n) = (x_{m+1}, \cdots, x_n)$. It also follows that if x is in the graph of g in $U \times V$, we know that

$$x = (x_1, \cdots, x_m, g(x_1, \cdots, x_m)) = i \circ \phi(x_1, \cdots, x_m) \in M \ .$$

The steps above demonstrate that any point on g's graph is in $M \cap (U \times V)$. Finally, we get the map g that we want because $M \cap (U \times V)$ is in the graph of g, which completes the **first** part of **4**.

4.2 Part B

We can show that this property holds by taking $g_1 : U_1 \longrightarrow V_1$ and $g_2 : U_2 \longrightarrow V_2$ be 2 maps. In particular, we will want g_1, g_2 to satisfy the property that we showed holds in **A**. With this property, we know that if $U_1 \cap U_2$, and if M is in the graph of g_1 and g_2, then $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \in M$ iff $g_1(x_1, \cdots, x_m) = g_2(x_1, \cdots, x_m) = (x_{m+1}, \cdots, x_n)$. Then, we know that $g_1(x_1, \cdots, x_m) = g_2(x_1, \cdots, x_m)$ because for all $(x_1, \cdots, x_m) \in U_1 \cap U_2$, then there exists a unique point in M with the coordinates $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n)$. Therefore, we have that $g_1 = g_2$ on all points $(x_1, \cdots, x_m)$ in $U_1 \cap U_2$, which finishes **B**.

4.3 Part C

We know that $g(0) = 0$ because $(0, 0) \subseteq M$, $0 \in U$ and $0 \in V$, which implies that $(0, 0) \subseteq U \times V$. So $(0, 0)$ is contained in the graph of g, which shows that the first property that we wanted is true, in which $g(0) = 0$ because the chart ϕ is centered at the origin and π_2 does not move the origin. Next, let $g = (g_{m+1}, \cdots, g_n)$. From the entries of g, we have that $D(g_j)_0 e_i = 0$, because the jth entry of $(i \circ \phi^{-1})'$, evaluated at e_i, is 1 if $j = i$ and 0 otherwise. However, we know that we are in the 'otherwise' case because we are looking at the projection onto the last $n-m$ coordinates, which in this case implies that $j \neq i$. Therefore the derivative must vanish. In short, by the chain rule we know that $Dg_0 = D(\pi_2 \circ \phi^{-1} \circ D(\phi)) = D\pi_2 \circ D\phi^{-1} \circ D(\phi)$ and that each of the terms vanish from what I have mentioned above.