towards a quantum variational algorithm for the Navier-Stokes equation

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overview of the approach: motivation and steps towards analyzing turbulent flow

to characterize ansatzae for physical interpretations of different turbulent trajectories that are dictated by the Navier-Stokes PDE, we will introduce

- a Hamiltonian $\mathcal{H}$ to measure the short, and long range, time evolution of different ansatzae for the solution,
- interpretations of the time evolution of different ansatzae,
- finally, quantum circuits for qubit preparation that are executable on the IBM Q - experience, in which the assignment of a 0 qubit corresponds to a flow whose turbulence persists on "shorter" scales, while the assignment of a 1 qubit corresponds to a flow whose turbulence persists on a "longer" scale (I will make the notions of "shorter" and "longer" precise with a parameter h that will be defined)

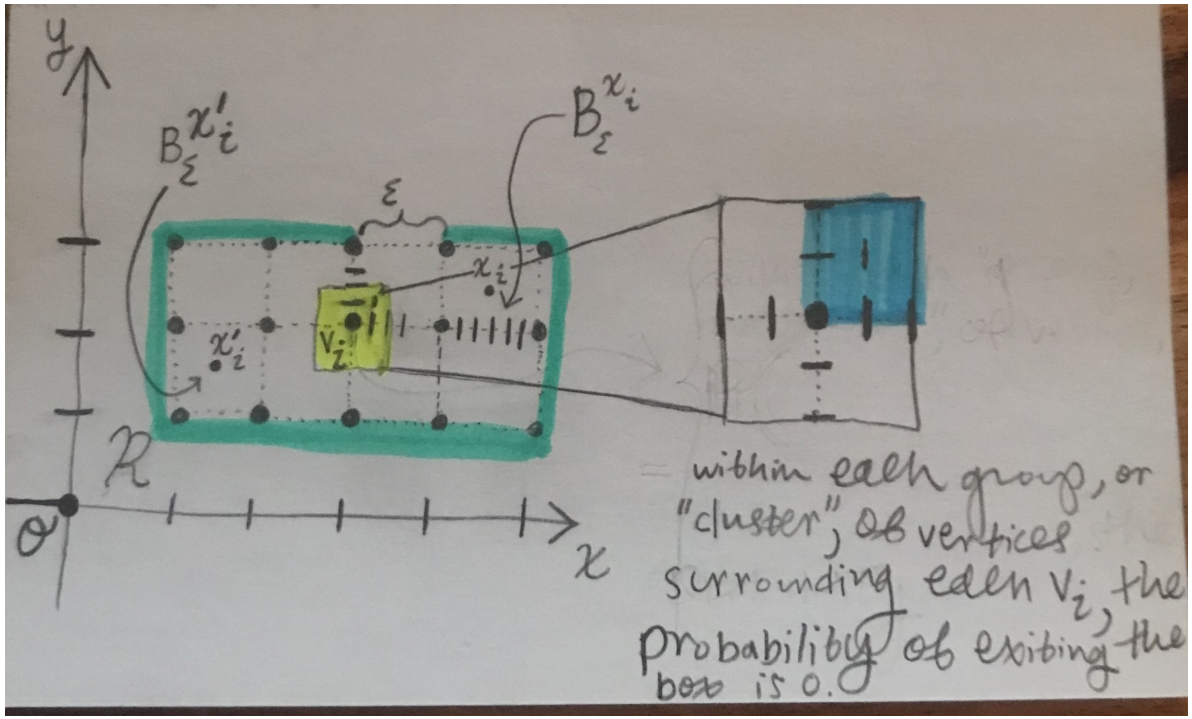
preliminaries: defining quantities of interest

before proceeding into further motivation and the technicalities of our approach, we will need to make use of the notions of

- a **trajectory**, which is regarded as a path of fluid flow whose direction (and time evolution) is determined by,
  - the location of the plane, $\mathbf{R}^2$, on which the random sample is collected,
  - the magnitude of the random sample at a particular location,
- the **Brownian Bridge**, which is a standard Brownian Motion, ie a collection of normally distributed random variables with 0 mean and nonzero variance, sufficiently small, **except** with the endpoints at the beginning and end of the time intervals fixed
  - for our implementation, the beginning and end points of each sampling process will be taken to equal 0, which is representative of imposing no boundary conditions on the system

a simplified visual heuristic

over any subset of $\mathbf{R}^2$, taking the region of fluid flow $\mathcal{R}$ (shown in green/teal) as the area of the plane over which turbulent flows will be analyzed, we introduce a suitable **partitioning** of $\mathcal{R}$ into disjoint $B_\epsilon^{x_i}$, namely boxes whose center is at x_i, each of which have sides of length ϵ, arbitrarily small, with a nonzero probability of the random walk visiting each of the 3 vertices in each cluster (shown in yellow), stays within the blue region



literature survey I: why should we look to solve this problem using a quantum variational algorithm?

