Synchronisation of Nonlinearly Coupled Oscillators

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Acknowledgements

Chaos

Phase and amplitude dynamics of nonlinearly coupled oscillators

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Coupled Oscillators in Nature

Oscillatory systems are common in nature and the physical sciences. For example:

- Fireflies in South East Asia,
- Circadian Rhythms,
- Neural Oscillations,
- Semiconductor Physics (Josephson Junctions),
- *Quantum Nanotechnology*.

When many individual oscillators are coupled, allowing each oscillator to influence some (or all) others, surprising emergent phenomenon can occur.

In particular such systems can exhibit *synchronisation*!

Synchronisation: a universal phenomenon.

Synchronisation occurs when mutual interactions cause a portion of the individual oscillators to move in unison.

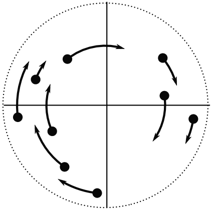


Figure: Unsynchronised motion: oscillators rotate at different angular velocities.

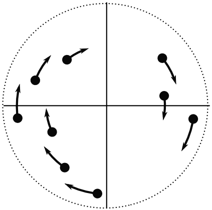


Figure: Synchronised motion: all oscillators rotate at the same angular velocity.

Nano-electromechanics: A future technology.

Nano-electromechanical systems(NEMS) have potential applications in the measurement of weak forces (gravitational waves, single atom charge/spin), quantum computing (switches, memory) and exploration below the quantum limit.

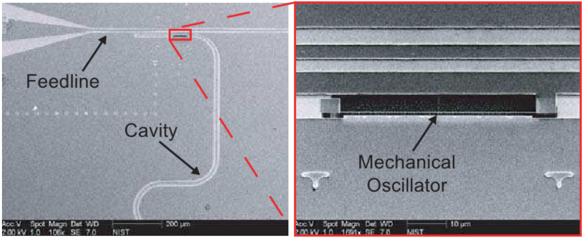


Figure: A Nano-electromechanical system.[1]

[1] Teufel, Regal, and Lehnert, 2008, *New Journal of Physics*.

NEMS as a coupled oscillator system.

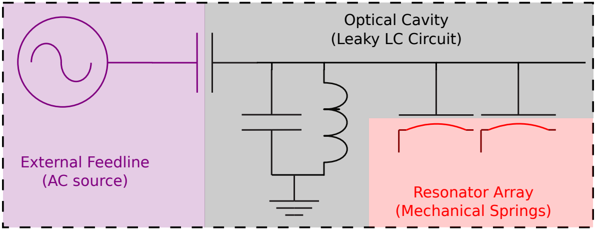


Figure: Semiclassical model of a NEMS.

- Mean spring displacement modulates the circuits resonant frequency.
- Photon-phonon interaction forces springs.
- *Nonlinear all-to-all coupling between mechanical oscillators*.

Oscillators in $\mathbb{C}$.

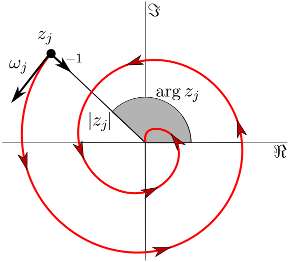
Consider a set (or population) of n damped harmonic oscillators $\{z_j\}$;

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (-1 + \mathrm{i}\omega_j)z_j. \qquad j = 1, \ldots, n$$

Each oscillator z_j has:

- Phase $\arg z_j$,
- amplitude $|z_j|$ and
- natural frequency ω_j.

Each ω_j is a real valued I.I.D random variable with density $g(\omega)$.



$$z_j(t) = z_j(0)\exp[(-1 + \mathrm{i}\omega_j)t]$$

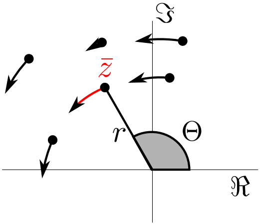
Population mean as a measure of coherence.

We can measure the state of the population by observing the population mean, or *order parameter*[2]:

$$z = \frac{1}{n}\sum_{j=1}^{n} z_j$$

We call:

- z the *mean field*.
- $r = |z|$ the mean field amplitude.
- $\Theta = \arg z$ the mean phase.
- $\Omega = \frac{\mathrm{d}\Theta}{\mathrm{d}t}$ the mean field velocity.



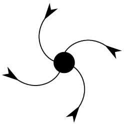
[2]Kuramoto, 1975, *International Symposium on Mathematical Problems in theoretical physics*.

