Tribe Simulation

Peter Fortuna, Bryce Hepner, Carter Landon, Caelan Osman, Tanner Wallace

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Abstract

There are several different models that examine population dynamics using differential equations, including predator-prey models and SIR models. We modify these models to examine population dynamics and interactions between populations of different tribes. Cases involving two tribes, three tribes, and the introduction of a new tribe are examined. Solution existence of the model is proven, and equilibrium conditions are examined in each case.

1 Background/Movitation

Throughout human history, individuals have found themselves in tribes. Naturally, the population size of each tribe is dynamic and is often affected by the tribes around it. We are interested in modeling this phenomenon.

1.1 The General Phenomenon

In order to understand tribal growth rates, we need to think about what factors affect the population count of any given tribe at a particular time. Tribal populations are heavily impacted by:

- Birth rates
- Death rates
- Entrance rates
- Exit rates

While any model by necessity makes simplifying assumptions, these factors create a theoretical framework to understand important population dynamics.

1.2 A (Brief) Literature Review

While this is a relatively novel modeling endeavor, there are a few studies that have considered the phenomenon. A 2011 study entitled "A Simple Differential Equation System for the Description of Competition among Religions", focuses on the competition between three groups. This study uses a modified predator/prey model to examine different growth dynamics, finding that the predator/prey model captures the phenomenon well. This paper also notes that for a system of three groups, there often exists an equilibrium, although the birth, death, and transition rates necessary to achieve that equilibrium can become unrealistic depending on initial conditions [3]. Another study supported these conclusions, while noting that one driving force behind growth (or decline) in a tribe's populations was through contact between groups[1].

There is already a decent foundation of work done regarding differential modeling of tribal population numbers. We intend to take these findings and use them, while also modifying our approach slightly to focus more on interactions between tribes and less on overall population growth like many numerical studies have.

2 Modeling

To begin our mathematical study of the subject at hand, we will start in a general setting, considering arbitrary tribes A, and B.

2.1 The Base Case: Tribes A and B

To understand the motivation behind our governing equations, let's consider the base case where there are only two tribes to think about, A and B. Let's first consider the population change in A for any given time period t.

With $P_A(t)$ referring to the population of A at time t, we have that:

$$P'_A = (\text{birth rate of } A - \text{death rate of } A) + (\text{conversions to } A - \text{conversion rate from } A)$$

Next, note that in this universe we have only two tribes, A and B. We can therefore simply our equation to:

$$P'_A = (\text{birth rate } A - \text{death rate of } A) + (\text{conv. rate to } A - \text{conv. rate to } B) \cdot (\text{A-B Interaction})$$

We consider the interaction between tribes A and B as the product of the two populations, $P_A(t)P_B(t)$. Finally, since the birth rate and death rate for each tribe can be simplified to one number, we can write the final equations as:

$$\begin{aligned} P'_A(t) &= P_A(t)(n_A + (C_{AB} - C_{BA})P_B(t)), \\ P'_B(t) &= P_B(t)(n_B + (C_{BA} - C_{AB})P_A(t)). \end{aligned} \tag{1}$$

With:

$$\begin{aligned} n_A &= \text{birth rate A - death rate A}, \\ C_{AB} &= \text{conversion rate to A from B}. \end{aligned} \tag{2}$$

2.2 Extension to the Nth Tribe

Expanding our universe past 2 tribes is fairly trivial. For N tribes, we have that for tribe i,

$$P'_i(t) = P_i(t)\left(n_i + \sum_{j=1}^{N}(C_{ij} - C_{ji})P_j(t)\right) \tag{3}$$

The only difference here is that we are summing over all $N-1$ interaction effects given by expanding our universe ($N-1$ instead of N because when $j=i$ we have $(C_{ii}-C_{ii})P_i(t)=0$, because a tribe does not meaningfully self-interact in this model).

We can also define a matrix C containing the conversion rates:

$$C = \begin{bmatrix} 0 & C_{AB} & C_{AC} & ... \\ C_{BA} & 0 & C_{BC} & ... \\ C_{CA} & C_{CB} & 0 & ... \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

The diagonal is all zeros for the reason stated in the previous paragraph–if $C_{ii} \neq 0$, it would make no difference for the model.

