### THE BROWNIAN SPATIAL COALESCENT

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ABSTRACT. We introduce the Brownian spatial coalescent, a class of Markov spatial coalescent processes on the d-dimensional torus in continuous space, that is axiomatically defined by the following property: conditional on the times and locations of all coalescence events, lineages follow independent Brownian bridges backwards in time along the branches of the coalescence tree. We prove that a Brownian spatial coalescent is characterised by a set of finite  $transition\ measures$  on the torus, in analogy to the transition rates that characterise a non-spatial coalescent.

We prove that a Brownian spatial coalescent is sampling consistent, in a novel sense, if and only if all transition measures are constant multiples of Lebesgue measure, and the multiples are the transition rates of a  $\Xi$ -coalescent, or a  $\Lambda$ -coalescent if simultaneous mergers are not allowed. This defines the *Brownian spatial*  $\Xi$ -coalescent, which we prove describes the genealogies of the  $\Xi$ -Fleming-Viot process—a generalisation of the well known Fleming-Viot process—at stationarity, and in fact describes its full time reversal if augmented with "resampling" steps at the times of coalescence events.

An important consequence of our results is that all spatial population models in which individuals follow independent Brownian motions and the branching mechanism depends non-trivially on the spatial distribution, for example through local regulation, have non-Markovian genealogies.

Byproducts of our results include explicit formulas for samples from the stationary distribution of a  $\Xi$ -Fleming-Viot process, Wright-Malecot type formulas, and a representation of the backward dynamics of lineages in terms of Brownian motions with coupled drift. This includes calculations of the drift that leads to multiple or even simultaneous mergers in any dimension.

#### 1. Introduction

Coalescent processes arise when describing the genealogies (family relations) of a biological population model by tracing the ancestries of a sample from the population backwards in time. If the population is *haploid*, which means that every individual has exactly one parent, then every individual has a single ancestral lineage going backwards in time, and the lineages associated with a set of individuals coalesce when their common ancestor is

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reached. We will not consider diploid populations, but remark that they can be treated as haploid populations if one is only interested in a single gene, see [18, Rem. 2.1]. The most basic coalescent process is Kingman's coalescent, introduced in 1982 [25], in which every pair of lineages coalesces at constant rate, independently of all other pairs.

In this paper, we are ultimately interested in the genealogies of *structured populations*, where every individual has an evolving type or location, but we begin with a brief overview of some classical theory in the unstructured case. The reader familiar with the field should feel free to skim Section 1.1.

1.1. Coalescents. Mathematically, a coalescent is typically described by giving disjoint, set-valued labels to the lineages, which merge when the lineages coalesce. Denote by  $\mathcal{P}$  the set of partitions of finite subsets of  $\mathbb{N}$ .

Definition 1.1. A coalescent is a  $\mathcal{P}$ -valued càdlàg Markov process  $(\Pi_t)_{t\geq 0}$  whose transitions can be obtained by merging partition elements. It is called *label invariant* if its law is invariant under changing the labels of the initial set of lineages.<sup>1</sup>

A precise definition is Definition 2.1. The label invariance condition is equivalent to requiring the counting process  $(|\Pi_t|)_{t\geq 0}$  to be Markovian. Another equivalent phrasing is that for any  $n,m\in\mathbb{N}$  and  $k_1\geq\ldots\geq k_m\geq 2$  with  $\sum_{i=1}^m k_i\leq n$ , every coalescence event in which m disjoint sets of lineages of sizes  $k_1$  to  $k_m$  merge simultaneously while there are currently n lineages—called an  $(n,\vec{k})$ -merger—happens at the same rate, which we denote by  $\lambda_{n,\vec{k}}=\lambda_{n,k_1,\ldots,k_m}$ .

Not every coalescent in the sense of Definition 1.1 arises from the genealogies of a population model, and a natural necessary condition is  $sampling\ consistency$ : if we construct the genealogical tree corresponding to a sample of size n+1, then the induced genealogical tree for a subsample of size n will have the same distribution as the tree obtained by constructing the coalescent directly from the subsample. An equivalent definition is that there exists a coalescent started from all of  $\mathbb N$  such that the coalescent started from a finite subset  $A \subset \mathbb N$  is its restriction to A.

**Theorem 1.2** ([13, 34, 35]). A label invariant coalescent with no simultaneous mergers, that is  $\lambda_{n,k_1,...,k_m} = 0$  whenever m > 1, is sampling consistent if and only if there exists a finite measure  $\Lambda$  on [0, 1] such that

$$\lambda_{n,k} = \int_0^1 p^{k-2} (1-p)^{n-k} \Lambda(dp), \qquad 2 \le k \le n.$$

It is called the  $\Lambda$ -coalescent.

The Kingman coalescent arises as a special case if  $\Lambda$  is the unit mass at zero. If  $\Lambda$  is the uniform measure on [0,1], then the  $\Lambda$ -coalescent corresponds to the clustering process constructed by Bolthausen and Sznitman [8]. A characterisation in the most general case of simultaneous multiple mergers is also known. Denote by  $\Delta = \{\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots) : \xi_1 \geq \xi_2 \geq \ldots \geq 0, |\boldsymbol{\xi}| \leq 1\}$  the infinite simplex, where  $|\boldsymbol{\xi}| = \sum_{i=1}^{\infty} \xi_i$ . Write  $(u, v) = \sum_{i=1}^{\infty} u_i v_i$  for  $u, v \in \Delta$ .

<sup>&</sup>lt;sup>1</sup>In such a way that does not necessarily preserve their size. For example, the substitution  $\{1,3\} \rightarrow \{3,4,9\}$  is allowed.

**Theorem 1.3** ([38, 29]). A label invariant coalescent is sampling consistent if and only if there exists a finite measure  $\Xi = a\delta_0 + \Xi_0$  on  $\triangle$ , where  $\delta_0$  is the unit mass at zero and  $\Xi_0$  has no atom at zero, such that  $\lambda_{n,2} = a$  for all  $n \geq 2$ , and for any  $(n, \vec{k}) \neq (n, 2)$ ,

$$\lambda_{n,\vec{k}} = \int_{\triangle} \sum_{l=0}^{s} {s \choose l} (1 - |\xi|)^{s-l} \sum_{i_1 \neq \dots \neq i_{m+l}} \xi_{i_n}^{k_1} \dots \xi_{i_m}^{k_m} \xi_{i_m+1} \dots \xi_{i_{m+l}} \frac{\Xi_0(\mathrm{d}\xi)}{(\xi, \xi)}, \tag{1.1}$$

where  $s = n - \sum_{i} k_{i}$ . It is called the  $\Xi$ -coalescent.

The  $\Lambda$ -coalescent arises as a special case if  $\Xi$  is the pushforward of  $\Lambda$  under the map  $p\mapsto (p,0,\ldots)$  from [0,1] to  $\triangle$ . We introduced sampling consistency as a necessary condition for a coalescent to describe the genealogies of some population, but it is also sufficient. Möhle and Sagitov [29, 36] give conditions for the genealogies of a haploid population evolving according to Cannings model—that is, a constant size population evolving in discrete generations with an exchangeable offspring mechanism—to converge as the population size tends to infinity when properly rescaled. The class of resulting coalescents is exactly the class of  $\Xi$ -coalescents. Conditions for the genealogies to converge to Kingman's coalescent were previously discovered by Kingman [26] with later improvements by Möhle [31, 32]. Analogous results for the  $\Lambda$ -coalescent were independently discovered by Donnelly and Kurtz [13] and Sagitov [35].

1.2. **Spatial Coalescents.** The  $\Lambda$ - and  $\Xi$ -coalescents appear as a universal limiting object for the genealogies of a wide range of population models [4, 14, 39, 37, 43, 10], including diploid populations [30, 7, 11] and populations where offspring sizes are not exchangeable [41]. However, they cannot fully describe the genealogies of a *structured* population, where every individual has a *type* or *location* evolving in some space E. Typically, type is the appropriate interpretation in a genetic model, and location is appropriate in a model of a geographically dispersing population. We usually refer to spatial locations from now on, but the genetic interpretation remains valid throughout. On top of the set of lineages  $\Pi_t$ , an associated coalescent process needs to keep track of the spatial location  $X_t(u)$  of every lineage  $u \in \Pi_t$ , which we summarise in a map  $X_t: \Pi_t \to E$ .

Definition 1.4. A spatial coalescent is a càdlàg Feller Markov process  $(\Pi_t, \mathbf{X}_t)_{t\geq 0}$  such that  $\Pi_t \in \mathcal{P}$  and  $\mathbf{X}_t \colon \Pi_t \to E$  for all  $t \geq 0$ , and all transitions in  $(\Pi_t)$  can be obtained by merging partition elements. It is called *label invariant* if its law is invariant under changing the labels of the initial set of lineages.

A precise definition is in Definitions 2.3 and 2.4. The Feller property is a form of continuity in the initial condition that implies the strong Markov property, and is automatic in the non-spatial setting. The label invariance assumption is equivalent to the requirement that the empirical measure  $\sum_{u \in \Pi_t} \delta_{\mathbf{X}_t(u)}$  is a Markov process.

The picture in the spatial setting is much less complete than in the non-spatial set-

The picture in the spatial setting is much less complete than in the non-spatial setting. Consider, for instance, a population model in which individuals move as independent Brownian motions on the d-dimensional (flat) torus  $E = \mathbb{R}^d/\mathbb{Z}^d$  forwards in time, while undergoing branching according to some unspecified (think general) mechanism. The well-known Fleming-Viot process and its  $\Lambda$ - and  $\Xi$ -generalisations [12, 5, 6], as well as the class of models considered by Donnelly and Kurtz in their lookdown papers [12, 13], when specialised to Brownian movement, fall into this category. Describing the genealogies of such

a population model essentially requires partially time reversing the process: given a finite sample from the population at present together with their spatial locations (with no knowledge of their history, or the spatial distribution of the remaining population), how do their lineages move through space backwards in time? Since Brownian motion is reversible, they will presumably follow Brownian motions, but there would have to be a coupled drift; if they were independent, lineages would never meet in dimensions  $d \geq 2$ , and even in d = 1 only in pairs. But what kind of drift leads to binary mergers in  $d \geq 2$ , or multiple, even simultaneous mergers in any dimension? These are interesting and basic questions, yet generally unanswered to the best of our knowledge. Our results will provide answers for a broad class of population models. The curious reader is welcome to take a peek at Example 1.17.

1.3. The Brownian Spatial Coalescent. The genealogies of a spatial model of the kind considered in the previous paragraph will be very complicated to describe in general; they depend on the initial condition of the process, and wouldn't be Markovian without conditioning at least on the total population size process (as is the case in the non-spatial setting, where genealogies are Markovian after a time change that depends on the total population size process, see e.g. [13, Thm. 5.1] for the Kingman coalescent). But if the population is at stationarity forwards in time, and the total population size is constant, then the genealogies are time homogeneous, and there is hope for them to be Markovian. If that is the case, they would be described by a spatial coalescent in the sense of Definition 1.4.

The goal of this paper is to identify a universal class of spatial coalescents that describe the genealogies of constant size spatial population models at stationarity in which individuals follow independent Brownian motions. We will do so by axiomatically defining a class of spatial coalescents through a property that we would expect the genealogies of all such populations to satisfy. To motivate it, note that if we condition a realisation of the forwards in time population model on the initial and final locations of all individuals, as well as the times and locations of branch events, then all individuals will follow independent Brownian bridges along the branches of the "genealogical" tree (or forest in general), regardless of the branching mechanism. Interpreting this backwards in time, it means for the associated coalescent that, conditional on the genealogical tree and the times and locations of what are now the coalescence events, lineages follow independent Brownian bridges along the branches of the tree backwards in time.

Before turning this into a definition, we point out a subtle difficulty that arises when starting such a coalescent process from an initial configuration with more than one particle in the same location. If  $d \geq 2$ , then forwards in time, almost-surely the only occasion where any pair of particles would ever be in the same location is when they were just born in the same branching event. The same is true in d=1 for more than two colliding particles. Thus the coalescent started from such a configuration would have to merge those particles instantaneously, which it cannot if we require right-continuous paths. One way out would be to work with left-continuous processes, but that would leave many technical special cases in dealing with instantaneous mergers at time zero. The simpler solution is to exclude those points from the state space. For  $\Pi \in \mathcal{P}$  denote by  $E^{\Pi}$  the set of maps  $\Pi \to E$ , and let

$$E_{\circ}^{\Pi} := \{ \boldsymbol{x} \in E^{\Pi} : \# \{ \{u, v\} \subset \Pi : u \neq v, \boldsymbol{x}_{u} = \boldsymbol{x}_{v} \} \leq \mathbb{1}_{\{d=1\}} \}$$
 (1.2)

denote the set of spatial decorations of  $\Pi$  in which no (if  $d \ge 2$ ) or at most one (if d = 1) pair of particles reside in the same location. We occasionally identify  $E_{\circ}^{\Pi}$  with an open subset of

 $E^{|\Pi|}$  by ordering the elements of  $\Pi$  in ascending order of their minima. Similarly in other contexts. Let

$$\mathcal{X}\coloneqq\left\{(\Pi, oldsymbol{x})\colon \Pi\in\mathcal{P}, oldsymbol{x}\in E^\Pi_\circ
ight\}.$$

Note that  $\Pi$  is the domain of x for  $(\Pi, x) \in \mathcal{X}$ , so we occasionally just write  $x \in \mathcal{X}$ .

