Math 7710 Report: Tzvetkov's 2015 *Quasi-Invariant Gaussian Measures for One Dimensional Hamiltonians PDEs*

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Contents

1 introduction

To begin our discussion, the aim of the paper is to construct gaussian measures and to also show that such measures are quasi invariant under the flow of a 1D Hamiltonian PDE. To this end, we recall that a measure μ is said to be quasi invariant if its image is absolutely continuous under the transformation $\Phi : X \longrightarrow X$. In terms of the specific equation that we will be studying, we know that the time evolution of a water wave is captured by the PDE,

$$\partial_t u + \partial_x u + \epsilon_1 \partial_x^3 u + \epsilon_2 \partial_x (u^2) = \mathcal{O}(\epsilon_1^2 + \epsilon_2^2) \ ,$$

where the parameters ϵ_1, ϵ_2 can be taken arbitrarily small, each of which physically represent the ratio of the depth of the fluid, in addition to the typical wavelength, as well as the ratio between the amplitude and depth of the wave, respectively. As in class discussions relating to other linear PDEs, the Duhamel formulation will be applied so that we can study the time evolution of the equation by incorporating steps under which the solution can be determined by taking the solution to the PDE at some fixed time, in addition to adding an integral term that will be defined in upcoming arguments. Before establishing these observations, we remark that taking the approximation that the first derivative with respect to time is comparable to the minus first derivative of the solution with respect to space, x, give a PDE of the form,

$$\partial_t u + \partial_x u - \partial_t \partial_x^2 u + \partial_x (u^2) = 0 \ .$$

From the PDEs introduced above, one can not only readily obtain the KdV equation by ignoring the error terms associated with ϵ_1 and ϵ_2, but also obtain the a formulation of the PDE for generalized dispersion, of the form,

$$\partial_t u + \partial_x u - |D_x|^\gamma \partial_x u + \partial_x (u^2) = 0 \ .$$

From here, we will proceed to study the behavior of the generalized dispersion of the KdV equation for different values of γ. More specifically, we will also want to study the generalization of the second PDE, which slightly differs from the one presented above, of the form,

$$\partial_t u + \partial_t |D_x|^\gamma u + \partial_x u + \partial_x (u^2) = 0 \ .$$

Now that we have listed all of the PDEs of interest, in addition to listing the values of γ in the above generalized PDE under which the construction of quasi invariant Gaussian measures will be exhibited, we will present more background before the main results of the work.

2 defining measure of interest, more background leading up to the main results

From the PDEs given in the previous section, define the measure induced by the random Fourier series,

$$\phi_s(\omega, x) = \sum_{n \neq 0} \frac{g_n(\omega)}{|n|^{s+\frac{\gamma}{2}}} e^{inx} ,$$

where the g_n are complex gaussians of the form,

$$g_n = \frac{1}{\sqrt{2}}(h_n + il_n) ,$$

satisfying the condition that $g_n = \overline{g_{-n}}$, and h_n and l_n are normally distributed random variables with mean 0 and variance 1. With such a measure, we will want to show that it is quasi invariant under the flow of a 1D Hamiltonian PDE. The result is as follows.

Theorem: For $\gamma > \frac{4}{3}$ and for every integer $s \geq \frac{\gamma}{2}$, the measure μ_s is quasi invariant under the flow $\Phi(t)$, for every $t \in \mathbf{R}$.

To prove the Theorem and several results leading up to it, we must also detail the manner by which the transformation Φ may be decomposed, by virtue of the Duhmamel formula. Besides being able to make use of the Duhamel formula to solve over linear PDEs, in this case, in tandem with the Cameron Martin type result which provides the transformation that the Wiener integral undergoes, given a translation of continuous functions $x(t)$ over $t \in [0, 1]$ which vanish at 0. Specifically, if we denote the free evolution associated to the most generalized PDE that is given as the last item in the previous section, then we know that the Duhmael formulation consists of expressing the image of u_0 under $\Phi(t)$ in terms of the contributions of $S(t)(u_0)$, in addition to smoother terms besides $S(t)(u_0)$ dependent on the u_0.

