STSCI 5080 Homework 3

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1 Problem: Rice 3.8.54 Modified

To show that the Gamma random variables Y_1 and Y_2 are independent, we recall that the gamma distribution has a pdf of the form

$$f(x) = \frac{1}{\alpha^{\beta_i}\Gamma(\beta_i)} x^{\beta_i - 1} e^{-\frac{x}{\alpha}} \ .$$

From the transformations Y_1 and Y_2 that are given, we know that these transformations are $1-1$, precisely because choosing different X_1, X_2 maps values uniquely to each other. Next, we will compute the Jacobian of the given random variables, in which, from the definition of the probability density function of gamma random variables,

$$\mathcal{J} = \begin{vmatrix} Y_2 & Y_1 \\ -Y_2 & 1 - Y_1 \end{vmatrix} = Y_2(1 - Y_1) + Y_1 Y_2 = Y_2 \ ,$$

from which we can apply the transformation $Y_1 = \frac{X_1}{X_1 + X_2}$ to obtain the joint probability density, which is of the form

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} x_1^{\beta_1 - 1} x_2^{\beta_2 - 1} e^{-\frac{(x_1 + x_2}{\alpha}} \ ,$$

for $x_1, x_2 > 0$. From here, we will compute the resulting joint probability density function from the one reproduced above, in which the probability density function of Y_1 and Y_2 is of the form,

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2))|\mathcal{J}| \\ &= \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} (y_1 y_2)^{\beta_1 - 1} (y_2(1 - y_1))^{\beta_2 - 1} e^{-\frac{y_2}{\alpha}} |y_2| \ , \end{aligned}$$

which holds for $0 < y_1 < 1, y_2 > 0$. Finally, we can calculate the marginal distributions of Y_1 and Y_2 below. First, we will calculate the marginal distribution $f_{Y_1}(y_1)$, which is carried out below as we integrate wrt y_2,

$$\begin{aligned} f_{Y_1}(y_1) &= \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty (y_1 y_2)^{\beta_1 - 1} (y_2(1 - y_0))^{\beta_2 - 1} e^{-y_2/\alpha} |y_2| \mathrm{d}y_2 \\ &= \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty y_1^{\beta_1 - 1} y_2^{\beta_1 - 1} y_2^{\beta_2 - 1} e^{-y_2/\alpha} y_2 (1 - y_1)^{\beta_2 - 1} \mathrm{d}y_2 \\ &= \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty y_2^{\beta_1 + \beta_2 - 1} e^{-y_2/\alpha} y_1^{\beta_1 - 1} (1 - y_1)^{\beta_2 - 1} \mathrm{d}y_2 \ , \end{aligned}$$

for $0 < y_1 < 1$. Substituting $r = \frac{y_2}{\alpha}$ and $\mathrm{d}y_2 = \alpha \mathrm{d}r$ gives a marginal distribution of the form,

$$f_{Y_1}(y_1) = \frac{1}{\alpha^{\beta_1+\beta_2}\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty (r\alpha)^{\beta_1+\beta_2-1} e^{-r} y_1^{\beta_1-1}(1-y_1)^{\beta_2-1} ,$$

for $0 < y_1 < 1$, which by the definition of the Gamma function is equivalent to

$$f_{Y_1}(y_1) = \frac{\Gamma(\beta_1+\beta_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} y_1^{\beta_1-1}(1-y_1)^{\beta_2-1} .$$

Carrying out the same steps for the marginal distribution of Y_2 gives an expression of the form

$$f_{Y_2}(y_2) = \frac{\Gamma(\beta_1+\beta_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} y_2^{\beta_1-1}(1-y_2)^{\beta_2-1} .$$