Reference: **Azad & Andallah**, Analytical Solutions of 2D Incompressible Navier-Stokes Equations .. (2014)

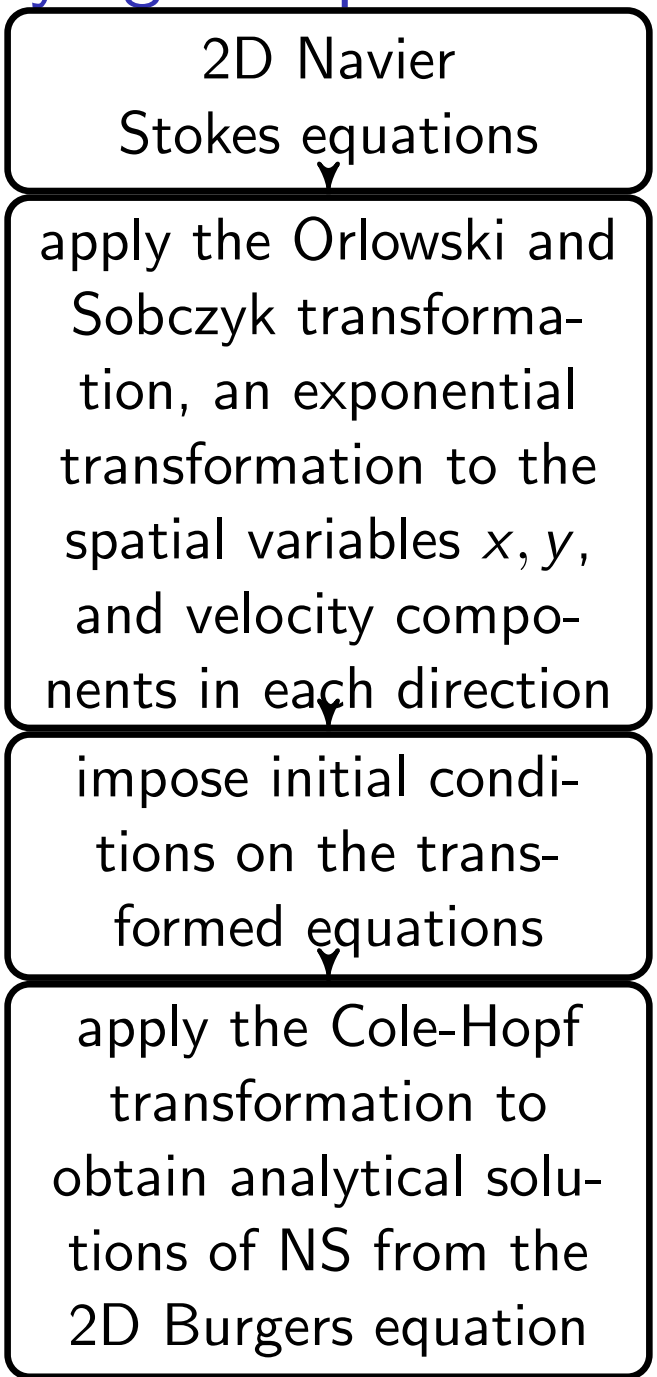
under the assumption of an exponentially decaying, time dependent, pressure gradient, 2D Navier-Stokes can be expressed as the Burgers equation, through the following sequence of transformations, which will be applied to the Navier Stokes equations, which by definition are of the form,

$$\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y^*} = 0 \text{ , continuit}$$

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \nu\left(\frac{\partial^2 u^*}{\partial (x^*)^2} + \frac{\partial^2 u^*}{\partial (y^*)^2}\right) \text{ , x moment}$$

$$\frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x^*} + \nu\left(\frac{\partial^2 v^*}{\partial (x^*)^2} + \frac{\partial^2 u^*}{\partial (y^*)^2}\right) \text{ , y moment}$$

applying a sequence of transformations to NS



literature survey II: why should we look to solve this problem using a quantum variational algorithm?