2.2.1 Existence and Uniqueness

Like all differential equations we have to question if a solution exists. Furthermore, if a solution exists, is it unique. Luckily we can use the Picard-Lindelöf theorem to answer this question.

Theorem 1. *Let $D \subset \mathbb{R}^N \times R$ be open. The N dimensional tribe growth initial value problem defined by*

$$\boldsymbol{P}'(t) = \boldsymbol{F}(\boldsymbol{P}(t), t), \tag{5}$$

$$\boldsymbol{P}(t_0) = \boldsymbol{P}_0, \tag{6}$$

where $\boldsymbol{F} : D \to \mathbb{R}^N$ and $(\boldsymbol{P}_0, t_0) \in D$ has a unique solution on some $I \subset D$.

Proof. It is easy to see that the function is continuous by the fact that $\boldsymbol{F}$ is differentiable. In fact $\boldsymbol{F} \in C^\infty(\mathbb{R}^N \times \mathbb{R}; \mathbb{R}^n)$. Using this, by the integral mean value theorem we know that $\boldsymbol{F}$ is uniformly Lipschitz and by the Picard-Lindelöf theorem we know that there exists some interval $I \subset D$ such that the solution is unique. $\square$

A nice consequence of this theorem is because we have that $\boldsymbol{F}$ is uniformly Lipschitz then we also know that there is continuous dependence among initial conditions. While this is a good start it is hardly enough to analyze our ODE, expecially when we would like to consider long term effects. We would like to consider questions like: will a solution ever not exist? Does our solution blow up in finite time? These still need to be answered before we begin analyzing numerical solutions and trust the results. The question becomes, does a global solution exist?

One property that our system has is that it is "close" to linear. Because the way the sum is written P_i is never multiplied by itself. This characteristic is what allows us to say that a Global solution exists.

Theorem 2. *Because of the close-to-linear property we described above the N dimensional initial value problem eq. (5) has a global solution on $[t_0, \infty)$. The proof of this is described in [2].*

Now that we can expect our results to make sense we can start analyze solutions to our system. We begin by identifying the assumptions we have made thus far.

2.3 Simplifying Assumptions

It is important to point out a few simplifying assumptions made in this model.

- We are assuming that all rates remain constant for each time period
- We are counting each baby born into a tribe as a member of that tribe.
- We are assuming that the net conversion rates between two tribes are generally proportional to the amount that members of the tribes interact.

Note that in this model we are claiming that all individuals must always belong to a tribe. While they are free to switch between tribes, there is no way to represent ”lone wolfs”.

2.4 Plan for Analysis

In order to analyze these governing equations, we will want to see how sensitive they are to very slight changes in initial conditions. We will also want to see how our system behaves as we slowly add in more tribes. We have shown that we can generalize this model to as many tribes as we want. However, in order to keep our analysis tenable, we will focus on a system with no more than 6 tribes. We start by exploring our base case (A and B) before seeking to generalize our findings to a full system. Finally, we explore the stability of our system and see if this tells us anything about the general relationship between tribes.

3 Results and Analysis

3.1 Case 1: Two Tribes

There are many ways to parametrize a two tribe system, but they are not always interesting. Many parametrizations lead to one tribe going to zero, and the other exponentially growing to infinity. One interesting balance is if one tribe has a positive n value (meaning it has more births than deaths), while the other tribe has a negative n (more deaths than births) but converts members from the other tribes without losing members to other tribes. Figure 1 is parametrized by

$$\mathbf{P}_0 = (100, 50), \quad \mathbf{n} = (0.1, -0.1), \quad C = \begin{bmatrix} 0 & 0 \\ 0.0005 & 0 \end{bmatrix}. \tag{7}$$