A more general model: The Hopf Bifurcation

The normal form of a Hopf Bifurcation at $\alpha = 0$ is given by

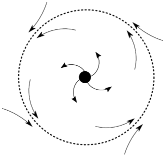
$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta\,|z_j|^2)z + \mathrm{i}\omega_j z_j, \qquad \beta > 0.$$

$\alpha < 0$



When $\alpha \leq 0$ the fixed point at $z_j = 0$ is stable.

$\alpha > 0$



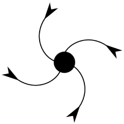
For $\alpha > 0$, the $z_j = 0$ state is unstable and a stable limit cycle exists with $|z_j| = \sqrt{\alpha/\beta}$

Coupled Oscillator Systems on either side of a Hopf

Linear Oscillator Model:

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha + \mathrm{i}\omega_j)z_j + \Gamma(z)$$

$$\alpha < 0,\ j = 1 \ldots n.$$

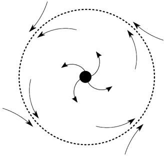


Decoupled system has a stable node and no stable limit cycles.

Limit Cycle Model:

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + \Gamma(z)$$

$$\alpha, \beta > 0, j = 1 \ldots n$$



Decoupled system has unstable node and a stable limit cycle.

Coupled Oscillator Systems on either side of a Hopf

Limit Cycle Model:

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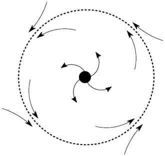


Decoupled system has unstable node and a stable limit cycle.

The Kuramoto Model.

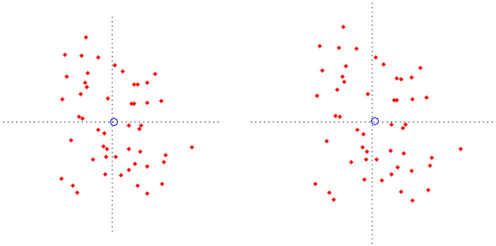
$$\frac{\mathrm{d}\theta_j}{\mathrm{d}t} = \omega_j + \frac{\gamma}{n}\sum_{k=1}^{N}\sin(\theta_k - \theta_j)$$

Literature includes work by

- Y. Kuramoto,
- S. Strogatz & R. Mirollo,
- M. Cross,
- A. Pikovsky & M. Rosenblum,
- G. Ermentrout and
- D. Aronson & more..

Synchronisation of Limit Cycle Oscillators

A common theme is that synchronisation occurs due to a *play-off between frequency spread and coupling strength*.



Linear oscillators with nonlinearly coupling.

We are concerned with nonlinear all-to-all coupling via the mean field

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = -(1 - \mathrm{i}\omega_j)z_j + zF(|z|), \qquad z = \frac{1}{n}\sum_{j=1}^{n} z_j$$

- ω_j are I.I.D. with a p.d.f given by $g(\omega)$.
- Coupling is linear in mean field phase;
- and nonlinear in mean field amplitude.
- Nonlinear coupling function $F : C^1[0,\infty) \to \mathbb{C}$.
- *Nonlinearly coupled oscillators are not well understood.*

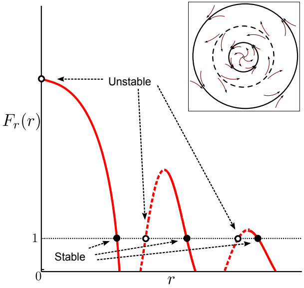
Identical oscillators follow mean field

It has been shown[3] that if $\omega_j = \omega$ then in the steady state $z_j = z$ and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -(1 - \mathrm{i}\omega)z + zF(|z|).$$

In polar form:

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= -r[1 - \operatorname{Re} F(r)], \\ \Omega &= \omega + \operatorname{Im} F(r). \end{aligned}$$



- Ω constant implies $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$.
- Fixed points at $r = 0$ & $\operatorname{Re} F(r) = 1$.
- *Phase and amplitude decouple!*

[3]Holmes, Meaney, and Milburn, 2012, *Phys. Rev. E.*

Non-identical case: does synchronisation occur?

Key questions for the non-identical case:

- Can nonlinear coupling destabilise the origin?
- Does this system have stable synchronised solutions?
- Can this system have *multiple* stable synchronised states?

The answer: yes![4] (but it would take more than ten minutes to explain).

[4] PC and Holmes, 2015, *Chaos*.

'Shape' of F and g are bifurcations parameters.