With the goal of constructing quasi-invariant maps μ_s which preserve the Sobolev regularity, as a consequence of the Cameron-Martin Theorem mentioned previously, the measure μ_s is quasi invariant, but in contrast to the previous Duhamel formulation, has higher order terms which are independent of u_0. To provide another point of comparison with the Duhamel formulas which have higher order terms rather than $S(t)(u_0)$ that are either dependent or independent of u_0, we may even consider a Duhamel formula for $\Phi(t)(u_0)$ which besides having the common term $S(t)(u_0)$, has a $1+\epsilon$ smoother part which could depdent on u_0 for arbitrary $\epsilon > 0$. Significantly, we mentioned all of the Duhamel formulas from which one can apply standard fixed point arguments to show that there exist unique solutions to a given linear partial differential equation with specified initial data.

To make progress towards applying such arguments, we must introduce further notation before studying the truncated version of the last generalized PDE given in the first section. From the usual Fourier series expansion of smooth f, in which $f(x) = \sum_{n \in \mathbf{Z}} \hat{f}(n) e^{inx}$, with f satisfying $f(n) = \overline{\hat{f}(-n)}$, we can define the Sobolve norm associated to the Fourier series to be of the form,

$$||f||_s = \left(\sum_{n \in \mathbf{Z}} < n >^{2s} |\hat{f}(n)|^2 \right)^{\frac{1}{2}} .$$

Additionally, we can equip the closed subspace $H^s \subset \mathcal{H}^s$ of the Hilbert space with the norm,

$$||f||_{H_s} = \frac{1}{\sqrt{2}} \left(\sum_{n \in \mathbf{Z}} |n|^{2s} |\hat{f}(n)|^2 \right)^{\frac{1}{2}} .$$

3 existence of dynamics for the truncated system

As with straightforward arguments which pertain to defining and making use of truncated Fourier series for analysis, one can define the truncation of the generalized PDE, which is of the form,

$$\partial_t u + \partial_t |D_x|^\gamma u + \partial_x u + \partial_x \pi_n ((\pi_n u)^2) = 0 .$$

From this truncated version of the general PDE on $\mathbf{T}_1$ with H^s initial data, we will make use of the following 2 lemma to establish a formulation of the product rule, in addition to a fixed point argument from the Duhamel formula.

Lemma 1 (*product formula*) : For $\gamma > 1$ and every $\sigma > 0$,

$$||(1 + |D_x|^\gamma)^{-1} \partial_x (uv)||_\sigma \leq C_\sigma (||u||_\sigma ||v||_{\frac{\gamma}{2}} + ||u||_{\frac{\gamma}{2}} ||v||_\sigma) .$$

Proof sketch: Apply the Sobolev embedding $\mathcal{H}^{\frac{\gamma}{2}} \subset L^\infty$, in addition to the classical product estimate which is of the form,

$$||uv||_\sigma \leq C_\sigma (||u||_\sigma ||v||_{L_\infty} + ||u||_{L_\infty} ||v||_\sigma) .$$

Lemma 2 (*uniqueness of solutions*): Fix $\sigma \geq \frac{\gamma}{2}$. For every $u(0) \in H^\sigma$ there exists a time $\tau > 0$, dependent only on $||u(0)||_{H^{\frac{\gamma}{2}}}$ and a unique solution of the truncated PDE over the space $C([-\tau,\tau],H^\sigma)$ with initial data $u(0)$. Moreover, $||u||_{L^\infty}([-\tau,\tau],H^\sigma) \leq 2||u(0)||_{H^\sigma}$.

