Homework 8

November 27, 2020

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1 Problem 1

A proof of the Extension Lemma is given in David Mehrle's 6520 notes online, on Page 60.

1.1 Counterexample

To provide a counterexample for the Generalized Extension Lemma, we will consider the following. The counterexample that we will give involves taking $f(x) = \frac{1}{x}$, with f defined on $\mathbf{R} - \{0\}$. With this function and the domain on which it is defined, in the counterexample we now have to provide a subset $\mathcal{A}$ of the open set $\mathcal{U}$ to which f cannot be extended. In particular, we see that there does not exist some $\mathcal{A} \subset \mathcal{U}$ for which our f can be extended at $x = 0$ because $f(x) \longrightarrow \infty$ as $x \longrightarrow 0$. The function's behavior, in which it grows arbitrarily large as we approach $x = 0$, disables us from picking some open set $\mathcal{A}$ of our open neighborhood $\mathcal{U}$ on which we can extend f to $x = 0$, which gives a counterexample for **1**.

2 Problem 2

2.1 Part A

In order to show that there exists a unique smooth structure such that the quotient map $p : M \longrightarrow M/R$ is a submersion, we will break up the proof with several claims. In particular, we observe from the symmetry of the equivalence relation R that we have chosen implies that the projection $(\pi_2)|_R : R \longrightarrow M$ is also a sumbersion onto M.

2.1.1 Claim

The projection map $p : M \longrightarrow M/R$ is open, and the quotient topology is Hausdorff.

The proof of this claim lies in taking some $U \subset M$, and observing that $p^{-1}(p(U)) = \pi_1((M \times M) \cap R)$. By hypothesis, $(\pi_1)|_R$ being a submersion implies that for an open set U, the set $p^{-1}(p(U))$ is open in M. Therefore, we have that $p(U)$ is open in M/R which gives that p is an open map. Finally, the quotient topology is Hausdorff follows from the fact that if the map $p : M \longrightarrow M/R$ is open, the quotient topology is Hausdorff.

For completeness, I have included a proof of this claim, from which we will move onto other steps.

2.1.2 Claim

If $p : X \longrightarrow X/R$ is open, X/R is Hausdorff, with the converse also true.

We know that p being open implies that $p \times p$ is also open. Furthermore, $X \times X/R$ being open implies that its image under $p \times p$ is also open. In particular, the image is equal to $(X/R \times X/R)/\Delta$ where Δ is the diagonal. But Δ is closed, which implies that X/R is Hausdorff and completes the proof of this claim.

2.1.3 Claim

Each equivalence class of R is a closed submanifold of M with dimension $\dim(R) - \dim(M)$.

We will make use of symmetry to show that $[x] = \pi_1((\pi_2)^{-1}(\{x\}))$ for every $x \in M$. In particular, we know that the projection map $\pi_2 : R \longrightarrow M$ is a submersion by symmetry, which implies that if we pick some $x \in M$, the preimage $(\pi_2)^{-1}(\{x\})$ is a smooth manifold on R with dimension $\dim(R) - \dim(M)$. It also follows that $(\pi_2)^{-1}(\{x\}) = (M \times \{x\}) \cap R$, which implies that $(\pi_2)^{-1}(\{x\})$ is a smooth manifold of $M \times \{x\}$. We conclude the proof of this claim by observing that the projection on the first coordinate is equal to $[x]$, which is a smooth manifold of M. This finishes the claim.

2.1.4 Claim

Take $a \in M$. There exists an open neighborhood $U \ni a$, a closed submanifold S of U as well as a submersion $q : U \longrightarrow S$ such that for every $x \in U$, the set $[x] \cap U$ intersects S at a single point $q(x)$.

For this claim, we begin by fixing a submanifold S' of M through a. More specifically, we want S' to satisfy the condition that $T_a S' \bigoplus T_a[a] = T_a M$. To this end, we observe that S' having codimension $\dim[a] = \dim(R) - \dim(M)$, as well as $(\pi_2)|_R$ being a submersion from R to M implies that we have a submanifold

$$Z = \pi_2(S') \cap R = (M \times M) \cap R$$

with codimension $\dim(R) - \dim(M)$ in R. So we have that Z has dimension equal to that of M. From the submanifold Z, we can also choose another submanifold $[a] \times \{a\}$, as well as $\Delta_{S'}$ of $S' \times S'$, where $\Delta_{S'}$ is the diagonal of S'. From our choice of these submanifolds, we observe that their dimensions together equal that of M. Also, we can look at the tangent spaces at (a, a), which are given by

$$T_a[a] \times \{0\} \ ,$$

and

$$\Delta_{T_a S'} \ ,$$

respectively. Observe that these tangent spaces have a trivial intersection if we look at their intersection, which gives that the tangent space of our submanifold Z can be expressed with the following direct sum

$$T_{(a,a)} Z = ([a] \times \{0\}) \bigoplus \Delta_{T_a S'} \ .$$

From this, we now have that the projection $(\pi_1)|_Z : Z \longrightarrow M$ is a local diffeomorphism at the point (a, a) at which we computed the tangent space. This allows us to conclude that there exists an open neighborhood of a which we will denote as $\mathcal{U}$, such that the projection pr_1 is a diffeomorphism. In particular, $\pi_1 : Z \times (\mathcal{U} \times \mathcal{U}) \longrightarrow \mathcal{U}'$, where $\mathcal{U}'$ is some open neighborhood of $a \in M$. From π_2, we can denote its inverse with s. The isomorphism entails that $\pi_1 \circ s = \mathrm{I}_{U'}$, and also that $q = \pi_2 \circ s$ is a submersion from $\mathcal{U}'$ to $\mathcal{V}$, where $\mathcal{V} \subset S' \cap \mathcal{U}$. Altogether, we now have that the inverse s of π_2 is given by

$$s(x) = (x, q(x)) \ ,$$

where in this equality we have $x \in \mathcal{U}'$, with $\mathcal{U}' \subset \mathcal{U}$. In particular, we have that $q(x) = x$ because if we take some $x \in \mathcal{U}' \cap S'$, we have that $(x, x) \in (\mathcal{U} \times \mathcal{U}) \cap Z$, which implies that

$$\pi_1(x, x) = \pi_1 s(x) \ ,$$

and so we have that

$$s(x) = (x, x) \ .$$

which gives that $q(q(x)) = q(x)$ for $q(x) \in \mathcal{U}'$. Next, will take sets U and S, with $U = \mathcal{U}' \cap q^{-1}(U' \cap \mathcal{U})$, and $S = S' \cap U$. We will assume that U and S satisfy the conditions that we have given in the claim, and we also observe that $S \subset \mathcal{U}$ implies that if we pick some $x \in \mathcal{U}$, we have that $q(x) \in \mathcal{U}'$. Hence $q(q(x)) = q(x) \in \mathcal{U} \cap S'$. But we know that $S' \cap U = S$ by assumption.

To finish off this claim, we will use the same $x \in U$ that we have already mentioned, which satisfies the condition that $g(x) \in S' \cap U = S$. In addition to this x, we will also take $y \in S$ satisfying that $y \in [x]$, from which it follows that

$$\begin{aligned}(x, y) \in (\mathcal{U} \times \mathcal{U}) \cap ((M \times S) \cap R) \\ \Rightarrow (x, y) \in (\mathcal{U} \times \mathcal{U}) \cap Z \ ,\end{aligned}$$

because $(\mathcal{U} \times \mathcal{U}) \cap ((M \times S) \cap R) = (\mathcal{U} \times \mathcal{U}) \cap Z$. This gives that $(x, y) = \pi_1(x, y)$, which implies that $y = q(x)$. Indeed, this demonstrates that the intersection of $[x] \cap U$ and S is exactly the single point $q(x)$.

After this result, we will define a slice.

2.1.5 Definition

A submanifold S which has properties from the previous Claim is called a slice through the equivalence classes of our relation R in U.