after solving a second order ODE, and applying the inverse OST, analytic solutions to NS are of the form,

$$u(x,y,t) = \frac{1}{a}e^{-b} - \frac{1}{a}e^{-at-b} + u'(x,y,t) \ ,$$
$$v(x,y,t) = \frac{1}{c}e^{-d} - \frac{1}{a}e^{-ct-d} + v'(x,y,t) \ ,$$

respectively for the x and y directions [1]

more broadly, making use of the transformations introduced in the aforementioned paper could allow for potential connections with the quantum variational algorithm introduced in **Lubasch et al** that is applied to the Burgers Equation

[1] u' , v' are quantities from the OST in the second step from the previous slide

a first step: assigning suitable ansatzae for 2-qubit quantum circuits

to ensure that a quantum variational algorithm for Navier-Stokes can be achieved, we must consider

- an appropriate partition of the region $\mathcal{R}$ over which the Navier-Stokes trajectory is to be analyzed,
- and a qubit assignment, on vertices of a graph on $\mathbf{Z}^2$, from which collections of random samples of a Brownian bridge will be constructed

in the upcoming slides, we will continue to present motivation before introducing plots to study the time evolution of $\mathcal{H}$

additional motivation: defining the qubit assignment to construct Brownian collections of random samples

in order to achieve a suitable tradeoff between classical and quantum measurements, the qubit assignment, corresponding to an initial point of time evolution, is of the form,

$$\varphi : V_i \longrightarrow \mathcal{H} : \{\forall v \in \partial B_\epsilon^{x_i}\} \mapsto \begin{cases} |0> & \text{w.p } \epsilon \\ |1> & \text{w.p } 1-\epsilon \end{cases} ,$$

begins the time evolution, and additionally, prepares quantum registers from boxes $B_\epsilon^{x_i}$ that partition $\mathcal{R}$

proceeding further: leveraging the initial qubit assignment towards constructing ansatzae

before feeding ansatzae into our quantum circuits, it is imperative that we illustrate more aspects of our strategy, including approaches towards

- formulating the Hamiltonian that will model the time evolution for our system of interest,
- plotting typical behavior of the Hamiltonian, given a fixed number of time steps and an initial position at which the collection of samples will be constructed, with **no** boundary conditions

defining the Hamiltonian

now, given the initial qubit assignment on vertices of the box, we define the Hamiltonian through an iterative decomposition of short and long range order terms, of the form,

$$\mathcal{H} = \sum_{i\sim j} \mathcal{J}_{ij} \mathbf{1}_{|x_i - x_j|<1} \exp(-(t_i + t_j)) + \sum_{i'\sim j'} \mathcal{J}_{i'j'} \mathbf{1}_{|x'_i - x'_j|>1} \exp(-(t'_i + t'_j)) \ ,$$

in which the couplings satisfy

$$\mathcal{J}_{ij} = \begin{cases} N|t_i - t_j| \exp(\mathbf{1}_{W(t_j)\geq h}|x_i - x_j|) \ , \text{ for } |x_i - x_j| \leq \frac{\epsilon}{n} \\ \frac{\exp(\mathbf{1}_{W(t)\leq h}|x_j|)}{|x_i - x_j|} \ , \text{ for } |x_i - x_j| > \frac{\epsilon}{n} \end{cases} \ ,$$

for arbitrary N (for the time being, we set $N \equiv 1$)

continued...

... where in the first term for short range interactions, random samples are collected from one box in which the time evolution is initiated, whereas for the second term, representative of long range order, interactions are represented through the given product of exponentials

heuristically, given that each term in $\mathcal{H}$ is dependent on the magnitude of the random sample collected with each additional increment in the time evolution, future steps in the approach will aim to compress classical observations, and in attempting to adequately represent such observations with qubits, will in turn influence our choice of ansatzae (to be defined)

defining the height parameter

besides our expression for $\mathcal{H}$, we also define a height h, which will be useful in

- distinguishing the scales over which short and long range dynamics persist (respectively, each of which can be physically realized as regions over which turbulent flows have 'degrees of freedom,' which will be used to compute the set of all possible trajectories from an initial configuration, due to turbulence being intrinsically unpredictable),
- while also making use of h to **compress** classical observations before passing them as qubits to a quantum circuit, which will be rigorously justified

studying the behavior of our Hamiltonian

to this end, h will be deterministically set in order to

- most efficiently separate the scales over which dynamics of interest occur,
- in addition to constructing unitary operations, through 2 qubit gates in circuits, from which simulations with distinct h will be enforced so as to study the time evolution

putting it together: a summary of steps before running simulations

thus far, we have described arguments for

- iteratively constructing a Hamiltonian, which is intended to describe the time evolution of a trajectory, which at consecutive time increments is dependent on the collection of Brownian random samples,
- analyzing different trajectories, insofar as to produce ansatzae for turbulent motion, dependent on the collection of random samples from the Brownian bridge that are gathered during a specific interval of the time evolution

upcoming steps: running simulations for different h and collections of random samples

with such goals from previous slides, we will now turn to the compression of classical observations into quantum observations, which is significant for