Definition 1.5. A spatial coalescent with state space  $\mathcal{X}$  is called a Brownian spatial coalescent if, conditional on the genealogical forest, and times and locations of all coalescence events, lineages follow independent Brownian bridges along the branches of the forest, and Brownian motions after the last coalescence event in which they are involved.

A precise definition is Definition 2.10. We remark here that the only real obstacle in replacing Brownian motion by a much more general motion lies in finding the points that have to be excluded from the state space. We go into more detail in Section 1.7.2.

Remark 1.6. The defining property of a Brownian spatial coalescent should be satisfied by the genealogies of any population model in which individuals follow independent Brownian motions forwards in time, regardless of the branching mechanism. This includes spatial interactions such as competition or local regulation. However, it will turn out that assuming the Markov property of the genealogies eliminates all branching mechanism that non-trivially depend on the spatial distribution of the population. See Section 1.7.1 below.

We do not yet require any kind of sampling consistency. If a parallel is drawn with the non-spatial setting, then the Brownian spatial coalescent should be compared with the non-spatial coalescent in the sense of Definition 1.1; both contain many examples that do not describe the genealogies of any population. Our first result is a full characterisation of the class of Brownian spatial coalescents, which shows that this comparison is more apt than one might expect. Write M for the set of tuples  $(n, \vec{k}) = (n, (k_1, \ldots, k_m))$  with  $n \geq 2$ ,  $1 \leq m \leq n$ ,  $k_1 \geq \ldots \geq k_m \geq 1$  and  $\sum_{i=1}^m k_i \leq n$ . Write  $\mathcal{M}_F(A)$  for the set of finite measures on a measurable space A.

**Theorem 1.7.** Every label invariant Brownian spatial coalescent is uniquely characterised by a family of measures  $\boldsymbol{\nu} = (\nu_{n,k_1,...,k_n} \in \mathcal{M}_F(E^m))_{(n,\vec{k})\in M}$ . We call it the Brownian spatial coalescent with transition measures  $\boldsymbol{\nu}$ .

The transition measures are reminiscent of the transition rates of a non-spatial coalescent, and in fact we will see in Remark 1.10 that if we formally replace E with a singleton, then the Brownian spatial coalescent with transition measures  $(\nu_{n,\vec{k}})$  is just the non-spatial coalescent with transition rates  $(\lambda_{n,\vec{k}} = |\nu_{n,\vec{k}}|)$ . Here  $|\nu|$  is the total mass of a measure  $\nu$ .

We refer to Theorem 2.14 for a formal version of Theorem 1.7 that applies without the assumption of label invariance, and gives a precise description of the law of a Brownian spatial coalescent with given transition measures  $\nu$  building on the notation introduced in Section 2. Here, we give an informal description in the case where there are no simultaneous mergers, so  $\nu = (\nu_{n,k}: 2 \le k \le n)$ , and we further assume that for every  $n \ge 2$  there is some  $k \in \{2, ..., n\}$  with  $\nu_{n,k} \ne 0$ . This ensures that almost-surely all lineages will merge eventually, so the genealogical forest is always a tree. Under these assumptions, the law of the Brownian spatial coalescent with transition measures  $\nu$ , started from initial locations  $x_1$  to  $x_n$  is as follows. For every possible shape (topology) T of the genealogical tree on n leaves, a measure  $\nu_T$  is defined on the space of time and spatial coordinates of coalescence

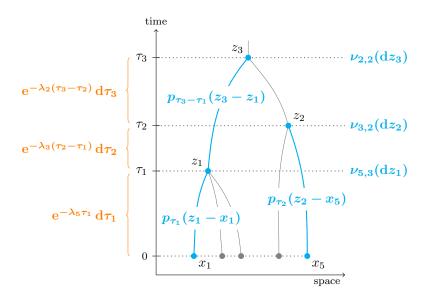


FIGURE 1. Illustration of the finite measure on the space and time decorations of a particular genealogical tree shape. Spatial factors associated with branches of the tree are only shown for three of the seven branches.

events in the tree, which is most easily described by picture (Fig. 1): for every branch in the tree there is a spatial factor, illustrated in cyan in Fig. 1, where  $(p_t: E \to (0, \infty))_{t>0}$  denotes the transition densities of a Brownian motion on E with periodic boundary conditions; and for every merge event there is a factor of the form  $e^{-|\nu_n|(\tau-\tau')}$ , where  $\tau$  and  $\tau'$  are the times of the merge event and the previous merge event (or zero), respectively, and

$$|\nu_n| := \sum_{2 \le k \le n} \binom{n}{k} |\nu_{n,k}|, \quad n \ge 2,$$

can be thought of as the analogue of the total event rate  $\lambda_n$  while at n lineages in the non-spatial setting. The density in the time coordinates is with respect to Lebesgue measure, and in the spatial coordinate z associated with an (n,k)-merger it is with respect to the transition measure  $\nu_{n,k}(\mathrm{d}z)$ . The measure  $\nu_T$  is not finite for arbitrary choice of  $\nu$ , but it will be for the transition measures of a Brownian spatial coalescent. Now,

- (i) Sample a tree shape, where the probability of sampling T is proportional to  $|\nu_T|$ .
- (ii) Given T, sample time and space coordinates of all merge events from  $\nu_T/|\nu_T|$ .
- (iii) Given T and time and space coordinates of all merge events, run independent Brownian bridges along the branches of the tree, starting from  $x_1$  to  $x_n$  at the leaves, and a Brownian motion after the final merge event.

Equivalently to (i) and (ii), we could sample a tree together with its time and space decorations directly from the normalised formal sum of all the measures  $\nu_T$ . The normalisation

$$N^{\nu}(\boldsymbol{x}) = \sum_{T} |\nu_{T}| \tag{1.3}$$

as a function of the initial condition  $\boldsymbol{x}=(x_1,\ldots,x_n)\in\mathcal{X}$  plays an important role. For example, it decides for a given family  $\boldsymbol{\nu}$  of transition measures whether an associated Brownian spatial coalescent exists.

**Theorem 1.8.** For a given family  $\boldsymbol{\nu} = (\nu_{n,k_1,\dots,k_m} \in \mathcal{M}_F(E^m))_{(n,\vec{k})\in M}$ , a Brownian spatial coalescent with transition measures  $\boldsymbol{\nu}$  exists if and only if  $N^{\boldsymbol{\nu}} \colon \mathcal{X} \to [0,\infty]$  is finite and continuous.

See Theorem 2.14. Continuity of  $N^{\nu}$  corresponds to the Feller property of the associated coalescent; for it to be well-defined, finiteness of  $N^{\nu}$  would suffice. The statement is not trivial, because the process defined through the transition measures  $\nu$  as described above does not obviously have the Markov property. We don't have a nice characterisation of the measures  $\nu$  for which  $N^{\nu}$  is finite and continuous, but a sufficient condition is that all transition measures are constant multiples of Lebesgue measure, see Lemma 2.12. Since  $N^{\nu}(x)$  is an integral over functions that are continuous in x, dominated convergence gives a sufficient integrability condition.

The function  $N^{\nu}$  also reveals a technical reason why it was necessary to exclude some points from the state space  $\mathcal{X}$ : If x approaches a point in the boundary of  $\mathcal{X}$ , then  $N^{\nu}(x)$  diverges if the transition measure associated with the coalescent event that would have to happen instantaneously is non-zero, see also Remark 2.13.

An example and some remarks are in order.

Example 1.9. Starting from two lineages at locations  $x_1$  and  $x_2$ , the probability that they coalesce at time t > 0 at location  $z \in E$  is proportional to

$$e^{-|\nu_2|t}p_t(x_1-z)p_t(x_2-z)\nu_{2,2}(dz) dt.$$

In particular, if  $\nu_{2,2}(dz) = |\nu_{2,2}| dz$  (which will turn out to be a consequence of sampling consistency, see Theorem 1.12), then the probability that they reach a common ancestor at time t > 0 is proportional to

$$|\nu_2| e^{-|\nu_2|t} p_{2t}(x_1 - x_2) dt$$

since  $|\nu_2| = |\nu_{2,2}|$ . This is a form of the Wright-Malecot formula. The normalisation is given by

$$N^{\nu}(x_1, x_2) = \int_0^{\infty} |\nu_2| e^{-|\nu_2|t} p_{2t}(x_1 - x_2) dt.$$

If  $r = |x_1 - x_2|$  is small, then  $N^{\nu}(x_1, x_2)$  is of order  $r^{2-d}$  if  $d \ge 3$ , of order  $\log \frac{1}{r}$  if d = 2, and of order 1 if d = 1. Similar calculations lead to higher-order versions of the Wright-Malecot formula.

Remark 1.10. (i) If we formally replace E with a singleton, and Brownian motion with the trivial Markov process on E, then the spatial factors in Fig. 1 vanish, the transition measures reduce to constants, and by a simple argument about competing exponential clocks, the remaining exponential structure exactly describes the law of

- the non-spatial coalescent with transition rates  $\lambda_{n,k} = |\nu_{n,k}|$ . The same is true if simultaneous mergers are allowed.
- (ii) This is a very "non-Markovian" description of a Markov process. The Markov property in terms of this description states that, if we sample the decorated coalescence tree at time zero and start running the particles along the corresponding Brownian bridges, and at some time t>0 we resample the decorated coalescence tree for the remaining lineages using their current locations as initial conditions, and run them along the Brownian bridges associated with the new tree, then this does not affect the overall law of the process.

The main work in the proof of Theorem 1.7 is to construct the measures  $\nu_{n,\vec{k}}$  using only the defining property (Definition 1.5) and the Markov property of a given Brownian spatial coalescent. The idea is to apply the Markov property at the (random) time of the first merge event, and proceed inductively over the number of initial particles, starting with two.

1.4. Sampling Consistency. Our main result is a characterisation of sampling consistency within the class of Brownian spatial coalescents, analogous to the characterisations of the  $\Lambda$ - and  $\Xi$ -coalescents in the non-spatial setting (Theorems 1.2 and 1.3). A characterisation of this form is, to our knowledge, without precedent in the literature on spatial coalescents.

Sampling consistency for spatial coalescents is classically taken to mean that the distribution of the coalescent started from some set of n initial locations is the same as that of the coalescent started with an additional n+1'st particle at any fixed location and projected back onto the first n particles. We would expect the genealogies of a forward in time population model to have this property, but only if knowing the location of an additional individual does not reveal any information about the population. This is the case, for example, if the population is spread homogeneously throughout space and time, as is the case for a number of spatial models whose genealogies have been studied successfully, which includes the spatial  $\Lambda$ -Fleming-Viot process, or populations living on a discrete lattice with positive population density at every deme. We review related literature in Section 1.6. But in our case, where the spatial distribution of the population fluctuates in time and space driven by its forward dynamics, the location of the additional particle will reveal information about the entire population. This will, in general, influence the genealogy of the first n particles. Instead, the additional particle has to be genuinely sampled from the (stationary) spatial distribution of the population, conditional on the n lineages whose locations are already known.

Definition 1.11. A spatial coalescent is sampling consistent if, for every choice of initial locations  $x_1$  to  $x_n$ , there exists a probability measure  $\mu_{x_1,...,x_n}$  on E such that the coalescent started from  $x_1$  to  $x_n$  has the same distribution as the coalescent induced on the first n particles when starting with an additional particle  $x_{n+1}$  sampled from  $\mu_{x_1,...,x_n}$ .

See Definition 2.5 in the next section for a precise definition. We will prove that, as the motivation suggests,  $\mu_{x_1,...,x_n}$  is the law of a sample from the stationary distribution of an associated forwards in time population model, conditional on having sampled  $x_1$  to  $x_n$  in n previous samples. The familiar reader may also note that Definition 1.11 can be cast in the language of the Markov mapping theorem:  $\mu_{x_1,...,x_n}$  is the kernel that recovers the state of

the "larger" Markov process (the coalescent started from n + 1 particles) from the state of the projected one (the coalescent started from n particles).

**Theorem 1.12.** A label invariant Brownian spatial coalescent is sampling consistent if and only if there exists a finite measure  $\Xi$  on  $\triangle$  such that the transition measures satisfy

$$\nu_{n,\vec{k}}(\mathrm{d}\boldsymbol{z}) = \lambda_{n,\vec{k}}\,\mathrm{d}\boldsymbol{z}, \qquad (n,\vec{k}) \in M,$$

where  $(\lambda_{n,\vec{k}})$  are the rates of the non-spatial  $\Xi$ -coalescent.

