Homework 11

November 27, 2020

---

1 Problem 1

1.1 Part A

1.1.1 From the Limit Definition

We will directly apply the limit definition of the derivative, which states that

$$\frac{d}{dt}\theta_t^*(fs) = \lim_{t'\to 0}\frac{\theta_{t+t'}^*(fs) - \theta_t^*(fs)}{t'} \ .$$

From this equality, splitting pullbacks of the product shows that

$$\Rightarrow \lim_{t'\to 0}\frac{\theta_{t+t'}^*(f)\theta_{t+t'}^*(s) - \theta_t^*(f)\theta_t^*(s)}{t'} \ ,$$

and in the final steps below we will prove the usual product rule. Altogether, the limit is

$$\begin{aligned} &\lim_{t'\to 0}\frac{\theta_{t+t'}^*(f)\theta_{t+t'}^*(s) + \theta_{t+t'}^*(f)\theta_t^*(s) - \theta_{t+t'}^*(f)\theta_t^*(s) + \theta_t^*(f)\theta_t^*(s)}{t'} \\ &= \lim_{t'\to 0}\theta_{t+t'}^*(f)\frac{\theta_{t+t'}^*(s) - \theta_t^*(s)}{t'} + \lim_{t'\to 0}\theta_{t+t'}^*(s)\frac{\theta_{t+t'}^*(f) - \theta_t^*(f)}{t'} \ , \end{aligned}$$

which is equivalent to

$$\theta_t^*(f)\frac{d}{dt}\theta_t^*(s) + \theta_t^*(s) + \frac{d}{dt}\theta_t^*(f) \ .$$

Finally, evaluating all of these terms at $t = 0$, while observing that θ_0 is the identity map, implies that

$$f\mathcal{L}_\xi(s) + s\mathcal{L}_\xi(f) \ ,$$

because we pulled back along the identity θ_0 which directly gives the expression above. This completes **A**.

1.1.2 From the Lie Bracket

We will show that the given formula for the product rule with the Lie Derivative holds. To do this, we begin by making use of the definition of the Lie Bracket, in which we know as a general fact that vector fields are derivations that map scalar fields to scalar fields. Moreover, if we have 2 scalar fields, we can compose them together with the function composition $U \circ V$, which in terms of the Lie Bracket can be expressed directly from the definition, in which $[U, V] = U \circ V - V \circ U$. From the definition of the bracket, we can also define the equalities $L_U V = [U, V]$ and $L_U f = Uf$ for vectors and scalars, respectively. From these observations, we can then deduce the ordinary Leibniz Rule for the Lie Derivative by looking at the expression $L_U(f \cdot V)x$. Conveniently, we can express this term with the Lie Bracket, in which we have that $L_U(f \cdot V)x = [U, f \cdot V]x$. Directly from this expression, we have that the ordinary Leibniz Rule holds, in particular the equality $Uf \cdot Vx + f \cdot U(Vx)$. In turn, we have that the ordinary Leibniz Rule holds because from this previous step, we conclude that $L_U(f \cdot V) = L_U f \cdot V + f \cdot L_U V$. This completes **A**.

1.2 Part B

1.3 With Bump Functions

We will show that such a smooth section s exists satisfying $s(b) = h$. To do this, we will introduce a vector bundle (U, φ), with $b \in U$, as well as a smooth function $f : U \to \mathbf{R}^n$ where $b \mapsto \varphi(h)$. To guarantee that $s(b) = h$, we will define the smooth section s, where $s : U \to E : x \mapsto \varphi^{-1}(x, f(x))$. In the most general case, we know that our smooth section s is defined on some open set U. To account for the possibility that every local section on a manifold cannot be extended to a global section, we will introduce some bump function g. With g, we will define it as a piecewise function that $= 1$ at b and for a neighborhood $\mathcal{A} \subset U$ of b. On $\mathcal{A}$, we will define g to be $= 0$.

Our bump function allows us to modify s, from which we conclude that

$$\begin{aligned} s(x) &= \varphi^{-1}(x, g(x)f(x)) \text{ for } x \in U \\ &= 0 \text{ for } x \notin U \ . \end{aligned}$$

From the expression of $s(x)$ above, we observe that it is smooth. We know this because looking at the overlap of the open sets U and V, namely $U \backslash V$, demonstrates that the smooth functions 0 and gs are equal. Furthermore, we also have another section $\tilde{s}$, where $\tilde{s}$ is precisely the zero section on $B \backslash V$. At b, $\tilde{s}$ is equal to s and is smooth. To conclude, we have that $s(b) = h$, which completes **B**.

1.4 Without Bump Functions

We observe that $E_b = \mathcal{F}(T_b B)$ for all $b \in B$. Next, we will introduce $v \in T_{b_0} B$ so that $\mathcal{F}(v) = h$, and we know that this v must exist because $\mathcal{F}$ is surjective. We will also introduce a smooth constant vector field τ, which is defined as

$$\tau : B \to TB : b \mapsto v_b \in T_b B \ .$$

From this expression for τ, to be clear, we observe that τ is a smooth constant vector field on B with $\tau \in \Gamma(B, TB)$. With τ, the image v_b of b has an equivalent expression at b as v does under another chart at b_0, with $h \in E_{b_0}$ fixed.

With a remark from Lecture on 11/19, we know that

$$\Gamma(B, \mathcal{F}(TB)) = \mathcal{F}(\Gamma(B, TB)) \ ,$$

for a smooth contravariant functor $\mathcal{F}$. Now, the section s can be expressed as

$$s = \mathcal{F}(t) : B \to E : b \mapsto \mathcal{F}(\tau(b)) \ ,$$

that is given by $s(b) = \mathcal{F}(\tau(b))$. From usual arguments, we know that s is smooth because $\mathcal{F}$ and τ are. Therefore,

$$s(b_0) = \mathcal{F}(\tau(b_0)) = \mathcal{F}(v) = h \ .$$

This completes **B**.

