Bond Graph Clinic: Part 2

Constitutive Relations

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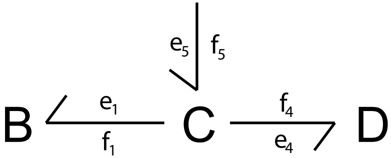
Previously...

Junction Structures

Storage and I/O

An example...

Network models of energetic systems



Last week we showed Bond Graphs capture:

- Energy transferred between B, C, D without loss via bonds.
- Power transfer represented by conjugate variables $P_i = e_i f_i$.
- Subsystem dynamics through constitutive relations; $\Phi_B(p, q, e, f) = 0$ for example.

Power Conservation Laws

Suppose this network conserves power, then

$$e_1 f_1 + e_2 f_2 = 0.$$

That is; the power into Φ must sum to zero. Two solutions: $e_1 \propto e_2$, or $e_1 \propto f_1$.

Transformer

$$\xrightarrow[f_1]{e_1} \underset{\rho}{\mathrm{TF}} \xleftarrow[f_2]{e_2}$$

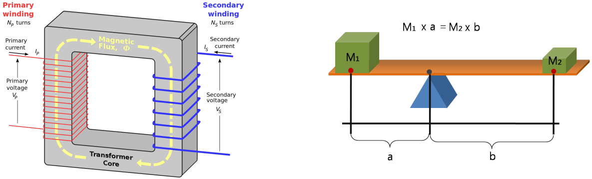


Figure: An electrical transformer and a mechanical transformer (ie, a lever)

Also includes: motors and generators, waterwheels, etc.

Transformer

$$\xrightarrow[f_1]{e_1} \underset{\rho}{\mathrm{TF}} \xleftarrow[f_2]{e_2}$$

For transformers we require $e_2 = \rho e_1$, so

$$e_1 f_1 + e_2 f_2 = 0 \implies e_1(f_1 + \rho f_2) = 0 \implies f_2 = -\frac{1}{\rho} f_1$$

Giving a constitutive relation

$$\Phi_{\mathrm{TF}} = \begin{pmatrix} e_2 - \rho e_1 \\ f_2 + \frac{1}{\rho} f_1 \end{pmatrix} = 0.$$

Transformer

$$\overset{e_1}{\underset{f_1}{\rightharpoondown}}\ \underset{\rho}{\mathrm{TF}}\ \overset{e_2}{\underset{f_2}{\leftharpoondown}}$$

For transformers we require $e_2 = \rho e_1$, so

$$e_1 f_1 + e_2 f_2 = 0 \implies e_1(f_1 + \rho f_2) = 0 \implies f_2 = -\frac{1}{\rho} f_1$$

Giving a constitutive relation

$$\Phi_{\mathrm{TF}} = \begin{pmatrix} -\rho & 0 & 1 & 0 \\ 0 & \rho^{-1} & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \\ e_2 \\ f_2 \end{pmatrix} = 0.$$

Gyrator

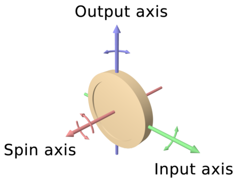


Figure: A gyroscope

Electrical gyrators were proposed by Tellengen in 1948, but a passive implementation hasn't yet been found.

Gyrator

For gyrators we require $e_2 = \rho f_1$, so

$$e_1 f_1 + e_2 f_2 = 0 \implies f_1(e_1 - \rho f_2) = 0 \implies f_2 = -\frac{1}{\rho} e_1.$$

Giving a constitutive relation

$$\Phi_{GY} = \begin{pmatrix} e_2 - \rho f_1 \\ f_2 + \frac{1}{\rho} e_1 \end{pmatrix} = 0.$$

Gyrator

$$\xrightarrow[f_1]{e_1} \mathrm{GY}_{\rho} \xleftarrow[f_2]{e_2}$$

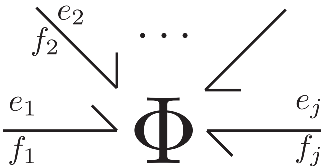
For gyrators we require $e_2 = \rho f_1$, so

$$e_1 f_1 + e_2 f_2 = 0 \implies f_1(e_1 - \rho f_2) = 0 \implies f_2 = -\frac{1}{\rho} e_1.$$

Giving a constitutive relation

$$\Phi_{\mathrm{GY}} = \begin{pmatrix} 0 & -\rho & 1 & 0 \\ \rho^{-1} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \\ e_2 \\ f_2 \end{pmatrix} = 0.$$

Network Conservation Laws

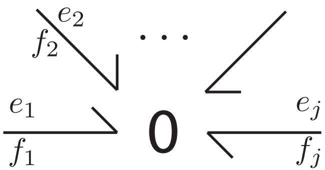


We require nodes to capture the distribution of power across many subsystems. These junction nodes capture network conservation laws which must satisfy

$$0 = \sum_{i=1}^{j} e_i f_i.$$

The two base cases are *common effort*, and *common flow*.