This gives rise to what we call the Brownian spatial  $\Xi$ -coalescent. If simultaneous mergers are not allowed, we get an analogous statement for what we call the Brownian spatial  $\Lambda$ -coalescent. If  $\nu$  are the transition measures of the Brownian spatial  $\Xi$ -coalescent, write  $N^{\Xi} = N^{\nu}$ .

**Proposition 1.13.** Let  $\Xi \neq 0$  be a finite measure on  $\triangle$ . Then the family of probability measures associated with the Brownian spatial  $\Xi$ -coalescent through Definition 1.11, which we denote by  $(\mu_{x_1,\ldots,x_n}^{\Xi})$ , is unique and

$$\mu_{x_1,\dots,x_n}^{\Xi}(\mathrm{d}y) = \frac{N^{\Xi}(x_1,\dots,x_n,y)}{N^{\Xi}(x_1,\dots,x_n)} \,\mathrm{d}y.$$
 (1.4)

Furthermore, there exists a unique random probability measure  $\mu^{\Xi}$  on E such that

$$\mathbb{E}\left[\mu^{\Xi}(\mathrm{d}x_1)\dots\mu^{\Xi}(\mathrm{d}x_n)\right] = N^{\Xi}(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n \tag{1.5}$$

as measures on  $E^n$  for every  $n \in \mathbb{N}$ .

Note that Proposition 1.13 reveals another meaning of the normalisation function  $N^{\Xi}$ : it describes the joint density of samples from a random realisation of  $\mu^{\Xi}$ .

Remark 1.14. A spatial coalescent process is sampling consistent in the classical sense if and only if it is sampling consistent in the sense of Definition 1.11 with respect to (w.r.t.) any choice of probability measures ( $\mu_{x_1,...,x_n}$ ). But since the probability measures in Proposition 1.13 are unique, there exists no Brownian spatial coalescent, except the trivial one, which is sampling consistent in the classical sense. This may explain why no non-trivial, sampling consistent spatial coalescent in which an individual lineage follows a Brownian motion has been found to this day in the setting of constant population density (where we would expect associated coalescents to be sampling consistent in the classical sense), except in one dimension where independent Brownian motions meet.

Eqs. (1.4) and (1.5) imply

$$\mathbb{E}\left[\mu^{\Xi}(\mathrm{d}x_1)\dots\mu^{\Xi}(\mathrm{d}x_n)\right] = \mathrm{d}x_1\mu_{x_1}^{\Xi}(\mathrm{d}x_2)\dots\mu_{x_1,\dots,x_{n-1}}^{\Xi}(\mathrm{d}x_n),$$

that is,  $\mu_{x_1,\dots,x_n}^{\Xi}$  is the distribution of a sample from  $\mu^{\Xi}$  conditional on having already sampled  $x_1$  to  $x_n$  in n previous, independent samples. This confirms the motivation behind Definition 1.11, provided we can find a forwards in time population model with stationary distribution  $\mu^{\Xi}$  whose genealogies are described by the Brownian spatial  $\Xi$ -coalescent. This will be the  $\Xi$ -Fleming-Viot process. Before showing that, we present a drift representation for the evolution of the Brownian spatial coalescent. For  $\nu \in R$  and  $\lambda \in R^{\circ}$  we write  $\nu(\mathrm{d}z) = \lambda \, \mathrm{d}z$  as a short hand for  $\nu_{n,\vec{k}}(\mathrm{d}z) = \lambda_{n,\vec{k}} \, \mathrm{d}z$  for all  $(n,\vec{k}) \in M$ .

**Theorem 1.15.** If a Brownian spatial coalescent with transition measures  $\nu$  of the form  $\nu(\mathrm{d}\mathbf{z}) = \lambda \, \mathrm{d}\mathbf{z}$  for some  $\lambda \in R^{\circ}$  is started from distinct initial locations  $x_1$  to  $x_n$ , then the paths  $\mathbf{Z}_t = (Z_t^1, \ldots, Z_t^n)$  of the lineages until just before the first merge event (or until just before the first time that any pair of lineages meet if d = 1) are described by the following stochastic differential equation:

$$d\mathbf{Z}_t = d\mathbf{B}_t + \nabla \log N^{\nu}(\mathbf{Z}_t) dt, \tag{1.6}$$

where  $(B_t)$  is an nd-dimensional standard Brownian motion on E with periodic boundary conditions.

In particular, Theorem 1.15 applies to the Brownian spatial  $\Xi$ -coalescent. It also holds for a general Brownian spatial coalescent provided  $N^{\nu}$  is differentiable and the order of the differentiation and the integrals defining  $N^{\nu}$  can be exchanged, in the way that is needed in the proof; see Section 3.4. The following remark and example apply only to Brownian spatial coalescents of the form assumed in Theorem 1.15.

- Remark 1.16. (i) If  $d \ge 2$ , then the drift representation determines the law of a Brownian spatial coalescent completely: run (1.6) until its maximal existence time, where the drift diverges and some set of lineages collide. Coalesce lineages that are in the same location and restart (1.6). Repeat until only one lineage is left (or a number k such that  $|\nu_k| = 0$ ).
  - (ii) If d=1, then lineages may meet in pairs without coalescing. At such points  $N^{\nu}$  is (continuous and) not differentiable, but it is possible to extend  $\nabla \log N^{\nu}$  to all of  $\mathcal{X}$  in such a way that the drift representation (1.6) holds up until just before the first merge event. But unlike in  $d \geq 2$ , binary mergers always happen before the maximal existence time of (1.6), making the construction in (i) invalid (except if binary mergers are impossible because  $\nu_{k,2}=0$  for the relevant values of k). We leave it open to find a description of the law with which binary mergers occur conditional on the solution of (1.6); calculations suggest that they do *not* happen at a rate proportional to the collision local time of pairs of lineages, as is common in models of coalescing Brownian motions.

The function  $N^{\nu}$  is a sum over forests, whose number explodes combinatorially as the number of initial locations increases, but we can gain some interesting insights in simple special cases.

Example 1.17. Let  $\nu \in R$  with  $\nu(\mathrm{d}z) = \lambda \, \mathrm{d}z$  for some  $\lambda \in R^{\circ}$ , and consider the Brownian spatial coalescent with transition measures  $\nu$ .

(i) If  $\lambda_{2,2} > 0$  and the Brownian spatial coalescent is started from two lineages, then the drift symmetrically pulls them towards each other,<sup>2</sup> and the strength of the drift as their separation r tends to zero is of order 1 if d = 1, otherwise of order 1/r. This is not surprising in light of the fact that the Brownian movement in  $d \geq 2$  effectuates a repulsive drift of the same order 1/r.

<sup>&</sup>lt;sup>2</sup>More precisely, the drift acting on a lineage located at x when there is another lineage at location y points in the same direction as  $\int_0^\infty \mathrm{e}^{-|\nu_2|t} \nabla_x p_t(x-y) \, \mathrm{d}t$ . On  $\mathbb{R}^d$ , this is the same direction as y-x, but on the torus it is more complex because of the periodicity. The difference becomes neglible for small separation.

- (ii) If  $\lambda_{3,3} > 0$  and the Brownian spatial coalescent is started from three individuals, and conditioned on a triple merger, then the drift pulls each particle in the direction of the midpoint of the other two.<sup>3</sup> Without the conditioning, the drift is mixed in a complicated way with pairwise binary attractions.
- (iii) If  $\lambda_{4,2,2} > 0$  and  $\lambda_{2,2} = 0$ , and the Brownian spatial coalescent is started from four individuals at locations  $x_1$  to  $x_4$ , and conditioned on a simultaneous binary merger of  $x_1$ ,  $x_2$ , and  $x_3$ ,  $x_4$ , then the drift symmetrically pulls  $(x_1, x_3)$  and  $(x_2, x_4)$  towards each other, and the strength of the drift is of order 1/r where r is their 2d-dimensional Euclidean separation. We put  $\lambda_{2,2} = 0$  because the drift is otherwise influenced in a complex way by the inevitable second merge event.

Remark 1.18. It is known in a variety of settings that the movement of a single lineage  $Z_t$  in a spatially evolving population is driven by an SDE of the form  $dZ_t = dB_t + \nabla \log N(Z_t) dt$ , where N is something akin to a population density (for example in [9] the population density N is a centred one-dimensional Gaussian and the drift is  $\nabla \log N(x) \propto -x$ ). In our case, we have more than one lineage, and the spatial distribution of the population is random, fluctuates, and does not admit a density except in one dimension (see e.g. [17, Chapter 2.9]). In that light it seems surprising that the same representation still holds, simply by replacing N with the joint density of an n-sample from the population's stationary distribution.

## 1.5. Associated Population Models.

1.5.1.  $\Xi$ -Fleming-Viot Process. Let us briefly define what has been called the  $\Xi$ -Fleming-Viot process [6], a probability measure valued Markov process on E (or on all of  $\mathbb{R}^d$ ) that generalises the "generalised (or  $\Lambda$ -)Fleming-Viot process" coined in [5]. The  $\Xi$ -Fleming-Viot process is dual to the non-spatial  $\Xi$ -coalescent [6], in the same sense that the classical Fleming-Viot process is dual to Kingman's coalescent [12].

Let  $\Xi = a\delta_0 + \Xi_0$  be a finite measure on  $\triangle$ , where  $\delta_0$  is the unit mass at zero and  $\Xi_0$  has no atom at zero. The  $\Xi$ -Fleming-Viot process is most easily defined using a lookdown construction [12, 13, 6]. Let  $\mathfrak{N}^{ij}$  for  $i, j \in \mathbb{N}, i < j$  be a family of independent, rate a Poisson processes on  $\mathbb{R}$ , and  $\mathfrak{M}$  a Poisson point process on  $\mathbb{R} \times \triangle$ , independent of the  $\mathfrak{N}^{ij}$ , with intensity

$$\mathrm{d}t\otimes \frac{\Xi_0(\mathrm{d}\boldsymbol{\xi})}{\langle \boldsymbol{\xi},\boldsymbol{\xi}\rangle}.$$

We define a dynamic on  $E^{\infty}$ , started from an initial configuration  $\mathbf{Y}(0) = (Y_1(0), Y_2(0), \ldots)$  as follows.

- (i) At a point t of the process  $\mathfrak{N}^{ij}$ , the j'th level particle "looks down" to the i'th level particle and copies its location.
- (ii) At a point  $(t, \boldsymbol{\xi})$  of the process  $\mathfrak{M}$ , we independently assign every level to a basket with label  $i \in \mathbb{N}$  with probability  $\xi_i$ , and to no basket with probability  $1 \sum_i \xi_i$ . Then every particle that has been assigned to a basket copies the location of the particle with the smallest label in the same basket.

 $<sup>^3</sup>$ As for the binary merger, this is technically only true if we were on  $\mathbb{R}^d$ , but the difference is neglible for small separations of the three lineages.

(iii) All levels follow independent Brownian motions between reproductive events.

There may be an infinite number of lookdown events in finite time, but any fixed level is only hit by non-trivial reproductive events at a finite rate.

Remark 1.19. If  $\xi_2 = \xi_3 = \ldots = 0$  for  $\Xi$ -a.e.  $\boldsymbol{\xi} \in \triangle$ , then  $\Xi$  reduces to a measure on [0,1] which is commonly denoted  $\Lambda = a\delta_0 + \Lambda_0$ , where  $\Lambda_0$  has no atom at zero. In terms of  $\Lambda$ , step (ii) above can be summarised as follows: At a point (t,p) of a Poisson point process with intensity  $\mathrm{d}t \otimes \frac{\Lambda_0(\mathrm{d}p)}{p^2}$  on  $\mathbb{R} \times [0,1]$ , select every level independently with probability p, and each selected particle copies the location of the selected particle with the lowest level. The resulting particle process is called the  $\Lambda$ -Fleming-Viot process.

The point in this construction lies in the fact that it preserves exchangeability: If  $\mathbf{Y}(0) = (Y_1(0), Y_2(0), \ldots)$  is exchangeable, then so is  $\mathbf{Y}(t) = (Y_1(t), Y_2(t), \ldots)$  for every t > 0 [12, 6]. We denote the associated de Finetti measure by

$$\mathcal{Y}_t = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \delta_{Y_i(t)}, \quad t \ge 0.$$

The  $\Xi$ -Fleming-Viot process started at  $(Y(0), \mathcal{Y}_0)$  is a càdlàg modification of the process  $(Y(t), \mathcal{Y}_t)_{t\geq 0}$  (only the second component has to be modified). For details of the construction see [6]. (The term " $\Xi$ -Fleming-Viot process" usually just refers to the process  $(\mathcal{Y}_t)$ , but we find it convenient to include the particle representation explicitly.) If  $\Xi = \delta_0$ , then  $(\mathcal{Y}_t)$  is the well-known Fleming-Viot process (this was the main subject of [12]), which can, for example, be obtained from a Dawson-Watanabe superBrownian motion by conditioning it on having constant, unit population size.