1.5 Part C

1.5.1 k - linearity

Immediately,

$$(\sigma_1 + \sigma_2)(b)s(b) = \sigma_1(b)s(b) + \sigma_2(b)s(b) \ ,$$

which shows that the image of the sum $\sigma_1 + \sigma_2$ of sections is equivalent to the sum of the individual images for σ_1 and σ_2, for any s, b. This equality holds because we can carry out addition in the usual way pointwise. This k-linearity is significant because in a similar way we have that

$$P^\sharp(f\sigma) = f(b)\sigma(b)s(b) = f(b)P^\sharp(\sigma) \ ,$$

for smooth f on B.

1.5.2 Injectivity

Suppose that $P^\sharp(\sigma_1) = P^\sharp(\sigma_2)$. To show that this assumption implies that $\sigma_1 = \sigma_2$, we observe that for all smooth sections s, we have that

$$b \mapsto \sigma_1(b)s(b) \ ,$$

and

$$b \mapsto \sigma_2(b)s(b) \ ,$$

define the same function. However, from **2B** we have conditions that our smooth section s must satisfy. With bump functions or not, we saw that $b \in B$ and $h \in E_b$ in the equality $s(b) = h$. This implies that

$$\sigma_1(b)(h) = \sigma_2(b)(h) \ ,$$

for all $h \in E_b$. We observe that this equality holds for all b, which implies that $\sigma_1 = \sigma_2$.

1.5.3 Surjectivity

To show surjectivity, we will begin with the case in which we have a trivial bundle. In this situation, we know that $\Gamma(B, E)$ and $\Gamma(B, E^*)$ as $C^\infty(B)$-modules are free. In particular, these $C^\infty(B)$-modules have rank r, which allows us to conclude that the vector bundle being trivial implies that we have a basis $s_1, \cdots, s_b$ of $\Gamma(B, E)$. We have this basis for $\Gamma(B, E)$ because our vector bundle being trivial gives us a global frame that we can make use of to get the basis $s_1, \cdots, s_b$. Next, we observe that the basis of $\Gamma(B, E)$ gives a basis of $\Gamma(B, E^*)$, because we can introduce a smooth $\overline{s}_i(b) : E_b \to \mathbf{R}$, that satisfies $\overline{s}_i(b)(s_j(b)) = \delta_{i,j}$. In this previous equality, observe that $\delta_{i,j}$ is the Kronecker δ; moreover, with the linear functional $\overline{s}_i(b)$, we can obtain a basis of $\Gamma(B, E^*)$ because we have a basis $s_1(b), \cdots, s_r(b)$ of E_b. Specifically, we have this basis for $\Gamma(B, E^*)$ because r of the basis elements are k-linearly independent. Similarly, taking $P^\sharp$ and applying it to the smooth $\overline{s}_i$ would give a composition, which for s, elementwise takes

$$s \mapsto (b \mapsto \overline{s}_i(b)s(b)) \ .$$

We observe that the elements are linearly independent because we get the Kronecker Delta $\delta_{i,j}$ after we substitute $s_i(b)$. Therefore, these arguments show us that $\Gamma(B, E^*)$ having the same dimension as the product $\dim_k \Gamma(B, E) \cdot \dim_k(k)$. Also, we know that the basis that we have specified for $\Gamma(B, E)$ is sent under $P^\sharp$ to a basis of $\Gamma(B, E^*)$. We have this property under $P^\sharp$ because for each $f \in \Gamma(B, E^*)$, there exists some s satisfying $P^\sharp(s) = f$. By injectivity of $P^\sharp$ we know that this s is unique, as mentioned in office hours. These arguments show that $P^\sharp$ is an isomorphism if we have a trivial vector bundle.

Otherwise, we can still make use of our previous arguments. We know that a vector bundle is locally trivializable, which implies that we can make use of a vector bundle atlas. On this atlas, there exists some section s that is mapped to f under $P^\sharp$, which we have already mentioned. Finally, we have that $P^\sharp$ is surjective because we can glue sections of the type that we have described together. With this construction, we can glue the sections together by observing that on the overlap of any of these charts, our section being unique by the surjectivity of $P^\sharp$ implies that s must agree on the overlap between the charts. To conclude, our work shows that $P^\sharp$ is an isomorphism, which completes **C**.

2 Problem 2

2.1 Part A

We can show that the annihilator subspace that is defined at the beginning is of the given form, in which we will first assume that α is some decomposable element. From α, we know by definition that this form can be expressed as $\alpha = e_1 \wedge \cdots \wedge e_m \wedge \alpha'$, where in this equality $\{e_1, \cdots, e_m\}$ is a basis and $\alpha' \in A^k(V)$. First of all, we know that the dimension of the annihilator subspace has k elements in it because a decomposable element α is of the form $\alpha = v_1 \wedge \cdots \wedge v_k$. This shows that the dimension of the annihilator, because of these k elements which comprise the decomposable α, is at least k. Now, we can show that the k dimensional subspace is of the given form, in which the dimension of the annihilator could be greater than k, by taking some $\alpha \in A^k(V)$, in addition to the linearly independent subset $\{e_1, \cdots, e_m\}$ of $\alpha^\perp$. From the annihilator, we know that its dimension could be larger than k if we can find other v that are linearly independent from v_i, satisfying $\alpha \wedge v = 0$.

