Synchronisation of Nonlinearly Coupled Oscillators

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Acknowledgements

Chaos

Phase and amplitude dynamics of nonlinearly coupled oscillators

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Origins

SynchronousPikovsky, Rosenblum, and Kurths, 2001

From: Greek χρόνος (*chronos*, meaning time) and σύν (syn, meaning *same*)

Translated: 'Sharing the common time' or 'sharing the same time'

Example: FirefliesYiu, 2017, *Inside Science*

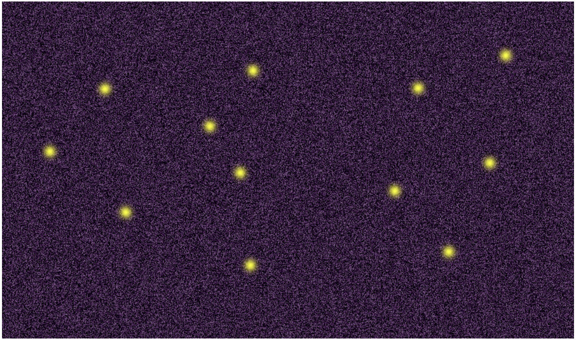


Figure: *Photonius carolinus* in Elkmont, Tennessee

A Definition

Synchronisation is the process by which weakly interacting oscillatory systems adjust their behaviour to form a collective rhythm.

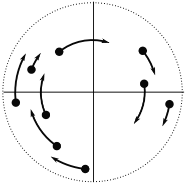


Figure: Unsynchronised motion: oscillators rotate at different angular velocities.

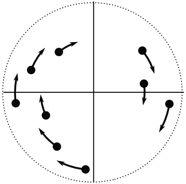


Figure: Synchronised motion: all oscillators rotate at the same angular velocity.

Nano-electromechanics: A future technology.

Nano-electromechanical systems(NEMS) have potential applications in the measurement of weak forces (gravitational waves, single atom charge/spin), quantum computing (switches, memory) and exploration below the quantum limit.

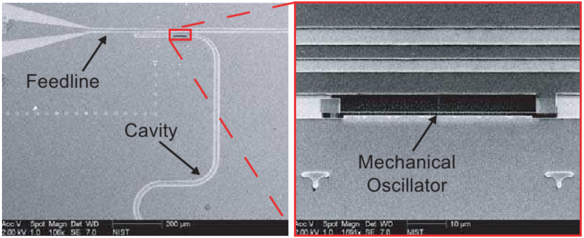


Figure: A Nano-electromechanical system.[1]

[1] Teufel, Regal, and Lehnert, 2008, *New Journal of Physics*.

NEMS as a coupled oscillator system.

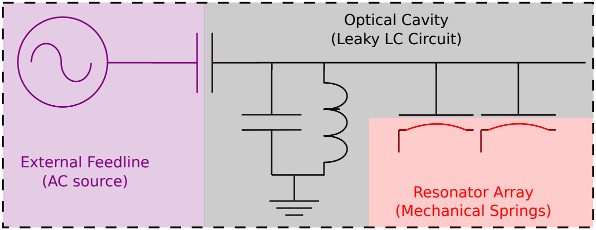


Figure: Semiclassical model of a NEMS.

- Mean spring displacement modulates the circuits resonant frequency.
- Photon-phonon interaction forces springs.
- *Nonlinear all-to-all coupling between mechanical oscillators.*

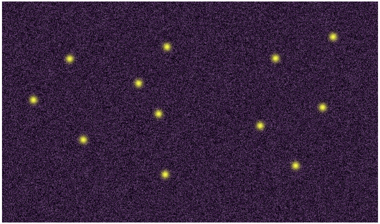
Coupled Oscillators in Nature

Oscillatory systems are common in nature and the physical sciences. For example:

- Fireflies in South East Asia,
- Circadian Rhythms,
- Neural Oscillations,
- Semiconductor Physics (Josephson Junctions),
- *Quantum Nanotechnology*.

When many individual oscillators are coupled, allowing each oscillator to influence some (or all) others, surprising emergent phenomenon can occur.
In particular such systems can exhibit *synchronisation*!
We wish to understand and control this behaviour!