**Proposition 1.20.** The stationary distribution of the  $\Xi$ -Fleming-Viot process (exists and) is equal to  $\mu^{\Xi}$ . It can be constructed to run bi-infinitely at stationarity, which we denote  $(\mathbf{Y}(t), \mathcal{Y}_t)_{t \in (-\infty, \infty)}$ .

More precisely, at stationarity the distribution of  $\mathcal{Y}_t$  is  $\mu^{\Xi}$ , and conditional on  $\mathcal{Y}_t$ , the vector  $\mathbf{Y}(t)$  is an i.i.d. sequence of samples from  $\mathcal{Y}_t$ . Note, in particular, that the joint law of the first n levels  $(Y_1(t), \ldots, Y_n(t))$  at stationarity is given by (1.5) for any  $n \in \mathbb{N}$ .

Remark 1.21. The stationary distribution  $\mu^{\Xi}$  is not reversible in the sense that the process  $(\mathbf{Y}(t), \mathcal{Y}_t)_{t \in (-\infty, \infty)}$  running at stationarity has the same law as  $(\mathbf{Y}(-t), \mathcal{Y}_{-t})_{t \in (-\infty, \infty)}$ . The Fleming-Viot process is only reversible in this sense if the underlying spatial motion (or, in the genetics setting, mutation process) is a pure jump process in which the destination of each jump is independent of the location just before the jump. See [19, 24, 27, 33, 40] for literature on this topic.

Remark 1.22. A byproduct of this result is explicit formulas for samples from the stationary distribution of a  $\Xi$ -Fleming-Viot process through (1.5), where  $N^{\Xi}$  can be calculated from (1.3). For instance, recalling Example 1.9, the density of a two-sample from the stationary distribution  $\mu^{\Xi}$  of the  $\Xi$ -Fleming-Viot process is

$$\mathbb{E}\left[\mu^{\Xi}(\mathrm{d}x_1)\mu^{\Xi}(\mathrm{d}x_2)\right] \propto \left(\int_0^\infty \mathrm{e}^{-|\Xi|t} p_{2t}(x_1 - x_2) \,\mathrm{d}t\right) \mathrm{d}x_1 \,\mathrm{d}x_2,$$

normalised such that the integral over  $x_1$  and  $x_2$  is one. Here we used that  $\lambda_2 = \lambda_{2,2}$ , which equals  $|\Xi|$  by (1.1). In particular, the distribution of a two-sample from a stationary  $\Xi$ -Fleming-Viot process only depends on  $|\Xi|$ . Formulas for higher order samples involve sums over an increasing number of trees, and depend on higher order rates of the  $\Xi$ -coalescent.

The particle representation (Y(t)) can be used to define the genealogies of the  $\Xi$ -Fleming-Viot process. Denote by  $L_k(t)$  the level of the ancestor at time -t of the particle at level k at time 0 (assigned at jump times in such a way that  $L_k$  is right-continuous). Let  $\Pi_t$  be the partition of  $\mathbb{N}$  induced by the equivalence relation

$$i \sim_t j \iff L_i(t) = L_i(t).$$

Note that  $L_i(t) = \min\{j: i \sim_t j\}$  by the definition of the lookdown construction, so  $(\Pi_t)$  contains all information about the processes  $(L_i(t): i \in [n])$ , where  $[n] = \{1, \ldots, n\}$ . Denote by

$$\boldsymbol{X}_t(u) = \lim_{s \uparrow -t} Y_{\min u}(s), \qquad t \ge 0, u \in \Pi_t,$$

the position at time -t of the common ancestor of the particles with levels in u at time zero (assigned again in such a way that X is right-continuous).

For  $n \in \mathbb{N}$  and  $t \geq 0$ , write  $\Pi_t^n$  for the partition induced by  $\Pi_t$  on [n], and  $\mathbf{X}_t^n$  for the restriction of  $\mathbf{X}_t$  to  $\Pi_t^n$ . Then  $(\Pi_t^n, \mathbf{X}_t^n)_{t\geq 0}$  is the coalescent process that describes the genealogies of a sample of n individuals from the  $\Xi$ -Fleming-Viot process at stationarity. The initial condition is random:  $\Pi_0 = \{\{1\}, \ldots, \{n\}\}$  by construction, and  $\mathbf{X}_0(\{i\}) = x_i$  is the location of the i'th level particle at time zero, so  $(x_1, \ldots, x_n)$  is an i.i.d. sample from  $\mathcal{Y}_0 \sim \mu^{\Xi}$ .

**Theorem 1.23.** The law of  $(\Pi_t^n, X_t^n)_{t\geq 0}$  is that of a Brownian spatial  $\Xi$ -coalescent started from the random initial condition  $(\Pi_0^n, X_0^n)$ .

In fact, the Brownian spatial  $\Xi$ -coalescent can be used to describe the full time-reversal of the  $\Xi$ -Fleming-Viot process.

**Theorem 1.24.** For any  $n \in \mathbb{N}$ , the law of the time reversal  $(Y_1(-t), \ldots, Y_n(-t))_{t \geq 0}$  at stationarity can be described as follows:

- (i) Sample initial points  $y_1, \ldots, y_n$  from a random realisation of  $\mu^{\Xi}$ .
- (ii) Evolve according to a Brownian spatial  $\Xi$ -coalescent.
- (iii) When a coalescence event occurs, resample points using the measures  $(\mu_{x_1,...,x_n}^{\Xi})$  so that the total number of lineages stays constant.

A precise statement, including how levels are reassigned at coalescence–resampling events, can be found in Section 3.5.1. As an example for rule (iii), if a ternary merger occurred and the remaining lineages are at locations  $y_1, \ldots, y_{n-2}$ , we sample  $y_{n-1}$  from  $\mu_{y_1, \ldots, y_{n-2}}^{\Xi}$ , and  $y_n$  from  $\mu_{y_1, \ldots, y_{n-2}, y_{n-1}}^{\Xi}$ , then resume the Brownian spatial  $\Xi$ -coalescent from  $y_1, \ldots, y_n$ .

1.5.2. Spatial Cannings Models. Next, we present a class of particle models whose forward evolution and genealogies converge in the limit of large population size, properly rescaled, to the  $\Xi$ -Fleming-Viot process and Brownian spatial  $\Xi$ -coalescent, respectively. We stress that the convergence of the forward evolution isn't novel, and is essentially contained in

[13]. We consider a spatial version of the Cannings model. For  $N \in \mathbb{N}$ , let  $(O_1, \ldots, O_N)$  be an exchangeable  $\mathbb{N}_0^N$  valued random variable with  $\sum_{i=1}^N O_i = N$ , and let  $T_N > 0$ . We define an  $E^N$ -valued process  $\mathbf{Y}^N(t) = (Y_1^N(t), \ldots, Y_N^N(t))$  as follows.

- (i) At rate  $T_N$ , draw a random realisation  $(o_1, \ldots, o_N)$  of  $(O_1, \ldots, O_N)$ , independent of previous realisations and current spatial locations of all particles. Partition [N] into sets of sizes  $o_1, \ldots, o_N$  uniformly at random. Then every individual copies the location of the individual with the smallest level in the same set.
- (ii) Individuals follow independent Brownian motions between reproduction events.

Step (i) may look more complicated than it is. It is equivalent to saying "individual i has  $o_i$  offspring", followed by a particular way of reassigning levels in such a way that preserves exchangeability. The version of this process with no spatial motions (and generations at discrete time steps rather than the points of a Poisson process) was studied in the coalescent context by many authors, including Möhle and Sagitov [29]. Following their line of reasoning, the probability that a particular  $(n, \vec{k})$ -merger occurs in a single generation is given by

$$p_{n,\vec{k}}^N = \frac{(N)_{n'}}{(N)_n} \mathbb{E}\left[ (O_1)_{k_1} \dots (O_m)_{k_m} O_{m+1} \dots O_{n'} \right],$$

where  $n' = n - \sum_{i=1}^{n} k_i + m$  is the number of lineages after the merge event, and  $(k)_a = k(k-1) \cdot \ldots \cdot (k-a+1)$  for  $a,k \in \mathbb{N}, \ a \leq k$  (see [29, Eq. (3)]). They give necessary and sufficient conditions under which  $T_N p_{n,\vec{k}}^N$  converges for every  $(n,\vec{k}) \in M$  as  $N \to \infty$ , in which case the limits are the rates  $\lambda_{n,\vec{k}}$  of a non-spatial  $\Xi$ -coalescent, to which the genealogies converge. We essentially prove a spatial analogue. Denote by

$$\mathcal{Y}_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{Y_i^N(t)}, \quad t \ge 0,$$

the empirical measure of  $\mathbf{Y}^{N}(t)$ .

**Proposition 1.25.** The process  $(\mathbf{Y}^N, \mathcal{Y}^N)$  has a stationary distribution, and can be constructed to run bi-infinitely at stationarity, which we denote  $(\mathbf{Y}^N(t), \mathcal{Y}_t^N)_{t \in (-\infty, \infty)}$ .

We can define the genealogies of  $\mathcal{Y}^N$  in the same way as we did previously:  $L_k^N(t)$  for  $t\geq 0$  and  $1\leq k\leq N$  denotes the level of the ancestor at time -t of the particle at level k at time 0. Then  $\Pi_t^N$  denotes the partition on [N] induced by the equivalence relation " $i\sim j\iff L_i^N(t)=L_j^N(t)$ ". Finally  $\boldsymbol{X}_t^N(u)=\lim_{s\uparrow-t}Y_{\min u}^N(s)$  for  $t\geq 0$  and  $u\in \Pi_t^N$ . Write  $\Pi_t^{N,n}$  for the partition induced by  $\Pi_t^N$  on [n], and  $\boldsymbol{X}_t^{N,n}$  for the restriction of  $\boldsymbol{X}_t^N$  to  $\Pi_t^{N,n}$ .

**Theorem 1.26.** For every  $n \in \{1, ..., N\}$ , the distribution of  $(\Pi_t^{N,n}, \mathbf{X}_t^{N,n})_{t\geq 0}$ , conditional on initial locations, is the Brownian spatial coalescent with transition measures

$$\nu_{n,\vec{k}}^N(\mathrm{d}\boldsymbol{z}) = T_N p_{n,\vec{k}}^N \,\mathrm{d}\boldsymbol{z}.$$

Furthermore, the following are equivalent.

(i) The process  $((Y_1^N(t), \dots Y_n^N(t)))_{t>0}$  converges weakly as  $N \to \infty$  for every  $n \in \mathbb{N}$ .

- (ii) The process  $(\Pi_t^{N,n}, \boldsymbol{X}_t^{N,n})_{t\geq 0}$  converges weakly as  $N \to \infty$  for every  $n \in \mathbb{N}$ .
- (iii) The process  $(\Pi_t^{N,n})_{t\geq 0}$  converges weakly as  $N\to\infty$  for every  $n\in\mathbb{N}$ ,
- (iv) The limit  $T_N p_{n,\vec{k}}^N$  as  $N \to \infty$  exists for all  $(n, \vec{k}) \in M$ .

In that case, there exists a unique finite measure  $\Xi$  on  $\triangle$  such that, for every  $n \in \mathbb{N}$ ,  $(Y_1^N, \ldots, Y_n^N)$  converges to the first n levels  $(Y_1, \ldots, Y_n)$  of the  $\Xi$ -Fleming-Viot process, and  $(\Pi^{N,n}, \boldsymbol{X}^{N,n})$  converges to the Brownian spatial  $\Xi$ -coalescent.

Here by weak convergence in (i), (ii), and (iii) we mean convergence in law in the Skorokhod topology. We repeat that the equivalence of (iii) and (iv) is proved in [29].

- Remark 1.27. (i) By arguments in [13], in the case of no simultaneous mergers where  $\Xi$  reduces to a measure  $\Lambda$  on [0,1], if (i) to (iv) hold, then also the weak convergence  $(\mathbf{Y}^N, \mathcal{Y}^N) \Longrightarrow (\mathbf{Y}, \mathcal{Y})$  to the  $\Lambda$ -Fleming-Viot process holds. We expect the corresponding result to be true in the general case of simultaneous mergers.
  - (ii) The proof that the genealogies of the spatial Cannings model are a Brownian spatial coalescent (the first statement in Theorem 1.26) is constructive, see Lemma 3.31. Studying this class of spatial population models could thus have been a way to define the Brownian spatial coalescent in the first place, but it would not have revealed its axiomatic universality.
  - (iii) Similar to the  $\Xi$ -Fleming-Viot process, the full time reversal at stationarity of the spatial Cannings model can be described by the Brownian spatial coalescent "with resampling", with the transition measures defined in Theorem 1.26. See Lemma 3.32.
- 1.6. Related Literature. The inclusion of space in population and coalescent models is the subject of a vast amount of literature. The classical approach is to consider populations that are subdivided into demes of large constant size, each situated at a vertex of a graph. Lineages can merge with other lineages that are currently in the same deme, and migrate between neighbouring demes. We refer to [1] for a short review of classical results in this setting, and point out [28, 42] and references therein for recent results. In reality, populations are often not subdivided but spread across a spatial continuum. One might be tempted to approximate this through deme-models with small granularity, but this breaks the crucial assumption of large population size in every deme, because local population size can be small (see also [18, Section 5]). A common approach that incorporates continuous space directly is to assume that genealogical trees can be constructed from independent Brownian motions (or more general spatial motions [21]) which coalesce when they meet, either instantaneously or at a rate proportional to their collision local time. This is inherently limited to one dimension, and a suggested extension to higher dimensions has lineages coalesce at rates depending on their separation. The position of the common ancestor is typically taken to be a Gaussian centred on the midpoint between the two lineages immediately before the coalescence event. Aside from the fact that this behaviour is biologically unnatural, there is no corresponding forwards in time model for the evolution of the population [1].