From this point onwards, our argument will be focused towards extending this linearly independent subset to form a basis of V. This can be done by observing that there exists an induced basis of the alternating algebra $A^k(V)$, which allows us to express our differential form α as

$$\alpha = \sum_{1<i_1<\cdots<i_k<n} a_{i_1\cdots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} \ ,$$

because the induced basis for $A^k(V)$ is given by $\{e_{i_1} \wedge \cdots \wedge e_{i_k} | 1 \leq i_q < \cdots < i_k \leq n\}$, with $a_{i_1 \cdots i_k}$ arbitrary coefficients. Altogether, computing the wedge product

$$e_1 \wedge \alpha = \sum_{1<i_1<\cdots<i_k<n} a_{i_1 \cdots i_k} e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_k} \ ,$$

which will help demonstrate that $a_{i_1 \cdots i_k} = 0$ for $1 < i_1 < \cdots < i_k < n$. Rearranging the term above more shows that

$$\begin{aligned}\sum_{1<i_1<\cdots<i_k<n} a_{i_1 \cdots i_k} e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_k} &= \sum_{1<i_1<\cdots<i_k<n} a_{i_1 \cdots i_k} e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_k} \\ &= \sum_{1<i_2<\cdots<i_k<n} a_{i2 \cdots i_k} e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \ .\end{aligned}$$

Observe that the equalities above hold because $\{e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_k} | 1 < i_1 < \cdots < i_k \leq n\}$ is a linearly independent subset of $A^{k+1}(V)$. Also, we have that $e_1 \wedge \alpha = 0$ implies that $a_{i_1 \cdots i_k} = 0$, for $1 < i_1 < \cdots < i_k \leq n$. Continuing in this fashion, these observations imply that

$$\begin{aligned}\alpha &= \sum_{m<i_{m+1}<\cdots<i_k<n} a_{1 \cdots m i_{m+1} \cdots i_k} e i_{m+1} \wedge \cdots \wedge e i_k \\ &= e_1 \wedge \cdots \wedge e_m \wedge \left(\sum_{m<i_{m+1}<\cdots<i_k<n} a_{1 \cdots m i_{m+1} \cdots i_k} e i_{m+1} \wedge \cdots \wedge e i_k \right) \\ &= e_1 \wedge \cdots \wedge e_m \wedge \alpha' \ .\end{aligned}$$

To reiterate, in the steps above we repeated the same argument in our computation that we originally introduced for α. The steps above are significant because they allow us to expand the original basis $\{v_1, \cdots, v_k, v\}$ to obtain a basis for V. Furthermore, we observe that $v_1 \wedge \cdots \wedge v_k \wedge v$ is a basis element of $A^{k+1}(V)$, with the basis elements $e_{i_1}, \cdots, e_{i_k}$ (or $v_{i_1}, \cdots, v_{i_k}$m, whichever notation you prefer). With this procedure, we observe that increasing multi-indices of length k , if the annihilator has dimension k with basis $v_1, \cdots, v_k$ of $\alpha^{\perp}$, implies that a basis for $A^k(V)$ would include $v_{i_1} \wedge \cdots \wedge v_{i_k}$. Hence

$$\alpha = \sum_I c_I v_I \ ,$$

with c_I and v_I arbitrary constant, and I is some index set that we use to index the basis above. Again, repeating the same procedure that we have given above, in which we compute the wedge products $v_1 \wedge \alpha$ (or equivalently, $e_1 \wedge \alpha$), allows us to expand our original basis to one for V. From the computation of these wedge products, we know that each term in our summation with $i_I = 1$ will vanish. From previous steps, we observe that

$$v_1 \wedge \alpha = \sum_{1<i_1<\cdots<i_k<n} c_I v_I \wedge v_I \ ,$$

because the elements $\{v_J\}$ form a basis of $A^{k+1}(V)$, for another index set J that we introduce to index the v_J. Hence by linear independent we have that all of the c_I vanish on all I such that $i_I \neq 0$. Rewriting elements from the wedge product shows that

$$\Rightarrow \alpha = \sum_{1=i_1<\cdots<i_k<n} c_I v_I = \sum_{1<i_2<\cdots<i_k} c_I v_1 \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \ .$$

From the equality above, we have an expression from the wedge product in which the first entry in our summation is v_1. From these terms, we carry out the argument at each step to alter the index set over which we are carrying out the summation to compute α with the wedge product $\wedge$. From the equalities above, observe that $\alpha' \in$, which is given by

$$\alpha' = \sum_{m<i_{m+1}<\cdots<i_k\leq n} a_{1\cdots m i_{m+1}\cdots i_k} e i_{m+1} \wedge \cdots \wedge e i_k \ .$$

Beyond the first step in the previous lines above, we evaluated summation term by term to recover the wedge product of the basis elements $e_1, \cdots, e_m$. Therefore, we have that α is m-decomposable, which indeed shows that the k-linear subspace that we have been identifying with the annihilator is of the form $\alpha^\perp$, where α is a that decomposable element. In short, we know that this process shows us that $v_i \wedge \alpha = 0$ for $i \leq k$, which implies that the decomposable α can be expressed in the form $v_1 \wedge \cdots \wedge v_k \wedge \beta$, with $\beta \in A^{n-k}(V)$. After carrying out the procedure with multi=indices of length k, of which there is only one multi-index with length k. This wedge will have the basis elements $v_1, \cdots, v_k$ in it, which allows us to express the summation as a single term. The wedge would then equal 1, which indeed shows that α is decomposable. This discussion completes both directions of the statement.