2 Problem On Sheet

2.1 Part A

To find the joint cdf and joint pdf of $(X_{(1)}, X_{(n)})$, we first observe that the random variables $X_1, \cdots, X_n$ are independent, and also that these continuous random variables have the same pdf f. Furthermore, to determine the joint pdf and cdf, we can show that the desired equality holds by rearranging terms, in which

$$\begin{aligned}
\mathbf{P}(X_{(1)} > x, X_{(n)} \leq y) &= \mathbf{P}(\text{all} x_i > x, \text{all } x_i \leq y) \\
&= \prod_{i=1}^n \mathbf{P}(x_i > x, x_i \leq y) \\
&= \prod_{i=1}^n \mathbf{P}(x < x_i \leq y) \\
&= \prod_{i=1}^n \Big(\mathbf{P}(x_i \leq y) - \mathbf{P}(x_i \leq x) \Big) \\
&= \prod_{i=1}^n \Big(F(y) - F(x) \Big) \\
&= \Big(F(y) - F(x) \Big)^n ,
\end{aligned}$$

from which further rearrangements give

$$\begin{aligned}
\mathbf{P}(X_{(1)} \leq x, X_{(n)} \leq y) &= \mathbf{P}\Big((\cup_{i=1}^n \{x_i \leq x\}) \cap (\cap_{i=1}^n \{x_i \leq y\}) \Big) \\
&\mathbf{P}\Big((\cap_{i=1}^n \{x_i \leq y\} \backslash (\cap_{i=1}^n \{x_i > x\} \Big) \\
&= \mathbf{P}(\cap_{i=1}^n \{x_i \leq y\}) - \mathbf{P}(\cap_{i=1}^n \{x < x_i \leq y\}) \\
&= \prod_{i=1}^n \mathbf{P}(x_i \leq y) - \prod_{i=1}^n \mathbf{P}(x < x_i \leq y) \\
&= (F_x(y))^n - (F_x(y) - F_x(x))^n ,
\end{aligned}$$

for $x \leq y$. Therefore, the joint pdf of $(X_{(1)}, X_{(n)})$ is of the form

$$\begin{aligned} f_{X_{(1)},X_{(n)}}(x,y) &= \frac{\partial^2}{\partial x \partial y} F_{X_{(1)}}(x,y) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} F(x,y)\right) \\ = \frac{\partial}{\partial x}\left(mF_x(y)^{n-1} f_x(y) - n\left(F_x(y) - F_x(x)\right)^{n-1} f_x(y)\right) \\ &= (-1) - n(n-1)(F_x(y) - F_x(x))^{n-2} f_x(y) f_x(x) \\ &= n(n-1)(F_x(y) - F_x(x))^{n-2} f_x(y) f_x(x) \ . \end{aligned}$$

2.2 Part B

With the common distribution as the uniform distribution $\mathcal{U}[0,1]$, we know that if $x_i \sim \mathcal{U}[0,1]$, then

$$f(x_i) = 1 \text{ for } 0 \leq x_i \leq 1 \ ,$$

and 0 otherwise. Altogether, we can determine the probability

$$F(x) = \mathbf{P}(X \leq x) = \int_0^x f(u) \mathrm{d}u = u|_0^x = x \ ,$$

which implies that the joint pdf of $X_{(1)}$ and $X_{(n)}$ is of the form

$$f(X_{(1)}, X_{(n)}) = n(n-1)(X_{(n)} - X_{(1)})^{n-2} \ ,$$

for $1 \geq X_{(n)} \geq X_{(1)} \geq 0$. Otherwise, the joint pdf takes a value of 0.

2.3 Part C

To compute the desired probability, we observe that $U \sim \mathcal{U}[0,1]$ is independent of $(X_{(1)}, X_{(n)})$. Therefore, $f_{U,X_{(1)},X_{(n)}}(u,x,y) = f_u(u) f_{X_{(1)},X_{(n)}}(x,y)$, from which the probability can be computed by observing that,

$$\begin{aligned} \mathbf{P}(X_{(1)} < U < X_{(n)}) &= \mathbf{P}(x < U < y | X_{(1)}, X_{(n)} = y) \mathbf{P}(X_{(1)} = x, X_{(n)} = y) \\ &= \mathbf{P}(x < U < y) \mathbf{P}(X)(1) = x, X_{(n)} = y) \ , \end{aligned}$$

because $X_{(1)}$, $X_{(n)}$ are independent. Simplifying a bit more gives the probability that we want,

$$\int_x^y 1 \mathrm{d}u f_{X_{(1)},X_{(n)}}(x,y) = (y-x)n(n-1)(y-x)^{n-2} = n(n-1)(y-x)^{n-1} \ .$$