To show that the desired unique and smooth structure exists, we will now present a few more simple results, from which we will conclude that $p : M \longrightarrow M/R$ is a submersion. But we first observe that the following diagram

commutes because if we have some $V \subset M$, the restriction of R to V, R_V, in terms of a graph is $R_V = (V \times V) \cap R$. With the equivalence relation R, the equivalence classes of R_V in V are sets of the form $[x] \cap V$, where $x \in V$. So with the diagram above, we can separate the inclusion map $V/R_V \longrightarrow M/R$ (arrow should be a hook again, sorry!). In the diagram above, we denote $p_V : V \longrightarrow V/R_V$.

2.1.6 Claim

Take $a \in M$. Then there exists an open neighborhood U of a in M such that U/R_U is a manifold.

Fix U, S and q as given from the previous Claim. We know that $q : U \longrightarrow S$ can be factored to the bijection $\overline{q} : U/R_U \longrightarrow S$, from which we have that $q = \overline{g} \circ p_U$. We can make use of the bijection $\overline{q}$ to guarantee that the manifold structure that we have on S can be transfered to a manifold structure on U/R_U, which would make p_U a submersion. This finishes the claim.

With this claim, we will take our manifold structure and extend it to open neighborhoods, as well as saturated open neighborhoods. Again for completeness, we define a saturated open neighborhood, and then return to the next Claim.

2.1.7 Definition

A subset $V \subset M$ is said to be saturated iff it can be expressed as a union of equivalence classes for R. Equivalently, $V = p^{-1}(p(V))$.

2.1.8 Claim

Let $U \subset M$ be an open subset such that U/R_U is a manifold. We will denote $V = p^{-1}(p(U))$. Then we have that $q : V \longrightarrow U/R_U$ is a surjective submersion.

This claim follows form defining the projection maps $p_U : U \longrightarrow U/R_U$ and $p_V : V \longrightarrow V/R_V$. With p_U and p_V, we have that the inclusion map $i : U \longrightarrow V$ can be factored into a bijection, with the bijection $J : U/R_U \longrightarrow V/R_V$. With the commutative diagram below, we see that the map q that we want can be expressed with the composition $j \circ q = p_V$.

From the diagram, we observe that the vertical arrows, namely the maps that I have drawn in with these arrows, are surjective submersions. Also, the left vertical arrow is a surjective submersion. At this point, applying the following result finishes the proof of this Claim. We are relying on the fact that if we have manifolds M, N, Z, and the following commutative diagram of sets

We have that the map β is smooth if α is a surjective submersion, which implies that f is smooth. Similarly, if β is a submersion, then f is a submersion. Finally, if β is surjective, then f is surjective as well. In this situation, applying this result directly gives that q is a surjective submersion, and finishes this Claim.

2.1.9 Claim

With U, V as given in the previous claim, if U/R_U is a smooth manifold, then V/R_V is also a smooth manifold.

We will show that p_V is a submersion. From the previous Claim, we will introduce the same bijection $j : U/R_U \longrightarrow V/R_V$, with $j \circ q = p_V$. If we equip V/R_V with the manifold structure that we want, we can endow V/R_V with this manifold structure by transporting it through the bijection j. This implies that p_V is a submersion, hence V/R_V is a smooth manifold which completes the Claim.

This last Claim implies that the map $r : V \longrightarrow S : [x] \cap S \mapsto \{r(x)\}$ is a smooth surjective submersion. From our definition of saturated neighborhoods, we also know that the slice S of U is also a slice for V. From the definition of the slice that we have introduced, we know that the slices are the same precisely because each of the manifolds have the necessary properties, namely that the intersection of the set $[x] \cap U$ with the closed submanifold S only consists of a single point. With this result, we will now show that the desired smooth structure on p exists, and also that p is a submersion.

To finish **A**, we will now make use of the fact that for every point $a \in M$, we know that this point is contained in an open saturated neighborhood V with V/R_V a manifold. With this observation, we observe that M can be covered with saturated open neighborhoods U_i such that U_i/R_{U_i} is a smooth manifold, for each i. Moreover, if we have the projection map $p : M \longrightarrow M/R$, we know that restricting the projection to the open neighborhood U_i gives a map of the form

$$p|_{U_i} : U_i \longrightarrow M/R \ ,$$

that can be factored into a bijection from the quotient U_i/R_{U_i} onto an open set $p(U_i) \subset M/R$. This will allow us to, intuitively speaking, transfer the manifold structure that we want from U_i/R_{U_i} to $p(U_i)$, which are precisely given from our restriction $p|_{U_i}$ of the projection p.

With this manifold structure in mind, we will proceed by taking indices i, j so that $p(U_i) \cap p(U_j) \neq \emptyset$. From our definition of the saturated set, we know that U_i and U_j are saturated which implies that $p(U_i \cap U_j) = p(U_i) \cap p(U_j)$. With this equality, we observe that the manifold structure from $p(U_i)$ induces a manifold structure $\mathcal{M}_i$ on $p(U_i) \cap p(U_j)$. It is important to note that this manifold structure depends on the projection map $p : U_i \cap U_j \longrightarrow p(U_i) \cap p(U_j)$ being a submersion. But we know that the smooth structure that we have been describing is unique because the manifold structure from $p(U_j)$ surely induces another manifold structure $\mathcal{M}_j$ on $p(U_i) \cap p(U_j)$, again which depends on $p : U_i \cap U_j \longrightarrow p(U_i) \cap p(U_j)$ being a submersion. From the indices i, j, we conclude that the manifold structure is unique because the manifold structures that we have described from the respective manifold structures $\mathcal{M}_i$ and $\mathcal{M}_j$ are the same. For the third (and last) time, recall that the commutative diagram

guarantees that we can get f to be either smooth, a submersion or a surjective mapping depending on whether β is smooth, a submersion or a surjective mapping. With the uniqueness of this smooth structure, we conclude that M/R has a unique manifold structure ensuring that the inclusion maps $U_i/R_{U_i} \longrightarrow M/R$ are diffeomorphisms onto open subsets. This manifold structure gives us that the projection map $p : M \longrightarrow M/R$ is a submersion, as desired and completes **A**.

2.2 Part B

We will show that each statement holds by stating the following Claim and incorporating its proof into **2B**.

2.2.1 Claim

Assume that M/R is a smooth manifold. Then R is a closed submanifold of M, and the projection maps $(\pi_j)|_R : R \longrightarrow M$ are surjective submersions for $j = 1, 2$.

We will show that our equivalence relation R is a submanifold of $M \times M$ first. We can do this by observing that by definition, the projection map p is a submersion. In particular, if we have a surjection between the manifolds M, N given by $p : M \longrightarrow N$, and M has a smooth manifold structure, we can define the identity map I. Specifically, we will introduce the manifolds N_j, for $j = 1, 2$, that have the same smooth manifold structure as N. With N_1, N_2, we define the identity map $I : N_1 \longrightarrow N_2$. With the identity, we then have that the map $I \times p : M \times M \longrightarrow M \times M/R$ is a submersion. This map being a submersion implies that the graph, which is by definition given by $\text{graph}(R) = \{(x, p(x)) | x \in M\}$, is a submanifold of $M \times M/R$. From properties of the graph of an equivalence relation R, we have the commuting diagram

More generally, we know that $\text{graph}(R) \subseteq M \times M$, where

$$\begin{aligned} \text{graph}(R) &= \{(a, b) \in M \times M | a \sim b\} \\ &= \{(a, b) \in M \times M | p(a) = p(b)\} \\ &= \{(a, b) \in M \times M | p \circ \pi_1(a, b) = p \circ \pi_2(a, b)\} \ . \end{aligned}$$

With this definition of the graph of R, as well as p being a submersion, suppose that we have smooth maps satisfying the following diagram

with g a submersion. From the diagram above, we know that $W = \{(x, y) \in X \times Y | f(x) = g(y)\}$ is a submanifold of $X \times Y$ with the help of this diagram. Also, it is clear that any submanifold of Z is a submanifold of Y, that is taken through g. As a special case, we know that taking $X = \{*\}$ and $z = f(*)$ implies that $W = \{(*, y) | g(y) = z\}$. This gives that $W = g^{-1}(z)$ so W is a submanifold of $X \times Y$ by the Regular Value Theorem.