Phenomenological Model of Fireflies



With each firefly we associate

- an index $j \in 1, \ldots, n$,
- a period between events T_j,
- a frequency $\omega_j = 2\pi / T_j$,
- and a phase $\theta_j \in [0, 2\pi)$ such that $\theta_j(t) = \theta_j(t + T_j)$.

Some comments:

- The natural frequency ω_j is rarely measurable.
- Functions of phase (waveforms) are often measurable.
- Noise (both physical and 'model' noise) is usually averaged out.
- Sometimes amplitudes need also be modelled.

Periodic motion maps really nicely into $\mathbb{C}$!

Example: Damped harmonic motion in $\mathbb{C}$.

Consider a set (or population) of n damped harmonic oscillators $\{z_j\}$;

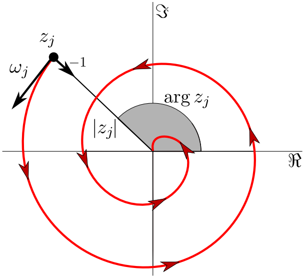
$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (-1 + \mathrm{i}\omega_j)z_j. \qquad j = 1, \ldots, n$$

Each oscillator z_j has:

- Phase $\arg z_j$,
- amplitude $|z_j|$ and
- natural frequency ω_j.

Each ω_j is a real valued I.I.D random variable with density $g(\omega)$.

$$z_j(t) = z_j(0)\exp[(-1 + \mathrm{i}\omega_j)t]$$



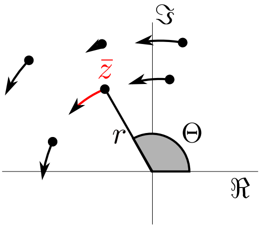
Population mean as a measure of coherence.

We can measure the state of the population by observing the population mean, or *order parameter* Kuramoto, 1975, *International Symposium on Mathematical Problems in theoretical physics*:

$$z = \frac{1}{n}\sum_{j=1}^{n} z_j$$

We call:

- z the *mean field*.
- $r = |z|$ the mean field amplitude.
- $\Theta = \arg z$ the mean phase.
- $\Omega = \frac{\mathrm{d}\Theta}{\mathrm{d}t}$ the mean field velocity.



Synchronised Vs Unsynchronised Motion

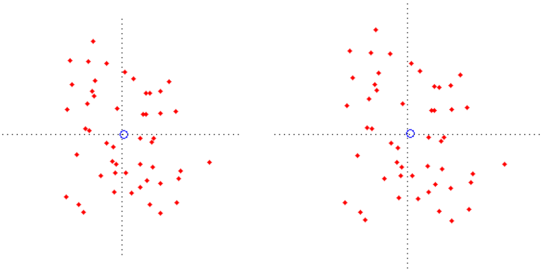


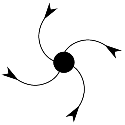
Figure: 25 Oscillators (in red) and the mean field (in blue).

A more general model: The Hopf Bifurcation

The normal form of a Hopf Bifurcation at $\alpha = 0$ is given by

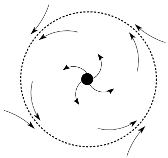
$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j, \qquad \beta > 0.$$

$\alpha < 0$



When $\alpha \leq 0$ the fixed point at $z_j = 0$ is stable.

$\alpha > 0$



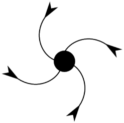
For $\alpha > 0$, the $z_j = 0$ state is unstable and a stable limit cycle exists with $|z_j| = \sqrt{\alpha/\beta}$

Coupled Oscillator Systems on either side of a Hopf

Linear Oscillator Model:

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha + \mathrm{i}\omega_j)z_j + \Gamma_j(\mathbf{z})$$

$$\alpha < 0, \; j = 1 \dots n.$$

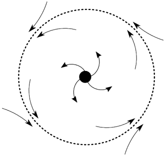


Decoupled system has a stable node and no stable limit cycles.

Limit Cycle Model:

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + \Gamma_j(\mathbf{z})$$

$$\alpha, \beta > 0, j = 1 \dots n$$



Decoupled system has unstable node and a stable limit cycle.