Proof sketch: To set up the fixed point argument, we can first rewrite the truncated solution of the PDE in the form,

$$u(t) = S(t)(u(0)) - \int_0^t S(t-\tau)((1+|D_x|^\gamma)^{-1}\partial_x \pi_n((\pi_n u(\tau))^2))\mathrm{d}\tau \ .$$

From the equality above, we will want to obtain a fixed point for the map defined by the right hand side. Besides this, we must also define the accompanying space, with the same conditions on the L^∞ norm as given in the Theorem statement, which is of the form,

$$E \equiv \{u \in C([-\tau,\tau]; H^{\frac{\gamma}{2}} : ||u||_{L^\infty([-\tau,\tau];H^{\frac{\gamma}{2}}} \leq 2||u(0)||_{H^{\frac{\gamma}{2}}}\} \ .$$

From this space, in addition to the fixed point mapping defined on the right hand side of $u(t)$ that we will denote as the map $F_{u(0)}$, we can obtain the bound,

$$||F_{u(0)}(u) - F_{u(0)}(v)||_{L_\infty([-\tau,\tau];H^{\frac{\gamma}{2}}}) \leq \frac{1}{2}||u-v||_{L^\infty([-\tau,\tau];H^{\frac{\gamma}{2}}}) \ ,$$

for $u, v \in E$, by taking smaller constants c, as given in the definition of τ. By establishing that the mapping $F_{u(0)}$ is a contraction on E, we can obtain a fixed point of the contraction to obtain a solution for the truncated PDE given at the beginning of the section.

However, rather than only having a solution of the truncated PDE from the fixed point argument, we can also establish uniqueness of solutions as follows. Suppose that u_1 and u_2 are each solutions of the truncated system in $C([-\tau,\tau];H^\sigma)$. Then, we know that for any times $[-\tau_1,\tau_1]$, where $\tau_1 \leq \tau$, we can measure the discrepancy between the solutions u_1 and u_2 with,

$$||u_1 - u_2||_{L^\infty([-\tau_1,\tau_1];H^\sigma)} \leq C_\sigma \tau_1 ||u_1 - u_2||_{L^\infty([-\tau_1,\tau_1];H^\sigma}||u_1 + u_2||_{L^\infty([-\tau,\tau];H^\sigma)} \ .$$

From the difference between the solutions u_1 and u_2 as measured from L^∞, furthermore we know that by making use of the **previous lemma** which gives an expression using the ordinary product formula gives a uniform bound for the truncation π_N on H^σ. We conclude that the solutions satisfy that $u_1 \equiv u_2$ on $[-\tau_1,\tau_1]$ because the difference between u_1 and u_2, as measured above with respect to the L^∞ norm, can be made arbitrarily small so that the two solutions are identical. As for the times τ_1, we know that there exists τ_1 satisfying

$$\tau_1 C_\sigma \left(||u_1||_{L^\infty([-\tau,\tau];H^\sigma)} + ||u_2||_{L^\infty([-\tau,\tau];H^\sigma)} \right) < \frac{1}{2} \ .$$

In words, the inequality above captures the fact that we can cover up the interval of existence $[-\tau, \tau]$ with intervals of size τ_1, repeatedly, so that $u_1 \equiv u_2$ on $[-\tau, \tau]$. More generally, this argument allows us to conclude that if we have the interval of existence $[-\tau, \tau]$ for a solution of the truncated PDE, then it is possible for us to deterministically obtain τ depending only on the magnitude of the solution at the initial time t_0 of existence, namely $||u(t_0)||_{H^{\frac{\gamma}{2}}}$.

Lemma 3 Statement: For a local solution u of the truncated PDE, the following equality holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(||u(t)||^2_{L^2} + 4\pi ||u(t)||^2_{H^{\frac{\gamma}{2}}} \right) = 0 \ .$$

Remark: The above Lemma will not only allow us to obtain global well-posedness of the solution in H^σ of the truncated PDE uniformly in N, but also makes use of the conservation law to obtain the following result:

Proposition: Fix $\gamma > 1$ and $\sigma \leq \frac{\gamma}{2}$. For every $u_0 \in H^\sigma$ there exists a unique, global solution of

$$\partial_t u + \partial_t |D_x|^\gamma u + \partial_x u + \partial_x(u^2) = 0 \ ,$$

in $C(\mathbf{R}; H^\sigma)$. Finally, if we denote the the flow of the PDE above by the $\Phi(t)$, then for every $t \in \mathbf{R}$, $\Phi(t)$ is a continuous bijection on H^σ.