From the commutative diagram and the special case, we can apply the arguments that we have started, we can apply the observations that I have given, setting $X = Y = M$ in the diagram above. Because $R = \alpha^{-1}(R)$, we know that our equivalence relation $R = (I \times p)^{-1}(\gamma)$, implying that R is a closed submanifold of $M \times M$. The special case and arguments that I have given are reflective of the more general fact that the pull back of a submersion is a submersion. In a more general circumstance, if we have some smooth $f : M \longrightarrow N$, and a submersion $p : Z \longrightarrow N$, the pull back $f^*Z = \{(m, z) \in M \times Z | f(m) = p(z)\}$. It is known that f^*Z is equal to the inverse image of $\text{graph}(f)$ under our submersion $I \times p$, from which we get a submersion $f^*p : f^*Z \longrightarrow M$.

Besides concluding that R is a closed submanifold of $M \times M$, we also observe that $(I \times p)_R : R \longrightarrow \gamma$ is submersion that is surjective, and because on M, we have that the composition, $\pi_1 \circ (I, p) = I$ we conclude that the projection $\pi_1 : \gamma \longrightarrow M$ is a surjective submersion. For the last few steps, we can obtain that π_1 is a surjective submersion from π_1 being a surjective submersion because the composition $\pi_1 \circ (I \times p)$ is also a surjective submersion on $R \longrightarrow M$. To conclude, this composition equals $(\pi_1)|_R$, which completes **B**.

2.3 Part C

I used this result for **A**. I have reproduced the claim and proof below.

2.3.1 Claim

Each equivalence class of R is a closed submanifold of M with dimension $\dim(R) - \dim(M)$.

We will make use of symmetry to show that $[x] = \pi_1((\pi_2)^{-1}(\{x\}))$ for every $x \in M$. In particular, we know that the projection map $\pi_2 : R \longrightarrow M$ is a submersion by symmetry, which implies that if we pick some $x \in M$, the preimage $(\pi_2)^{-1}(\{x\})$ is a smooth manifold on R with dimension $\dim(R) - \dim(M)$. It also follows that $(\pi_2)^{-1}(\{x\}) = (M \times \{x\}) \cap R$, which implies that $(\pi_2)^{-1}(\{x\})$ is a smooth manifold of $M \times \{x\}$. We conclude the proof of this claim by observing that the projection on the first coordinate is equal to $[x]$, which is a smooth manifold of M. This finishes the claim.

This completes **C**.

2.4 Part D

2.4.1 Shortest Way

We will make use of the following general fact that a topological space X is Hausdorff iff the diagonal $\Delta = \{(x, x)|x \in X\}$ is a closed subspace of $X \times X$.

$\Rightarrow$ Suppose that X is Hausdorff. In order to show that the complement of the diagonal Δ is open, we will take some $(x, y) \in \Delta^c := X \times X \backslash \Delta$. In particular, we know that X is Hausdorff, which implies that there are open disjoint sets U, V that respectively contain x and y. Also, we know from the definition of the product topology that $U \times V$ is an open subset of $X \times X$, which implies that Δ^c is open because $U \times V \subset \Delta^c$. Otherwise, if $U \times V$ were not a subset of Δ^c would contradict that the open set U and V are disjoint. We know that U and V must be disjoint because X is Hausdorff; so we have shown that $\Delta = \{(x, x)|x \in X\}$ is a closed subspace of $X \times X$ which completes the forward direction.

$\Leftarrow$ Suppose that Δ is closed, which by the definition of a closed set means that the complement Δ^c is open. If we take distinct $x, y \in X$, we can show that X is Hausdorff from the definition, in which there should be disjoint open neighborhoods surrounding each of these points. So we have that $(x, y) \in \Delta^c$, which implies that there is an open set $U \times V \subset \Delta^c$ that contains (x, y). Because U and V are open sets, we know that these open sets are in fact disjoint and contain the points $x, y \in X$, respectively. This completes the reverse direction.

With this general fact, **D** is finished. In particular, from this general fact we can take $M = X$ and the given equivalence relation R, from which the desired claim follows.

2.4.2 Forward Direction With a Contradiction, and Reverse Direction with Submersion Theorem

$\Rightarrow$ Suppose that M/R is Hausdorff. To show that R is a closed subset of $M \times M$, we observe the following. First, from the equivalence relation R, observe that the quotient map $q : M \longrightarrow M/R : m \mapsto [m]$ is defined, and also that the topology on M/R can be defined by looking at the inverse image and making sure that the inverse image is open for an open set $\mathcal{O} \subset M/R$. With these properties, if we have that M/R is Hausdorff, we can show that R is closed by appealing to this definition; we will fix some $(x, y) \in (M \times M)/R$. In order to show that the set $M \times M - R$ is open, we will give a proof by contradiction.

We will take (m, m') in the set, we can look at the respective images $p(m)$ and $p(m')$, from which we know that $p(x) \neq p(m')$ because $(m, m') \notin R$. By assumption, we know that M/R is Hausdorff which by definition implies that we can take disjoint open sets of $p(m)$ and $p(m')$, which we will respectively denote as $\mathcal{U}$ and $\mathcal{U}'$. We will also denote $V = p^{-1}(\mathcal{U})$, and $V' = p^{-1}(\mathcal{U}')$. With everything, we observe that if $(V \times V') \cap R \neq \emptyset$, then there would exist $(v, v') \subset V \times V'$ for which $p(v) = p(v')$. From what we have mentioned previously, if $p(v) = p(v')$, then $p(v) \in \mathcal{U}$ and $p(v') \in \mathcal{U}'$. But this contradicts our assumption that $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, and demonstrates that our equivalence relation R is indeed closed in $M \times M$, because in the product topology $V \times V'$ is open, and also because $x \in V \times V'$. Finally $V \times V'$ is also disjoint with R, which completes the forward direction.

$\Leftarrow$ Suppose that R is a closed subset of $M \times M$. We will show that M/R is Hausdorff by observing that from the Regular Value Theorem, which is proven using the Submersion Theorem, that the map h is a submersion. Specifically, we will define $h = (f, g)$ where

$$h : X \times Y \longrightarrow Z \times Z \ .$$

If we take some $W \subseteq X \times Y$, we observe that, as previously mentioned, h is a submersion because each of its components are submersions. From the basic definition of the diagonal Δ_Z, we can identify W with the inverse image $h^{-1}(\Delta_Z)$. With this observation, we then know that the result from the reverse direction holds with the following result. With manifolds M, N, we know that if $F : M \longrightarrow N$ is a submersion and P is a submanifold of N, then the inverse image $F^{-1}(P)$ is a submanifold of M. With the Submersion Theorem, we know that this result holds, and applying this statement directly gives the result that we want for the reverse direction. With the inverse image $F^{-1}(P)$ being a submanifold of M, we have that M/R is Hausdorff.

3 Problem 3

3.1 Part A

3.1.1 The Sjamaar Way

We will show that $\theta : G \times M \longrightarrow M$ is proper. By definition, the map being proper would imply that the image of (g, a) under θ would be closed, in which $\theta(g, a) = (g, a)$. To check that this property holds, we will take $C \subseteq G \times M$ and show that $\theta(C)$ is closed. In particular, we can take a subsequence $(a_i, g_i a_i)$ in $\theta(C)$. With this subsequence, suppose that it converges to (a, ga). If this is the case, then we know that G being compact implies that we can put an invariant Riemannian metric on M, and with the sequence $(a_i, g_i a_i)$ and θ, we know that the sequence converges precisely when each of its components do. From the definition of compactness, we can replace the term g_i in the second component of $\theta(C)$ with a convergent subsequence. So with one of the definitions of compactness, we can obtain a convergent subsequence, and if we have that $\lim_{i \longrightarrow \infty} g_i = g$, and also that $\lim_{i \longrightarrow \infty} g_i a_i = ga$, the smoothness of our left action θ allows us to conclude that the second limit holds. In particular, we have that $\lim_{i \longrightarrow \infty} g_i a_i = ga$ because our smooth left action θ is continuous. The continuity of θ implies that we can multiply the limits of g_i and a_i individually as $i \longrightarrow \infty$ to determine the behavior of the limit in the second component $g_i a_i$. For the first component of the limit (a, ga), we have that the first limit converges to a as $i \longrightarrow \infty$ because the manifold M is second countable, an assumption added to the HW which guarantees that the limits that we have computed are unique in a Hausdorff space, and this completes **A** with all of our assumptions.

