Thoughts, GFF 1

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1 Setting Everything Up

1.1 Background

Under similar assumptions that we have discussed, in order to 'generalize' the algorithm for integrating out the randomness from some environment, we will try to generalize the edge densities p_{xy}, etc with the following. Please note that although I have listed the same expressions for h and $\vec{v}_x^n$ and $\vec{v}_y^n$, there constants would have to be determined by part of the stochastic domination arguments that we don't know.

1.2 Edge Densities From Page 11

Let $G = (V, E)$ be a finite graph, and $\Lambda \subset V$. For $\vec{v} \in \mathbf{S}^{n-1}$, we will decompose the random vector as

$$\vec{v} = \sum_{n \geq 0} \vec{v}^n \ ,$$

with edge densities

$$\begin{aligned} \vec{p}_{xy} &= \vec{q} \ , \\ \vec{p}_{\overline{xy}} &= 1 - \exp\left(\vec{h} - \sum_{k>n} (\vec{v}_x^k)^2 + (\vec{v}_y^k)^2\right) \ , \\ \vec{p}_{x_1 y} &= 1 - \exp(-(\vec{v}_x^n)^2 \mathbf{1}_{\vec{v}_x^n \geq \lambda}) \ , \\ \vec{p}_{x y_1} &= 1 - \exp(-(\vec{v}_y^n)^2 \mathbf{1}_{\vec{v}_y^n \geq \lambda}) \ , \end{aligned}$$

for $x, y \in V$. We define the probability measure $\mathbf{P}_{\vec{q}, \vec{h}, \lambda}$ on G. Under the canonical identification of the probability measure on G with a tensor product,

$$\mathbf{P}_{\vec{q}, \vec{h}, \vec{\lambda}} = \otimes_{i=1}^n \mathbf{P}_{q_i, h_i, \lambda_i} \ .$$

In $\mathbf{Z}^d$ for $d \geq 2$, arranging all possible configurations, per hyperplane of the lattice, fixing both $n \leq d$, gives the random $n \times nd$ matrix A

$$\begin{pmatrix} v_1^1 & v_2^1 & \cdots & \cdots & v_n^1 & \cdots & \cdots & v_d^1 \\ v_1^2 & v_2^2 & \ddots & & \vdots & & & \vdots \\ \vdots & \vdots & & \ddots & \vdots & & & \vdots \\ v_1^n & & & & v_n^n & & & v_d^n \end{pmatrix} \ ,$$

whose rows denote individual Gaussian random variables. We observe that the entries of this matrix conforming to a normal distribution with some mean and variance, implies that $A \sim \mathcal{N}(\mu, \sigma^2)$. From A, taking the maximum and minimum v^i_j, for $1 \leq i \leq n$ and $1 \leq j \leq nd$, over columns, respectively gives the Gaussian random variables v_{Max_j} and v_{Min_j}, again for each j.

There exists real q_i with

$$\vec{q} = (q_1, \ldots, q_n) \ ,$$

which represent, from the viewpoint of the algorithm that is integrating out the randomness from the GFF, the individual constants q_i that account for each of the random Gaussian variables in each of the components of $\vec{v}$.

1.3 Stochastic Domination Thoughts from the J_{xy}

For stochastic domination, we know that the coupling constants $J_{xy}(\psi) = |\psi_x||\psi_y|$ for the $O(N)$ model, after rearranging terms from the GFF measure to get

$$\frac{1}{Z(\psi)} \exp\bigg(\sum_{x \sim y} J_{xy}(\psi) \sigma_x \cdot \sigma_y \bigg) G(|\psi|) \exp\bigg(-\frac{1}{2} \sum_{x \sim y} |\psi_x|^2 + |\psi_y|^2 \bigg) \ .$$

To ensure that we can obtain a random process that always demonstrates at least the same amount of 'randomness' that is in the Hamiltonian above with the coupling constants $J_{xy}(\psi)$, we want another coupling constant $\mathcal{J}_{xy}$ that, probabilistically, is just as likely to exhibit strong, or weak, couplings as the $J_{xy}(\psi)$ would. To do this, we set

$$\mathcal{J}_{xy} = \max_{x,y} \bigg(|\psi_x|, |\psi_y|, |\psi_x||\psi_y| \bigg) \ ,$$

which absolutely guarantees, from the slightly different Hamiltonian,

$$\frac{1}{Z'(\psi)} \exp\bigg(\sum_{x \sim y} \mathcal{J}_{xy}(\psi) \sigma_x \cdot \sigma_y \bigg) G'(|\psi|) \exp\bigg(-\frac{1}{2} \sum_{x \sim y} |\psi_x|^2 + |\psi_y|^2 \bigg) \ ,$$

that our original $O(N)$ model is stochastically dominated, because the coupling constant $\mathcal{J}_{xy}$, in the case that the GFF is larger at either x or y, would more strongly correlate the ψ_x and ψ_y with each other. Similarly, in the case that the ψ_x and ψ_y, by coincidence, are comparable, the coupling $\mathcal{J}_{xy}$ would, under the Hamiltonian above, would stochastically dominate the Hamiltonian with the couplings J_{xy}.

Therefore,

$$\mathbf{E}_\Lambda[\sigma_x \cdot \sigma_y]_{\mathcal{J}_{xy}} \geq \mathbf{E}_\Lambda[\sigma_x \cdot \sigma_y]_{J_{xy}} \ ,$$

for all $x, y \in \Lambda$.