4 defining global flows on H^σ, establishing the result for $\gamma > 2$ from Ramer

Define $\Phi(t)$ and $\Phi_N(t)$ to be the global flows for H^σ, for fixed $\sigma \geq \frac{\gamma}{2}$, each of which respectively correspond to

$$\partial_t u + \partial_t |D_x|^\gamma u + \partial_x u + \partial_x(u^2) = 0 \ ,$$

and

$$\partial_t u + \partial_t |D_x|^\gamma u + \partial_x u + \partial_x \pi_n((\pi_n u)^2) = 0 \ .$$

With the regular and approximated flows, we will now state a few results to build up to the proof of the main result of quasi invariance of the Gaussian measures, through making use of properties the measure evolution in addition to properties of the measure primarily relying on the image, under the measure, of the global flow of a Borel measurable set.

Proposition 2.5 (*upper bound of the global and approximated flows*): Fix $\sigma \geq \frac{\gamma}{2}$. For every $R > 0$ there exists a constant C such that for every $v \in H^\sigma$, with $||v||_{H^{\frac{\gamma}{2}}} \leq R$ and every $N \geq 1$, the following upper bound holds,

$$||\Phi(t)(v)||_{H^\sigma} + ||\Phi_N(t)(v)||_{H^\sigma} \leq \exp(C(1+|t|))||v||_{H^\sigma} \ .$$

Proof sketch: Iterate the local bound for t, where the inequality is of the form $||u(t)||_{H^\sigma} \leq 2^{[\frac{|t|}{\tau}+1}||u(0)||_{H^\sigma}$.

The result above is useful for comparing characteristics of the global and approximated flows. Below, the next result is useful for comparing the difference between the global and approximated flows of the Hamiltonian PDE.

Proposition 2.7(*intuitive result, "closeness" of global and approximated norms with respect to the H^σ norm*: Fix $\sigma \geq \frac{\gamma}{2}$. For $t \in \mathbf{R}$, same $R > 0$ as in the previous proposition and compact $K \subset B_{R,\sigma}$, where $B_{R,\sigma}$ is the ball of radius R given the aforementioned choice of σ, we have the following inequality,

$$||\Phi(t)(v) - \Phi_N(t)(v)||_{H^\sigma} < \epsilon \ ,$$

for every $\epsilon > 0$ and N such that $N \geq N_0$, for all $v \in K$.

Proof sketch: Take solutions u and u_N corresponding to the generalized and truncated PDEs, respectively, each with the same initial data $v \in K$. With these solutions, we can naturally defined $w_N = u - \pi_N u_N$, from which the truncated PDE takes the form, after rearranging terms,

$$u^2 - \pi_N((\pi_N u_N)^2) = (1-\pi_N)(u^2) + \pi_N(w_N(u + \pi_N u_N)) \ .$$

From the product rule lemma given in the previous section, there exists a constant C, dependent only on σ, t and R, so that the inequality in the proposition holds.

5 further background: variable change formula, the Lebesgue and image measures

After having applied Ramer's result to study the case in which $\gamma > 2$, we now proceed to study the real vector space spanned by $\{\cos(nx), \sin(nx)\}_{1 \leq n \leq N}$. From such a basis, we can naturally equip it with the scalar product, as well as the Lebesgue measure by setting $u_n = a_n + ib_n$, for $(a,b) \in \mathbf{R}^2$, from which we obtain

$$(\pi_N u)(x) = \sum_{n=1}^{N} \bigg(a_n(2\cos(nx)) + b_n(-2\sin(nx)) \bigg) \ ,$$

if we denote

$$(\pi_N u)(x) = \sum_{0<|n|\leq N} u_n \exp(inx) \ ,$$

satisfying $u_n = \overline{u_{-n}}$. With the Lebesgue measure L_N on the orthogonal compelement $E_N^{\perp}$ of the vector space given above in H^s, the measure $\mu_{s;N}^{\perp}$ is of the form,

$$\mu_{s;N}^{\perp} : \omega \mapsto \sum_{|n|>N} \frac{g_n(\omega)}{|n|^{s+\frac{\gamma}{2}}} e^{inx} \ .$$