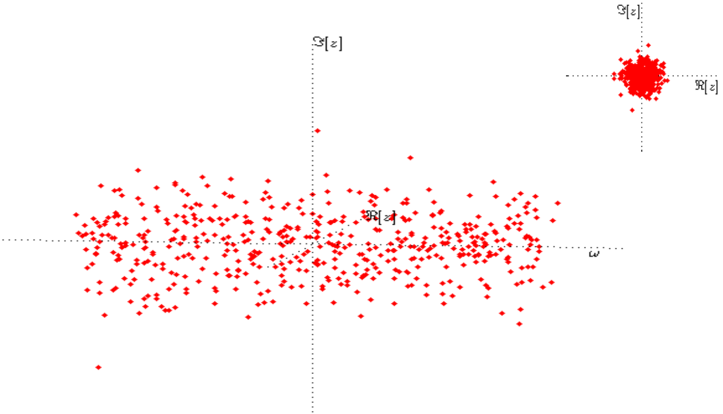
We can, with generality, identify existence, stability and bifurcations of synchronised solutions in the non-identical case.

A bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behaviour.

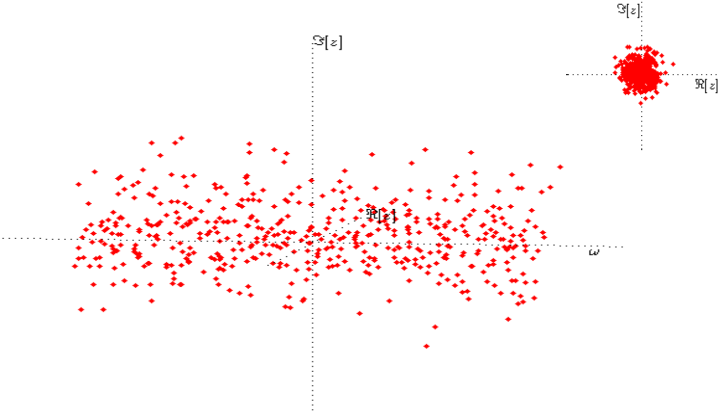
Bifurcation parameters for this system:

- The 'shape' of coupling function $F(r)$ (e.g.; coefficients of nonlinearities).
- The 'shape' of the distribution $g(\omega)$ (e.g.; spread, number of modes).

'Unsynchronised' Motion



Synchronisation



Density on real line to density on a circle

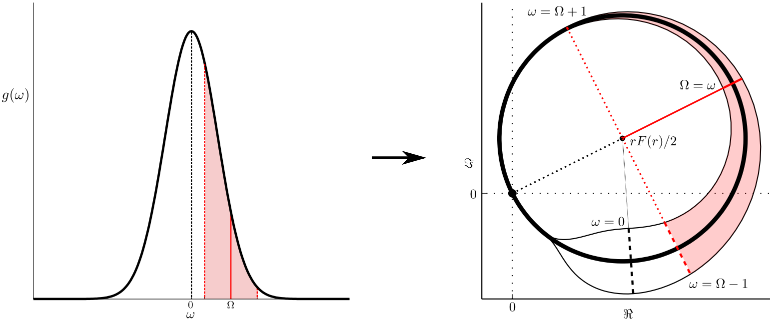
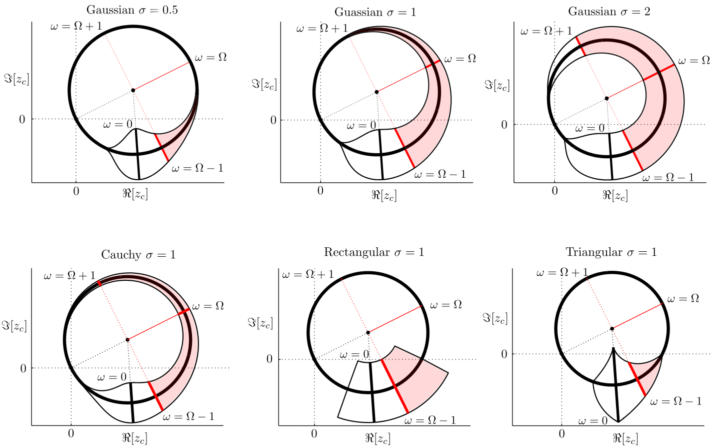


Figure: In the fixed state, the density g is mapped to a density on the circle represented by the shaded area. Oscillators with $\omega \in (\Omega - 1, \Omega + 1)$ are mapped to the shaded region furthest from the origin.

Some different distributions



Conclusions

Synchronised states exist systems consisting of linear oscillators with nonlinear coupling.
These system are different to much of those in coupled oscillator literature as:

- The amplitude dynamics are important.
- The individual oscillators are linear.
- The coupling is nonlinear.

In particular, synchronise solutions exist on a regular pattern, and this pattern is related to the distribution of oscillator frequencies.

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