3.1.2 The Lee Way

From Lee, we know that the following statement holds.

3.1.3 Lemma 7.1, Page 147

Suppose a Lie Group G acts smoothly on a smooth manifold M. The action is proper if and only if for every compact subset $K \subset M$, the set $G_k = \{g \in G : (g \cdot K) \cap K \neq \emptyset\}$ is compact.

$\Rightarrow$ Suppose that the action is proper. Then, we begin by defining the map $\Theta : G \times M \longrightarrow M \times M : (g, p) \mapsto (g \cdot p, p)$. Assume that Θ is proper, and we then have that for any compact set K, we have that

$$
\begin{aligned}
G_k &= \{g \in G : \text{ there exists } p \in K \text{ such that } g \cdot p \in K\} \\
\Rightarrow G_k &= \{g \in G : \text{ there exists } p \in M \text{ such that } \Theta(g,p) \in K \times K\} \\
&= \pi_G(\Theta^{-1}(K \times K)) \ .
\end{aligned}
$$

In the steps above, observe that $\pi_G : G \times M \longrightarrow G$ is the projection map. We conclude that G_k is compact.

$\Leftarrow$ Suppose that G_k is compact for every set $K \subset M$. If we have a compact set $L \subset M \times M$, we can denote $K = \pi_1(L) \cup \pi_2(L) \subset M$, where the maps π_1 and π_2 denote the respective projections onto the first and second factors, which is given by

$$\pi_1, \pi_2 : M \times M \longrightarrow M \ .$$

We have that

$$
\begin{aligned}
\Theta^{-1}(L) &\subset \Theta^{-1}(K \times K) \\
\subset \{(g,p) : g \cdot p \in K \text{ and } p \in K\} &\subset G_k \times K \ .
\end{aligned}
$$

By continuity, we know that $\Theta^{-1}(L)$ is closed. Also, we have that $\Theta^{-1}(L)$ is a closed subset of $G_k \times K$, which is compact. It follows that $\Theta^{-1}(L)$ is compact, which completes the reverse direction.

With this claim, we have an equivalent formulation for an action being proper. Therefore, we can show that G being compact with a Hausdorff Manifold M implies that the set G_k is compact. To show that this holds, we will make use of the following.

3.1.4 Claim

Any smooth action by a compact Lie Group on a smooth manifold is proper.

As usual, we will define G to be our compact Lie Group that is acting smoothly on the manifold M. If we take some compact set $K \subset M$, we know that G_K is closed in G by continuity. Hence, G_k is compact, and from the equivalence this completes **A**.

3.2 Part B

If we suppose that θ is a free action, we can show that Θ is an immersion. In particular, we suppose that θ is free because G acts freely on M, which implies that we can only have $(g^{-1}g') \cdot p = p$ hold if $g^{-1}g' = e$, implying that $g' = g$. With $\theta^{(p)}$ injective because $g' = g$, we also know that $\theta^{(p)}$ is equivariant with respect to left translation of G, and also with respect to θ. Recall from the **Equivariant Map Theorem** that the following diagram must commute

Therefore,

$$
\begin{aligned}
\theta^{(p)}(g'g) &= (g'g) \cdot p \\
&= g' \cdot (g \cdot p) \\
&= g' \cdot \theta^{(p)}(g) \ .
\end{aligned}
$$

Again, these equalities demonstrate that $\theta^{(p)}$ is equivariant with respect to the smooth left action θ on the Lie Group G. By assumption, G acting transitively on itself ensures that $\theta^{(p)}$ has constant rank, in which the following Proposition from Lee is helpful.

3.2.1 Proposition 5.17, Page 111

Let $F : M \longrightarrow N$ be a smooth map with constant rank. If F is injective, then it is an immersion.

So we have that $\theta^{(p)}$ is an immersion because of the following. We know that $\theta^{(p)}$ having constant rank as well as being injective implies that it is an immersion, by Proposition 5.17 in Lee. This completes **B**.

3.3 Part C

We will show that orbits of the smooth left action are embedded submanifolds of M. In conjuction with the smooth left action θ, we will fix $p \in M$ and again use the smooth map $\theta^{(p)} : G \longrightarrow M : g \mapsto g \cdot p$. We have defined $\theta^{(p)}$ so that the image $\theta^{(p)}(p)$ is the orbit of p. Now, we will show that $\theta^{(p)}(p)$ is an embedding, observing that

$$
\theta^{(p)}(g') = \theta^{(p)}(g) \Rightarrow g' \cdot p = g \cdot p \ ,
$$

from which we conclude that $(g^{-1}g') \cdot p = p$. At this point, we are repeating the arguments that we have given in **B**, in which we showed that the smooth action θ is an immersion from Proposition 5.17 in Lee. With θ being an immersion, we also show that $\theta^{(p)}$ is an **embedding** by taking a compact set $K \subset M$, from which we know that $(\theta^{(p)})^{-1}$ is closed in G because the inverse map $(\theta^{(p)})^{-1}$ is continuous. The image $(\theta^{(p)})^{-1}(K)$ is closed. From **A**, we know that our equivalent formulation of a Lie Group action being proper implies that

$$
(\theta^{(p)})^{-1}(K) \subset G_k \ \{g \in G : (g \cdot K) \cap K \neq \emptyset\} \ .
$$

By Lemma 7.1 in Lee, we know that $(\theta^{(p)})^{-1}(K)$ is compact because the set G_k is compact, which is precisely given by the first Claim that I gave in **A**. Next, we observe that $\theta^{(p)}$ is a proper map, and from this properness, we have that $\theta^{(p)}$ is an embedding because from our work in **B**, $\theta^{(p)}$ is an injective immersion. From Lee, Proposition 5.4(b) states that a proper injective immersion is an embedding with a closed image. The map being an embedding with a closed image necessarily gives that the orbits are embedded submanifolds.

In short, we have that the orbits are in fact embedded submanifolds because if we take some $p \in P$, we know that the Isotropy Group, as defined in Lee, that is given by $G_p = \{g \in G : g \cdot p = p\}$ is a closed subgroup of the Lie Group G. With our smooth left action θ, we know that G acts transitively on M which gives that the quotient G/G_p has a smooth manifold structure that is unique. With the orbit map $\theta^{(p)}$ that we have defined, we can obtain another smooth map $F_p : G/G_p \longrightarrow M$. It is important to note that F_p is a bijection from the quotient G/G_p to the orbit $G \cdot p$ under the action. Also, F_p being equivariant implies that it has constant rank. So by restricting the smooth map $G \times M \longrightarrow M$, the restriction map to the embedded submanifold $G \times \{p\} \subseteq G \times M$ is given by the smooth map $G = G \times \{p\} \longrightarrow M$. The induced map $G/G_p \longrightarrow M$, because G_p is a constant map, implies that $G \longrightarrow G/G_p$ is a submersion. We conclude that F_p is a smooth imersion with its image $G \cdot p$ a smooth immersed submanifold, which completes **C**.

3.4 Part D

3.4.1 Quotient Manifold Theorem, Theorem 7.10 from Lee, Page 153

To prove the uniqueness of the smooth structure, first suppose that M/G has two different smooth structures for which the projection $\pi : M \longrightarrow M/G$ is a smooth submersion. We will denote these first and second structures as $(M/G)_1$ and $(M/G)_2$, respectively. From Proposition 5.19 in Lee, which states that any map $F : N \longrightarrow P$ is smooth iff $F \circ \pi$ is smooth, with $\pi : M \longrightarrow N$ a surjective submersion, we conclude that the identity map from $(M/G)_1$ to $(M/G)_2$ is smooth. From Proposition 5.19, we have the commuting diagram (top of **112** in Lee)

with M, N, P smooth manifolds. Directly applying Proposition 5.19 then diagrammatically entails that (top of **153** in Lee)

Repeating the same arguments to the inverse of the identity map implies that it is smooth in the opposite direction. Hence the 2 smooth structures that we have introduced are the same, which guarantees uniqueness.