2.2 Part B

2.2.1 With Decomposable Elements

We will introduce bases $\{v_i\}_{i=1,\cdots,k}$, $\{u_i\}_{i=1,\cdots,k}$. For the first part of **B**, we know from previous work in **A**, that $W = \alpha^\perp$, because we have precisely shown that the annihilator is a subspace of V that has dimension at least k, from properties of the wedge product and our discussion. For **B**, we will form the decomposable elements

$$\alpha = v_1 \wedge \cdots \wedge v_k \in A^k(V) \ ,$$

and

$$\alpha' = u_1 \wedge \cdots \wedge u_k \in A^K(V) \ .$$

We have that

$$u_i = \sum_{j=1}^{k} a_{ij} v_i \ ,$$

from implies that

$$\alpha' = (\sum_{j=1}^{k} a_{ij}v_i) \wedge \cdots \wedge (\sum_{j=1}^{k} a_{kj}v_i) \ .$$

From properties of the wedge product, we are able to collect k^2 terms, k of which are nonzero. Altogether, the steps below exhibit the multiplicative constant C, in which

$$\alpha' = (\sum_{j=1}^{k} c_k)v_1 \wedge \cdots \wedge v_k = C(v_1 \wedge \cdots \wedge v_k) = C\alpha \ .$$

In short, we made use of our steps to rewrite the second basis to get an expression, as linear combination, for elements in the first basis. Then, we made use of the multilinearity to expand and rearrange terms amongst the wedge products, in which we know that each term in our summation is a wedge of k terms. Again from properties of the wedge product, we made use of linearity to recover a single constant C which is the unique constant that we want. This is one way to complete the last part of **B**.

2.2.2 With Linear Functionals

To show that the k dimensional subspace $W \subset V$ is uniquely determined up to a multiplicative constant, we begin by observing that if we have 2 linear functionals $f, g \in V^*$, then the multiplicative constant for the 0 linear functional $g = 0$ would be $c = 0$. Directly from the statement that we want to prove, if $g = cf$, then $c = 0$. If $g \neq 0$, then we can similarly show that g is determined up to a unique multiplicative constant by observing that for the linear functionals f, g, we have that $N_f \subseteq N_g$. To show that $g = cf$ for some $g \neq 0$, in which $\dim(N_g) = \dim(N_f) = n - 1$. Next, we will pick some $e \notin N$, from which we have that $V = N + \mathbf{F}e$, with N another subspace of V because $\dim(V) = n$. From our choice of e, observe that $f(e), g(e) \neq 0$. With $c = \frac{g(e)}{f(e)}$, examining the following expression for some $x = y + \lambda e$, with $y \in N$ and $\lambda \in \mathbf{F}$, below shows that

$$\begin{aligned} g(x) &= g(y + \lambda e) \\ &= g(y) + \lambda g(e) \\ &= \lambda g(e) \\ &= \lambda c f(e) \\ &= c(f(y + \lambda e)) \\ &= cf(x) \ . \end{aligned}$$

In the steps above, we observe that $g(e) = cf(e)$. Alternatively, we can show that the unique constant satisfies this condition by

This completes **B** because we have shown that the k dimensional is uniquely determined up to the constant $c \in \mathbf{F}$, from either approach in the steps above.

2.3 Part C

$\Rightarrow$ Suppose that α and β are decomposable, and that $\alpha^{\perp} \subseteq \beta^{\perp}$. We will denote $\beta^{\perp} = W$ as a subspace of V, from which we know that $W = \alpha^{\perp} \bigoplus U$ for another subspace U of V. From the previous part, we know that every k dimensional subspace W of V is of a given form for a decomposable α, which is uniquely determined up to a unique multiplicative constant C. We will introduce $\beta = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_l$, which implies that $\{\omega_i\}_{i=1,\cdots,l}$ is a basis of W. With the ω_i, without loss of generality we will assume that $\omega_1, \cdots, \omega_l$ is a basis for $\alpha^\perp$. We will assume that the remaining terms from our basis, $\omega_{k+1}, \cdots, \omega_l$, is a basis for U. From **2B**, we know that $\alpha^\perp$ can be determined in the stated way. But from the remaining $\omega_{k+1}, \cdots, \omega_l$ in our basis, we know that U determines these basis elements up to another unique constant. Altogether, we know that β can be expressed with the given wedge product because

$$\beta = \omega_1 \wedge \cdots \wedge \omega_l = \alpha \wedge \gamma \ ,$$

which completes the forward direction because $\gamma \in A^{l-k}(V) \subseteq A(v)$.

$\Leftarrow$ Suppose that $\beta = \alpha \wedge \gamma$, for some $\gamma \in A(V)$. Then, we know that because α is decomposable, we can express β as $\beta = \alpha \wedge \gamma = \alpha \wedge (\sum c_i \gamma_i)$, with γ_i decomposable of fixed degree. As a result, we can rearrange terms from the equality $\beta = \alpha \wedge \gamma$ by distributing the wedge product in the summation, from which we have that $\beta \wedge v = \alpha \wedge \gamma \wedge v = -\alpha \wedge v \wedge \gamma = 0 \wedge \gamma = 0$. Therefore, the annihilator of α is contained in the annihilator of β, which is precisely what we need; it directly implies that $\alpha^\perp \subseteq \beta^\perp$, and finishes the reverse direction.