0-Junction

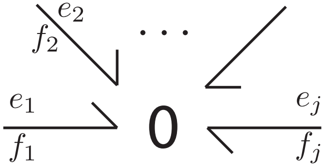


For common effort junctions $e_i = e_k, \ \forall 1 \leq i, k \leq j$. Hence

$$\sum_{i=1}^{j} e_i f_i = 0 \implies \sum_{i=1}^{j} f_i = 0$$

This is Kirchoff's Current Law

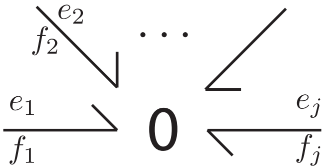
0-Junction



The constitutive relation is given by:

$$\Phi_0 = \begin{pmatrix} e_1 - e_2 \\ \dots \\ e_{j-1} - e_j \\ f_1 + f_2 + \dots + f_j \end{pmatrix} = 0$$

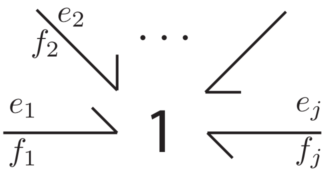
0-Junction



The constitutive relation is given by:

$$\Phi_0 = \begin{pmatrix} 1 & 0 & -1 & 0 & \dots & 0 \\ & \ddots & & \ddots & & \\ \dots & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & \dots & & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \\ \vdots \\ e_j \\ f_j \end{pmatrix} = 0$$

1-Junction

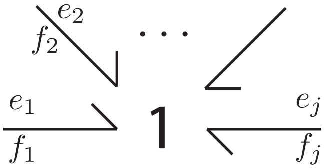


Similarly for common flow junctions $f_i = f_k,\ \forall 1 \leq i, k \leq j$. Hence

$$\sum_{i=1}^{j} e_i f_i = 0 \implies \sum_{i=1}^{j} e_i = 0$$

This is Kirchhoff's Voltage Law

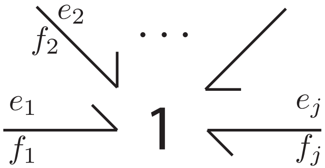
1-Junction



The constitutive relation is given by:

$$\Phi_1 = \begin{pmatrix} f_1 - f_2 \\ \dots \\ f_{j-1} - f_j \\ e_1 + e_2 + \ldots + e_j \end{pmatrix} = 0$$

1-Junction



The constitutive relation is given by:

$$\Phi_1 = \begin{pmatrix} 0 & 1 & 0 & -1 & \dots & 0 \\ & & \ddots & & \ddots & \\ \dots & & 0 & 1 & 0 & -1 \\ 1 & 0 & \dots & & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \\ \vdots \\ e_j \\ f_j \end{pmatrix} = 0$$

Physics review

Energy stored is a function of local storage co-ordinates q, p

$$H(q,p) = E_0 + \int P_{\text{in}}\,\mathrm{d}t.$$

The canonical co-ordinates q, p are taken such that $q \in X$ where X is some manifold, and $p \in X^*$ where X^* is the dual space of X.

In the case where $X = \mathbb{R}^n$, the dual space X^* is the set of column vectors acting on X. That is if $q, x \in \mathbb{R}^n$ then there exists a dual vector (a linear functional) $p \in X^*$ such that $p(q) = x^T \cdot q$.

As an aside, in physics p is often written as $\langle p|$, so that in the above example $\langle p|q\rangle = x^T q$.

Physics review (cont.)

If $P_{\text{in}} = ef$ then

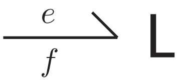
$$H(q,p) = E_0 + \int P_{\text{in}}\,\mathrm{d}t \implies \frac{\partial H}{\partial q}\mathrm{d}q + \frac{\partial H}{\partial p}\mathrm{d}p = ef\,\mathrm{d}t.$$

We want to:

- associate e with the tangent space of X^*, ideally by saying something like $e = \frac{\mathrm{d}p}{\mathrm{d}t}$ so that $e\,\mathrm{d}t = \mathrm{d}p$.
- associate f with the tangent space of X, ideally by saying that $f = \frac{\mathrm{d}q}{\mathrm{d}t}$ so that $f\,\mathrm{d}t = \mathrm{d}q$.

We can do this easily if we consider individual components whose stored energy depends only on generalised position or momentum, but not both.