We must also show that M/G is a topological manifold. We can do this by taking an adapted chart (U, ϕ) of M for the G action. From this action, it is possible that if we have a specific type of chart, say a cubical one, with coordinate functions $(x, y) = (x^1, \cdots, x^k, y^1, \cdots, y^k)$, we could then have the intersection orbit from the action of the Lie Group with U is either a slice of the form $(y^1, \cdots, y^n)$ or the empty set. If we want to show that we have an adapted chart centered at p for each $p \in M$, it is enough to make use of our result in **C**. With the orbits of the action being embedded submanifolds, we know that these orbits are diffeomorphic to G.

To this end, we will take some $D \subset TM$ with

$$D = \bigcup_{p \in M} D_p \ ,$$

where $D_p = T_p(G \cdot p)$. From this union, we know that each of the submanifolds (which are the orbits) have dimensions k so each D_p has dimension k. In particular, we can argue that D has a smooth distribution by taking some $X \in \mathbf{g}$, where again $\mathbf{g}$ is the Lie Algebra, and $\hat{X}$ is the vector field on M. Specifically, we define this vector field with a generator of the flow $(t, p) \mapsto (\exp tX) \cdot p$. So if we are given a basis $X_1, \cdot, X_k$ of $\mathbf{g}$, we are able to conclude that D is smooth because we have a global frame $\hat{X}_1, \cdots, \hat{X}_k$. But the orbits of the Lie Group action are closed, which implies that a connected component of the orbit from the group action is a leaf of foliations. Altogether, these observations allow us to argue that with our cubical subset of U, we can argue that the restriction of our smooth left action $\theta : G \times Y \longrightarrow M$, where Y is an n dimensional submanifold of M which has points in U of the form $(0, y)$ for each i, is a diffeomorphism. More specifically, Y must also satisfy that $Y_i = U_i \cap Y$ for each i.

We can show that this diffeomorphism holds by fixing some $p \in M$ and taking a smooth chart (U, ϕ) on M that is centered at p. In particular, we will want this smooth chart to be flat in D, and also that it have coordinate functions $(x, y) = (x^1, \cdots, x^k, y^1, \cdots, y^k)$. With these conditions, the intersection of any orbit of our Lie Group action with U is either a countable union of constant slices, or the empty set. More specifically, the coordinate functions $(x, y) = (x^1, \cdots, x^n, y^1, \cdots, y^n)$ that we have chosen are adapted to the Lie Group Action, which gives that the intersection of the orbit with U could be a slice that has the form $y^i = c^i$ for all $1 \le i \le n$. In future arguments for $\mathbf{D}$, we will introduce a slice chart centered at p, and will look at coordinate functions so that the intersection $(G \cdot p) \cap U$ is a slice with $v^i = 0$ for all $1 \leq i \leq n$. If we take a submanifold S of U with $u^i = 0$ for $1 \leq i \leq n$, we know that the direct sum decomposition for T_pM allows us to look at the orbit map $\theta^{(p)}$. Altogether, we also want a subset $U_0 \subset U$, centered at p, of our cubical set which has points that intersect each orbit of the action at a single slice.

If there is no subset U_0 satisfying these properties, then for each i we begin by defining U_i as the cubical subset of U. This subset U_i is merely a collection of all of coordinates that are less than $\frac{1}{i}$ in absolute value. Next, if we denote Y as a n dimensional submanifold of M as we have mentioned previously, we are able to observe that each k slice that is our subset U_i that intersects Y_i at exactly one point implies that we can get distinct $p_i, p_i' \in Y_i$ for each i. These p_i, p_i' would satisfy the condition that $g_i \cdot p_i = p_i'$ where $g_i \in G$ which would imply that p_i, p_i' belong to the same orbit of the Lie Group Action. But it is also important to note that the inverse images of p_i and p_i' under Θ (which is proper) lie in some compact set $\mathcal{L} \subset G \times M$. By assumption in $\mathbf{D}$, we demand that our action is proper, which enables us to pass to a subsequence. We can obtain such a subsequence because from our construction of the subsets $\{Y_i\}$, we have that the sequences $p_i, p_i' = g_i \cdot p_i \longrightarrow p$ as $i \longrightarrow \infty$. In short, we have that $\lim_{i \longrightarrow \infty} g_i \cdot p_i = \lim_{i \longrightarrow \infty} p_i' = p = g \cdot p$ because our Lie group action is free and by continuity. Hence $g = e$.

We will now define the restriction of our smooth left action θ to $G \times Y$. It is given by $\theta^Y : G \times Y \longrightarrow M$, and observe that the dimensions of $G \times Y$ and M add up to m because $k + n = m$. The restriction θ^Y is important because we can restrict the map $\{e\} \times Y$. In fact, restricting to $\{e\} \times Y$ gives a map that is the same as the inclusion $Y \longrightarrow M$, and we have that the map $d(\theta^Y)_{(e,p)}$ is an isomorphism because $T_pM = T_p(G \cdot p) \bigoplus T_pY$. Therefore, looking at $\theta^Y|_W$ for some neighborhood W of $(e, p) \in G \times Y$ implies that $\theta^Y|_W$ is a diffeomorphism. From the uniqueness of the sum decomposition $T_p(G \cdot p) \bigoplus T_pY$ for T_pM, we also know this restriction is a local diffeomorphism, which is detailed in the **Other Way**. Also, observe that in this direct sum decomposition $T_p(G \cdot p)$ is the span of $\frac{\partial}{\partial u^i}$ and that T_pS is the span of $\frac{\partial}{\partial v^i}$. However, $\theta^Y|_W$ is also injective which contradicts the fact that $\theta^Y(g_i, p_i) = \theta^Y(e, p_i') = p_i'$ for i sufficiently large so that $(g_i, p_i), (e, p_i') \in W$. We initially assumed that p_i and p_i' are distinct elements of Y_i.

We will also show that adapted coordinate charts even exist. We will choose diffeomorphisms $\alpha : \mathbf{B}^k \longrightarrow X$ and $\beta : \mathbf{B}^n \longrightarrow Y$, with $\mathbf{B}^k, \mathbf{B}^n$ the open unit balls in $\mathbf{R}^k$ and $\mathbf{R}^n$, respectively, we will also define $\gamma : \mathbf{B}^k \times \mathbf{B}^n \longrightarrow U : (x, y) \mapsto \theta_{\alpha(x)}(\beta(y))$. From the bottom of **155** in Lee, γ is equal to the following composition of diffeomorphisms

and is therefore also a diffeomorphism. Also, the inverse γ^{-1} is a coordinate map on U, and to show that adapted charts exist, there must exist an adapted G-action from our Lie Group. From the definition for an adapted G-action, we know that each slice y that is constant is contained in an orbit of the Lie Group action. Specifically, it is of the form $\theta(X \times \{p_0\}) \subset \theta(G \times \{p_0\} = G \cdot p_0$, with $p_0 \in Y$ a point whose y coordinate is equal to a constant. So if the orbit does intersect U, then this intersection is preserved if we take a union of slices where y equals some constant. Therefore, adapted coordinate charts exist because from our action being G-adapted, each orbit intersects U at only one slice it there is an intersection.

To show that M/G is a manifold, we will again make use of the proper map $\Theta : G \times M \longrightarrow M \times M : (g, p) \mapsto (g \cdot p, p)$. As a more general fact, we know that for any open set $U \subset M$, $\pi^{-1}(\pi(U))$ can be expressed as a union of sets of the form $\theta_g(U)$ for $g \in G$. We know that θ_p is a diffeomorphism because any set of this type is open, which implies that $\pi^{-1}(\pi(U))$ is also open. But π is an open map because the image $\pi(U)$ is open in M/G, because π is a quotient map. The openess of π will help demonstrate that there is a basis for the quotient topology on M/G, and also that M/G is second countable.

