

Previously...



Junction Structures



Storage and I/O



Table of Constitutive Relations



Bond Graph Clinic: Part 2

Constitutive Relations

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Previously...

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Junction Structures

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Table of Constitutive Relations

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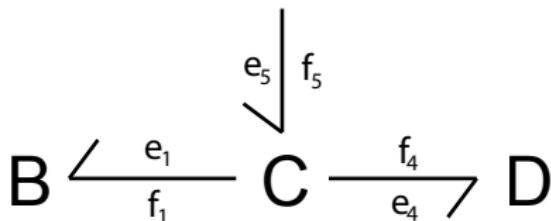
Previously...

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Network models of energetic systems



Last week we showed Bond Graphs capture:

- ▶ Energy transferred between B, C, D without loss via bonds.
- ▶ Power transfer represented by conjugate variables $P_i = e_i f_i$.
- ▶ Subsystem dynamics through constitutive relations;
 $\Phi_B(p, q, e, f) = 0$ for example.

Power Conservation Laws



Suppose this network conserves power, then

$$e_1 f_1 + e_2 f_2 = 0.$$

That is; the power into Φ must sum to zero. Two solutions:
 $e_1 \propto e_2$, or $e_1 \propto f_1$.

Transformer

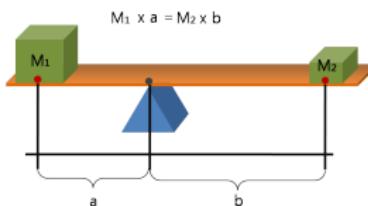
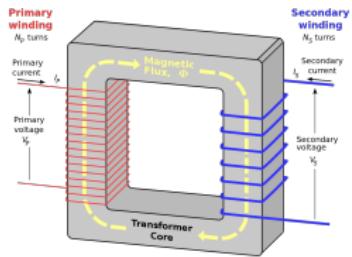


Figure: An electrical transformer and a mechanical transformer (ie, a lever)

Also includes: motors and generators, waterwheels, etc.

Transformer



For transformers we require $e_2 = \rho e_1$, so

$$e_1 f_1 + e_2 f_2 = 0 \implies e_1(f_1 + \rho f_2) = 0 \implies f_2 = -\frac{1}{\rho} f_1$$

Giving a constitutive relation

$$\Phi_{TF} = \begin{pmatrix} e_2 - \rho e_1 \\ f_2 + \frac{1}{\rho} f_1 \end{pmatrix} = 0.$$

Transformer



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Giving a constitutive relation

$$\Phi_{TF} = \begin{pmatrix} -\rho & 0 & 1 & 0 \\ 0 & \rho^{-1} & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \\ e_2 \\ f_2 \end{pmatrix} = 0.$$

Gyrator

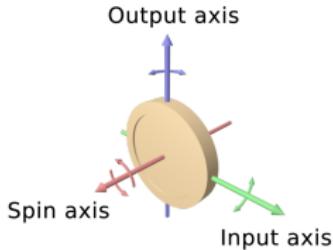


Figure: A gyroscope

Electrical gyrators were proposed by Telligen in 1948, but a passive implementation hasn't yet been found.

Gyrator



For gyrators we require $e_2 = \rho f_1$, so

$$e_1 f_1 + e_2 f_2 = 0 \implies f_1(e_1 - \rho f_2) = 0 \implies f_2 = -\frac{1}{\rho} e_1.$$

Giving a constitutive relation

$$\Phi_{GY} = \begin{pmatrix} e_2 - \rho f_1 \\ f_2 + \frac{1}{\rho} e_1 \end{pmatrix} = 0.$$

Gyrator



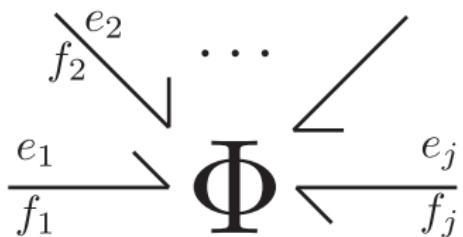
For gyrators we require $e_2 = \rho f_1$, so

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Giving a constitutive relation

$$\Phi_{GY} = \begin{pmatrix} 0 & -\rho & 1 & 0 \\ \rho^{-1} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \\ e_2 \\ f_2 \end{pmatrix} = 0.$$

Network Conservation Laws

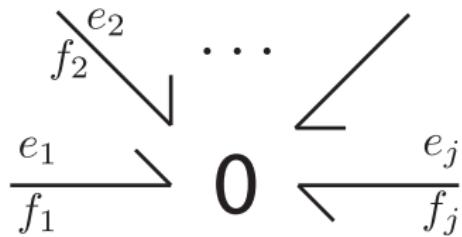


We require nodes to capture the distribution of power across many subsystems. These junction nodes capture network conservation laws which must satisfy

$$0 = \sum_{i=1}^j e_i f_i.$$

The two base cases are *common effort*, and *common flow*.