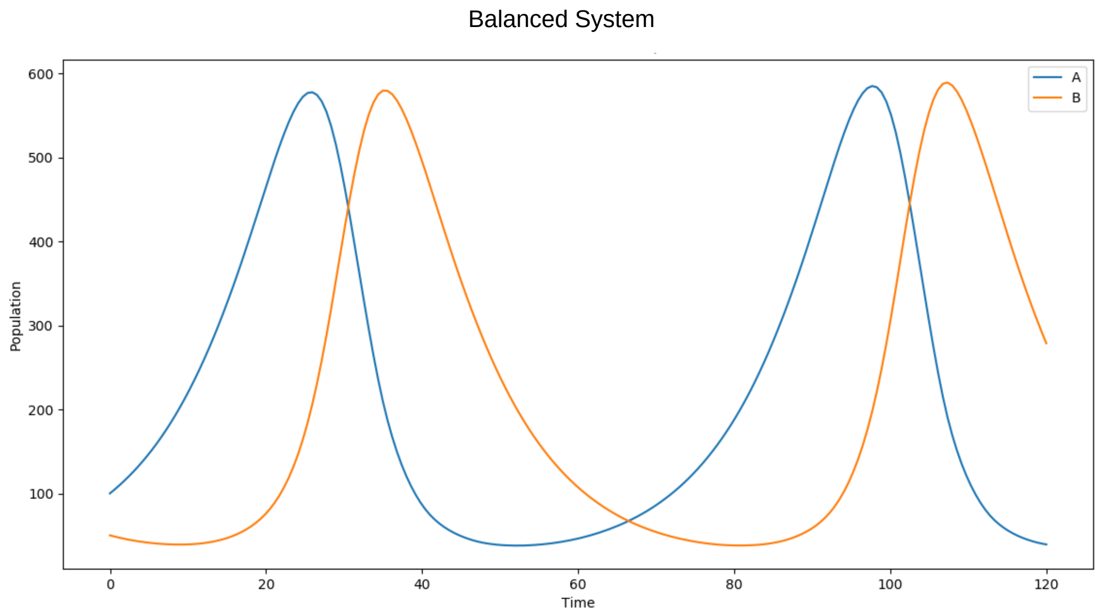

Figure 1: 'Balanced' System with Two Tribes

The behavior is very interesting, with repetitive behavior being shown, as P_A grows due to its birth rate, then P_B grows due to conversion which also pushes P_A down, then P_B decreases due to its death rate, allowing P_A to grow and the cycle repeats.

3.2 Case 2: Three Tribes with Different Birth and Conversion Rates

Different tribes have different birth rates and conversion rates, so we tried to narrow in on these features to see how they interacted.

We first began by modeling how the different birth rates affected each tribe. Since without conversion this would be simple exponential growth, a conversion factor was added to each tribe. A larger birth rate was added to the first group, whereas the other two were modeled after less populous groups. This trend can be seen in society, with different sub groups having different birth rates and people going from one group to another. The parametrization for fig. 2 became

$$\mathbf{P}_0 = (10, 10, 10), \quad \mathbf{n} = (0.2, -0.01, -0.02), \quad C = \begin{bmatrix} 0 & 0.001 & 0.001 \\ 0.01 & 0 & 0.01 \\ 0.01 & 0.01 & 0 \end{bmatrix}. \tag{8}$$

