McMahon Lab, Post 2

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1 Linearizing Navier-Stokes in 2 dimensions, holding ρ constant

By definition, the equations in 2 dimensions have the form,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \ ,$$

which stipulates that the fluid flow that we are considering is incompressible, in addition to second order derivatives of u which represent the shear and normal stresses, which satisfy

$$\begin{aligned} \rho\bigg(\frac{\partial u_x}{\partial t} + u\frac{\partial u_x}{\partial x} + v\frac{\partial u_x}{\partial y}\bigg) &= -\frac{\partial p}{\partial x} + \mu\bigg(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\bigg) \ , \\ \rho\bigg(\frac{\partial u_y}{\partial t} + v\frac{\partial u_y}{\partial x} + v\frac{\partial u_y}{\partial y}\bigg) &= -\frac{\partial p}{\partial y} + \mu\bigg(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\bigg) \ , \end{aligned}$$

for $\rho, \mu > 0$. From these expressions, we will linearize to separate higher and lower order terms, in which substituting $u = \bar{u} + \hat{u}$ eliminates all higher order terms from the equations, which can then be expressed, in terms of $\bar{u}$, as the relations

$$\begin{aligned} \frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_y}{\partial y} &= 0 \ , \\ \rho\bigg(\frac{\partial \bar{u}_x}{\partial t} + u\frac{\partial \bar{u}_x}{\partial x} + v\frac{\partial \bar{u}_x}{\partial y}\bigg) &= -\frac{\partial p}{\partial x} + \mu\bigg(\frac{\partial^2 \bar{u}_x}{\partial x^2} + \frac{\partial^2 \bar{u}_x}{\partial y^2}\bigg) \ , \\ \rho\bigg(\frac{\partial \bar{u}_y}{\partial t} + v\frac{\partial \bar{u}_y}{\partial x} + v\frac{\partial \bar{u}_y}{\partial y}\bigg) &= -\frac{\partial p}{\partial y} + \mu\bigg(\frac{\partial^2 \bar{u}_y}{\partial x^2} + \frac{\partial^2 \bar{u}_y}{\partial y^2}\bigg) \ , \end{aligned}$$

in which one of the components of the solution u is of the form

$$u_x = \frac{g}{u_y}\sin\bigg(hy - \frac{y^2}{2}\bigg) \ ,$$

for fluid flow restricted to the y direction.

2 Defining infinitesimal regions of fluid flow

For the region $\mathcal{R}$ of fluid flow, denote $B^{x_i}_\epsilon = [x_i, x_i+\epsilon] \times [y_i, y_i+\epsilon]$ for $\epsilon > 0$ small enough, with $\cup_{x_i} B^{x_i}_\epsilon = \mathcal{R}$ for $x_i \in \mathcal{R}$. To measure the time evolution across each degree of freedom in $\mathcal{R}$ along the coordinate axes, we will introduce unitary operators $\mathcal{U}^x_\epsilon$ and $\mathcal{U}^y_\epsilon$. After applying these unitary operators to quantum states that will be defined in the **next** section, we can verify that the proposed operators are indeed unitary, and are also dependent on the direction in which the stress is exerted, in which

$$\mathcal{U}^{x_i+\epsilon}_\epsilon = \mathcal{U}^{x_i}_\epsilon |\varphi> \quad ,$$
$$\mathcal{U}^{y_i+\epsilon}_\epsilon = \mathcal{U}^{y_i}_\epsilon |\varphi> \quad ,$$

for $x_i, y_i \in B^{x_i}_\epsilon$ and wave function φ, that will assign values in a mapping to be introduced next. In particular, the unitary operators will be chosen from $\mathrm{U} \in U(2)$, so that $\mathrm{U}^T\mathrm{U} = \mathrm{I}$.