0-Junction

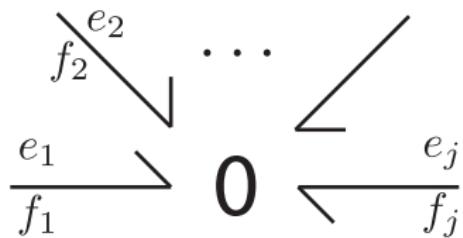


For common effort junctions $e_i = e_k$, $\forall 1 \leq i, k \leq j$. Hence

$$\sum_{i=1}^j e_i f_i = 0 \implies \sum_{i=1}^j f_i = 0$$

This is Kirchoff's Current Law

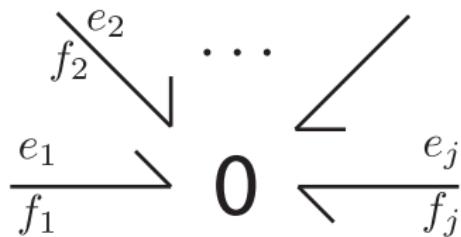
0-Junction



The constitutive relation is given by:

$$\Phi_0 = \begin{pmatrix} e_1 - e_2 \\ \dots \\ e_{j-1} - e_j \\ f_1 + f_2 + \dots + f_j \end{pmatrix} = 0$$

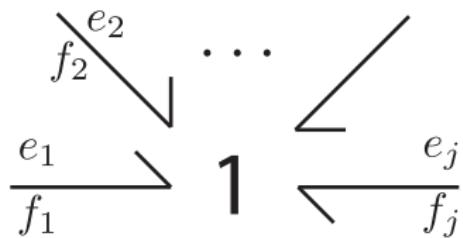
0-Junction



The constitutive relation is given by:

$$\Phi_0 = \begin{pmatrix} 1 & 0 & -1 & 0 & \dots & 0 \\ \ddots & & \ddots & & & \\ \dots & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & \dots & & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \\ \vdots \\ e_j \\ f_j \end{pmatrix} = 0$$

1-Junction

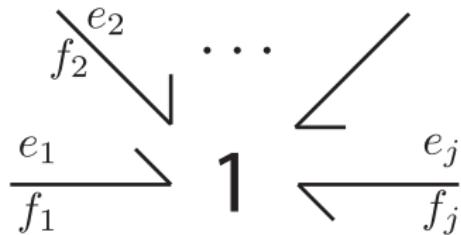


Similarly for common flow junctions $f_i = f_k$, $\forall 1 \leq i, k \leq j$. Hence

$$\sum_{i=1}^j e_i f_i = 0 \implies \sum_{i=1}^j e_i = 0$$

This is Kirchhoff's Voltage Law

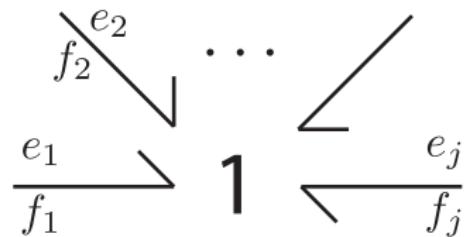
1-Junction



The constitutive relation is given by:

$$\Phi_1 = \begin{pmatrix} f_1 - f_2 \\ \dots \\ f_{j-1} - f_j \\ e_1 + e_2 + \dots + e_j \end{pmatrix} = 0$$

1-Junction



The constitutive relation is given by:

$$\Phi_1 = \begin{pmatrix} 0 & 1 & 0 & -1 & \dots & 0 \\ & \ddots & & \ddots & & \\ \dots & & 0 & 1 & 0 & -1 \\ 1 & 0 & \dots & & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ f_1 \\ \vdots \\ e_j \\ f_j \end{pmatrix} = 0$$

Physics review

Energy stored is a function of local storage co-ordinates q, p

$$H(q, p) = E_0 + \int P_{\text{in}} dt.$$

The canonical co-ordinates q, p are taken such that $q \in X$ where X is some manifold, and $p \in X^*$ where X^* is the dual space of X .

In the case where $X = \mathbb{R}^n$, the dual space X^* is the set of column vectors acting on X . That is if $q, x \in \mathbb{R}^n$ then there exists a dual vector (a linear functional) $p \in X^*$ such that $p(q) = x^T \cdot q$.

As an aside, in physics p is often written as $\langle p |$, so that in the above example $\langle p | q \rangle = x^T q$.

Physics review (cont.)

If $P_{\text{in}} = ef$ then

$$H(q, p) = E_0 + \int P_{\text{in}} dt \implies \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp = ef dt.$$

We want to:

- ▶ associate e with the tangent space of X^* , ideally by saying something like $e = \frac{dp}{dt}$ so that $edt = dp$.
- ▶ associate f with the tangent space of X , ideally by saying that $f = \frac{dq}{dt}$ so that $f dt = dq$.

We can do this easily if we consider individual components whose stored energy depends only on generalised position or momentum, but not both.