2.4 Part D

$\Rightarrow$ Suppose that $\alpha^\perp \cap \beta^\perp = \{0\}$. To show that the wedge product $\alpha \wedge \beta \neq 0$, in which case the equality $(\alpha \wedge \beta)^\perp = \alpha^\perp \bigoplus \beta^\perp$, we begin by observing that for α and β decomposable we know that $\alpha = v_1 \wedge \cdots \wedge v_k$, and also that $\beta = \omega_1 \wedge \cdots \omega_l$. From previous work in **2A**, we know that $\alpha^\perp = \text{span}\{v_i\}_{i=1,\cdots,k}$, and also that $\beta^\perp = \text{span}\{\omega_j\}_{j=1,\cdots,l}$. With these conditions from **A**, we know that $\alpha \wedge \beta \neq 0$ because $v_i \notin \text{span}(\omega_i)$, which implies that $v_i \wedge \beta \neq 0$ for all i. So the wedge must be nonzero, ie $\alpha \wedge \beta \neq 0$, because the basis element ω_i cannot be in $\alpha^\perp$, which would give a contradiction. This finishes the **first** part of the forward direction for **D**.

To show that $(\alpha \wedge \beta)^\perp = \alpha^\perp \bigoplus \beta^\perp$, we observe that

$$\alpha \wedge \beta = v_1 \wedge \cdots \wedge v_k \wedge \omega_1 \wedge \cdots \wedge \omega_l \ ,$$

implies that

$$\begin{aligned}(\alpha \wedge \beta)^\perp &= \text{span}(v_i, \omega_j)_{i=1,\cdots,k \text{ and } j=1,\cdots,l} \\ &= \text{span}(v_i)_{i=1,\cdots,k} \bigoplus \text{span}(\omega_j)_{j=1,\cdots,l} \ ,\end{aligned}$$

which readily implies the last equality that is necessary for the forward direction.

$\Leftarrow$ Suppose that $\alpha \wedge \beta \neq 0$. To show that $\alpha^\perp \cap \beta^\perp = \{0\}$, we observe that this condition would require that $\dim V > k + l$. By decomposability of the element α and β, we know that

$$\alpha = v_1 \wedge \cdots \wedge v_k, \text{ and } \beta = \omega_1 \wedge \cdots \wedge \omega_l \ .$$

Next, we see that collecting terms from the decomposable α and β above $\{v_1, \cdots, v_k, \omega_1, \cdots, \omega_l\}$ are linearly independent. Hence $v_i \notin \text{span}\{\omega_j\} j = 1, \cdots, l$ for all i, which implies that $\alpha^\perp \cap \beta^\perp = \{0\}$ because $\alpha^\perp = \text{span}(v_i)$.

This completes the reverse direction, in which we showed that the intersection of the annihilators for α and β is the empty set.

Another way is that we can begin by observing that the condition that $(\alpha \wedge \beta)^{\perp} = \alpha^{\perp} \bigoplus \beta^{\perp}$ implies that we have a decomposition for our finite dimensional vector space V in terms of the subspaces X and Y, in which $W = X \bigoplus Y$. With the subspaces $X, Y \subset V$, we are able to show that the annihilators for the decomposable elements α and β are disjoint by first recalling that an arbitrary λ is said to be in the annihilators $\alpha^{\perp}$ and $\beta^{\perp}$ if it satisfies the property entailing that

$$\lambda \in \alpha^{\perp} \Leftrightarrow \lambda \wedge x = 0 \text{ for } x \in X \ ,$$

and also that

$$\lambda \in \beta^{\perp} \Leftrightarrow \lambda \wedge y = 0 \text{ for } y \in Y \ .$$

Assume that $\lambda \in \alpha^{\perp} \cap \beta^{\perp}$. From the definition of the annihilator, it suffices to show that $\lambda = 0$ if we can intersection $\alpha^{\perp} \cap \beta^{\perp} = \{0\}$. We observe that from the stipulations above that $x \wedge \lambda = 0$, and also that $y \wedge \lambda = 0$, for all $x \in X$ and $y \in Y$, respectively. Altogether, we then have that there exists some $z \in V$ so that $z = x + y$, for $x \in X$ and $y \in Y$, with these arguments demonstrating that $\lambda \in \alpha^{\perp} \cap \beta^{\perp}$, which in turn confirms that $\lambda = 0$. To conclude, this shows that $\alpha^{\perp} \cap \beta^{\perp} = \{0\}$, and completes the reverse direction.

This finishes **D**.

2.5 Part E

We will show that every nonzero element of $A^{n-1}(V)$ is decomposable by induction. To begin, we observe that any element of $A^{n-1}(V)$ can be expressed as $\sum_I \lambda_I e^I$, where in this summation each of the summands $\lambda_I e^I$ is decomposable. So we can show that every nonzero element is decomposable by taking $\alpha, \beta \in A^{n-1}(V)$ and showing that $\alpha + \beta$ is also decomposable. In the more general case, we will induct on the number of summands.

If we have $\alpha = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n$ and $\beta = \omega'_1 \wedge \omega'_2 \wedge \cdots \wedge \omega'_n$, with $\alpha, \beta \neq 0$, we will introduce $W = \text{span}\{\omega_1, \cdots, \omega_{n-1}\}$ and $W' = \text{span}\{\omega'_1, \cdots, \omega'_{n-1}\}$. With $U = W \cap W'$, we know that $\dim(U) \geq n-2$ because $\dim V = n$, and also because $\dim W = \dim W' = n-1$. Furthermore, if we have that $\dim U = n-1$ then the space $W = W'$. Also, we have that the forms α and β are proportional because from W and W' that we have chosen above, we can chose another basis $\omega'_1, \cdots, \omega'_n$ of W. From the basis $\omega'_1, \cdots, \omega'_n$, we know that α is proportional to β because $\omega'_1 \wedge \cdots \wedge \omega'_n = \det A \cdot \omega_1 \wedge \cdots \wedge \omega_n$. In this equality, A is a transition matrix taking $\omega_1, \cdots, \omega_n$ to $\omega'_1, \cdots \omega'_n$. From this statement, we then have that $\alpha + \beta$ is a multiple of α, which shows that $\alpha + \beta$ is decomposable.