Besides this additional background for the change of variable formula, we present 2 "facts" as necessary ingredients for the quasi-invariance of the measure. Without proof, they include:

Fact 1 (*Proposition 4.1, change of variables rule*): For Borel $A \subset H^s$, we have,

$$\mu_{s,r}(\Phi_N(t)(A)) = \int_{\Phi_N(t)(A)} \chi_r(u) \mathrm{d}\mu_s(u) = \gamma_N \int_A \chi_r(u) \exp\bigg(- ||\pi_N(\Phi_N(t)(u)||^2_{H^{s+\frac{\gamma}{2}}} \prod_{i=1}^{N} \mathrm{d}u_i \mathrm{d}\mu_{S;N}^{\perp} \bigg) \ .$$

Remark: The terms given in can be obtained from a product measured on $E_N \times E_N^{\perp}$, in addition to most terms in the power of the exponential.

Fact 2 (*Lemma 4.2, invariance of the measure*): The product measure $\mathrm{d}u_1 \cdots \mathrm{d}u_N$ is invariant under the flow $\tilde{\Phi}_N(t)$, where $\tilde{\Phi}_N(t)$ is the well defined flow of the ODE

$$\partial_t u + \partial_t |D_x|^\gamma u + \partial_x u + \partial_x \pi_N(u^2) = 0 \ ,$$

for $u(0,x) \in E_N$.

6 on the energy estimate

Moreover, besides results concentrated only on the measure associated with E_N and $E_N^{\perp}$, we will briefly present results surrounding the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}||\pi_N(t)||^2_{H^{s+\frac{\gamma}{2}}}$. Because we know that $\pi_N u$ is a solution of

$$\partial_t \pi_N u + \partial_t |D_x|^{\gamma} \pi_N u + \partial_x \pi_N u + \partial_x \pi_N((\pi_N u)^2) = 0 \ ,$$

from which several lines of technical computations we can make use of Hoelder's inequality to conclude that,

$$||\pi_N u||_{H^s} \leq ||\pi_N u||^{\theta}_{H^{\frac{\gamma}{2}}} ||\pi_N u||^{1-\theta}_{H^{s+\frac{\gamma}{2}-\frac{1}{2}-\epsilon}} \ ,$$

for $\epsilon > 0$ and every N. By establishing the connection that

$$||\pi_N u||_{H^{s+\frac{\gamma}{2}-\frac{1}{2}-\epsilon}} \lesssim |||D_x|^{s+\frac{\gamma}{2}-\frac{1}{2}-\epsilon} \pi_N u||_{L^{\infty}} \ .$$

By making similar rearrangements with Hoelder's inequality and by then introducing an upper bound for the $H^{s+\frac{\gamma}{2}-\frac{1}{2}-\epsilon}$ norm of $\pi_N u$ as given above, will be applied to prove the main result of the work in tandem with the measure evolution property. For other necessary ingredients, we remark:

Lemma 5.2: Fix $\sigma \in [\frac{\gamma}{2}, s + \frac{\gamma}{2} - \frac{1}{2} - \epsilon]$. For $\theta \in [0, 1]$ satisfying

$$\sigma > \theta \frac{\gamma}{2} + (1 - \theta)\left(s + \frac{\gamma}{2} - \frac{1}{2} - \epsilon \right) \ ,$$

one obtains the bound,

$$||\partial_x^{\sigma} u||_{L^p} \lesssim ||u||^{\theta}_{H^{\frac{\gamma}{2}}} |||D_x|^{s+\frac{\gamma}{2}-\frac{1}{2}-\epsilon} u||^{1-\theta}_{L^{\infty}} \ ,$$

for sufficiently small $\epsilon > 0$, in addition to $p = \frac{2}{\theta}$.