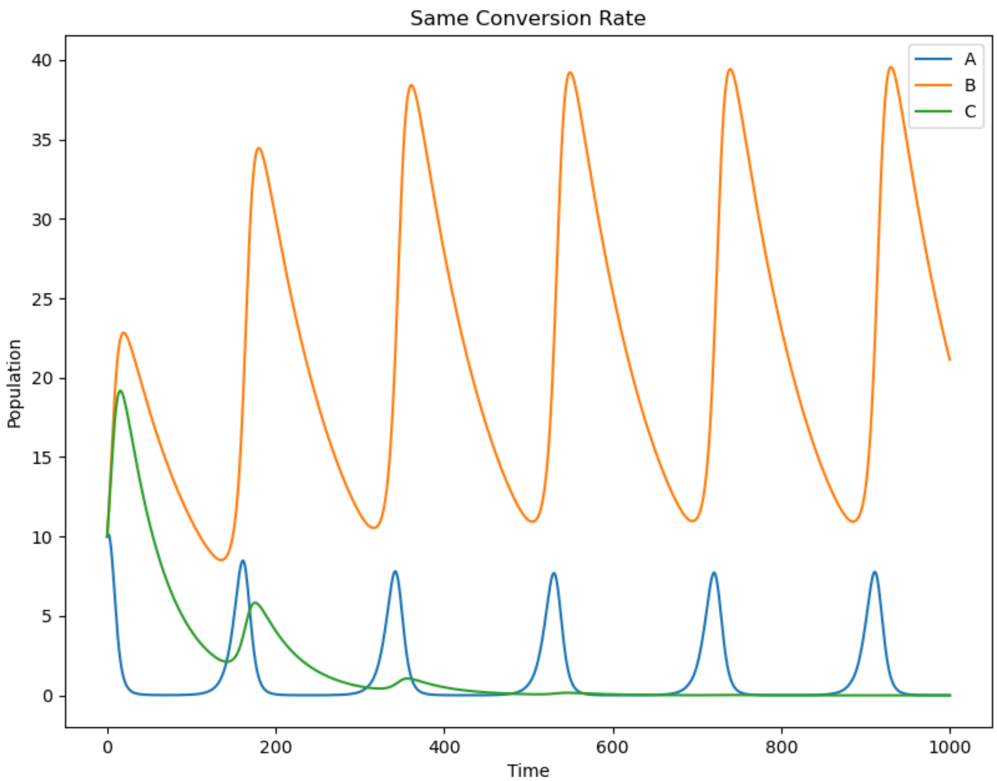

Figure 2: Same Conversion Rate

As evidenced by the graph, a small change in the death rate has a large impact on the population in the long run. Tribe A is the group with positive population growth, and even though it adds 20 percent of its population to itself every generation, it quickly dies down as many members convert to the other two tribes. However, due to the negative birth rate found in the other two, those tribes quickly diminish. Later, tribe A is able to grow without as many people to pull them away. The interesting part here comes between tribes B and C. The difference between the two birth rates is only 1% of the population; however, this causes one group to almost die out while the other thrives. Although it is technically nonzero, it never spikes up, never growing above the bottom of the graph.

Using the similar parameters as above with a slight change in the conversion from B to C, we have

$$\mathbf{P}_0 = (10, 10, 10), \quad \mathbf{n} = (0.2, -0.01, -0.02), \quad C = \begin{bmatrix} 0 & .001 & .001 \\ 0.01 & 0 & 0.01 \\ 0.01 & 0.011 & 0 \end{bmatrix}. \tag{9}$$

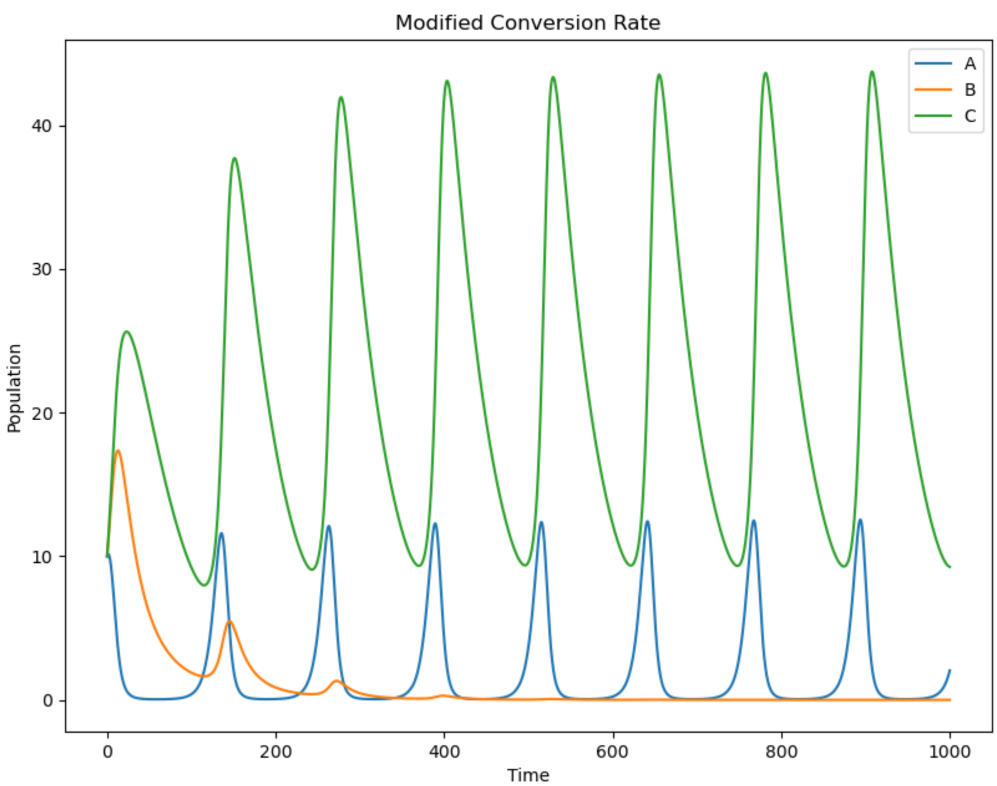

Figure 3: Modified Conversion Rate

We find fig. 3 to have a very different shape. Tribe B has been replaced by C as the largest tribe for most of the time by very slightly changing the conversion from B to C. Even though it was changed by just 1%, this was a large enough change B from being the largest to it almost going extinct over time. Time was run for 1000 time steps, which represents 1000 generations which may seem unrealistic, but it still useful to see convergence over time.