1.4 Back to the Random Matrix A

From the v_{Max_j} and v_{Min_j} for each j, we can introduce the 'maximum' and 'minimum' Gaussian Free Fields as the span of the random vectors

$$\mathcal{G}_{\mathrm{Max}} \propto (v_{\mathrm{Max}_1}, 0, \cdots, 0) + (0, v_{\mathrm{Max}_2}, \cdots, 0) + \cdots + (0, \cdots, 0, v_{\mathrm{Max}_n}) \ ,$$

and

$$\mathcal{G}_{\mathrm{Min}} \propto (v_{\mathrm{Min}_1}, 0, \cdots, 0) + (0, v_{\mathrm{Min}_2}, \cdots, 0) + \cdots + (0, \cdots, 0, v_{\mathrm{Min}_n}) \ .$$

With these random linear combinations, we will attempt in the next Section to show that Long Range Order exists (or if any of these ideas are helpful), by determining if there is a suitable constant c satisfying

$$\frac{1}{|\Lambda|^2} \sum_{x,y} \mathbf{E}[\sigma_x \cdot \sigma_y]_{J_{xy}} \geq c \ .$$

2 Attempt at Long Range Order

Using $\mathcal{G}_{\mathrm{Min}}$ and $\mathcal{G}_{\mathrm{Max}}$, we will try to analyze the expression above for LRO. For any $x, y \in \Lambda \subset V$, calculating the expectation

$$\mathbf{E}_\Lambda[\sigma_x \cdot \sigma_y] \ ,$$

amounts to determining the correlations between different random vectors that we have arranged in A. First, we obviously know that the expectation above can be expressed as,

$$\sum_{x,y \in \Lambda} \sigma_x \cdot \sigma_y \ ,$$

because for the discrete GFF we are not integrating continuously across paths in the lattice $\mathbf{Z}^d$, with the measure $\mathrm{d}\mathbf{P}_{\vec{q},\vec{h},\vec{\lambda}}$.

I was trying to relate this expectation to the correlations that we would want to calculate for LRO.

In particular, for any 2 random vectors in $\mathbf{S}^{n-1}$ that are represented by fixing a column and looking at the Gaussian variables in all rows for that fixed column, we know that for all possible choices of vertices $x, y \in \Lambda \subset V$, calculating each possible pair-wise correlation between these random vectors that we have arranged along the columns j of A, can be expressed with, for all possible pairings of the columns j, j' of A, satisfying $1 \leq j, j' \leq nd$, as

$$\sum_{j,j'} \mathrm{Corr}(A_j, A_{j'}) \ .$$

To evaluate this summation with all possible cross-correlations from A, we recall,

$$\mathrm{Corr}(A_j, A_{j'}) = \frac{\mathrm{Cov}(A_j, A_{j'})}{\sqrt{\mathrm{Var}(A_j)}\sqrt{\mathrm{Var}(A_{j'})}} \ ,$$

which fortunately can be used to evaluate terms from the expectation listed at the beginning of the section. Specifically, we would want to rearrange terms from the expectation in terms of the correlations between different columns of A, from which we could provide a lower bound, as listed in the end of Section 1, by making use of properties of the distribution $\mathcal{N}(\mu, \sigma^2)$. For the random Gaussian variables that are normally distributed under $\mathcal{N}$, it is possible for us to, conditionally on the level $\lambda > 0$, where $\lambda = \min_{1 \leq i \leq n} \lambda_i$, of the Gaussian Free Field that we initially proposed for the measure $\mathbf{P}_{\vec{q},\vec{h},\vec{\lambda}}$, to provide a lower bound for each of the correlations.

Again, for all possible combinations of different columns j, j' of A, we could evaluate the correlation between two columns of A, which as random column vectors have correlations that can be calculated by calculating the correlation between the Gaussian random variables in each respective component. Furthermore, with an interest towards obtaining the desired lower bound, calculating all possible correlations between different columns of A, for some fixed j, is precisely given by calculating the covariance between the Gaussian random variables, and then normalizing the covariance by the square root of the variances of each random variable.

In order to provide a lower bound for these correlations, which will collectively give a lower bound for LRO, we could look to the Gaussian distribution $\mathcal{N}$. In particular, from this normal distribution, either one of the conditional probabilities

$$\mathbf{P}_{\vec{q},\vec{h},\vec{\lambda}}(|v_i^j - v_i^{j'}| < \sigma || \vec{v}_i^j \leq \lambda, \text{ and } \vec{v}_{i'}^{j'} \leq \lambda) \ ,$$

or,

$$\mathbf{P}_{\vec{q},\vec{h},\vec{\lambda}}(|v_i^j - v_i^{j'}| < \sigma^2 || \vec{v}_i^j \leq \lambda, \text{ and } \vec{v}_i^{j'} \leq \lambda) \ ,$$

captures the following. In the conditioning, we want $1 \leq i \leq n$ fixed, as well as $1 \leq j \neq j' \leq nd$, so that we can provide a lower bound for each of the possible cross correlations by not only measuring how much the particular random vector that we are looking at deviates from the mean of $\mathcal{N}$, but also how much the random vectors are correlated to each other, from each of the n entries.

I have more but I want to wait..

3 Thoughts/Questions

- **Question 1**: For Stoachastic domination, don't we want to introduce a coupling constant that is always larger than J_{xy}?
- **Question 2**: More discussion about dealing with the stochastic domination for the couplings J_{xy}?
- **Question 3**: More discussion about computing the correlations for long range order?