Kinetic Storage



For kinetic energy, the stored energy depends only on momentum, $\frac{\partial H}{\partial q} = 0$, so we can define $e\,\mathrm{d}t = \mathrm{d}p$.

$$\frac{\partial H}{\partial q}\mathrm{d}q + \frac{\partial H}{\partial p}\mathrm{d}p = ef\,\mathrm{d}t \implies \left(\frac{\partial H}{\partial p} - f\right)\mathrm{d}p = 0$$

Hence we have a constitutive relation

$$\Phi_L = \frac{\partial H}{\partial p} - f = 0, \qquad \text{where} \qquad \dot{p} = e.$$

Kinetic Storage (cont.)

$$\xrightharpoondown[f]{e} \ \mathsf{L}$$

In the case of linear kinetic storage; we have a Hamiltonian

$$H(q,p) = \frac{1}{2L}p^2,$$

which is the energy stored in a L-henry inductor; in the linear motion of an L-kg object.

This gives rise to the familiar constitutive relation:

$$\Phi_L = \frac{\partial H}{\partial p} - f = \frac{1}{L}p - f \qquad \Longrightarrow \qquad Lf - \int e \,\mathrm{d}t = 0.$$

Kinetic Storage (cont.)

$$\xrightharpoonup[f]{e} \; \mathrm{L}$$

This gives rise to the familiar constitutive relation:

$$\Phi_L = \frac{1}{L}p - f$$

Which, if we define the integration operator $\mathcal{I}x = \int_0^t x(t)\,\mathrm{d}t$, then we have

$$\Phi_L = (\mathcal{I}, -L)\begin{pmatrix} e \\ f \end{pmatrix} \quad \overset{\mathcal{L}}{\Longrightarrow} \quad \hat{\Phi}_L(s) = (s^{-1}, -L)\begin{pmatrix} \hat{e} \\ \hat{f} \end{pmatrix}$$

Potential Storage

$$\overset{e}{\underset{f}{\rightharpoonup}}\ \mathrm{C}$$

Similarly for potential, the stored energy depends only on position, $\frac{\partial H}{\partial p} = 0$, so we define $f\,\mathrm{d}t = \mathrm{d}q$.

$$\frac{\partial H}{\partial q}\mathrm{d}q + \frac{\partial H}{\partial p}\mathrm{d}p = ef\,\mathrm{d}t \implies \left(\frac{\partial H}{\partial q} - e\right)\mathrm{d}q = 0$$

Hence we have a constitutive relation:

$$\Phi_C = \frac{\partial H}{\partial q} - e = 0, \qquad \text{where} \qquad \dot{q} = f.$$

Potential Storage (cont.)

$$\xrightarrow[f]{e} \mathrm{C}$$

Storage of potential energy has more variety. In the linear case (represented by the C node above) we have

$$H(q,p) = \frac{1}{2C}q^2$$

which is the power stored in a C-farad capacitor, the elastic energy stored in a spring displaced by q from equilibrium, etc.

$$\Phi_C = \frac{\partial H}{\partial q} - e = \frac{1}{C}q - e \qquad \Longrightarrow \qquad Ce - \int f\,\mathrm{d}t = 0$$

Potential Storage (cont.)

$$\xrightharpoonup[f]{e} \mathsf{C}$$

Hence we have a constitutive relation:

$$\Phi_C = \frac{1}{C}q - e$$

Which, if we define the differentiation operator $Dx = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, then we have

$$\Phi_C = (C, -D)\begin{pmatrix} e \\ f \end{pmatrix} \qquad \overset{\mathcal{L}}{\Longrightarrow} \qquad \hat{\Phi}_C(s) = (C, -s)\begin{pmatrix} \hat{e} \\ \hat{f} \end{pmatrix}$$

up to a constant.

Linear Dissipation

$$\xrightharpoondown[f]{e} \ \mathrm{R}$$

Dissipation can be captured by introducing a Dirac structure $\mathcal{D}(\dot{q})$, and choosing $f = \dot{q}$ such that

$$\mathcal{D}(\dot{q})\,\mathrm{d}q = e\,\mathrm{d}q \implies \Phi_R = e - \mathcal{D}(f) = 0$$