As a brief side remark, we know from the bottom of **154** in Lee that the orbit map $\theta^{(p)}$ can be expressed in terms of the composition

From **C**, $\theta^{(p)}$ being an embedding with image the orbit $G \cdot p$ implies that the image

$$\theta_*^{(p)}(T_e G) = T_p(G \cdot p) \subset T_p M \ .$$

Therefore, the image

$$\phi_* : T_{(e,p)}(G \times S) \longrightarrow T_p M \ ,$$

satisfies the condition that $T_p(G \cdot p) \subset T_p M$. In a similar way, the embedding $j_e : S \longrightarrow G \times S : q \mapsto (e, q)$and inclusion $i : S \longrightarrow M$ (Sorry, I don't know how to do a hook on i, but I'm writing this to let you know that it should be there!) imply that we have the composition

These compositions are important for the product neighborhood X and Y that we will choose for later arguments.

We will now turn to verifying that we indeed have a basis for the topology of M/G. By definition, this means that if we have a countable basis for the topology of M, say $\{U_i\}$, we can obtain a countable collection of open subsets of M/G. We know that this holds because if we have a basis for the original topology on M, we can obtain a topology for the quotient M/G by observing that the image $\pi(\{U_i\})$ covers M/R. We know this because the open sets $\{U_i\}$ that we have chosen cover our manifold M. Also, we have that intersecting the images $\pi(\mathcal{U})1)$ and $\pi(\mathcal{U}_2)$ is open, in which we recall that $\mathcal{U}_1$,$\mathcal{U}_2$ are chosen from the collection of open set $\{U_i\}$. Additionally, we observe that $\mathcal{O} = \pi(\mathcal{U})1) \cap \pi(\mathcal{U}_2)$ is open because q is continuous. Therefore, we have a basis for the quotient topology M/G; we observe that q being surjective implies that the union $\mathcal{O}' = \pi(\pi^{-1}(\mathcal{O}')) = \pi(\mathcal{O}) = \cup_\delta \pi(B_\delta)$ of basis elements, with $\delta > 0$ arbitrary, implies that for all $x \in \mathcal{O}'$, $x \in \pi(B_\delta) \subseteq \mathcal{O}'$. Furthermore, we can take some $\pi(p) = q \in M/G$ and an adapted chart (U, ϕ). With this point and chart, we would like for the image $\phi(U)$ to equal an open Euclidean balls in $\mathbf{R}^k \times \mathbf{R}^n$ by shrinking the individual components X and Y of the product neighborhood, if necessary. Next, we can express $\phi(U) = U' \times U_1'$ with U', U_1' are open cubes in $\mathbf{R}^k$ and $\mathbf{R}^n$, respectively. Again from π being an open map, we have that $\pi(U) = V$ is also open. So if we take coordinate functions of ϕ, which we will denote with $(x^1, \cdots, x^k, y^1, \cdots, y^n)$, taking the submanifold $Y \subset U$ with $x^1, \cdots, x^k \equiv 0$ gives a bijective map $\pi : Y \longrightarrow V$. The bijectivity of π follows from the fact that we took an adapted chart (U, ϕ) of M centered at p.

But for $W \subset Y$ open, we know that the image

$$\pi(W) = \pi(\{(x, y) : (0, y) \in W\}) \ ,$$

is open in M/G because restricting the map π to Y, namely $\pi|_Y : Y \longrightarrow V$, is a homeomorphism. This map exhibits a smooth mapping between the topological spaces Y and V, admitting a smooth inverse $(\pi|_Y)^{-1} : V \longrightarrow Y$. To proceed, we will denote the map $\sigma : V \longrightarrow Y \subset U$ as a local section of π that is given by $\sigma = (\pi|_Y)^{-1}$. We will define another map $\eta : V \longrightarrow U_1'$ that sends the equivalence class of (x, y) to y, ie $(x, y) \mapsto y$. As a brief remark, we observe that η is well defined because it we choose two representatives of the points $(x, y), (x', y')$, from the definition of the adapted chart we know that $\eta((x, y)) = \eta((x', y'))$ necessarily because the points $(x, y), (x', y')$ would be sent to exactly the same equivalence class, under η. Moreover, these properties of η imply that η can be expressed in terms of the composition

$$\eta = \pi'' \circ \phi \circ \sigma \ ,$$

where $\pi'' : U' \times U_1' \longrightarrow U_1' \subset \mathbf{R}^n$ is the projection onto the second factor. In the **notation** given for **2** and **3**, clearly $\pi_2 = \pi''$ are the projections onto the second coordinate. Now, we know from this composition for η that σ being a homeomorphism implies that M/G is locally Euclidean. This demonstrates that we have a manifold that has dimension $= n$, and with our previous verification that we have a basis for M/G, we also conclude that M/G is second countable.

From here, we need to show that M/G is Hausdorff. This follows from the fact that we can define an orbit relation with the following from Lee.

3.4.2 Definition

The orbit relation $O \subset M \times M$ is given by

$$\begin{aligned} O &= \Theta(G \times M) \\ &= \{(g \cdot p, p) \in M \times M : p \in M, g \in G\} \ , \end{aligned}$$

where $\Theta : G \times M \longrightarrow M \times M$, as given.

With the orbit relation $\mathcal{O}$, we are able to conclude that O is a closed subset of $M \times M$ because proper maps are closed. Furthermore, we naturally have that p and q lie in distinct orbits of the Lie Group action if $\pi(q)$ and $\pi(p)$ are distinct points in M/G, which immediately implies that $(p, q) \notin O$. Also, we have that for a product neighborhood $U \times V$ of $(p, q) \in M \times M$ disjoint from O, the images $\pi(U)$ and $\pi(V)$ under π are disjoint. Moreover, $\pi(U)$ and $\pi(V)$ are open subsets of M/G that respectively contain $\pi(p)$ and $\pi(q)$. This gives that M/G is Hausdorff.

Finally, we can show that the quotient map is a submersion with the following. First of all, we will introduce the atlas that consists of all charts (V, η) that we have previously given. With this atlas, we know that we have a coordinate representation from the projection map, in which $\pi(x, y) = y$ for π. Furthermore, with this representation, we know that the image y of (x, y) under π is taken with respect to any chart for M/G that corresponds to an adapted chart for M. We also know that the projection onto the second coordinate, π_2, is a smooth submersion.

With this characteristic of π_2, it suffices to show that taking any 2 charts with the properties that we have described are smoothly compatible. To do this, we will fix two adapted charts (U, ϕ), $(\tilde{U}, \tilde{\phi})$ on M. With the adapted charts for M, we will denote their corresponding charts with (V, η) and $(\tilde{V}, \tilde{\eta})$ on the quotient M/G, respectively. From this set up, it is possible that the adapted charts could be centered at the same point p, or that they could not be. In the former case, the charts being centered at a common base point p implies that the adapted coordinates are (x, y) and $(\tilde{x}, \tilde{y})$, respectively. Then, looking at the adapted coordinates implies that two points having equal y coordinates means that these points belong to the same orbit of our Lie Group action. We know that this statement holds because we have specifically adapted our smooth left action, from which we conclude that the points must not only have the same y coordinate, but also the same $\tilde{y}$ coordinate. Directly, we can express the transition map with these adpated charts; between the coordinates (x, y) and $(\tilde{x}, \tilde{y})$, we know that the coordinate $(\tilde{x}, \tilde{y})$ can be expressed in terms of $A(x, y) = \tilde{x}$, and $B(y) = \tilde{y}$. In these expressions for $A(x, y)$ and $B(y)$, observe that A and B are merely smooth maps. In terms of the construction, we define the maps A and B on a neighborhood of $(0, 0)$ that we are free to choose. Explicitly, the transition map $\tilde{\eta} \circ \eta^{-1}$ is given by $B(y) = \tilde{y}$. The image of y under the map B is certainly smooth, which concludes the first possibility.