A major breakthrough in the theory of spatial population models was the introduction of the Spatial  $\Lambda$ -Fleming-Viot process (SLFV) by Barton, Etheridge and Véber in [18, 1], see

also [2] for a review. It models the genetic composition of a spatially structured population by a measure on  $\mathbb{R}^d \times [0,1]$ , where  $\mathbb{R}^d$  is the geographical space and [0,1] is the space of genetic types. The population density is modelled to be constant so that the spatial marginal is always Lebesgue measure. The population evolves through a sequence of events, each of which replaces a certain proportion of the population in a randomly chosen ball by offspring of an individual chosen at random within the same ball. If the radius is small, the event is interpreted as an ordinary reproduction event subject to local regulation, and if it is large one may think of it as an extinction-recolonisation event. The genealogies of the SLFV (also called its dual) have a simple description: backwards in time, whenever a lineage is hit by an event, with a certain probability (equal to the proportion of the population that was hit forwards in time) it jumps to a randomly chosen location within the same ball, otherwise it stays put. If a number of lineages are affected by the same event, they jump to the same location and coalesce. The dual of the SLFV was the first tractable model for the ancestry of a population evolving in a two-dimensional spatial continuum that was sampling consistent, had a natural corresponding forward in time population model, and in which independently evolving lineages fail to meet. A variant of the SLFV allows for more than one parent per event, leading to the possibility of simultaneous mergers in the dual.

Despite its success, the SLFV has weaknesses. It only models populations with perfectly homogeneous spatial distribution, which in real populations fluctuates strongly in time and both geographical and type space [18, Chapter 7]. Secondly, individual lineages in the SLFV evolve through a sequence of correlated jumps, but in many applications it would be more natural if they followed Brownian motions, or the paths of other continuous Markov processes. The  $\Xi$ -Fleming-Viot processes are a class of spatial population model that showcase both characteristics.

A different approach to spatial coalescents is to replace geographical space with a tree-like structure, such as the hierarchical group which is often used to mimic higher-dimensional spaces. A spatial variant of the  $\Lambda$ -coalescent on the hierarchical group is the subject of [23]. In [22], Freeman also works on a space with a hierarchical structure, on which he considers a coalescent model called the segregated  $\Lambda$ -coalescent.

- 1.7. Conclusion. We introduced the Brownian spatial coalescent, a class of coalescent processes in continuous space of any dimension in the most general case of simultaneous multiple mergers. It has the following three crucial properties, all of which are, to the best of our knowledge, novel in the literature on coalescent models in continuous space in  $d \geq 2$ . First, an individual lineage evolves as a Brownian motion. Second, they describe the genealogies of a class of spatial population models whose spatial distribution is non-homogeneous and fluctuates randomly in time and space, driven by its forward dynamics. Third and most outstandingly, they can be axiomatically characterised through a notion of sampling consistency in direct analogy to the non-spatial  $\Lambda$  and  $\Xi$ -coalescents. In the following paragraphs, we mention some consequences of our results, and possible further research directions.
- 1.7.1. Spatial Population Models with Spatially Interactive Branching. The Brownian spatial coalescent was axiomatically defined through a property that the genealogies of any spatial population model in which individuals follow independent Brownian motions should satisfy. This includes models with spatially interactive branching mechanisms such as competition or

local regulation. Yet Section 1.5.2 shows that the full set of sampling consistent Brownian spatial coalescents is exhausted by spatial models in which the branching mechanism is oblivious to spatial locations. The only other assumptions made in the definition of Brownian spatial coalescents are time-homogeneity and the Markov property. If the spatial model has a stationary distribution, which will certainly be the case for some interesting spatially interactive branching mechanisms, then their genealogies will be time homogeneous, which means they cannot be Markovian. Thus, describing the genealogies of populations whose branching mechanisms have interesting spatial interactions, at least in the present setting, requires the study of non-Markovian coalescent processes. Since we expect most of our results to remain true for much more general spatial motions (see next paragraph), this observation is not limited to population models with Brownian movement.

1.7.2. General Spatial Motions. The definition of the Brownian spatial coalescent, as well as most theorems and proofs (with the exception of the drift representation Theorem 1.15) work with little modification when Brownian motion on the torus is replaced by another well-behaved Markov process  $(Y_t)$  on some compact Polish space E with a stationary distribution  $\lambda$  and continuous transition densities  $(q_t)$  w.r.t.  $\lambda$ , and such that its time reversal w.r.t.  $\lambda$  is also a well-behaved Markov process. Then  $\lambda$  takes the place of Lebesgue measure in the main theorems. Examples include random walks,  $\alpha$ -stable motion or diffusions on fractals. The only major difficulty is understanding which points have to be excluded from the state space. One approach could be to exclude those points where the normalisation  $N^{\nu}$  diverges, but then it remains to show that these points will almost-surely never be reached by the evolution of the coalescent. This comes down to characterising whether or not k independent copies of the spatial motion have a positive probability of meeting simultaneously, and if so in which points. In fact, by studying the normalisation of the " $(Y_t)$ -spatial  $\Lambda$ -coalescent" with  $\Lambda = \delta_1$  (in which only (n, n)-mergers for  $n \in \mathbb{N}$  are possible), this gives rise to the following conjecture.

Conjecture 1.28. Under some regularity conditions on  $(Y_t)$ , the following are equivalent for every  $k \in \mathbb{N}$ ,  $k \geq 2$ .

- (i) If k independent copies of Y are started from any set of pairwise disjoint initial locations, then with positive probability there exists a time at which all k copies are in the same location.
- (ii) For  $\lambda$ -a.e.  $x \in E$ ,  $\int_0^1 \int_E q_s(x,z)^k \lambda(\mathrm{d}z) \, \mathrm{d}s < \infty.$

A simple case is if the motion is a random walk on some finite graph E. Then no points have to be excluded, and all of our results (except Theorem 1.15) apply with obvious modifications.

1.7.3. The Brownian Spatial Coalescent on  $\mathbb{R}^d$ . We have some unpublished results about Brownian spatial coalescents evolving on all of  $\mathbb{R}^d$ , which can be defined analogously to Definition 1.5. Corresponding to each is again a family of transition measures  $\boldsymbol{\nu} = (\nu_{n,\vec{k}})$ , where  $\nu_{n,k_1,\dots,k_m}$  is a locally finite measure on  $(\mathbb{R}^d)^m$ . If  $\boldsymbol{\lambda} = (\lambda_{n,\vec{k}})$  are the rates of a  $\Xi$ -coalescent, then  $\nu_{n,\vec{k}}(\mathrm{d}\boldsymbol{z}) = \lambda_{n,\vec{k}} \, \mathrm{d}\boldsymbol{z}$  is again a valid choice, and the corresponding Brownian

spatial coalescent describes the genealogies of a  $\Xi$ -Fleming-Viot process evolving on  $\mathbb{R}^d$ , at "stationarity modulo translation" (cmp. [12, Thm. 2.9]).

If d > 3, then the class of Brownian spatial coalescents that describe the genealogies of some population is larger than that: for each Brownian spatial coalescent on  $\mathbb{R}^d$  that we described in the previous paragraph, there exists another whose definition is the same except that all exponential factors in the densities on tree decorations (of the form  $e^{-|\nu_n|(\tau-\tau')}$ ) are removed. If  $\nu_{n,2}(\mathrm{d}z) = \mathrm{d}z$  for all  $n \in \mathbb{N}$ , and all other transition measures are equal to zero, then this coalescent describes the genealogies of the infinite-mass superBrownian motion on  $\mathbb{R}^d$  at stationarity (which exists iff d > 3 [17, Chapter 2.7]). It has the property that, from any initial configuration with at least two particles, the probability that no further merge event happens is strictly between zero and one. The existence of coalescents with this property on  $\mathbb{R}^d$  is directly linked to the existence of infinite-mass stationary distributions for associated Dawson-Watanabe superprocesses on  $\mathbb{R}^d$ , which is known to correspond to transience of the underlying spatial motion [15, 16]. This connection can also be seen from the perspective of the Brownian spatial coalescent: Brownian motion is transient if and only if  $\int_C p_t(x) dx$  is integrable over  $t \in (0, \infty)$  for every compact  $C \subset \mathbb{R}^d$  (this characterisation holds mutatis mutandis for more general spatial motions), which is certainly necessary for the normalisation  $N^{\nu}(x)$  to be finite, since integrability at large  $\tau$  is no longer guaranteed by the exponential factors.

Finally, it is not clear how to define sampling consistency on  $\mathbb{R}^d$  in such a way that it captures the coalescents associated with infinite-mass populations, because their stationary distributions are random measures of infinite mass that cannot be normalised to be probability measures.

1.8. Structure of the Paper. In Section 2, we set up the notation necessary to formalise some of the definitions and statements from the introduction, and introduce the framework used for the proofs, which are in Section 3. Appendix A contains proofs of a number of technical lemmas used in Sections 2 and 3, and Appendix B contains the proof of Lemma 2.12, which states that the normalisation function  $N^{\nu}$  is continuous under certain conditions; it is conceptually straight-forward but long and technical.

### 2. Setup

2.1. Non-spatial Coalescents. Recall that  $\mathcal{P}$  denotes the (countable) set of partitions of finite subsets of  $\mathbb{N}$ , which we equip with the discrete topology. For  $\Pi, \Pi' \in \mathcal{P}$  write  $\Pi \leq \Pi'$  if  $\Pi'$  is a refinement of  $\Pi$ , and define

$$\Omega_0 := \{ \Pi \in D([0, \infty), \mathcal{P}) : \Pi_s \ge \Pi_t \forall 0 \le s \le t \},$$

where  $D([0,\infty), \mathcal{P})$  is the space of càdlàg  $\mathcal{P}$ -valued paths, with the usual Skorokhod topology. We equip  $\Omega_0$  with its Borel  $\sigma$ -algebra  $\mathcal{F}^{\Pi} := \sigma(\Pi_t : t \geq 0)$ , and filtration  $\mathcal{F}_t^{\Pi} := \bigcap_{s>t} \sigma(\Pi_r : r \leq s)$  for  $t \geq 0$ , where  $(\Pi_t)_{t\geq 0} := \mathrm{id}_{\Omega_0}$  denotes the canonical process in  $\Omega_0$ . All three spaces  $\mathcal{P}$ ,  $D([0,\infty), \mathcal{P})$ , and  $\Omega_0$  are Polish (see Appendix A).

Definition 2.1. A (non-spatial) coalescent is a time-homogeneous  $\Omega_0$ -valued Markov process  $(\mathbb{P}^\Pi)_{\Pi \in \mathcal{P}}$ . It is called *label invariant* if it is independent of particle labels in the sense that any transition that is the result of merging m disjoint sets of lineages of sizes  $k_1 \geq \ldots \geq k_m$ , while at n particles, occurs at the same rate  $\lambda_{n,k_1,\ldots,k_m}$ .

In general, a coalescent is characterised by its transition rates  $\lambda_{\Pi,\Pi'} \geq 0$  for  $\Pi > \Pi'$ , which induces a bijection between the set of coalescents and the set  $R^{\circ}$  of non-negative arrays  $\boldsymbol{\lambda} = (\lambda_{\Pi,\Pi'} : \Pi > \Pi')$ . For  $\Pi \in \mathcal{P}$  and  $u \subset \mathbb{N}$ , write  $\Pi \setminus u := \{v \setminus u : v \in \Pi, v \neq u\}$  for the partition induced by  $\Pi$  on  $\mathbb{N} \setminus u$ .