In the case in which $\dim U = n-2$, we can show that the same result holds by fixing a basis $u_1, \cdots, u_{n-2}$ of U. From this basis, we can find some $w \in W$ so that $u_1, \cdots, u_{n-2}, w$ is a basis of W. Again from the observations that we have included with the transition matrix A in the previous paragraph, we know that we can rescale ω by some constant. After the rescaling, we have the equality

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{n-1} = u_1 \wedge u_2 \wedge \cdots \wedge u_{n-1} \wedge \omega \ .$$

Moreover, we can determine a $\omega' \in W'$ so that

$$\omega'_1 \wedge \omega_2 \wedge \cdots \omega'_{n-1} = u_1 \wedge u_2 \wedge \cdots \wedge u_{n-1} \wedge \omega' \ .$$

From this equality, we have that

$$\alpha+\beta = u_1 \wedge u_2 \wedge \cdots \wedge u_{n-1} \wedge \omega + u_1 \wedge u_2 \wedge \cdots \wedge u_{n-1} \wedge \omega' \\ = u_1 \wedge u_2 \wedge \cdots \wedge u_{n-1} \wedge (\omega+\omega') \ ,$$

which shows that $\alpha+\beta$ is decomposable. Generally speaking, we know that the subspace that we have found all of the decomposable elements for has dimension $n-1$, in which determining some v in the annihilator is the same as solving a linear equation with n variables. Therefore, from the induction and the separate cases that I have mentioned, this completes **E**.

3 Problem 3

3.1 Part A

$\Rightarrow$ Suppose that M is connected. To show that it is path connected, we will consider the manifold M^n and some $x \in M$. In particular, we also begin by observing that our manifold M being connected implies that M is locally path-connected, from definitions in lecture. Together this implies that M is path-connected. To show that path-connectedness implies C^∞ path-connectedness, we will proceed as follows. We know that the interval $[0,1]$ is compact. From this compactness, we will want to define some path $\gamma : [0,1] \to M$ to show that path-connectedness holds. With γ, we know that the compactness of the unit interval implies that we can make use of finitely many open sets to cover the image of γ.

We can achieve such a covering by picking points on γ that overlap in the open sets that we use to cover the image of γ. More specifically, on the overlap of these open sets that we use to cover the image of γ, we know that we can replace the path γ with another path γ' that is piecewise linear. Altogether, this then allows us to show path-connectedness because from γ and γ' we observe that the smoothness of linear functions implies that there are finitely many points on which γ' is not smooth.

From properties of our manifold, we can make γ' smooth on these finitely many points by taking a small neighborhood of each of these points. With M, we are able to make γ' smooth on these points by making sure that the open neighborhood of each of these points is the domain of some chart (U, φ). With this chart, we assume without loss of generality that the open neighborhood U can be taken small enough. This is significant because it allows us to isolate each of the points on which γ' may not be smooth. As a result, we know that these points are of the form $\gamma'(t) = x$, which is contained in U. Furthermore, our manifold is assumed to be Hausdorff so shrinking these neighborhoods for problematic points on which γ' is not smooth will give us respective neighborhoods for each of these points that are disjoint.

To this end, we will look at the composition $\varphi \circ \gamma'$. From these terms, we know that the composition is smooth except at the points $\varphi \circ \gamma'(t)$. However, we have that $\varphi \circ \gamma'$ is continuous, which implies that in $\mathbf{R}^n$ we are able to take some neighborhood of V that is centered around $\varphi \circ \gamma'(t)$. With V, we stipulate that $\varphi \circ \gamma'(t)$ is entirely contained in $\varphi(U)$, and also that $\varphi \circ \gamma'(t)$ is equal to the composition $\varphi \circ \gamma'$ outside of V. So with the open set V, we have shown path-connectedness as a result of C^∞ path-connectedness because we can replace the path γ' with another path c. With the other path c, we are able to suitably patch the inverse images under φ together. Suitably connecting the inverse images together entails that we look at the inverse images under ϕ of the smooth path in $\mathbf{R}^n$, and that we replace these inverse images with the smooth linear segments that we have described at length. Therefore, we will obtain a smooth path between the 2 points. To conclude, these arguments imply that there exists a path connecting x and y, which completes the forward direction.

$\Leftarrow$ Suppose that M is C^∞ path connected. To show that M is connected, we begin by observing that by the definition of M being path connected we know that for every $x, y \in M$, we can construct a smooth path connecting x and y. Directly from the definition of path connectedness, we have that M is connected because C^∞ path connected implies regular path connected. In turn, path connected implies connected, which completes the reverse direction.

3.2 Part B

We will follow the hint. If we have a manifold M that is connected, we can show that there exists a diffeomorphism satisfying the given properties that connects the points x and y together. With the hint and discussion from OH, in which we want to show that there exists some diffeomorphism, which we can denote with θ_1, that connects the points x and y. We will show this in detail below.

To show that the property from the previous paragraph holds, we need to show that between the 2 points that we have given, we can construct a vector field ξ with $\xi(t, c(t)) = c'(t)$ that is compactly supported and time independent. To begin, we will consider $[0, 1] \times M$. From this start, we already know that $(t, c(t))$ is an embedded submanifold because the derivative never vanishes, and also because it is injective. More profoundly, on the embdedded submanifold $(t, c(t))$ we can also introduce vector field $\xi = (\frac{d}{dt}, c'(t))$. With ξ, we observe that as a consequence of a general fact, we cannot extend the vector field globally because not every manifold is parametrizable.