3 Assigning qubits to fluid flow in the x and y directions

After regarding $B^{x_i}_\epsilon$, with $B^{x_i}_\epsilon \cap \mathcal{R} \neq \emptyset$, for all $x_i \in \mathcal{R}$ as a subgraph $G_i = (V_i, E_i)$ of the graph $\cup_i G_i = G$, the correspondence

$$\varphi : V_i \longrightarrow \mathcal{H} : \{\forall v \in \partial B^{x_i}_\epsilon\} \mapsto \begin{cases} |0> & \text{w.p } \epsilon \\ |1> & \text{w.p } 1-\epsilon \end{cases} \quad ,$$

maps, albeit not $1-1$, to the Hilbert space $\mathcal{H}$, and furthermore, assigns qubit states at random to arbtirary initial positions of displacement, for arbitrary ϵ. From such assignments, one can quantify the time evolution of a unitary operator U by identifying matrices from $U(2)$. Because each qubit for a grid point is a superposition of each possible state before a definitive observation has been made, the *state matrix*, on $x \in [x_i, x_i + \frac{\epsilon}{n}]$ for one application of the unitary operator, in which

$$\begin{pmatrix} |00> + |01> & |01> + |00> \\ |10> + |11> & |11> + |10> \end{pmatrix} \quad ,$$

or,

$$\begin{pmatrix} |10> + |11> & |11> + |10> \\ |11> + |10> & |10> + |11> \end{pmatrix} \quad ,$$

depending on whether $|0>$ or $|1>$ is initially assigned to the position $x_i = v_i \in V_i$, with each random matrix occuring independently with probability ϵ. It appears that after a fixed number of samples of the Brownian bridge for $t_i \neq t_j$, the quantum registers for storing these qubits can also scale polynomially, with exponentially fewer resources, if for each additional time step on which a Brownian brdige is independently sampled, each entry of the state matrix above can be decomposed as a tensor product of each one dimensional qubit.

Consistent with turbulent motion, one can adjust the time increment on which independently sampled, distinct Brownian bridges are taken, so that regardless of the initial position of the flow, the time evolution of the initial position, in comparison to the time evolution of another position, are independent. From the 3 parameter family $U(2)$, the time evolution operator $\exp(-i\frac{Ht}{h})$ for the Schrodinger equation, with $H = \sum_{i\sim j} \mathcal{J}_{ij} \bigg(\mathbf{1}_{|x_i - x_j| < 1} \exp(-i(t_i + t_j)) + \mathbf{1}_{|x_i - x_j| > 1} \exp(-it_i) \exp(-it'_j) \bigg)$, where t_i, t_j are the times for which one independently sampled Brownian bridge $W(t)$ is taken, while t_i' are the times at which another independently sampled Brownian bridge is taken, the time evolution of $\exp(-i\frac{Ht}{\hbar})$ will be compared to the time evolution of the unitary operator $\mathcal{U}_\epsilon^{x_i \text{or} y_i}$ applied to $\bar{u}_x$ or $\bar{u}_y$, depending on the direction of fluid flow. The couplings are given by

$$\mathcal{J}_{ij} = \begin{cases} N\frac{|t_i - t_j|}{\hbar}\exp(\mathbf{1}_{W(t_j)=\beta}|x_i - x_j|) \text{ , for } |x_i - x_j| \leq \frac{\epsilon}{n} \\ \frac{\exp(\mathbf{1}_{W(t)\neq\beta}|x_j|)}{|x_i - x_j|} \text{ , for } |x_i - x_j| > \frac{\epsilon}{n} \end{cases} ,$$

for suitable $N, \beta > 0$, with n the number of subintervals on $B_\epsilon^{x_i}$. For some i, j, again each $\mathcal{J}_{ij}$ ensures that, given the long and short term interactions, the sampled Brownian bridges are independent, in which for $|i - j| \leq$, $\mathcal{J}_{ij}$ is a bump function with height N for short range interactions, and otherwise, vanishes exponentially in the distance. From the couplings assigned according to this rule, the long and short term interactions in the hamiltonian permit quantum entanglement, as long as another Brownian bridge is sampled for some $x_j \neq x_i$. Given a sufficiently high variance of the sampled Brownian bridge, which can be qualitatively controlled for by calculating the Fischer information, the expectation $\mathbf{E}_{W(t)}(-\partial_\theta^2 \frac{\log f(x|\theta)}{\hbar})$, where $f$is the pmf, given a partitioning of $[x_i, x_i + \epsilon]$ and $[y_i, y_i + \epsilon]$ into n subintervals of each length, so that each sampled Brownian bridge has a sufficiently high variance to decay sufficiently fast within atomic scales. Independent samples w from each Brownian bridge are collected, per the law $\mathbf{P}_{W(t)}$.