The coupling function Γ

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + \Gamma_j(\mathbf{z}), \qquad j = 1, \ldots, n.$$

Network models usually assume linear coupling

$$\Gamma_j(\mathbf{z}) = \sum_k w_{jk} z_k.$$

Some common choices include:

- All-to-all coupling: $w_{jk} = K/n$
- Nearest-neighbour: $w_{jk} = K/n$ iff $k = j \pm 1$
- Small world networks,
- Random networks.

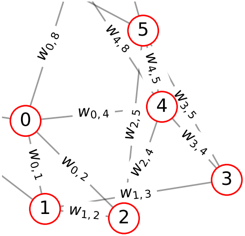


Figure: Part of a small world network

The coupling function Γ

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + \Gamma_j(\mathbf{z}), \qquad j = 1, \dots, n.$$

Nonlinear Γ must commute with $\mathrm{e}^{\mathrm{i}\zeta}$. (why?)

We restrict ourselves to linear and nonlinear all-to-all coupling of the form

$$\Gamma(\mathbf{z}) = zF(|z|) \qquad \text{where} \qquad z = \frac{1}{n}\sum_{k=1}^{n} z_k.$$

Assume $F : \mathbb{R}_+ \to \mathbb{C}$ is smooth, bounded and $F(|z|) \to 0$ as $|z| \to \infty$.

For example: $\frac{1}{1+|z|^2}$, $\mathrm{e}^{-|z|}$ or Bessel functions $J_m(|z|)$ of the first kind.

Coupled Limit Cycle Oscillators

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \ldots, n.$$

with

$$z = \frac{1}{n}\sum_{k=1}^{n} z_k.$$

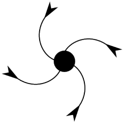
To recap:

- Oscillators are modelled as points z_j rotating in $\mathbb{C}$,
- α, β are fixed parameters,
- ω_j is sampled from an *even symmetric* distribution $g(\omega)$,
- F is smooth, bounded, and $F(|z|) \to 0$ as $|z| \to \infty$,
- The mean field z is often a useful measure of coherence.

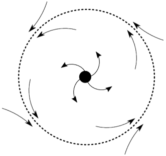
Limit Cycle Dynamics

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \ldots, n.$$

$\alpha < 0$



$\alpha > 0$



Deriving the Kuramoto Model

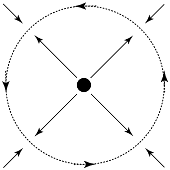
$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \ldots, n.$$

Kuramoto Kuramoto, 1975, *International Symposium on Mathematical Problems in theoretical physics* considered the limit $\alpha, \beta \to \infty$ with $\alpha/\beta \to 1$, and $F(r) = K$ is real-valued and constant. Let $z_j = r_j \exp(\mathrm{i}\theta_j)$. It follows that

$$\frac{1}{z_j}\frac{\mathrm{d}z_j}{\mathrm{d}t} = \frac{\mathrm{d}\log z_j}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\ln r_j + \mathrm{i}\theta_j) = \alpha - \beta r_j^2 + \mathrm{i}\omega_j + K\frac{r}{r_j}\mathrm{e}^{\mathrm{i}(\psi - \theta_j)}$$

we thus have

$$\begin{aligned} \frac{1}{\beta}\frac{\mathrm{d}r_j}{\mathrm{d}t} &= r_j(\frac{\alpha}{\beta} - r_j^2) + \frac{K}{\beta}\frac{r}{r_j}\cos(\psi - \theta_j) \\ \frac{\mathrm{d}\theta_j}{\mathrm{d}t} &= \omega_j + K\frac{r}{r_j}\sin(\psi - \theta_j) \end{aligned}$$



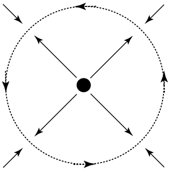
Deriving the Kuramoto Model

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \dots, n.$$

Kuramoto Kuramoto, 1975, *International Symposium on Mathematical Problems in theoretical physics* considered the limit $\alpha, \beta \to \infty$ with $\alpha/\beta \to 1$, and $F(r) = K$ is real-valued and constant.