As mentioned, it is also possible that the adapted charts do not have the same base point p. If this is the case, we can take $p \in U$ and $\tilde{p} \in \tilde{U}$ to form the adapted charts (U, ϕ) and $(\tilde{U}, \tilde{\phi})$, respectively, with $\pi(p) = \pi(\tilde{p})$. From these charts, we can translate them so that they are centered at the points p and $\tilde{p}$ that we have introduced. Also, arranging the charts in this way implies that we have $g \cdot p = \tilde{p}$ because p and $\tilde{p}$ lie in the same orbit of our action. But $\theta_p : M \longrightarrow M$ is a diffeomorphism that sends orbits of the action to orbits, which implies that $\phi' = \tilde{\phi} \circ \theta_g$ for another chart centered at p. We know that $\tilde{\sigma}' = \theta_g^{-1} \circ \tilde{\sigma}$ is a local section that corresponds to $\tilde{\phi}'$. So we have that

$$\begin{aligned} \tilde{\eta}' &= \pi'' \circ \tilde{\phi}' \circ \tilde{\sigma}' \\ &= \pi'' \circ \tilde{\phi} \circ \theta_g \circ \theta_g^{-1} \circ \tilde{\sigma} \\ &= \pi'' \circ \tilde{\phi} \circ \tilde{\sigma} \\ &= \tilde{\eta} \ . \end{aligned}$$

From the composition above, the only difference between these 2 possibilities is no longer existent. With the compositions that we have given above, see that forming this composition with the projection $\pi'' = \pi_2$ (from Sjamaar's notation) implies that $\tilde{\eta}' = \tilde{\eta}$. This is demonstrative of the fact that adding in 2 extra factors with θ_g and θ_g^{-1} in the second line above implies that the maps $\tilde{\eta}$ and $\tilde{\eta}'$ are equal. Now, our arguments from the previous case allow us to conclude that the projection onto the second coordinate is again a smooth submersion, and the same remarks that I have already given apply.

With both of these cases, we have that the quotient map is a smooth submersion, and we have shown that all of the statements hold for **3D**.

3.4.3 Other Way

To show that the quotient map $M \longrightarrow M/G$ is a submersion, we will begin by observing that the action being free implies that the map ev_m is bijective. In this statement, for completeness, we reiterate that the map ev_m is the evaluation map, with $\text{ev}_m : G \longrightarrow M : g \mapsto g \cdot m$. With this evaluation map ev_m, we will also introduce the derivative of the evaluation map, $(\text{dev}_m)_e$. From the notation, observe that we are merely taking the derivative of our function evaluated at m, and then evaluating the value of this derivative at e. We have that $(\text{dev}_m)_e$ is injective because $\ker(\text{dev}_m)_e = \{X \in \mathbf{g} | X_M(m) = 0\} = \text{Lie}(G_m) = \{0\}$. From *Sard's Theorem*, we then know that $(\text{dev}_m)_g$ must be surjective for all g, because ev_m has constant rank. But $(\text{dev}_m)_e$ is invertible which implies that we have the isomorphism $T_m(G \cdot m) \simeq \mathbf{g}$, where $\mathbf{g}$ is the Lie Algebra.

Next, with the evaluation map $\text{ev} : G \times M \longrightarrow M$, we will choose a submanifold $S \subseteq M$ with $m \in S$ satisfying the condition that $T_m(G \cdot m) \bigoplus T_m S = T_m M$. We want to look at ev because we would like to verify that ev restricts to another evaluation map $\text{ev}_1 : G \times S \longrightarrow M$ which has the following tangent map at the point (e, m)

$$(\text{dev})_{e,m}(X, Y) = (\text{dev}_m)_e X_e + Y_m \ .$$

In fact, this tangent map at (e, m) is invertible. This observation allows us to check the formula that we have given by taking any $f \in C^\infty(M)$, and observing that the following holds

$$\begin{aligned} \text{dev}_{e,m}(X, Y)(f) &= (X, Y)_{e,m}(f \circ \text{ev}) \\ &= X(f \circ \text{ev}_m) + Y(f \circ \tau_e) \\ &= (\text{dev}_m)_e X(f) + Y(f) \ , \end{aligned}$$

from which we conclude that the invertibility of ev_1 follows from the fact that the unique sum decomposition $T_m(G \cdot m) \bigoplus T_m S = T_m M$ is unique. Also, the invertibility follows from the map $(\text{dev}_m)_e : \mathbf{g} \longrightarrow T_m(G \cdot m)$ being invertible. So the tangent map dev is invertible at the point (e, m'), for $m' \in S$ by continuity, as long as we choose m' close to m. With a choice of small S, we conclude that the dev is invertible at any point in S, by choosing that point close enough. From the commutative diagram below, we have that the evaluation map $\mathrm{ev}_1 : G \times S \longrightarrow M$ has an invertible tangent map at every point,

which implies that it ev_1 is a local diffeomorphism onto its image. More specifically, we can guarantee that this local diffeomorphism holds with the following claim. If we are not able to find S sufficiently small, we can choose 2 sequences (g_1^k, m_1^k) and (g_2^k, m_2^k) with $m_1^k \longrightarrow m$, $m_2^k \longrightarrow m$, and finally, $g_1^k \cdot m_1^k = g_2^k \cdot m_2^k$. For convenience, we can set $g^k = (g_1^k)^{-1} g_2^k$, which implies that $g^k \cdot m_2^k = m_1^k$. We can introduce the following conditions on g^k because ev_1 is bijective near the point (e, m), and also because there exists a neighborhood $\mathcal{U}$ of e with $g^k \notin \mathcal{U}$ for all k. So we have that the set $\{(g^k \cdot m_2^k, m_2^k) \cup \{(m, m)\}\}$ is compact in $G \times G$.

From ordinary properties of compactness, we can argue that the action being proper implies that the sequence $\{(g^k, m_2^k)\}$, which is contained in a compact set, has a convergent subsequence. Equivalently, $\{g^k\}$ must have a convergent subsequence. However, this conclusion contradicts the fact that the limit point of $\{g^k\}$ has to equal e because the limit point has to send m to m, as well as the action being free from our assumption. Altogether, this raises a contradiction because we have that $g^k \notin \mathcal{U}$ from the bijectivity of ev_1. From this, we will denote the image of $G \times S$ with $\mathcal{V}$ under ev_1 in M. Therefore, we have the desired smooth manifold structure because the diffeomorphism $\mathrm{ev}_2 : G \times S \longrightarrow \mathcal{V} \subseteq M$ allows us to identify V/G with S. Specifically, this diffeomorphism provides a smooth structure of M/G near the orbit $G \cdot m$. Hence the quotient map $M \longrightarrow M/G$ is a submersion because we have a differential map that is surjective and linear. Besides this, we also have that M/G is Hausdorff because of the following claim.

3.4.4 Claim

For any continuous action of a topological group G on a topological space M, the quotient map $\pi : M \longrightarrow M/G$ is open.

We can show that this claim holds with the following. We will take $g \cdot U \subset M$, with $g \in G$ and $U \subseteq M$. Specifically, $g \cdot U$ is defined as

$$g \cdot U = \{g \cdot x : x \in U\} \ .$$

From this expression, we know that $\pi^{-1}(\pi(U))$ can be expressed with a union of sets with the form $g \cdot U$ with g ranging over G, for $U \subset M$ open. In this union, the sets are open because $p \mapsto g \cdot p$ is a homeomorphism, which by definition implies that the composition $\pi^{-1}(\pi(U))$ is open in M. Hence π is open because π is a quotient map, with the same arguments applying to $\pi(U)$ in which we can show that this image is open which implies that π is open.

3.4.5 Claim

If a Lie Group acts continuously and properly on a manifold, then the orbit space is Hausdorff.