3 Problem: Rice 4.7.4 Modified

For the random variable $f(x) - \alpha x^{-\alpha-1}$, for $x \geq 1$, we know that we can determine such a $k \in \mathbf{Z}^+$ so that $\mathbf{E}(X^{k-1}) < \infty$, and $\mathbf{E}(X^k) = \infty$, by observing that the integrals

$$\mathbf{E}(X^{k-1}) = \int_{x=1}^{\infty} \alpha x^{k-1} x^{-\alpha-1} \mathrm{d}x = \alpha \bigg(\frac{x^{k-\alpha-1}}{k-\alpha-1} \bigg)_1^{\infty} = \frac{\alpha}{k-\alpha-1} \lim_{n \longrightarrow \infty} (x^{k-\alpha-1})_1^n$$
$$= \frac{\alpha}{k-\alpha-1} \lim_{n \longrightarrow \infty} (n^{k-\alpha-1} - 1) \ ,$$

and

$$\mathbf{E}(X^k) = \int_{x=1}^{\infty} \alpha x^k x^{-\alpha-1} \mathrm{d}x = \int_{x=1}^{\infty} x^{k-\alpha-1} \mathrm{d}x = \lim_{n \longrightarrow \infty} \frac{x^{k-\alpha}}{k-\alpha} = \frac{\alpha}{k-\alpha} \lim_{n \longrightarrow \infty} (n^{k-\alpha} - 1) \ ,$$

represent the $k-1$, and k moments of the given random variable. More specifically, we know that the integrals above can be discretely approximated with a series whose summation indices exactly coincide with the limits of integration shown above, and furthermore, that, for appropriate choices of k and α, that the series can be interpreted as geometric with terms involving higher powers of x. But from the computations of the integrals above alone, we know that we can impose the conditions on k and α so that we can find the integer k so that the k^{th} moment is infinite. In the computation of the first integral, we observe that the $(k-1)$ th moment is finite iff $k \leq \alpha + 1$, while the infinite moment that we have calculated in the second integral impose the restriction that $k > \alpha$. Altogether, the 2 conditions that we have obtained from each of the moment calculations above that $k \in (\alpha, \alpha+1]$.

4 Problem: Variance of X

To show that the variance of x is equal to the given equality, we evaluate the integral below, each of which represent the individual terms in the formula for $\mathrm{Var}(X)$,

$$\mathbf{E}(X) = \int_{-\infty}^{\infty} x f)_X(x) \mathrm{d}x = \frac{1}{b-a} \int_a^b x \mathrm{d}x = \frac{1}{b-a} \frac{x^2}{2} |_a^b = \frac{1}{2} \frac{b^2-a^2}{b-a} = \frac{a+b}{2} \ ,$$

and also that

$$\mathbf{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) \mathrm{d}x = \frac{1}{b-a} \int_{x=a}^b x^2 \mathrm{d}x = \frac{1}{b-a} \frac{x^3}{3} |_a^b = \frac{1}{3} \frac{b^3-a^3}{b-a} = \frac{b^2+ab+a^2}{3} \ .$$

Putting everything together,

$$\mathrm{Var}(X) = \mathbf{E}(X^2) - (\mathbf{E}(X))^2 = \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12} \ .$$

5 Problem: Mean and Variance of PDF

To find the mean and variance of the given PDF, we observe that the expectation below can be calculated, as

$$\mathbf{E}(x) = \int_{-1}^{1} xf(x)\mathrm{d}x = \int_{-1}^{1} x(1-|x|)\mathrm{d}x = \int_{-1}^{0} x(1+x)\mathrm{d}x + \int_{0}^{1} x(1-x)\mathrm{d}x \ ,$$

from which integrating each one of the terms gives an expression of the form,

$$\Rightarrow \left(\frac{x^2}{2}+\frac{x^3}{3}\right)_{-1}^{0} + \left(\frac{x^2}{2}-\frac{x^3}{3}\right)_{0}^{1} = (-\frac{1}{2}+\frac{1}{3}) + (\frac{1}{2}-\frac{1}{3}) = 0 \ .$$