However, it is possible to obtain some type of extension that we want by using bump functions. Before that, we will make use of the compactness of $(t, c(t))$, which allows us to produce a covering of $(t, c(t))$ with finitely many sufficiently small neighborhoods $\mathcal{U}_1, \cdots, \mathcal{U}_n$. With these open neighborhoods, we will also define $\xi(t, x) = c'(t)$ over each of the open neighborhoods $\mathcal{U}_1, \cdots, \mathcal{U}_n$. With the aid of local charts, we can repeat this procedure in smaller neighborhoods than the ones $\mathcal{U}_1, \cdots, \mathcal{U}_n$ that we chose initially. After carrying out this procedure on smaller neighborhoods, we know from the Extension Lemma for Vector Fields defined on closed embedded submanifolds that we are able to take a sufficiently small neighborhood $\mathcal{V}$ of $(t, c(t))$, so that for each $\mathcal{U}_i$ that we chose above, we have that $\mathcal{V} \cap \mathcal{U}_i \, \mathcal{U}_i$. Also, even from a similar construction in HW 9, we will want that the closure of V is compact. We can make sure that this second condition of compactness is fulfilled by taking r to be the minimum diameter of the open $\mathcal{U}_i$. Also, it is necessary to take $\mathcal{V}$ to be a tubular neighborhood with a fixed radius $\frac{r}{4}$.

Finally, we will now use the bump function to get a function that equals 1 on $(t, c(t))$, and otherwise equals 0 outside of $\mathcal{V}$. With this standard construction, we will denote the function that we get from our bump function as f. In that case, the extension that we will define with f would be $\xi(x, t) = f(x)(\frac{d}{dt}, c'(t))$, with $(x, t) \in \mathcal{U}_i$ for any i. Otherwise, the global extension would vanish and $= 0$. Again from standard arguments, we know that on the overlap $\mathcal{U}_i \backslash V$ the 2 functions would agree. Also, we would have that the functions agreeing on the overlap would precisely allow us to define a global smooth vector field. On $I \times M$, we have a compactly supported vector field, which makes the vector field complete. From the diffeomorphism θ_1 that we described, we know that such a diffeomorphism because we can take the first slice to get a diffeomorphism on M that precisely takes x to y. This is achieved from the flow to time 1, which completes **B**.

4 Problem 4

4.1 Using the Limit Definition of the Lie Derivative

We know from the limit definition that

$$\lim_{h \to 0} \frac{\theta^*_{t+h}(\alpha_{t+h}) - \theta^*_t(\alpha_t)}{h} \ ,$$

which after simple rearrangements implies that the expressions in the limit, after adding and subtracting $\theta^*_{t+h}(\alpha_t)$ in the numerator, are of the following form

$$\lim_{h \to 0} \left(\theta^*_{t+h}(\frac{1}{h}(\alpha_{t+h} - \alpha_t)) + \frac{1}{h}(\theta^*_{t+h} - \theta^*_t)\alpha_t \right) \ ,$$

which allows us to now apply a characteristic of the flow law for time-dependent vector fields. In particular, this observation entails that for some fixed time t, we are able to apply the flow law so that we can rearrange terms in the limit some more, which implies that

$$\Rightarrow \theta_t^*\bigg(\lim_{h\to 0}\frac{1}{h}(\alpha_{t+h}-\alpha_t)+\lim_{h\to 0}\frac{1}{h}(\theta_h^*\alpha_t-\theta_0^*\alpha_t)\bigg)\ .$$

Next, we conclude that these terms give exactly what we want, because from the terms that we have given above, we observe the following. We know that the first term from our limit definition is $\dot{\alpha}_t$, and also that the second term is the Lie Derivative $\mathcal{L}_{\xi_t}(\alpha_t)$ along the vector field ξ_t. Altogether, this completes **4**.

4.2 Using Flows

Besides the limit definition, we can also show that the given equality holds. Generally speaking, we recall the fact that the isotopy of diffeomorphisms θ_t on M induces induces a flow Θ_s on $M\times I$, with I an open interval that contains $[0,1]$. We observe that this flow is induced because we have that θ_t is generated by integrating an s-independent vector field, in which we have that $\frac{d\Theta_s}{ds}(x,t)=X_t(x)+\frac{\partial}{\partial t}$. In the context of **4**, observe that $X_t(x)=\xi_t$.

With the flow Θ_s, we know that the flow $\Theta_t : M\times\{0\}\to M\times\{t\}$, where $(x,0)\mapsto(\theta_t(x),t)$ shows us that restricting $\Theta_t^*\alpha$ to $M\times\{0\}$ is $\theta_t^*\alpha_t$. Also, we know that we have the independent t-form θ_t^* on M which is an s-independent form on $\Omega(x,t)=\alpha_t$ on $M\times I$. Now, we will calculate the given derivative on the left hand side, in which we begin by observing that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\theta_t^*\alpha_t)=\frac{\mathrm{d}}{\mathrm{d}t}(\Theta_t^*\Omega|_{M\times\{0\}})\ ,$$

from which we rearrange terms further to conclude that

$$\begin{aligned}\Rightarrow \Theta_t^*(Y\rfloor d_{M\times I}\Omega+d_{M\times I}(Y\rfloor\Omega))|_{M\times\{0\}} &= \Theta_t^*(Y\rfloor(d_M\alpha_t+d_I\Omega)+d_{M\times I}(\xi_t\rfloor\alpha_t))|_{M\times\{0\}}\\ &= \Theta_t^*(X_t\rfloor d_M\alpha_t+\left(\frac{\partial}{\partial t}\right)\rfloor d_I\alpha_t+d_M(\xi_t\rfloor\alpha_t))|_{M\times\{0\}}\\ &= \Theta_t^*\left(\mathcal{L}(\xi_t)\alpha_t+\frac{d\alpha_t}{dt}\right)|_{M\times\{0\}}\\ &= \theta_t^*\left(\mathcal{L}(\xi_t)\alpha_t+\frac{d\alpha_t}{dt}\right)\\ &= \theta_t^*\Big(\mathcal{L}(\xi_t)\alpha_t+\dot{\alpha}_t\Big)\ .\end{aligned}$$

This establishes the formula we want and completes **4**.