The system becomes a 'phase oscillator'

$$\frac{\mathrm{d}\theta_j}{\mathrm{d}t} = \omega_j + Kr\sin(\psi - \theta_j)$$



- The effective coupling strength is Kr!
- There is a Hopf at $K = 2/[\pi g(0)]$.
- When $K > K_c$, $r \to r_\infty$.

Amplitude Dynamics

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \dots, n.$$

Matthews, Mirollo and StrogatzMatthews, Mirollo, and Strogatz, 1991, *Physica D* showed that in the $\alpha, \beta \approx O(1)$ regime there is a wide variety of dynamical states even for uni-modal g. In addition to synchronised states and incoherence, they found:

- amplitude death,
- quasi-periodic states,
- multi-stability,
- period doubling cascades and
- chaos

Amplitude Death

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \ldots, n.$$

Amplitude death occurs when increasing the spread of frequencies causes the origin to become an attracting.

Ermentrout**Ermentrout90** showed that amplitude death

- occurs in a wide variety of systems,
- does not depend special symmetries
- or infinite-range coupling

Amplitude Death

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \ldots, n.$$

Mirollo and Strogatz **Mirollo90** defined $f(\mu) = \int_{-\infty}^{\infty} \frac{g(\omega)}{\mu - \mathrm{i}\omega} \mathrm{d}\omega$

Theorem (**Mirollo90**)

Let $\alpha = 1 - K$, $F(|z|) = K$ and assume that the density $g(\omega)$ is an even function which is nonincreasing on $[0, \infty)$. Then:

- A *Amplitude death is stable with probability $\to 1$ as $n \to \infty$, $f(K-1) < \frac{1}{K}$*
- B *Amplitude death is unstable with probability $\to 1$ as $n \to \infty$, $f(K-1) > \frac{1}{K}$*

Demonstration of amplitude death.

...some more examples.

The Dispersion Function

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \ldots, n.$$

Matthews, Mirollo and Strogatz Matthews, Mirollo, and Strogatz, 1991, *Physica D* considered the dispersion function

$$f(\mu) = \int_{-\infty}^{\infty} \frac{g(\omega)}{\mu - \mathrm{i}\omega}\,\mathrm{d}\omega.$$

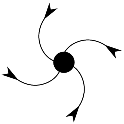
for real valued μ.
When g is symmetric and unimodal

- $f(\mu)$ is real valued iff μ real valued.
- f is odd and strictly decreasing on $[0, \infty)$
- f is discontinuous at $\mu = 0$ and $\lim_{\mu \to 0^+} = \pi g(0)$.

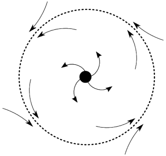
Stable Node Dynamics

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \ldots, n.$$

$\alpha < 0$



$\alpha > 0$



Amplitude Death

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \ldots, n.$$

Consider the case where $\alpha = -1, \beta = 0$, F smooth, bounded and $F \to 0$ as $|z| \to \infty$.

It follows from a fixed point argument that amplitude death and full synchronisation are the only limiting sets.

In particular, z_j is fixed iff z is fixed up to a constant $\exp(\mathrm{i}\Omega t + \Theta_0)$

Amplitude Death

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \ldots, n.$$

Consider the case where $\alpha = -1, \beta = 0$, F smooth, bounded and $F \to 0$ as $|z| \to \infty$.

Theorem (PC and Holmes)

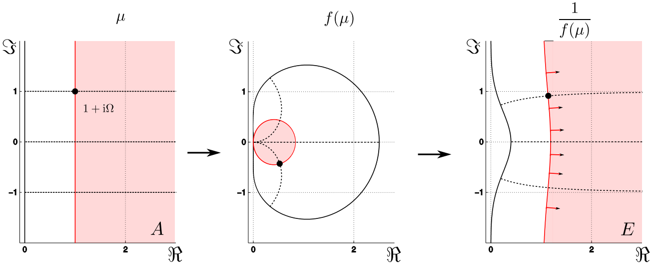
Let g be a even symmetric probability distribution, non-increasing on $[0, \infty)$, let $A = \{\lambda : \mathrm{Re}[\lambda] > 1\} \subset \mathbb{C}$ and $E = 1/f(A) \subset \mathbb{C}$. Then amplitude death is unstable with probability $\to 1$ as $n \to \infty$ if and only if $F(0) \in E$.