3.3 Case 3: Variable C Matrix and New Tribes

One of our simplifying assumptions was that the conversion rates represented in the C matrix remained constant. However, we wanted to see what behavior would emerge if we actively went against that assumption. Specifically, we wanted to use a C matrix that is dependent on time that would help model the emergence of a new tribe, where any new converts to that tribe remained in said tribe until a certain amount of time passed. Our first attempt at modeling this was to have a piecewise matrix-valued function $C(t)$, where in this case we have

$$C(t) = \begin{bmatrix} 0 & 0.1 & 0.001 & 0 \\ 0 & 0 & 0 & 0 \\ 0.01 & 0.001 & 0 & 0 \\ 0.01 & 0.01 & 0.01 & 0 \end{bmatrix} \text{ if } t < 2; \quad \begin{bmatrix} 0 & 0.1 & 0.001 & 0.01 \\ 0 & 0 & 0 & 0.01 \\ 0.01 & 0.001 & 0 & 0.01 \\ 0.01 & 0.01 & 0.01 & 0 \end{bmatrix} \text{ if } t \geq 2. \tag{10}$$

Our initial conditions were $\mathbf{P}_0 = (100, 100, 100, 1)$, and we had $\mathbf{n} = (0, 0.5, 0.01, 0)$. Here, tribe D represents the new tribe introduced, and the different conversion and birth/death rates for the other tribes are only meant to be varied enough as to make this case somewhat interesting. Figure 4 shows the results:

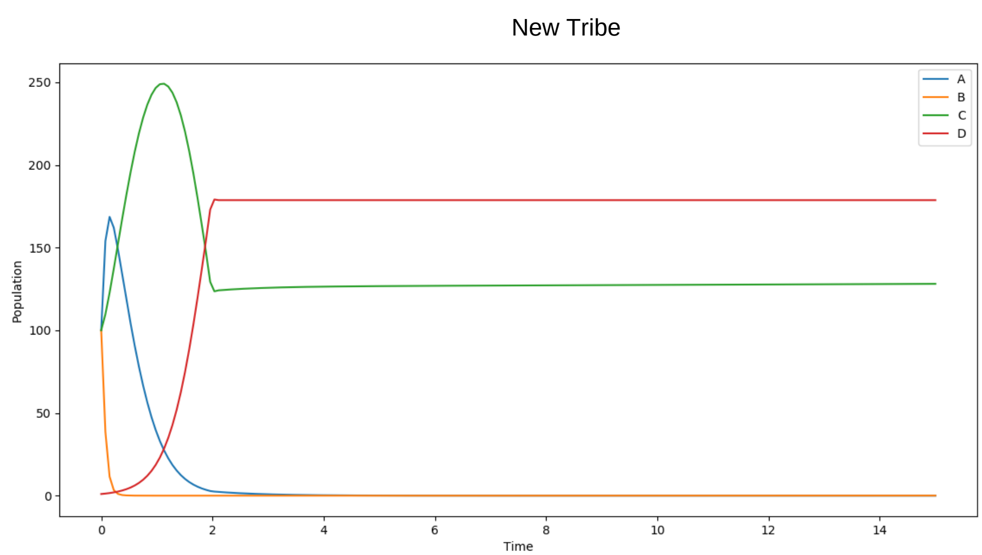

Figure 4: Behavior of model with a variable C matrix, and with tribe D representing a newly introduced tribe

This first attempt at modeling a variable C matrix is a good proof of concept, but the value of $C(t)$ is nearly symmetric for $t \geq 2$, especially for the relative populations of tribes at that point in time, which means that the solution is nearly steady. A more interesting modification of our function would be for nearly-extinct tribe B to have a higher rate of converting people from tribe D (i.e. change $C_{BD}(t \geq 2)$ to be 0.02 instead of 0.01).