Definition 2.2. A coalescent is called sampling consistent if, for any  $\Pi \in \mathcal{P}$  with  $|\Pi| \geq 2$  and  $u \in \Pi$ ,

$$\mathbb{P}^{\Pi}((\Pi_t \setminus u)_{t>0} \in \cdot) = \mathbb{P}^{\Pi \setminus u}((\Pi_t)_{t>0} \in \cdot).$$

2.2. **Spatial Coalescents.** Recall that  $E = \mathbb{R}^d/\mathbb{Z}^d$  denotes the d-dimensional square flat torus with periodic boundary conditions. It is a Polish space with the Euclidean metric

$$\rho(x + \mathbb{Z}^d, y + \mathbb{Z}^d) = \inf\{|x - y + \vec{k}| : \vec{k} \in \mathbb{Z}^d\}, \quad x, y \in \mathbb{R}^d.$$
 (2.1)

Recall the definitions of  $E^{\Pi}$  and  $E^{\Pi}_{\circ}$  for  $\Pi \in \mathcal{P}$  (see (1.2)), and  $\mathcal{X}$ . The identification of  $E^{\Pi}_{\circ}$  with an open subset of  $E^{|\Pi|}$  defines on it a topology and Lebesgue measure, and then  $\mathcal{X}$  is a Polish space with the topology inherited from  $E^{\Pi}_{\circ}$ ,  $\Pi \in \mathcal{P}$ , in which  $(\Pi_n, \boldsymbol{x}_n) \to (\Pi, \boldsymbol{x})$  iff  $\Pi_n = \Pi$  for all but finitely many n, and  $\boldsymbol{x}_n \to \boldsymbol{x}$ . Since  $\Pi = \text{dom}(\boldsymbol{x})$  is the domain of  $\boldsymbol{x}$  for  $(\Pi, \boldsymbol{x}) \in \mathcal{X}$ , we are justified to occasionally just write  $\boldsymbol{x} \in \mathcal{X}$ . Denote

$$\Omega := \{(\Pi_t, \mathbf{X}_t) \in D([0, \infty), \mathcal{X}) : \Pi_t \leq \Pi_s \forall t \geq s \geq 0\},$$

which we equip with its Borel  $\sigma$ -algebra  $\mathcal{F}^{\mathbf{X}} := \sigma(\mathbf{X}_t : t \geq 0)$ , and the filtration  $\mathcal{F}_t^{\mathbf{X}} := \bigcap_{s>t} \sigma(\mathbf{X}_r : r \leq s)$  for  $t \geq 0$ , where  $(\mathbf{X}_t)_{t\geq 0} := (\Pi_t, \mathbf{X}_t)_{t\geq 0} := \mathrm{id}_{\Omega}$  denotes the canonical process in  $\Omega$ . We prove that  $\Omega$  is Polish in Appendix  $\mathbf{A}$ .

Definition 2.3. A spatial coalescent is a time-homogeneous Ω-valued Markov process  $(\mathbb{P}^x)_{x \in \mathcal{X}}$  with the Feller property.

The Feller property in this context states that the semigroup, defined by

$$S_t(\boldsymbol{x}, A) := \mathbb{P}^{\boldsymbol{x}}(\boldsymbol{X}_t \in A) \tag{2.2}$$

for  $x \in \mathcal{X}$ ,  $t \geq 0$ , and  $A \subset \mathcal{X}$  measurable, satisfies  $S_t f \in C_b(\mathcal{X})$  for every  $f \in C_b(\mathcal{X})$ , the set of bounded continuous functions  $\mathcal{X} \to \mathbb{R}$ . The Feller property is a form of continuity in the initial condition which is automatic in the non-spatial setting, and it implies the strong Markov property: if T is an  $(\mathcal{F}_t^{\mathcal{X}})$ -stopping time, then

$$\mathbb{P}^{\boldsymbol{x}}\left(T<\infty,(\boldsymbol{X}_{T+t})_{t\geq0}\in\cdot\,\big|\,\mathcal{F}_T^{\boldsymbol{X}}\right)=\mathbb{1}_{\{T<\infty\}}\mathbb{P}^{\boldsymbol{X}_T}((\boldsymbol{X}_t)_{t\geq0}\in\cdot),\qquad\mathbb{P}^{\boldsymbol{x}}\text{-a.s.}$$

for every  $x \in \mathcal{X}$ .

The formal definition of label invariance in the spatial setting is a bit more technical than in the non-spatial setting, because a convenient formulation in terms of transition rates is not available. (Within the class of Brownian spatial coalescents, Lemma 3.12 provides an analogous characterisation.) Suppose that  $\Pi_0, \Pi_1 \in \mathcal{P}$  with  $|\Pi_0| = |\Pi_1|$ . Then the law of a label invariant spatial coalescent should not be affected if we change the labels of the initial set of lineages with a bijection  $\iota \colon \Pi_0 \to \Pi_1$ . To apply this change of labels to the entire process  $(\Pi_t)_{t\geq 0}$ , we extend  $\iota$  to a map  $\bigcup_{\Pi\leq \Pi_0}\Pi \to \bigcup_{\Pi\leq \Pi_1}\Pi$  with the property that  $\iota(u\cup v)=\iota(u)\cup\iota(v)$  whenever  $u,v,u\cup v$  are in the extended domain of  $\iota$  (this extension is unique). To perform the change of labels in the spatial variables, we extend  $\iota$  for every  $\Pi\subset \bigcup_{\Pi'\leq \Pi_0}\Pi'$  to a map  $E_o^\Pi\to E_o^{\iota(\Pi)}$  through  $\iota(x)=x\circ\iota^{-1}$ . Denote by  $\mathcal{I}(\Pi_0,\Pi_1)$  the set of bijections  $\iota\colon \Pi_0\to \Pi_1$  that are extended in the way described above.

Definition 2.4. A spatial coalescent  $(\mathbb{P}^{\boldsymbol{x}})_{\boldsymbol{x}\in\mathcal{X}}$  is called *label invariant* if for any two sets  $\Pi_0, \Pi_1 \in \mathcal{P}$  of equal size,  $\iota \in \mathcal{I}(\Pi_0, \Pi_1)$ , and any  $\boldsymbol{x} \in E_{\circ}^{\Pi_0}$ ,

$$\mathbb{P}^{\iota(\boldsymbol{x})}((\boldsymbol{X}_t)_{t\geq 0} \in \cdot) = \mathbb{P}^{\boldsymbol{x}}((\iota(\boldsymbol{X}_t))_{t\geq 0} \in \cdot). \tag{2.3}$$

We now make precise the notion of sampling consistency for spatial coalescents introduced informally in Definition 1.11. If  $\boldsymbol{x} \in E^A$  and  $\boldsymbol{y} \in E^B$  for disjoint sets A and B, we can join the two maps and denote the result by  $\boldsymbol{x}\boldsymbol{y} \in E^{A \cup B}$ , so

$$xy \colon A \cup B \to E; \ u \mapsto \begin{cases} x(u), & u \in A, \\ y(u), & u \in B. \end{cases}$$
 (2.4)

For  $(\Pi, \boldsymbol{x}) \in \mathcal{P}$ ,  $u \subset \mathbb{N}$  with  $u \cap \Pi = \emptyset$  and  $y \in E$ , write  $\boldsymbol{x}y \colon \Pi \cup \{u\} \to E$  for the extension of  $\boldsymbol{x}$  to  $\Pi \cup \{u\}$  with  $u \mapsto y$ , where u is suppressed in the notation and will be clear from context. Then for  $(\Pi, \boldsymbol{x}) \in \mathcal{X}$  and any such  $u \subset \mathbb{N}$ , and  $\Pi' \in \mathcal{P}$  disjoint from  $\Pi$ , write

$$E_{\boldsymbol{x}} := \{ y \in E \colon \boldsymbol{x}y \in E_{\circ}^{\Pi \cup \{u\}} \},$$

$$E_{\boldsymbol{x}}^{\Pi'} := \{ \boldsymbol{y} \in E_{\circ}^{\Pi'} \colon \boldsymbol{x}\boldsymbol{y} \in E_{\circ}^{\Pi \cup \Pi'} \}.$$

$$(2.5)$$

(The first definition does not depend on the choice of u.) For  $(\Pi, \mathbf{x}) \in \mathcal{X}$  and any  $u \in \Pi$ , define  $(\mathbf{x} \setminus u) : (\Pi \setminus u) \to E$  by  $v \setminus u \mapsto \mathbf{x}(v)$ . Write  $\mathcal{M}_1(S)$  for the set of probability measures on a measurable space S.

Definition 2.5. A spatial coalescent  $(\mathbb{P}^{\boldsymbol{x}})_{\boldsymbol{x}\in\mathcal{X}}$  is called sampling consistent if there exists a family of probability measures  $(\mu_{\boldsymbol{x}}\in\mathcal{M}_1(E_{\boldsymbol{x}})\colon\boldsymbol{x}\in\mathcal{X})$  such that for any  $\Pi\in\mathcal{P}$ ,  $|\Pi|\geq 2$ , and  $u\in\Pi$ ,  $\boldsymbol{x}\in E_{\circ}^{\Pi\setminus u}$ ,

$$\int_{E_{-}} \mathbb{P}^{xy}((\boldsymbol{X}_{t} \setminus u)_{t \geq 0} \in \cdot) \mu_{\boldsymbol{x}}(\mathrm{d}y) = \mathbb{P}^{\boldsymbol{x}}((\boldsymbol{X}_{t})_{t \geq 0} \in \cdot). \tag{2.6}$$

With these definitions, the projection of a (label invariant or sampling consistent) spatial coalescent onto  $\mathcal{P}$  is a (label invariant or sampling consistent) coalescent.

2.3. Brownian Spatial Coalescents. The law of a Brownian spatial coalescent will be defined using densities over time and space decorations of all possible shapes of the genealogical forest, as outlined in the introduction. We introduce the necessary notation.

Definition 2.6. A forest is a strictly decreasing (necessarily finite) sequence in  $\mathcal{P}$ .

The notation for a generic forest is  $F = (\Pi_0^F, \dots, \Pi_m^F)$ , where  $\operatorname{lf}(F) \coloneqq \Pi_0^F$  is the set of leaves,  $\Pi_i^F$  for  $i \in [m]$  is the set of nodes (or vertices) immediately after the ith merge event, and  $\operatorname{rt}(F) \coloneqq \Pi_m^F$  is the set of roots. Note that this notion of a forest encodes its topology and the order and possibly simultaneousness of merge events (but no branch lengths or spatial information). This is important because already in the non-spatial setting, forests with the same topology but different order of merge events may have different probability. Write  $\mathbb F$  for the set of all forests, and for  $\Pi, \Pi' \in \mathcal P$  with  $\Pi' \leq \Pi$ ,

$$\mathbb{F}(\Pi) = \{ F \in \mathbb{F} \colon \operatorname{lf}(F) = \Pi \},$$

$$\mathbb{F}(\Pi, \Pi') = \{ F \in \mathbb{F} \colon \operatorname{lf}(F) = \Pi, \operatorname{rt}(F) = \Pi' \}.$$

We call  $F \in \mathbb{F}$  a tree if  $|\operatorname{rt}(F)| = 1$ , and denote by  $\mathbb{T}$  and  $\mathbb{T}(\Pi)$  for  $\Pi \in \mathcal{P}$  the set of all trees, and the set of trees with leaves  $\Pi$ , respectively. Write  $\operatorname{nd}(F) := \bigcup_{i=0}^m \Pi_i$  for the set of

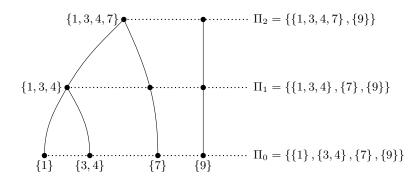


FIGURE 2. Illustration of the notation used for forests. In this example,  $\operatorname{ch}_F(\{1,3,4\}) = \{\{1\},\{3,4\}\}, \text{ and } \operatorname{pr}_F(\{7\}) = \{1,3,4,7\}, \text{ and } \operatorname{ch}_F(\{9\}) = \operatorname{pr}_F(\{9\}) = \emptyset.$ 

all nodes, and  $\operatorname{nd}^{\circ}(F) := \operatorname{nd}(F) \setminus \operatorname{lf}(F)$  for the non-leaf nodes of F. Call F trivial if m = 0, that is if  $\operatorname{rt}(F) = \operatorname{lf}(F)$ . We occasionally identify F with either of the sets  $\{\Pi_0^F, \dots, \Pi_m^F\}$  or  $\{(\Pi_0^F, \Pi_1^F), \dots, (\Pi_{m-1}^F, \Pi_m^F)\}$ . The maps

$$\operatorname{ch}_F \colon \operatorname{nd}(F) \to 2^{\operatorname{nd}(F)}, \quad \operatorname{pr}_F \colon \operatorname{nd}(F) \to \operatorname{nd}(F) \cup \{\emptyset\}$$

assign to a node its children and its parent, respectively, where  $\operatorname{ch}_F(u) := \emptyset$  for  $u \in \operatorname{lf}(F)$  and  $\operatorname{pr}_F(u) := \emptyset$  for  $u \in \operatorname{rt}(F)$ . Figure 2 illustrates this notation.