Clearly picking $\mathcal{D}(\dot{q}) = R\dot{q}$ gives both Ohms law and friction so that

$$\Phi_R = e - Rf = 0$$

Effort and Flow sources

$$\xrightarrow[f]{e} \mathsf{Se} \qquad\qquad \xrightarrow[f]{e} \mathsf{Sf}$$

To get power in and out of the system, we add control nodes. These impose a control value $u(t)$ on e (for Se) or f (for Sf), and leave the other variable free, under the assumption that there is enough power on tap to maintain the control value. For the effort source Se we have

$$\Phi_{\mathrm{Se}} = e - u.$$

and similarly for the flow source Sf

$$\Phi_{\mathrm{Sf}} = f - u.$$

Table of Constitutive Relations

Node	Constitutive Relation
R	$\Phi_{\mathrm{R}} = e - Rf$
L	$\Phi_{\mathrm{L}} = \int e\,\mathrm{d}t - Lf$
C	$\Phi_{\mathrm{C}} = Ce - \int f\,\mathrm{d}t$
Se	$\Phi_{\mathrm{Se}} = e - u$
Sf	$\Phi_{\mathrm{Sf}} = f - u$
TF	$\Phi_{\mathrm{TF}} = \begin{pmatrix} e_2 - \rho e_1 \\ f_2 + \frac{1}{\rho} f_1 \end{pmatrix}$
GY	$\Phi_{\mathrm{GY}} = \begin{pmatrix} e_2 - \rho f_1 \\ f_2 + \frac{1}{\rho} e_1 \end{pmatrix}$

Node	Constitutive Relation
0	$\Phi_0 = \begin{pmatrix} e_1 - e_2 \\ \dots \\ e_{j-1} - e_j \\ f_1 + f_2 + \dots + f_j \end{pmatrix}$
1	$\Phi_1 = \begin{pmatrix} f_1 - f_2 \\ \dots \\ f_{j-1} - f_j \\ e_1 + e_2 + \dots + e_j \end{pmatrix}$

RLC Example

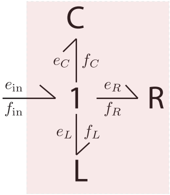


Figure: RLC Bond Graph

$$\Phi_R = e_R - Rf_r \tag{1}$$

$$\Phi_C = Ce_C - \int f_C \, \mathrm{d}t \tag{2}$$

$$\Phi_L = \int e_L \, \mathrm{d}t - Lf_L \tag{3}$$

$$\Phi_1 = \begin{pmatrix} f_{\text{in}} - (-f_R) \\ f_{\text{in}} - (-f_C) \\ f_{\text{in}} - (-f_L) \\ e_{\text{in}} + e_R + e_C + e_L \end{pmatrix} \tag{4}$$

Our goal is to find $\Phi_{\text{RLC}}(e_{\text{in}}, f_{\text{in}}) = 0$; that is the equivalent relation for the entire network.

RLC Example

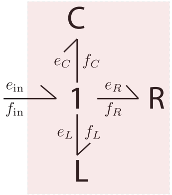


Figure: RLC Bond Graph

$$
\begin{aligned}
\Phi_R &= e_R - Rf_r && (1)\\
\Phi_C &= Ce_C - \int f_C \,\mathrm{d}t && (2)\\
\Phi_L &= \int e_L \,\mathrm{d}t - Lf_L && (3)\\
\Phi_1 &= \begin{pmatrix} f_{\mathrm{in}} - (-f_R) \\ f_{\mathrm{in}} - (-f_C) \\ f_{\mathrm{in}} - (-f_L) \\ e_{\mathrm{in}} + e_R + e_C + e_L \end{pmatrix} && (4)
\end{aligned}
$$

Observing the first three rows of Φ_1, in combination with Φ_R, Φ_C, Φ_L respectively; we have

$$
e_R = -Rf_{\mathrm{in}}, \qquad e_c = -\frac{1}{C}\int f_{\mathrm{in}}\,\mathrm{d}t, \qquad e_L = -\frac{1}{L}\frac{\mathrm{d}f_{\mathrm{in}}}{\mathrm{d}t}
$$

RLC Example

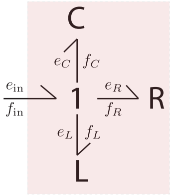


Figure: RLC Bond Graph

$$
\begin{aligned}
\Phi_R &= e_R - Rf_r && (1)\\
\Phi_C &= Ce_C - \int f_C \, \mathrm{d}t && (2)\\
\Phi_L &= \int e_L \, \mathrm{d}t - Lf_L && (3)\\
\Phi_1 &= \begin{pmatrix} f_{\text{in}} - (-f_R) \\ f_{\text{in}} - (-f_C) \\ f_{\text{in}} - (-f_L) \\ e_{\text{in}} + e_R + e_C + e_L \end{pmatrix} && (4)
\end{aligned}
$$

Substituting into the fourth row of Φ_1 gives our result:

$$\Phi_{\text{RLC}} = e_{\text{in}} - \left(\frac{1}{C} \int f_{\text{in}} \, \mathrm{d}t + Rf_{\text{in}} + \frac{1}{L} \frac{\mathrm{d}f_{\text{in}}}{\mathrm{d}t} \right) = 0$$