Lemma 5.3: For every $\gamma > \frac{4}{3}$ there exists some $\theta > \frac{4}{3}$ such that for suitable u satisfying $\hat{u}(0) = 0$, we have

$$||\partial_x u||_{L^3} \lesssim ||u||^{\theta}_{H^{\frac{\gamma}{2}}} |||D_x|^{\frac{1}{2}+\frac{\gamma}{2}-\epsilon} u||^{1-\theta}_{L^{\infty}} \ ,$$

for sufficiently small $\epsilon > 0$.

With the 2 Lemmata above, we will demonstrate how we can make use of such bounds, in addition to the measure evolution that will be introduced in the next section, to prove quasi invariance of the measure.

7 the measure evolution property

7.1 introducing the first lemma for the measure evolution

With properties of the invariance of the measure discussed in **Section 5**, in addition to the previous lemmata in **Section 6** which give Holder type inequality relationships between u and D_x, we can demonstrate from properties of the flow that $\mu_s(\Phi(t)(A)) = 0$. To begin, we will introduce the following Lemma.

Lemma 7.1: There exists $0 \leq \beta < 1$ such that for every $r > 0$ there exists a constant $C > 0$ such that for every $p \geq 2$ and every Borel $A \subset H^s$, for every $N \geq 1$, one has,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_{s,r}(\Phi_N(t)(A)) \leq Cp^{\beta}(\mu_{s,r}(\Phi_N(t)(A)))^{1-p}$$

Proof sketch: From properties of the approximated flow $\Phi_N(t)$ that we have introduced,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_{s,r}(\Phi_N(t)(A))|_{t=\bar{t}} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Phi_N(t)(A)} \xi_r(u)\mathrm{d}\mu_{s,r}(u)|_{t=\bar{t}} \ ,$$

from which we can express the above integral in terms of the produce measure over the vector space E with trigonometric basis functions,

$$\gamma_N \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Phi_N(\bar{t})(A)} \xi_r(u) e^{-||\pi_N(\Phi_N(t)(u))||^2_{H^{s+\frac{\gamma}{2}}}} \prod_i \mathrm{d}u_i \mathrm{d}\mu^{\perp}_{s;N}|_{t=0} \ .$$

By virtue of the product measure above, we can obtain an upper bound for the integral above with Hoelder's inequality,

$$\begin{aligned} &- \int_{\Phi_N(\bar{t})(A)} \left(\frac{\mathrm{d}}{\mathrm{d}t}||\pi_N(\Phi_N(t)(u))||^2_{H^{s+\frac{\gamma}{2}}}|_{t=0}\right)\mathrm{d}\mu_{s,r}(u) \\ &\leq ||\frac{\mathrm{d}}{\mathrm{d}t}||\pi_N(\Phi_N(t)(u))||^2_{H^{s+\frac{\gamma}{2}}}|_{t=0}||_{L^p(\mu_{s,r}(u))}(\mu_{s,r}(\Phi_N(\bar{t})(A)))^{1-\frac{1}{p}} \ , \end{aligned}$$

from which it suffices to prove that there exists an upper bound for the quantity above, with the inequality of the form

$$||\frac{\mathrm{d}}{\mathrm{d}t}||\pi_N(\Phi_N(t)(u))||^2_{H^{s+\frac{\gamma}{2}}}|_{t=0}||_{L^p(\mu_{s,r}(u))} \leq Cp^\beta \ ,$$

for $C > 0$ and $0 \leq \beta < 1$.

7.2 making use of the result to prove quasi invariance

With a bound of the type given above for the specified C and β, we will complete the key parts of the following argument to demonstrate that μ_s is quasi invariant by the flow $\Phi(t)$ for all real t.