Kinetic Storage

$$\frac{e}{f} \searrow L$$

For kinetic energy, the stored energy depends only on momentum,
 $\frac{\partial H}{\partial q} = 0$, so we can define $edt = dp$.

$$\frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp = ef dt \implies \left(\frac{\partial H}{\partial p} - f \right) dp = 0$$

Hence we have a constitutive relation

$$\Phi_L = \frac{\partial H}{\partial p} - f = 0, \quad \text{where} \quad \dot{p} = e.$$

Kinetic Storage (cont.)

$$\frac{e}{f} \searrow L$$

In the case of linear kinetic storage; we have a Hamiltonian

$$H(q, p) = \frac{1}{2L} p^2,$$

which is the energy stored in a L -henry inductor; in the linear motion of an L -kg object.

This gives rise to the familiar constitutive relation:

$$\Phi_L = \frac{\partial H}{\partial p} - f = \frac{1}{L} p - f \quad \Rightarrow \quad Lf - \int e dt = 0.$$

Kinetic Storage (cont.)

$$\frac{e}{f} \searrow L$$

This gives rise to the familiar constitutive relation:

$$\Phi_L = \frac{1}{L} p - f$$

Which, if we define the integration operator $\mathcal{I}x = \int_0^t x(t) dt$, then we have

$$\Phi_L = (\mathcal{I}, -L) \begin{pmatrix} e \\ f \end{pmatrix} \quad \xrightarrow{\mathcal{L}} \quad \hat{\Phi}_L(s) = (s^{-1}, -L) \begin{pmatrix} \hat{e} \\ \hat{f} \end{pmatrix}$$

Potential Storage

$$\frac{e}{f} \searrow C$$

Similarly for potential, the stored energy depends only on position,
 $\frac{\partial H}{\partial q} = 0$, so we define $f dt = dq$.

$$\frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp = ef dt \implies \left(\frac{\partial H}{\partial q} - e \right) dq = 0$$

Hence we have a constitutive relation:

$$\Phi_C = \frac{\partial H}{\partial q} - e = 0, \quad \text{where} \quad \dot{q} = f.$$

Potential Storage (cont.)

$$\frac{e}{f} \searrow C$$

Storage of potential energy has more variety. In the linear case (represented by the C node above) we have

$$H(q, p) = \frac{1}{2C}q^2$$

which is the power stored in a C -farad capacitor, the elastic energy stored in a spring displace by q from equilibrium, etc.

$$\Phi_C = \frac{\partial H}{\partial q} - e = \frac{1}{C}q - e \quad \Rightarrow \quad Ce - \int f dt = 0$$

Potential Storage (cont.)

$$\frac{e}{f} \searrow C$$

Hence we have a constitutive relation:

$$\Phi_C = \frac{1}{C}q - e$$

Which, if we define the differentiation operator $Dx = \frac{d}{dt}x(t)$, then we have

$$\Phi_C = (C, -D) \begin{pmatrix} e \\ f \end{pmatrix} \quad \xrightarrow{c} \quad \hat{\Phi}_C(s) = (C, -D) \begin{pmatrix} \hat{e} \\ \hat{f} \end{pmatrix}$$

up to a constant.

Linear Dissipation

$$\frac{e}{f} \searrow R$$

Dissipation can be captured by introducing a Dirac structure $\mathcal{D}(\dot{q})$, and choosing $f = \dot{q}$ such that

$$\mathcal{D}(\dot{q})dq = edq \implies \Phi_R = e - \mathcal{D}(f) = 0$$

Clearly picking $\mathcal{D}(\dot{q}) = R\dot{q}$ gives both Ohms law and friction so that

$$\Phi_R = e - Rf = 0$$

Effort and Flow sources

$$\frac{e}{f} \searrow \text{Se} \qquad \frac{e}{f} \searrow \text{Sf}$$

To get power in and out of the system, we add control nodes.

These impose a control value $u(t)$ on e (for Se) or f (for Sf), and leave the other variable free, under the assumption that there is enough power on tap to maintain the control value. For the effort source Se we have

$$\Phi_{\text{Se}} = e - u.$$

and similarly for the flow source Sf

$$\Phi_{\text{Sf}} = f - u.$$

Table of Constitutive Relations

Node	Constitutive Relation
R	$\Phi_R = e - Rf$
L	$\Phi_L = \int e dt - Lf$
C	$\Phi_C = Ce - \int f dt$
Se	$\Phi_{Se} = e - u$
Sf	$\Phi_{Sf} = f - u$
TF	$\Phi_{TF} = \begin{pmatrix} e_2 - \rho e_1 \\ f_2 + \frac{1}{\rho} f_1 \end{pmatrix}$
GY	$\Phi_{GY} = \begin{pmatrix} e_2 - \rho f_1 \\ f_2 + \frac{1}{\rho} e_1 \end{pmatrix}$

Node	Constitutive Relation
0	$\Phi_0 = \begin{pmatrix} f_1 - f_2 \\ \dots \\ f_{j-1} - f_j \\ e_1 + e_2 + \dots + e_j \end{pmatrix}$
1	$\Phi_1 = \begin{pmatrix} f_1 - f_2 \\ \dots \\ f_{j-1} - f_j \\ e_1 + e_2 + \dots + e_j \end{pmatrix}$