On the other hand, we calculate the expectation below as well, in which

$$\mathbf{E}(x^2) = \int_{-1}^{1} x^2 f(x)\mathrm{d}x = \int_{-1}^{1} x^2(1-|x|)\mathrm{d}x = \int_{-1}^{0} x^2(1+x)\mathrm{d}x + \int_{0}^{1} x^2(1-x)\mathrm{d}x = \int_{-1}^{0} (x^2+x^3)\mathrm{d}x + \int_{0}^{1} (x^2-x^3)\mathrm{d}x \ ,$$

from which similar simplifications give the result

$$(\frac{1}{3}-\frac{1}{4}) + (\frac{1}{3}-\frac{1}{4}) = \frac{1}{6} \approx 0.16667...$$

Finally,

$$\mathrm{Var}(X) = \mathbf{E}(X^2) - (\mathbf{E}(X))^2 \approx 0.16 \ ,$$

from which we also have that the mean is 0. So we have calculated the mean and variance of the given PDF.

6 Problem: Chebyshev's Association Inequality

6.1 Part A

To show that the given inequality holds, we will follow the hint, in which we observe that, given some random variable X, we know that for real valued functions g, h,

$$\mathbf{E}(|g(x)h(x)|) < \infty \ ,$$

given that

$$\mathbf{E}(|g(x)|) < \infty \ , \ \mathbf{E}(|h(x)|) \leq \infty \ .$$

So if we have random samples Ythat are independent of X, the inequalities above demonstrate that Y would have the same pdf and pmf as X. Furthermore, from the hint, if g and h are non decreasing, then by definition $g(x) \leq g(y)$ for any $x \leq y$, and the same inequality holds for h because it is also non decreasing. Altogether, the given expectation is positive because

$$\{g(X) - g(Y)\}\{h(X) - h(Y)\} \geq 0 \ ,$$

implying that

$$\mathbf{E}\Big(\{g(X) - g(Y)\}\{h(X) - h(Y)\}\Big) \geq 0$$

Specifically, distributing terms gives

$$\mathbf{E}\Big(g(X)h(X) - g(X)h(Y) - g(Y)h(X) - g(Y)h(Y)\Big) \ \geq 0 \ ,$$

and because X, Y are independent,

$$\mathbf{E}\Big(g(X)h(X)\Big) + \mathbf{E}\Big(g(Y)h(Y)\Big) \geq \mathbf{E}(g(X))\mathbf{E}(h(Y)) + \mathbf{E}(g(Y))\mathbf{E}(h(X)) \ .$$

Finally, we have that $\mathbf{E}(g(x)h(x)) \geq \mathbf{E}(g(X))\mathbf{E}(h(X))$, from

$$2\mathbf{E}(g(x)h(x)) \geq 2\mathbf{E}(g(x))\mathbf{E}(h(x)) \ ,$$

resulting from the fact that

$$\begin{aligned}
\mathbf{E}(h(X)) &= \mathbf{E}(h(Y)) \ , \\
\mathbf{E}(g(X)) &= \mathbf{E}(g(Y)) \ , \\
\mathbf{E}(g(X)h(X)) &= \mathbf{E}(g(Y)h(Y)) \ .
\end{aligned}$$

This demonstrates that the first inequality holds.

6.2 Part B

To show that the second inequality holds, we observe that by definition if h and g are non-increasing functions, then

$$g(X) \leq g(Y) \ ,$$

for any $X \leq Y$, and

$$h(X) \geq h(Y) \ ,$$

for any $X \leq Y$. Trivially rearranging the inequalities above implies that

$$g(X) - g(Y) \leq 0 \ ,$$

and

$$h(X) - h(Y) \leq 0 \ ,$$

with each of the inequalities remaining valid for the same choice of X and Y. Altogether, the inequalities above imply that

$$\begin{aligned} \mathbf{E}(\{g(X) - g(Y)\}\{h(X) - h(Y)\}) \leq 0 \\ \mathbf{E}(g(X)h(X) + g(Y)h(Y) - g(X)h(Y) - g(Y)h(X)) \leq 0 \\ \mathbf{E}(g(X)h(X)) + \mathbf{E}(g(Y)h(Y)) \ , \end{aligned}$$