Proof of Amplitude Death

To prove the stability result we require:

Theorem (PC and Holmes)

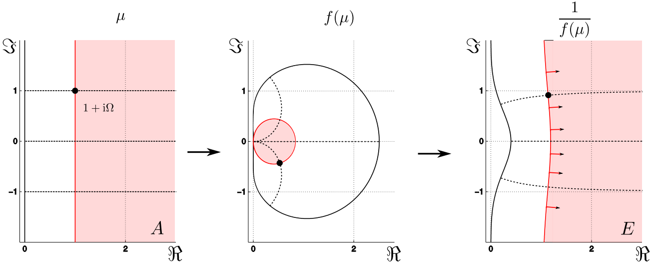
Let $g(\omega)$ be an even symmetric probability distribution, non-increasing on $[0, \infty)$. Then the dispersion function $f : \mathbb{C}_+ \to Y \subset \mathbb{C}_+$ is bijective and holomorphic.



Proof of Amplitude Death

Sketch of the proof:

- Linearise the system about $z_j = 0$..
- Show that the limiting operator has has continuous spectrum along $\mathrm{Re}[\lambda] = -1$, and a discrete spectrum at $F(0) = \frac{1}{f(\lambda+1)}$.
- Use the conformal property of f to show $F(0) \in E$ implies $\lambda + 1 \in A$ and hence fixed point is unstable.



Synchronised States

$$\frac{\mathrm{d}z_j}{\mathrm{d}t} = (\alpha - \beta|z_j|^2)z_j + \mathrm{i}\omega_j z_j + zF(|z|), \qquad j = 1, \dots, n.$$

Consider the case where $\alpha = -1, \beta = 0$, F smooth, bounded and $F \to 0$ as $|z| \to \infty$.

There are also results on stability and bifurcationsPC and Holmes, 2015, *Chaos* which are expressed in terms of a playoff between

- Population heterogeneity (via $f(\mu) = \int_{-\infty}^{\infty} \frac{g(\omega)}{\mu - \mathrm{i}\omega}\,\mathrm{d}\omega$ and it's derivatives) and
- coupling topology (via the 'shape' of F)

In Summary

Hopefully i've convinced you that:

- Synchronisation is an example of emergence,
- resulting from mutual interactions between parts of a larger system.
- The different coupling mechanisms and population variability can lead to differing synchronisation properties,
- which can be simulated and, in some cases, solved explicitly.
- There are also deep mathematical properties buried not far from the surface.

References I

Kuramoto, Y. (1975). "Self-Entrainment of a population of coupled non-linear oscillators". In: *International Symposium on Mathematical Problems in theoretical physics* Lecture Notes in Physics, pp. 420–422.

Matthews, P.C., R.E. Mirollo, and S.H. Strogatz (1991). "Dynamics of a large system of coupled nonlinear oscillators". In: *Physica D* 52, pp. 293–331.

PC and C. A. Holmes (2015). "Phase and amplitude dynamics of nonlinearly coupled oscillators". In: *Chaos* 25.2, 023110. DOI: http://dx.doi.org/10.1063/1.4908604. URL: http://scitation.aip.org/content/aip/journal/chaos/25/2/10.1063/1.4908604.

Pikovsky, A., M. Rosenblum, and J. Kurths (2001). *Synchronization: A universal concept in nonlinear sciences*. Vol. 12. Cambridge Nonlinear Sciences Series. Cambridge, U.K: Cambridge Universty Press.

References II

Teufel, J.D., C.A. Regal, and K.W. Lehnert (2008). "Prospects for cooling nanomechanical motion by coupling to a superconducting microwave resonator". In: *New Journal of Physics* 10.095002.

Yiu, Yuen (June 2017). "How to Synchronize Like Fireflies". In: *Inside Science*. URL: https://www.insidescience.org/news/how-synchronize-fireflies.