Lemma 8.1: For any fixed real t, $R, \delta > 0$, there exists $C > 0$ such that for every Borel $A \subset B_{R,s}$ in H^s, $\mu_{s,r}(\Phi(t)(A)) \leq C(\mu_{s,r}(A))^{1-\delta}$.

Proof: To show that the statement above holds, which in turn will be used to prove the quasi invariance of the gaussian measure under the global flow $\Phi(t)$, we take arbitrary $\epsilon > 0$, and $N \geq N_0$ sufficiently larger so that

$$\begin{aligned}\mu_{s,r}(\Phi(t)(A)) &\leq \mu_{s,r}(\Phi_N(t)(A + B_{\epsilon,S})) \ , \\ &\leq C(\delta, t, r)(\mu_{s,r}(A + B_{\epsilon,s}))^{1-\delta} \ ,\end{aligned}$$

which in words captures the fact that the image of the global flow $\Phi(t)$ under the gaussian measure not only has an upper bound of the image of the approximated flow $\Phi_N(t)$ under the same measure, but also that there exists a multiplicative constant C that is itself an upper bound for the image of the approximate flow of the measure. With this sequence of inequalities, we pass to the limit as $\epsilon \longrightarrow 0$ to obtain that $\mu_{s,r}(\Phi(t)(A)) \leq C(\delta, t, r)(\mu_{s,r}(A))^{1-\delta}$.

By virtue of the regularity of $\mu_{s,r}$ we are moreover able to also pass to the limit as $n \longrightarrow \infty$, in which there exists a sequence of compact set $\{K_n\}_{n=1}^{\infty}$ such that $\lim_{n \longrightarrow \infty} \mu_{s,r}(K_n) = \mu_) s, r(\Phi(t)(A))$ and also that $K_n \subset \Phi(t)(A)$ for each n.

Next, by taking advantage of the fact that $\Phi(-t)$ is a continuous map, in which case that for the set

$$K_n \subset \Phi(t)(\Phi(-t)(K_n)) \ ,$$

every $x \in K_n$ implies that

$$x = \Phi(t)(\Phi(-t)(K_n)) \in \Phi(t)(\Phi(-t)(K_n)) \ ,$$

so that F_n is compact. Besides obtaining that F_n is compact and to show that $F_n \subset A$, we know that if there exists n such that $K_n \subset \Phi(t)(A)$, this implies that $y \in \Phi(t)(A)$, and also that there must exist $z \in A$ such that $y = \Phi(t)(z)$. However, from properties of compact sets that we observed previous in our proof of the Lemma, namely that the compact Borel set A gives the existence of the limit for $\epsilon \longrightarrow 0$ by the dominated convergence theorem implies that a similar result holds,

$$\mu_{s,r}(\Phi(t)(F_n)) \leq C(\delta, t, r)(\mu_{s,r}(F)n))^{1-\delta} \leq C(\delta, t, r)(\mu_{s,r}(A))^{1-\delta} \ ,$$

with F_n instead of A.

7.3 overview

- With the result in **Lemma 8.1**, we can demonstrate quasi invariance of the gaussian measure by making use of a similar argument to show that $\mu_{s,r}(A) = 0$.
- Particularly, for Borel $A \subset H^s$, we know that $\mu_s(A) = 0$, which implies that for every $r, R > 0$, we have that the intersection under the gausssian measure,

  $$\mu_{s,r}(A \cap B_{R,s}) = 0 \ .$$

  From this observation, we can progress further by claiming, as from the Lemma in the previous section, that $\mu_{s,r}(\Phi(t)(A \cap B_{R,s}) = 0$ for every $R, r > 0$, We then pass to the limit as $r \longrightarrow \infty$ by the dominated convergence theorem.
- Finally, we can express $\Phi(t)(A)$ as a union over all R of $\Phi(t)(A \cap B_{R,s})$, from which we obtain that $\mu_s(\Phi(t)(A)) = 0$, in turn proving the quasi invariance of the measure.

8 references

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