from which interchanging x with y in the non-decreasing function h implies,

$$\mathbf{E}(g(X)h(Y)) + \mathbf{E}(g(Y)h(X)) \ .$$

Furthermore,

$$\begin{aligned} \mathbf{E}(g(X)h(X)) + \mathbf{E}(g(Y)h(Y)) \leq \mathbf{E}(g(X))\mathbf{E}(h(Y)) + \mathbf{E}(g(Y))\mathbf{E}(h(X)) \\ \Rightarrow \mathbf{E}(g(X)h(X)) \leq \mathbf{E}(g(X))\mathbf{E}(h(X)) \\ \Rightarrow \mathbf{E}(g(X)h(X)) \leq \mathbf{E}(g(X))\mathbf{E}(h(X)) \ . \end{aligned}$$

From the steps above, observe that we can rearrange the terms in this way because the X and Y are independent, and also because $\mathbf{E}(h(Y)) = \mathbf{E}(h(X))$, $\mathbf{E}(g(Y)) = \mathbf{E}(g(X))$, and also that $\mathbf{E}(g(Y)h(Y)) = \mathbf{E}(g(X)h(X))$. Finally, we have that the last equality holds after cancelling the 2's from each side of the inequality.

7 Problem: Rice 4.7.53 Modified

7.1 Part A

To prove that X and Y are not independent, it suffices to show that $\mathbf{P}(XY) \neq \mathbf{P}(X)\mathbf{P}(Y)$, or equivalently, that given the joint density of XY, this joint density is not equal to the product of the marginal densities $f_X(x)$ and $f_Y(y)$. To this end, we know that the uniform random vector over the unit disk with radius $r \equiv 1$, for some random point (x, y) in the disk, is of the form $f(x, y) = \frac{1}{\pi R^2}$, again where $R \equiv 1$ so that we are in the unit disk. Otherwise, $f(x, y) = 0$.

From this definition, we immediately know that, whether we directly calculate the densities or not, that the range of one of the variables that we fix, say X, is dependent on the other random variable Y, because if we parametrize the circle, each of the x and y depend on the unit radius $R \equiv 1$ and the angle θ. As we increase θ monotonically from 0 to 2π, the random variables X and Y coordinates, each of which depends on θ and R, change continuously with respect to the rotation by θ. More rigorously, this can be formulated by looking at the indicator from the density of X and Y, in which the indicator takes on a value of 1 inside the unit disk $\mathcal{D}$ and otherwise vanishes. With this restriction, we can directly see that the joint density of $f_{XY}(x, y)$ does not factor into the product of the marginal densities, because as previously mentioned, the support and range of the random variables X and Y depend on each other.

7.2 Part B

From our previous expression and discussion of the uniform random vector on the disk, it is directly clear that, from the random vector (X, Y), X and Y are absolutely not independent because if we look at the expectation of the random variables X, Y,

$$\mathbf{E}[XY] = \frac{1}{\pi} \int \int_{\mathcal{D}} xy \mathrm{d}x \mathrm{d}y \ ,$$

where $\mathcal{D}$ is the unit disk, we know that $\mathbf{E}[x] = \mathbf{E}[Y] = 0$. Moreover, the fact that these expectations vanish implies that $\mathrm{Cov}(X, Y) = 0$ as well, because

$$\mathrm{Cov}(X, Y) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y) \ ,$$

again because from the expectation of the random variable XY, the terms $\mathbf{E}[Y]$ and $\mathbf{E}[X]$ both vanish because of symmetry with respect to the y and x-axes, respectively, while the expectation of XY also vanishes because in the four quadrants of $\mathbf{R}^2$ on which the uniform random vector is defined, we know that the random quantity xy is symmetric in each of the 4 quadrants, with alternating sign. Therefore, we know that the first term in the covariance between X and Y also vanishes because of alternating signs in the 4 quadrants of the plane, in which the joint density, represented with the integral $\int xy \cdot f_{XY}(x, y) \mathrm{d}x \mathrm{d}y$, will have 4 contributions from each of the quadrants of $\mathbf{R}^2$ all of which cancel.