Under $\mathbf{P}_{W(t)}$, this method of collecting samples entails that new samples are generated past the maximum time of some partition of $[x_i, x_i + \epsilon]$ by continuing to sample the Brownian bridge up to the maximum point of time within the subinterval, from which the time interval of the next Brownian bridge can be sampled by superimposing another independently sampled Brownian bridge to the previous one.

4 Representing the solution to Navier-Stokes with Fourier Series, for each direction of fluid flow

Substituting $\hat{u}_x = \sum_{n\in\mathbf{Z}} \exp(-inx)u_x$, and $\hat{u}_y = \sum_{n'\in\mathbf{Z}} \exp(-in'y)u_y$ yields,

$$\Rightarrow (-in)\frac{\partial(u_x \exp(-inx))}{\partial x} + (-in)\frac{(\partial u_y \exp(-in'y))}{\partial y} = 0 ,$$

$$\Rightarrow (-in)\rho\bigg(\frac{\partial u_x \exp(-inx)}{\partial t} + (-in)u\frac{\partial(u_x \exp(-inx))}{\partial x} +$$

$$(-in)v\frac{\partial(u_x \exp(-in'y))}{\partial y}\bigg) = -\frac{\partial p}{\partial x} + \mu\bigg((-in)^2\frac{\partial^2(u_x \exp(-inx))}{\partial x^2} + (-in)^2\frac{\partial^2(u_x \exp(-in'y))}{\partial y^2}\bigg) ,$$

$$\Rightarrow \rho(-in)\bigg(\frac{\partial(u_y \exp(-in'y)}{\partial t} + v(-in)\frac{\partial(u_y \exp(-in'y))}{\partial x} + v\frac{\partial \bar{u}_y}{\partial y}\bigg) = -\frac{\partial p}{\partial y} + \mu\bigg((-in)^2\frac{\partial^2(u_y \exp(-inx))}{\partial x^2}$$

$$+(-in)^2\frac{\partial^2 u_y \exp(-in'y)}{\partial y^2}\bigg) ,$$

for each $n, n' \in \mathbf{Z}$.

With this Fourier series substituion, we will proceed to study additional higher order terms from the general N-S pde, which will allow for more understanding of how quantum entanglement, which partly accounts for the unpredictable motion and diffusion processes, can be elucidated from a proxy for quantum entanglement by sampling independent copies of the Brownian bridge. In line with **Lubasch** et al, from the exponential basis $\{\exp(-i\frac{t_i H}{\hbar})\}_{i\in\mathbf{N}}$, the time evolution for all times $t > t' = min_{k\in\mathbf{N}}\{t_k \in \mathbf{N} : W(t_k) = \beta\}$, for a threshold $\beta > 0$, represent contributions from the nonlinearized turbulent flow. With exponentially fewer resources, the projection

$$\Phi : \frac{\{\alpha, k \in \mathbf{N} : W(x_k) = \alpha\}}{\sim} \longrightarrow \bar{\mathcal{H}} : t_k \mapsto \begin{cases} \alpha|1> & , \text{w.p } \delta \\ \alpha|0> & , \text{w.p } 1-\delta \end{cases} \qquad ,$$

for $\delta > 0$, where $\sim$ means $x_i \sim x_j \Leftrightarrow$ for $\beta, t_i \neq t_j > 0$, $W(x_i) = W(x_j) = \beta$. Particularly, Φ assigns points at which samples are collected for which the Brownian bridge achieves the same height β.

5 Goals

- Test out characteristics of random sampling for early steps, ie low n, of random, independent samples of the Brownian bridge
- Determine the effects of distinctly sampled Brownian bridges on higher order terms, namely terms multiplied by the indicator $\mathbf{1}_{|x_i - x_j| > 1}$, for establishing simulations of quantum entanglement
- Generalize steps of the procedure that have been presented so far to non constant pressure $\rho = \rho(x, y)$, for $x, y \in \mathcal{R}$