Definition 2.7. A time decoration of a forest F is a map  $\tau \colon F \to (0, \infty)$  with  $\tau(\operatorname{lf}(F)) = 0$  and  $\tau(\Pi) < \tau(\Pi')$  for  $\Pi > \Pi'$ . Write  $\operatorname{tm}(F)$  for the set of time decorations of F, and

$$\operatorname{tm}(\mathbb{F}) = \{ (F, \tau) \colon F \in \mathbb{F}, \tau \in \operatorname{tm}(F) \}$$

for the set of time decorated forests. Define tm(S) for  $S \subset \mathbb{F}$  analogously.

Since  $F = \operatorname{dom}(\tau)$  for  $(F, \tau) \in \operatorname{tm}(\mathbb{F})$  we occasionally just write  $\tau \in \operatorname{tm}(\mathbb{F})$ . If  $F = (\Pi_0^F, \dots, \Pi_m^F) \in \mathbb{F}$  and  $\tau \in \operatorname{tm}(F)$ , then for  $i \in [m-1]$  we write  $\tau(\Pi_i^F)^+ := \tau(\Pi_{i+1}^F)$ , and  $\tau(\Pi_m^F)^+ := \infty$ . We extend  $\tau$  to  $\operatorname{nd}(F)$  by writing

$$\tau_u := \min \{ \tau(\Pi) \colon u \in \Pi \in F \}, \quad u \in \mathrm{nd}(F),$$

for the time of a nodes "birth", and we set  $\tau_{\emptyset} = \infty$  so that  $\tau_{\operatorname{pr}_F(u)} = \infty$  if u has no parent. See Fig. 3 for an illustration. If F is non-trivial, we can define a topology and Lebesgue measure on  $\operatorname{tm}(F)$  using the obvious bijection with an open subset of  $\mathbb{R}^m$ . If F is trivial then  $\operatorname{tm}(F)$  is a singleton whose only element is defined by  $\tau_{\operatorname{lf}(F)} = 0$ , and we equip  $\operatorname{tm}(F)$  with the measure that assigns mass one to the single member. In both cases we denote the measure on  $\operatorname{tm}(F)$  by  $\mathrm{d}\tau$ . The set  $\operatorname{tm}(\mathbb{F})$  inherits its topology from  $\operatorname{tm}(F)$ , noting it can be identified with a discrete union.

Definition 2.8. A spatial decoration of a forest F is a map  $\xi$ :  $\operatorname{nd}^{\circ}(F) \to E$  such that  $\xi|_{\Pi} \in E_{\circ}^{\Pi}$  for all  $\Pi \in F$ . Write  $\operatorname{sp}(F)$  for the set of spatial decorations of F.

If F is non-trivial, we define a topology and Lebesgue measure on  $\operatorname{sp}(F)$  using the obvious bijection with an open subset of  $E^{|\operatorname{nd}^{\circ}(F)|}$ . If F is trivial then  $\operatorname{sp}(F)$  is a singleton comprised

of the unique map  $\xi \colon \emptyset \to E$ , and we equip  $\operatorname{sp}(F)$  with the measure that assigns mass one to the single member. In both cases write  $\operatorname{d}\xi$  for the measure on  $\operatorname{sp}(F)$ .

The following maps assign to a path in  $\Omega_0$  its associated undecorated and time decorated coalescence forest, respectively.

Fr: 
$$\Omega_0 \to \mathbb{F}$$
;  $(\Pi_t) \mapsto \{\Pi_t : t \ge 0\}$  (the range of the path  $(\Pi_t)$ )  
Tm:  $\Omega_0 \to \operatorname{tm}(\mathbb{F})$ ;  $\omega = (\Pi_t) \mapsto \left(\operatorname{Fr}(\omega), \left[\Pi \mapsto \inf\{t > 0 : \Pi_t = \Pi\}\right]\right)$ 

We can think of a (non-spatial) coalescent as a random, time decorated forest, and indeed Tm is a bijective and in fact bimeasurable map. In particular, the laws  $\mathbb{P}^{\Pi}$  of a coalescent are determined by the pushforward laws  $\operatorname{Tm} \# \mathbb{P}^{\Pi}$  on the set of time-decorated forests, which can be calculated explicitly using simple arguments of competing exponential clocks, see Lemma 3.16 below. We identify Fr and Tm with their extensions to  $\Omega$  (obtained by composing with the projection  $\Omega \to \Omega_0$ ).

In the spatial setting, we can assign to a path in  $\Omega$  a forest decorated with space in addition to time coordinates. For  $\omega = (\boldsymbol{x}_t) \in \Omega$  with  $F = \operatorname{Fr}(\omega)$ , we write

$$\operatorname{Sp}(\omega) := \left[ u \mapsto \boldsymbol{x}_{\operatorname{Tm}(\omega)_u}(u) \right] \in \operatorname{sp}(F)$$

for the associated space decoration of F. Let  $dec(F) = tm(F) \times sp(F)$  for  $F \in \mathbb{F}$ , and write

$$\operatorname{dec}(\mathbb{F}) = \{ (F, \tau, \xi) \colon F \in \mathbb{F}, \tau \in \operatorname{tm}(F), \xi \in \operatorname{sp}(F) \}$$

for the set of time and space decorated forests, and define  $\operatorname{dec}(S)$  for  $S \subset \mathbb{F}$  analogously. We write  $F^* = (F, \tau, \xi)$  for a generic element of  $\operatorname{dec}(\mathbb{F})$ . The topology on  $\operatorname{dec}(\mathbb{F})$  is inherited from that of  $\operatorname{dec}(F)$ ,  $F \in \mathbb{F}$ , noting it can be identified with a discrete union.

Since  $F = \operatorname{dom}(\tau)$  for  $(F, \tau, \xi) \in \operatorname{dec}(\mathbb{F})$  we occasionally just write  $(\tau, \xi) \in \operatorname{dec}(\mathbb{F})$ . The following map assigns to a path in  $\Omega$  its associated decorated forest, see Fig. 3 for an illustration.

Dec: 
$$\Omega \to \operatorname{dec}(\mathbb{F}); \quad \omega = (\boldsymbol{x}_t) \mapsto \Big(\operatorname{Tm}(\omega), \big[u \mapsto \boldsymbol{x}_{\operatorname{Tm}(\omega)_u}(u)\big]\Big).$$
 (2.7)

It encodes all information about the path  $\omega$  except the spatial motion of lineages in between merge events. Brownian spatial coalescents are exactly those for which this motion is Brownian conditional on the decorated forest, which means that its laws  $\mathbb{P}^x$  are determined completely by the laws  $\operatorname{Dec} \#\mathbb{P}^x$  on the space of decorated coalescence forests. To make this precise, we introduce a stochastic kernel  $K_x(F^*,\cdot) \in \mathcal{M}_1(\Omega)$  describing the law of a Brownian spatial coalescent conditional on its decorated forest  $F^* \in \operatorname{dec}(\mathbb{F})$  and initial condition  $x \in E_0^{\operatorname{lf}(F)}$  (see Lemma 2.9). Then a spatial coalescent is a Brownian spatial coalescent if and only if  $K_x$  is a conditional probability of  $\mathbb{P}^x$  given  $\operatorname{Dec}$ , that is if  $\mathbb{P}^x = (\operatorname{Dec} \#\mathbb{P}^x) \otimes K_x$  for all  $x \in \mathcal{X}$ . See Definition 2.10 below. Measurability of the maps Fr, Tm, Sp, and Dec is proved in Appendix A.

The following (continuous) maps capture the motion of particles along each branch of the coalescence forest:

$$\begin{aligned} \operatorname{Path}_{u}^{F^{\star}} \colon \{ \operatorname{Dec} &= F^{\star} \} \ \to \ D([\tau_{u}, \tau_{\operatorname{pr}_{F}(u)}), E), \\ (\boldsymbol{x}_{t}) &\mapsto (\boldsymbol{x}_{t}(u))_{\tau_{u} \leq t < \tau_{\operatorname{pr}_{F}(u)}}, \end{aligned}$$

for  $F^* = (F, \tau, \xi) \in \text{Dec}(\mathbb{F})$  and  $u \in \text{nd}(F)$ , where  $\{\text{Dec} = F^*\} = \{\omega \in \Omega \colon \text{Dec}(\omega) = F^*\}$ . See again Fig. 3. Denote the law of a standard Brownian motion in E (with periodic

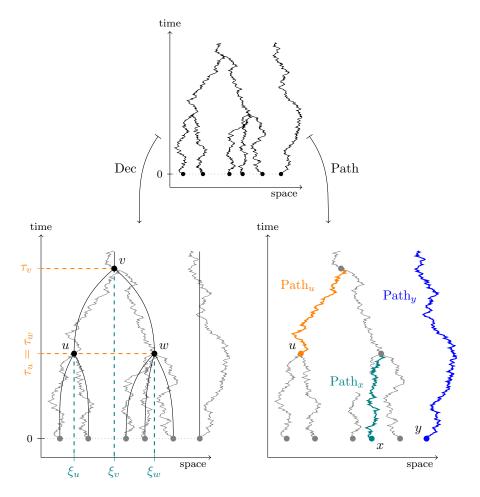


FIGURE 3. At the top is an illustration of an element  $\omega \in \Omega$ . The map Dec (bottom left) extracts the abstract coalescence forest as well as times and spatial locations of the merge events. The Path maps (bottom right) extract the motion of particles along branches of the coalescence forest.

boundary conditions) started at some x at time  $t \geq 0$  by  $B^{(t,x)+}$ , the law of a Brownian bridge started at time  $s \geq 0$  at  $x \in E$ , ending at time t > s at  $y \in E$  by  $B^{(s,x) \to (t,y)}$ , the law of the same bridge followed by a Brownian motion started at time t at y by  $B^{(s,x) \to (t,y)+}$  etc. Recall our notation for joining maps: if  $\mathbf{x} \in E_{\circ}^{\mathrm{lf}(F)}$  and  $\xi \in \mathrm{sp}(F)$  then  $(\mathbf{x}\xi)$  is the map  $\mathrm{nd}(F) \to E$  that extends  $\xi$  to  $\mathrm{lf}(F)$  using  $\mathbf{x}$ .

**Lemma 2.9.** Given  $F^* = (F, \tau, \xi) \in \operatorname{Dec}(\mathbb{F})$  and  $\boldsymbol{x} \in E_{\circ}^{\operatorname{lf}(F)}$ , there is a unique law  $K_{\boldsymbol{x}}(F^*, \cdot) \in \mathcal{M}_1(\Omega)$  under which  $\boldsymbol{X}_0 = \boldsymbol{x}$  and  $\operatorname{Dec} = F^*$  a.s., and

$$\left(\operatorname{Path}_{u}^{F^{\star}}(\boldsymbol{X}) \colon u \in \operatorname{nd}(F)\right)$$

is a family of independent random variables such that the law of  $\operatorname{Path}_{u}^{F^{\star}}(\boldsymbol{X})$  is  $B^{(\tau_{u},(\boldsymbol{x}\xi)_{u})+}$  if  $\operatorname{pr}_{F}(u)=\emptyset$ , and  $B^{(\tau_{u},(\boldsymbol{x}\xi)_{u})\to(\tau_{\operatorname{pr}_{F}(u)},\xi_{\operatorname{pr}_{F}(u)})}$  if  $\operatorname{pr}_{F}(u)\neq\emptyset$ . There exists an extension of these laws to a family  $(K_{\boldsymbol{x}}(F^{\star},\cdot)\colon \boldsymbol{x}\in\mathcal{X},F^{\star}\in\operatorname{dec}(\mathbb{F}))$  such that

$$\mathcal{X} \times \operatorname{dec}(\mathbb{F}) \to \mathcal{M}_1(\Omega); \quad (\boldsymbol{x}, F^{\star}) \mapsto K_{\boldsymbol{x}}(F^{\star}, \cdot)$$

is continuous. In particular,  $(x, F^*) \mapsto K_x(F^*, A)$  is measurable for any measurable  $A \subset \Omega$ .

The proof of Lemma 2.9 is in Appendix A. The exact definition of  $K_{\boldsymbol{x}}(F^{\star},\cdot)$  when  $\boldsymbol{x} \notin E_{\circ}^{\mathrm{lf}(F)}$  is irrelevant, we only need this extension to ensure  $K_{\boldsymbol{x}}(F^{\star},\cdot)$  can be integrated over  $F^{\star} \in \mathrm{dec}(\mathbb{F})$  for fixed  $\boldsymbol{x} \in \mathcal{X}$ , like in (2.8) below.

Definition 2.10. A spatial coalescent  $(\mathbb{P}^x)_{x \in \mathcal{X}}$  is called a Brownian spatial coalescent if  $K_x$  is a conditional probability of  $\mathbb{P}^x$  given Dec, that is if

$$\mathbb{P}^{\boldsymbol{x}}(\cdot) = P^{\boldsymbol{x}} \otimes K_{\boldsymbol{x}}(\cdot) = \int_{\operatorname{dec}(\mathbb{F})} K_{\boldsymbol{x}}(F^{\star}, \cdot) P^{\boldsymbol{x}}(\mathrm{d}F^{\star}), \tag{2.8}$$

for every  $x \in \mathcal{X}$ , where  $P^x := \operatorname{Dec} \# \mathbb{P}^x \in \mathcal{M}_1(\operatorname{dec}(\mathbb{F}))$ .