This difference in conversion rates gives quite a different behavior of the system, as shown in fig. 5.

The end behavior looks as though it is about to blow up due to the exponential increase in population, but there is very interesting behavior between $t = 7.5$ and $t = 10$. The increase in P_B is enough for the very large $n_B = 0.5$ to drastically increase P_B even further. However, tribe A

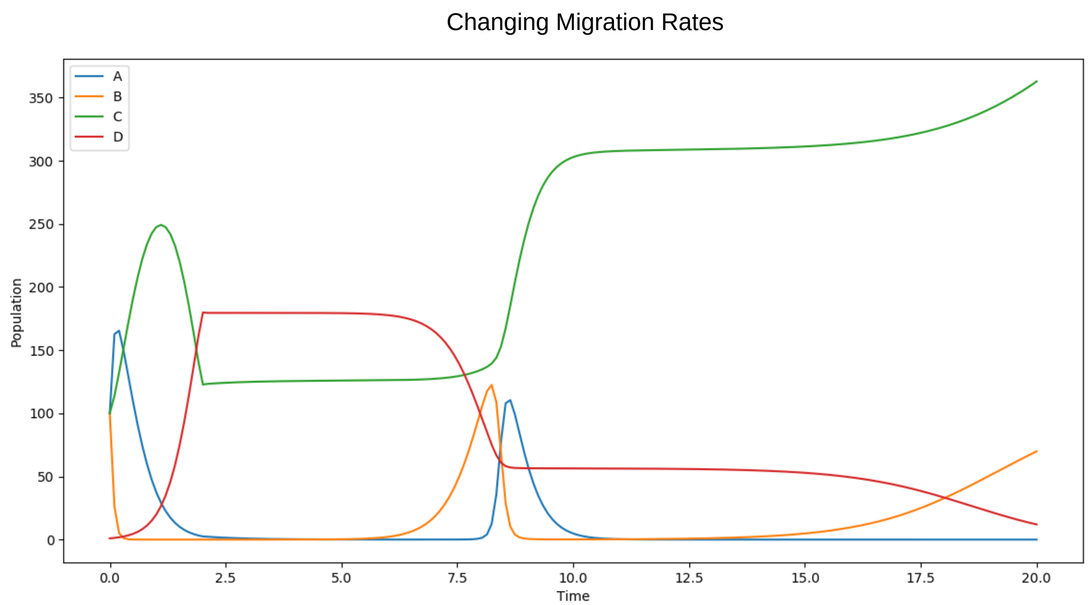

Figure 5: Modification of the model to decrease D's retention, which allows B to grow until it collapses again

converts a large portion of tribe B, rapidly decreasing P_B, and then much of tribe A converts to tribe C. Then, for $t = 10$ to $t = 15$ the system is nearly steady, with tribe C dominating, D faring well, and A and B nearly extinct. The behavior past this seems to be dominated by the positive birth/death rates of tribes B and C, and seem to be on the point of exponential explosion.

3.4 Case 4: Death Rate Interactions and Tribal Wars

To model tribal wars, we added an additional interaction matrix D that represents conflict between two groups. This interaction matrix is shown below.

$$D = \begin{bmatrix} 0 & D_{AB} & D_{AC} & \dots \\ D_{BA} & 0 & D_{BC} & \dots \\ D_{CA} & D_{CB} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{11}$$

These interaction terms are multiplied with the populations of other tribes and summed, as shown in the equation below. They represent the rate at which interaction between tribe A and tribe B results in death in each population, These terms can be thought of as a way to modify a tribe's death rate in response to other tribes.

$$P_i'(t) = P_i(t)\left(n_i + \sum_{j=1}^{N} D_{ij}P_j(t) + \sum_{j=1}^{N}(C_{ij} - C_{ji})P_j(t)\right) \tag{12}$$

In fig. 6, a war between two groups is modeled. Both tribes start with the same population and the same growth rates. For simplicity, we assume there is no conversion between the two groups. The first tribe is violent tribe while the second tribe is peaceful, resulting in a higher death rate