- constructing ansatzae that reflect higher degree of freedom in turbulent motion, through constructing bonds on the lattice, which can be realized through resampling regions of the lattice over which the Brownian bridge achieves values **greater** than the threshold specified by the height parameter h,
- introducing fewer samples for regions of the lattice for the classical observations that are assigned to a 0 qubit in the bit string of the time evolution of our system

plotting behavior of the Hamiltonian: observing typical behavior for a fixed number of time steps

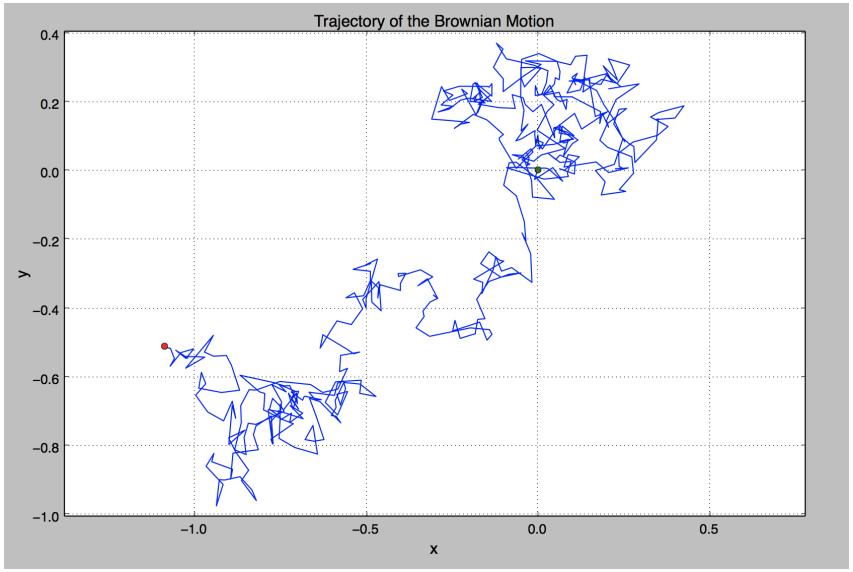
function setup in the implementation

to compute terms in the previously defined $\mathcal{H}$, one must specify

- a $B_\epsilon^{x_i}$ on which samples for the short range order terms are to be sampled,
- another box $B_\epsilon^{x_i'}$, over which long range order samples are to be collected

$\star$ given that analytical solutions to 2D Navier-Stokes are of the form of a decaying exponential, it is natural to model the time evolution of the Hamiltonian, iteratively, with a normally distributed random variable, whose PDF is of the form $\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$

simulating typical behavior of the Hamiltonian: observing the time evolution for a fixed number of time steps, by first generating collection of Brownian samples



future steps: We will relate the magnitude of the Brownian samples for simulated trajectories, of this type, to then study the behavior of $\mathcal{H}$. Directly, we will make use of the randomly generated samples to simulate the random behavior of the Hamiltonian.

live demonstration of the code: observing the relationship between the directions on the plane in which

besides merely enforcing a fixed number of random samples, at each step, for the time evolution, we can also observe how the trajectory of the Brownian bridge would change as we vary

- the total number of steps that we demand the Brownian bridge take,
- the height h that we enforce to distinguish between short and long range order dynamics

overview of how the quantum circuits will be constructed

to this end, given the arguments and background that have been presented, we will construct quantum circuits, after a suitable compression from bits to qubits, by

- **first** implementing a rotation gate, where the angle θ is dependent on whether the trajectory is of short, or long range, order,
- **second**, implementing a series of intermediate gates, before the read out, to account for potential interaction from long range order terms in $\mathcal{H}$ (I am still thinking about **which** sequence of gates would be optimal, but I think that we would include a CNOT gate)
- **third** , reading out the resulting qubit

future plans

- I will post the reference PDF to Slack
- I am interested in trying out different Hamiltonians, and circuits to compress the classical observations before assigning qubits, in addition to implementing symmetries with the Ricci curvature mentioned in my PDF, as well as trying out computations that I put forth for the coverage $\mathcal{C}$
- An update on my **travel plans**: I will stay in Ithaca after all of the undergraduates leave, but given that most of the campus will likely be closed very soon, I am arranging to fly back to Los Angeles by the end of the month when there is a smaller volume of students leaving campus