8 Problem: Rice 4.7.55 Modified

8.1 Part A

To find the covariance and correlation of X and Y, we observe that if we let A denote the support of Y, we have that

$$f(x) = \int_A f(x, y) \mathrm{d}y = 2 \int_x^1 \mathrm{d}y = 2(1 - x) \text{ for } , 0 \leq x \leq 1 \ ,$$

and 0 otherwise. Next, if we denote B as the support of x, then the expectation

$$\mathbf{E}(X) = \int_0^1 2x(1 - x) \mathrm{d}x = \frac{1}{3} \ ,$$

from which we conclude that

$$\int_0^y f(x, y) \mathrm{d}x = 2y \ ,$$

for $0 \leq y \leq 1$. Altogether,

$$\mathbf{E}(XY) = 2 \int_0^1 \int_x^1 xy \mathrm{d}y \mathrm{d}x = \int_0^1 x(1 - x^2) \mathrm{d}x = \frac{1}{4} \ .$$

Finally, we can also calculate the desired covariance, with

$$\begin{aligned}\text{Cov}(X,Y) = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y) &= \frac{1}{4} - (\frac{1}{3} \times \frac{2}{3}) \\ &= \frac{1}{4} - \frac{2}{9} \\ &= \frac{1}{36} \ .\end{aligned}$$

8.2 Part B

To determine each of the conditional expectations, we observe that

$$\mathbf{E}(X|Y) = \frac{1}{y}\int_0^1 x\mathrm{d}x = \frac{1}{2y} \ ,$$

because if we denote $f(X|Y)$ as the PDF of x given y, we know that

$$f(X|Y) = \frac{f(x,y)}{f(y)} = \frac{1}{y} \ ,$$

for $0 \leq x \leq 1$ and also that

$$\mathbf{E}(Y|X) = \frac{1}{1-x}\int_0^1 y\mathrm{d}y = \frac{1}{2(1-x)} \ .$$

if we denote the PDF of y given x as

$$f(Y|X) = \frac{f(x,y)}{f(x)} = \frac{2}{2(1-x)} \ ,$$

for $0 \leq y \leq 1$.

9 Problem: Poisson Coin Flips

If we are given that we toss a coin $N \sim \text{Po}(\lambda)$, then the pmf by definition of Poisson random variables is of the form

$$\mathbf{P}(N = n) = e^{-\lambda}\frac{\lambda^n}{n!} \ ,$$

$n = 0, 1, \cdots$. From this distribution, the number of heads, denoted Y, each of which as an individual event occurs with probability $\frac{1}{2}$, can be accounted for as a binomial random variable, in which the expectation

$$\mathbf{E}(Y|N) = \frac{N}{2} \ ,$$

and also that,

$$\mathrm{Var}(Y|N) = \frac{N}{2}(1 - \frac{1}{2}) = \frac{N}{4} \ .$$

By definition, the equality of expectations $\mathbf{E}(\mathbf{E}(Y|N)) = \mathbf{E}(Y)$ demonstrates that the Law of the Iterated Variance and Expectation holds, which states

$$\mathrm{Var}(Y) = \mathrm{Var}(\mathbf{E}(Y|N)) + \mathbf{E}(\mathrm{Var}(Y|N)) \ .$$

From the Law above, we also know that the mean and variance of the Poisson distribution are

$$\mathbf{E}(N) = \lambda \ ,$$

and

$$\mathrm{Var}(N) = \lambda \ .$$

Immediately, we can calculate the expectation and variance, in which

$$\mathbf{E}(Y) = \mathbf{E}(\mathbf{E}(Y|N)) = \mathbf{E}(\frac{N}{2}) = \frac{1}{2}\mathbf{E}(N) = \frac{\lambda}{2} \ ,$$

and

$$\mathrm{Var}(Y) = \mathrm{Var}(\mathbf{E}(Y|N)) + \mathbf{E}(\mathrm{Var}(Y|N)) - \mathrm{Var}(\frac{N}{2}) + \mathbf{E}(\frac{N}{4}) = \frac{1}{4}\mathrm{Var}(N) + \frac{1}{4}\mathbf{E}(N) = \frac{\lambda}{4} + \frac{\lambda}{4} = \frac{\lambda}{2} \ .$$