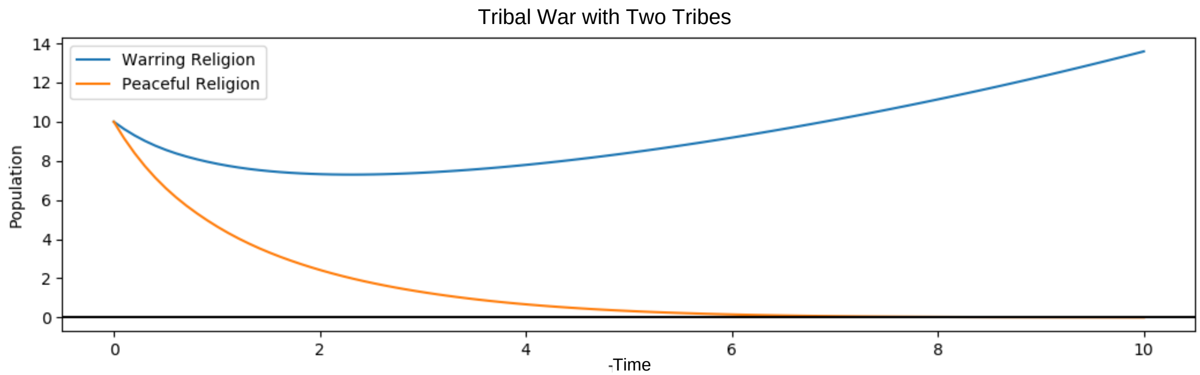

Figure 6: Base Case Tribal War

for the peaceful tribe than for the warring tribe. The warring tribe's population first decreases while fighting the peaceful tribe, then grows exponentially. The peaceful tribe is decreased to near extinction.

$$\mathbf{P}_0 = (10, 10), \quad \mathbf{n} = (0.1, 0.1), \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -0.05 \\ -0.1 & 0 \end{bmatrix} \tag{13}$$

In fig. 7, a second warring tribe is added. Warring tribes decrease each other's populations at a high rate, and the peaceful tribe initially thrives. However, as time goes on, a single warring tribe gains the largest population. Parameters are written below.

$$\mathbf{P}_0 = (1, 0.9, 1), \quad \mathbf{n} = (0.1, 0.1, 0.1), \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & -0.2 & -0.05 \\ -0.2 & 0 & -0.05 \\ -0.1 & -0.1 & 0 \end{bmatrix}. \tag{14}$$

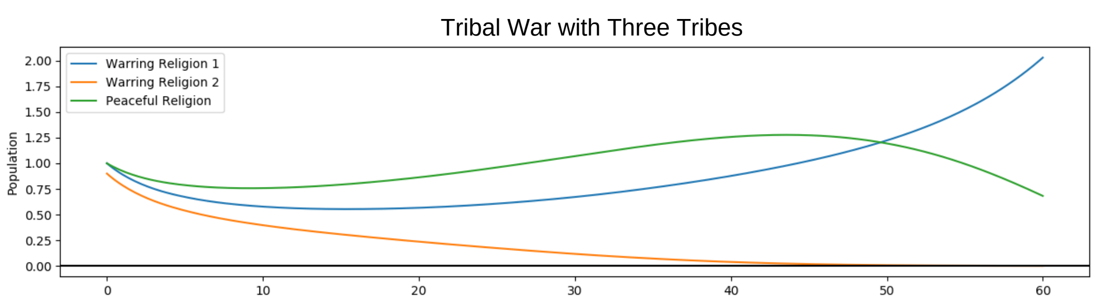

Figure 7: Second Case Tribe War

4 Conclusion

There are a few key takeaways from these cases that give us insight into the general phenomenon of tribe population dynamics. These are:

- The case with two tribes shows that by 'balancing' death rates and conversion rates, the populations can fall into population cycles in tandem with each other.

- Subtle changes to model parameters can dramatically affect the outcome of the model; this modeling approach is relatively unstable.

- Each case had a different result, with about half of the cases displaying a periodic solution and half of the cases displaying near asymptotic stability.

These are interesting results. We suggest continued research in this area, focusing on actual population numbers over time. An interesting challenge for a future researcher would be an attempt to 'back-test' this approach, by verifying the model's validity using historic population data.

References

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[3] Thomas Wieder. A simple differential equation system for the description of competition among religions. In *International Mathematical Forum*, volume 6, pages 1713–1723, 2011.