5 Problem 5

5.1 Part A

$\Rightarrow$ Suppose that $\mathrm{div}(\xi)=0$. We have that

$$\begin{aligned}\mathcal{L}_\xi \mu &= \frac{d}{dt}(\theta_t^*\mu)|_{t=0} \\ &= \frac{d}{dt}|_{t_0}(\theta_t^*\mu) \\ &= \theta_0^*(\mathcal{L}_\xi \mu) \\ &= \theta_0^*(\mathrm{div}(\xi)\mu) \\ &= \theta_0^*(0) \\ &= 0 \ ,\end{aligned}$$

which completes the first direction because $\theta_t^*\mu$ is constant in time, in which we have that $\theta_0^*\mu = I^*\mu = \mu$ because $\theta_0 : \mathrm{Id} : M \to M$. From the limit definition of the Lie Derivative, we can confirm that the steps above hold because

$$\frac{d}{dt}\theta_t^*\mu = \theta_t^*\mathcal{L}_\xi \mu \ ,$$

which demonstrates that the LHS of this expression must vanish because the Lie Derivative vanishes. In turn, the derivative vanishing implies that $\theta_t^*\mu = \theta_s^*\mu$ for all s, t, which then implies that $\theta_t^*\mu = \mu$ for $s = 0$. In particular, this claim holds because from the term $\frac{d}{dt}\theta_t^*\mu$, we can rearrange terms by observing that placing an extra term for the pullback, which we will evaluate at 0, implies that we would then have $\frac{d}{dt}\theta_t^*\theta_r^*\mu|_{r=0}$.

Moreover, the definition of the pullback that was given in lecture then implies that $\theta_t^*\mu = F(\theta_t)\circ\mu\circ\theta_t$. Taking the limit with respect to r and expanding terms inside the pullback of θ_t then shows us that $\theta_t^*(\frac{d}{dr}\theta_r^*\mu)|_{r=0}$. But we know that $\theta_t^*(\frac{d}{dr}\theta_r^*\mu)|_{r=0} = \theta_t^*\mathcal{L}_\xi\mu$, which indeed shows that $\theta_t^*\mu = \mu$ for all t, which completes the forward direction.

$\Leftarrow$ Conversely, suppose that $\theta_t^*\mu = \mu$. We have that

$$\begin{aligned}\mathrm{div}(\xi)\mu &= \mathcal{L}_\xi \mu \\ &= \frac{\partial}{\partial t}(\theta_t^*\mu)_{t=0} \\ &= \frac{\partial}{\partial t}\mu = 0 \ ,\end{aligned}$$

which implies that $\mathrm{div}(\xi) = 0$. Even from the limit definition of the Lie Derivative that we have used several times, we have that

$$\lim_{h\to 0}\frac{1}{h}(\theta_{t+h}^*\mu - \theta_t^*\mu)$$

will vanish for all nonzero h. This then implies that $\mathrm{div}(\xi) = 0$, as shown in our first steps above, which shows that the Lie Derivative vanishes. Either way completes the remaining direction.

5.2 Part B

In order to show that the Divergence Theorem holds for a compact manifold M we will make use of Cartan's Magic Formula that is given. From this formula, we know that all forms are compactly supported, which in turn implies that the Divergence Theorem holds because

$$\int_M \mathrm{div}(\xi)\mu = \int_M \mathcal{L}_\xi(\mu) \ ,$$

which by the Magic Formula is equivalent to

$$\Rightarrow \int_M [d, i(\xi)]\mu = \int_M di(\xi)\mu + \int_M i(\xi)d\mu \ .$$

By Stokes' Theorem, we know that the first term is the same as $\int_{\partial M} i(\xi)\mu$. As for the remaining terms, we know that the second term vanishes exactly because $d\mu$ is an $n+1$ form. We know that our manifold has dim $= n$, which implies that $d\mu = 0$. After these observations, we exactly have the Divergence Theorem which completes **B**.

5.3 Part C

As a general fact, we know that if we have a complete vector field, we have the orientation preserving diffeomorphism $\theta_t : A \to \theta_t(A)$. Indeed, pulling back along this diffeomorphism preserves the value of an integral that we can calculate, which implies that

$$\frac{d}{dt}\int_{\theta_t(A)} f_t\mu = \int_A \frac{d}{dt}\theta_t^* f_t\mu \ .$$

Applying the equality from **4** implies that

$$\int_A \theta_t^*(\mathcal{L}(\xi_t)f_t\mu + (\dot{f_t\mu})) \ ,$$

from which we now apply the Leibniz and Product Rules, at the same time observing that μ is constant in time, to conclude that

$$\int_A \theta_t^*((\mathcal{L}(\xi_t)(f) + f_t\mathrm{div}(\xi) + \dot{f}_t)\mu) \ ,$$

also after substituting divergence for the Lie Derivative. Finally, we recall that θ_t is an orientation preserving diffeomorphism, which implies that we have the equality because

$$\int_{\theta_t(A)} (\mathcal{L}(\xi_t)(f) + f_t\mathrm{div}(\xi) + \dot{f}_t)\mu \ .$$

6 References

I heard a lot of things in both OHs and I spoke with several people. I also consulted PDFs from past HWs that I have already mentioned.