We can show that this claim holds by considering some Lie Group G which acts properly and continuously on M. From the action being proper, we can associate a map $\Theta : G \times M \longrightarrow M \times M$ to this action, with $\Theta(g, p) = (g \cdot p, p)$. We will also define the orbit relation $\mathcal{O} \subset M \times M$. It is given by

$$\begin{aligned} \mathcal{O} &= \Theta(M \times M) \\ &= \{(g \cdot p, p) \in M \times M : p \in M, g \in G\} \ . \end{aligned}$$

By definition, we know that proper continuous maps are closed, which implies that $\mathcal{O}$ is closed in $M \times M$. At this point, our previous Claim implies that the orbit space is Hausdorff. We have this result because the quotient map $\pi : M \longrightarrow M/G$ being open implies that M/G is indeed Hausdorff. Besides defining the orbit relation with slightly different notation, we can apply the same arguments from our **previous** method to conclude that the desired smooth submersion holds. This completes **3D**.

4 Problem 4

We will prove that the statement holds by first observing that the Lie Subgroup H of G acts on G from the right. This characteristic of the action allows us to apply the Quotient Manifold Theorem on the Lie Subgroup H acting on G, which gives a natural smooth structure on the set of cosets G/H. Towards this goal, we specify the operation which is given by

$$H \times G \longrightarrow G : (h, g_1) \mapsto g_1 h \ ,$$

and for convenience we will denote $g_2 = g_1 h$. This map is particularly helpful, because we observe from the Closed Subgroup Theorem, which states that if we have a closed subgroup H of a Lie Group G, then H is an embedded Lie Group which has a topology that is the same as topology of G. Applying this closed Subgroup Theorem from Cartan implies that H is a properly embedded subgroup of G because H is a closed Lie subgroup by assumption. Roughly, the Closed Subgroup Theorem ensures that the group and relative topologies are the same. Next, we observe that the right action of H on G that we have described is smooth because this operation is the same as restricting the multiplication operation of G. Namely, this entails that the multiplication operation that is given by

$$m : G \times G \longrightarrow G \ ,$$

is the same as restricting the map m to H, which is given by

$$m|_H : H \times G \longrightarrow G \ .$$

So we know that the natural G-action on the coset G/H by left translations can be shown to be differentiable. In particular, this action is differentiable because if we take a locally trivial bundle $p_i : X_i \longrightarrow Y_i$ with fiber Z_i, we know that the map $p_1 \times p_2 : X_1 \times X_2 \longrightarrow Y_1 \times Y_2 : p_1 \times p_2 \mapsto (p_1(x_1), p_2(x_2))$ is also a locally trivial bundle with fiber $Z_1 \times Z_2$. In particular, this result is helpful to show that the action is differentiable because we can define the G-action on G/H with

$$\lambda : G \times G/H \longrightarrow G/H : (g', gH) \mapsto g'gH$$

plays nicely with the commutative diagram

with μ for multiplication for G. We know that $\mathrm{id} \times p$ is a quotient map because it is a locally trivial bundle, and making use of the fact that the quotient enters the commutative diagram with Z a differential manifold and q a differentiable map, which implies that ϕ is differentiable. With this diagram, we know that λ is indeed a differentiable mapping, which demonstrates that the described right action of H on G is smooth. We also have that G/H is a Lie Group from the commutative diagram

which implies that the multiplication for the Lie Subgroup μ_H that is naturally inherited from the multiplication μ on G is also differentiable. The quotient G/H has the multiplication μ_H, where H has a normal subgroup, and is also a Lie Group by itself.

Besides these qualities, we also observe that the right action on G by H is free because fixing $g \in G$ and $h \in H \subset G$ implies that

$$gh = g \Rightarrow h = e \ .$$

Furthermore, we also have that this action is proper because we can fix respective sequences $(g_i)_i$ and $(h_i)_i$ in G and H. We want these sequences to converge in G, and the continuity of the inverse map implies that the sequence $h_i = g_i^{-1}(g_i h_i)$ converges in G. But from the Closed Subgroup Theorem, we know that H being a Lie Subgroup of G implies that the sequence $(h_i)_i$ converges in H due to the fact that the subspace topology that H has guarantees that the sequence converges in H because it does in G. So far, this shows that our action is smooth and proper.

With these arguments, we have that an action of a Lie Group H on our manifold M is smooth, free and proper, and this enables us to apply the Quotient Manifold Theorem to conclude that the space of left cosets G/H is a topological manifold with $\dim(G) - \dim(H)$, with a unique smooth structure with the quotient map $G \longrightarrow G/H$ a smooth submersion. Therefore, applying the Quotient Manifold Theorem directly allows us to observe the following about the smooth structure from the set of cosets G/H with the commutative diagram below

The diagram above describes the action of G on the set of cosets G/H, with $\theta : G \times M \longrightarrow M : (g, p) \mapsto g \cdot p$. We will denote the image $g \cdot p$ as $\theta_g(p)$. In this diagram, $\mathrm{Id}_G \times \pi$ is a smooth submersion as a product, and is given by

$$\begin{aligned} \theta : (g_1, g_2 H) &\mapsto g_1 \cdot (g_2 H) \\ &= (g_1 g_2) H \ . \end{aligned}$$

To show that π is the submersion that we want, we will apply the following claim that ensures that such a θ and π exist with a Universal Mapping Property, which will allow us to homogenize the space $(G, G/H)$.

4.1 Claim

Let M, N, P be smooth manifolds and $\pi : M \longrightarrow N$ a surjective smooth submersion. If P is a smooth manifold and $F : M \longrightarrow P$ is a smooth map that is constant on the fibers of π [ie $\pi(p) = \pi(q) \Rightarrow F(p) = F(q)$], there exists a unique smooth map $\theta : N \longrightarrow P$ with $\theta \circ \pi = F$.

With this theorem, it suffices to show that the composition

$$\pi \circ m : G \times G \longrightarrow G \longrightarrow G/H$$

is constant on the fibers of $\mathrm{Id}_G \times \pi$, which we can show by taking $g_1, g_2 \in G$. We have that

$$(\mathrm{Id}_G \times \pi_1)(g_1, g_2) = (g_1, g_2 H) \ ,$$

and that taking the image of (g_1, g_2) under the inverse map $(\mathrm{Id}_G \times \pi)^{-1}$ is given by

$$(\mathrm{Id}_G \times \pi)^{-1}(g_1, \overline{g_2}) = \{(g_1, g) | gH = g_2 H\} \ .$$

But this set is precisely $(G \times G)_{(g_1, g_2)}$, and if we consider the following composition

$$\pi \circ m|_{(G\times G)_{(g_1,g_2)}} : (G \times G)_{(g_1,g_2)} \longrightarrow G/H \ ,$$

which elementwise is defined with

$$(g_1, g) \mapsto \pi(g_1 g) = g_1 g H \text{ for all } g \ ,$$

which implies that $\pi \circ m$ is constant on the fibers of $\mathrm{Id}_G \times \pi$. Certainly, θ being unique in this claim implies that we have the smooth structure that we wanted; this follows from the fact that the map itself is also well defined, and that θ specifies the group action (regardless of whether it is right or left multiplication) with the diagram above. Finally, we observe that this group action is also transitive because if we take $(g_1, g_2) \in G \times G$, we know that $g_2 g_1^{-1} \in G$ satisfies

$$\theta(g_2 g_1^{-1}, g_1 H) = (g_2 g_1^{-1}) \cdot g_1 H = g_2 H \ ,$$

which shows that the action is transitive. So the claim, as well as our discussion before it, shows that the quotient map is a submersion in addition to the smooth property for the set of cosets, in which we have made $(G, G/H)$ into a homogeneous space. This completes **4**.

5 References

[1] Lee, Smooth Manifolds

[2] Notes on proper actions: https://www.mathi. uni-heidelberg.de/ lee /StephanSS16.pdf , https://www.staff.science uu.nl/ ban00101/foliations2006/actions.pdf , http://webmath2.unito.it/paginepersonali/sergio.console/lee.pdf.

[3] I talked with a few students around OH about ideas in these problems.

[4] David's Notes for 6520 from 2016: http://pi.math.cornell.edu/ dmehrle/notes/cornell/16fa/6520notes.pdf.