In particular, a Brownian spatial coalescent is fully determined by the laws  $P^x$ , so we can essentially think of it as a random, time and space decorated forest. This is a useful simplification, and the main work in the characterisation of Brownian spatial coalescents Theorem 2.14 is to understand which families of laws  $(P^x)_{x\in\mathcal{X}}$  give rise to a Markov process on  $\Omega$  through (2.8).

We close by formalising the characterisation of a Brownian spatial coalescent in terms of its transition measures. Recall the heuristic description we provided in Section 1.3: for every tree (forest in general) we define a density on its spatial and time decorations, comprised of exponential factors for every merge event, and spatial factors associated with each branch of the forest. The density is w.r.t. Lebesgue measure in the time coordinates, and w.r.t. the transition measures in the spatial coordinates.

Definition 2.11. A family of transition measures is a collection  $\boldsymbol{\nu} = (\nu_{\Pi,\Pi'}: \Pi' < \Pi)$ , where  $\nu_{\Pi,\Pi'}$  is a finite measure on  $E_{\circ}^{\Pi'\setminus\Pi}$ . Write R for the set of all families of transition measures.

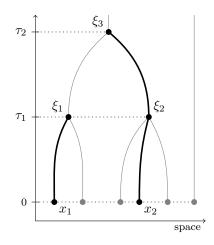
The spatial factors do not depend on the transition measures, and are given for a non-trivial forest  $F \in \mathbb{F}$ ,  $\tau \in \text{tm}(F)$  and  $\boldsymbol{x} \in E_{\circ}^{\text{lf}(F)}$ , by

$$f_{\mathrm{sp}}(\xi \mid \tau, \boldsymbol{x}) := \prod_{u \in \mathrm{nd}(F) \backslash \mathrm{rt}(F)} p(\tau_{\mathrm{pr}_F(u)} - \tau_u, (\boldsymbol{x}\xi)_u - \xi_{\mathrm{pr}_F(u)}), \qquad \xi \in \mathrm{sp}(F).$$
 (2.9)

See Fig. 4 for an illustration. Here and in the following, we sometimes write p(t,x) instead of  $p_t(x)$  if it benefits legibility. If F is trivial we put  $f_{\rm sp}|_{{\rm sp}(F)}\equiv 1$ . The dependence on F is implicit in  $\xi$  and  $\tau$ , but we will occasionally write  $f_{\rm sp}^F$  to make it explicit. Given transition measures  $\nu \in R$  and  $\Pi \in \mathcal{P}$ , write

$$|
u_{\Pi}| \coloneqq \sum_{\Pi' \subset \Pi} |
u_{\Pi,\Pi'}|,$$

which is zero if  $|\Pi| = 1$ . If the process is label invariant, then  $|\nu_{\Pi}|$  only depends on  $n = |\Pi|$ , and we write  $|\nu_{n}|$  like we did in the introduction. It turns out that, as in the non-spatial setting, if  $|\nu_{\Pi}| = 0$  then  $\Pi$  is an absorbing state (in the sense that almost-surely no further



$$f_{\rm sp}(\xi \mid \tau, \boldsymbol{x}) = p_{\tau_1}(\xi_1 - x_1)$$

$$\times p_{\tau_1}(\xi_2 - x_2)$$

$$\times p_{\tau_2 - \tau_1}(\xi_3 - \xi_2)$$

$$\times \dots$$

FIGURE 4. Illustration of  $f_{\rm sp}$  defined in (2.9). Note that only spatial factors associated with three of the seven branches (those highlighted on the left) are written out in the formula on the right.

coalescence events occur), and if  $|\nu_{\Pi}| > 0$  then almost-surely there is at least one more coalescence event. In particular, given an initial state  $(\Pi, \boldsymbol{x}) \in \mathcal{X}$ , only forests  $F \in \mathbb{F}(\Pi)$  with  $|\nu_{\mathrm{rt}(F)}| = 0$  are possible. Given such a forest, the full (unnormalised) density on  $\mathrm{dec}(F)$  is

$$f_{\boldsymbol{\nu}}(\tau,\xi \mid \boldsymbol{x}) := f_{\mathrm{sp}}(\xi \mid \tau, \boldsymbol{x}) \prod_{(\Pi,\Pi') \in F} e^{-|\nu_{\Pi}|(\tau_{\Pi'} - \tau_{\Pi})}, \qquad (\tau,\xi) \in \mathrm{dec}(F).$$
 (2.10)

If  $F \in \operatorname{dec}(\mathbb{F})$  is otherwise then we put  $f_{\boldsymbol{\nu}}(\cdot | \boldsymbol{x})|_{\operatorname{dec}(F)} \equiv 0$ , which defines  $f_{\boldsymbol{\nu}}(\cdot | \boldsymbol{x})$  on all of  $\operatorname{dec}(\mathbb{F})$ . Note if F is trivial and  $|\nu_{\operatorname{rt}(F)}| = 0$  then  $f_{\boldsymbol{\nu}}(\cdot | \boldsymbol{x})|_{\operatorname{dec}(F)} \equiv 1$ . The density  $f_{\boldsymbol{\nu}}$  is w.r.t. Lebesgue measure in the time coordinates, and w.r.t. the relevant transition measures in the spatial coordinates: if  $F \in \mathbb{F}$  is non-trivial, write

$$\nu_F(\mathrm{d}\xi) := \prod_{(\Pi,\Pi')\in F} \nu_{\Pi,\Pi'}(\mathrm{d}\xi_{\Pi,\Pi'}),\tag{2.11}$$

where  $\xi_{\Pi,\Pi'} := \xi|_{\Pi'\setminus\Pi}$  for  $\xi \in \operatorname{sp}(F)$  and  $(\Pi,\Pi') \in F$ , so that  $\nu_F$  is a finite measure on  $\operatorname{sp}(F)$ . If F is trivial then  $\nu_F$  denotes the unique probability measure on the singleton  $\operatorname{sp}(F)$ . It remains to define the normalisation. For  $(\Pi, \boldsymbol{x}) \in \mathcal{X}$  and  $F \in \mathbb{F}(\Pi)$  write

$$N_F^{\boldsymbol{\nu}}(\boldsymbol{x}) \coloneqq \int_{\operatorname{dec}(F)} f_{\boldsymbol{\nu}}(\tau, \xi \,|\, \boldsymbol{x}) \boldsymbol{\nu}_F(\mathrm{d}\xi) \, \mathrm{d}\tau, \qquad N^{\boldsymbol{\nu}}(\boldsymbol{x}) \coloneqq \sum_{F \in \mathbb{F}(\Pi)} N_{\boldsymbol{\nu}}^F(\boldsymbol{x}),$$

defining functions respectively on  $E_{\circ}^{\mathrm{lf}(F)}$  and  $\mathcal{X}$ . These quantities are not obviously, and will not generally, be finite. The following lemma gives a sufficient condition. If  $\lambda \in R^{\circ}$  then we formally write  $\nu(\mathrm{d}\xi) = \lambda \, \mathrm{d}\xi$  if  $\nu_{\Pi,\Pi'}(\mathrm{d}\xi) = \lambda_{\Pi,\Pi'} \, \mathrm{d}\xi$  for all  $\Pi' < \Pi$ . We write  $\nu(\mathrm{d}\xi) \sim \mathrm{d}\xi$  if there exists  $\lambda \in R^{\circ}$  with  $\nu(\mathrm{d}\xi) = \lambda \, \mathrm{d}\xi$ .

**Lemma 2.12.** If  $\nu(d\xi) \sim d\xi$  then  $N^{\nu}$  is continuous (in particular finite).

Remark 2.13. Under the assumption of Lemma 2.12,  $N^{\nu}(x) \to \infty$  as x approaches  $E^{\Pi} \setminus E_{\circ}^{\Pi}$  for some  $\Pi \in \mathcal{P}$ .

Given a measure P on  $\operatorname{dec}(\mathbb{F})$ , it will be a convenient notation to write  $P(F, \operatorname{d}\tau, \operatorname{d}\xi)$  for the measure on  $\operatorname{dec}(F)$  defined by  $\int_A P(F, \operatorname{d}\tau, \operatorname{d}\xi) := P(\{F\} \times A)$  for  $A \subset \operatorname{dec}(F)$ . Analogously in similar contexts. With this notation, a formal statement of Theorems 1.7 and 1.8 is as follows

**Theorem 2.14.** A spatial coalescent is a Brownian spatial coalescent if and only if there exists  $\nu \in R$  such that  $N^{\nu}$  is continuous and

$$P^{\boldsymbol{x}}(F, d\tau, d\xi) = \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} f_{\boldsymbol{\nu}}(\tau, \xi \mid \boldsymbol{x}) \boldsymbol{\nu}_F(d\xi) d\tau.$$
 (2.12)

We call it the Brownian spatial coalescent with transition measures  $\nu$ . In that case,  $x \mapsto \mathbb{P}^x$  is continuous w.r.t. the topology of weak convergence.

If a Brownian spatial coalescent is label invariant, then the family of transition measures reduces to a family  $(\nu_{n,\vec{k}}:(n,\vec{k})\in M)$  as explained in the introduction, see Lemma 3.12.

### 3. Proofs

3.1. Characterisation of Brownian Spatial Coalescents. In this section we prove Theorem 2.14.

3.1.1. "If" Direction of Theorem 2.14. Fix  $\nu \in R$  throughout, denote by  $(P^{\boldsymbol{x}})_{\boldsymbol{x} \in \mathcal{X}}$  the laws defined in (2.12), and  $\mathbb{P}^{\boldsymbol{x}} = P^{\boldsymbol{x}} \otimes K_{\boldsymbol{x}}$  for  $\boldsymbol{x} \in \mathcal{X}$ . We need to show that the coalescent process defined by  $(\mathbb{P}^{\boldsymbol{x}})_{\boldsymbol{x} \in \mathcal{X}}$  is Markov. Recall that the associated semigroup is denoted  $(S_t)_{t \geq 0}$ , see (2.2). For fixed  $\Pi' \leq \Pi$  and  $A \subset E_0^{\Pi'}$  we write

$$S_t((\Pi, \boldsymbol{x}), (\Pi', A)) := S_t((\Pi, \boldsymbol{x}), \{(\Pi', \boldsymbol{y}) : \boldsymbol{y} \in A\}),$$

which defines a sub-probability measure on  $E_{\circ}^{\Pi'}$  with total mass  $\mathbb{P}^{(\Pi,\boldsymbol{x})}(\Pi_t = \Pi')$ . We introduce some additional notation. Let  $F \in \mathbb{F}$  be a (possibly trivial) forest,  $\boldsymbol{x} \in E_{\circ}^{\mathrm{lf}(F)}$ ,  $\tau \in \mathrm{tm}(F)$ ,  $\xi \in \mathrm{sp}(F)$ , and  $t > \tau_{\mathrm{rt}(F)}$ ,  $\boldsymbol{y} \in E_{\circ}^{\mathrm{rt}(F)}$ . Then

$$f_{\boldsymbol{\nu}}(\tau, \xi, t, \boldsymbol{y} \mid \boldsymbol{x}) := f_{\boldsymbol{\nu}}(\tau, \xi \mid \boldsymbol{x}) e^{-|\nu_{\text{rt}(F)}|(t - \tau_{\text{rt}(F)})} \prod_{u \in \text{rt}(F)} p_{t - \tau_{u}}(\boldsymbol{y}_{u} - (\boldsymbol{x}\xi)_{u})$$

If  $t \leq \tau_{\mathrm{rt}(F)}$  we put  $f_{\nu}(\tau, \xi, t, \boldsymbol{y} \mid \boldsymbol{x}) \coloneqq 0$ .

**Lemma 3.1.** If  $(\Pi, x) \in \mathcal{X}$  and  $\Pi' \leq \Pi$ , then

$$S_t((\Pi, \boldsymbol{x}), (\Pi', d\boldsymbol{y})) = \frac{N^{\boldsymbol{\nu}}(\boldsymbol{y})}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} \sum_{F \in \mathbb{F}(\Pi, \Pi')} \left( \int_{\text{dec}(F)} f_{\boldsymbol{\nu}}(\tau, \xi, t, \boldsymbol{y} \,|\, \boldsymbol{x}) \boldsymbol{\nu}_F(d\xi) \, d\tau \right) d\boldsymbol{y}.$$

We introduce notation for the proof. If  $\Pi_1 \geq \Pi_2 \geq \Pi_3$  and  $F \in \mathbb{F}(\Pi_1, \Pi_2), F' \in \mathbb{F}(\Pi_2, \Pi_3)$ , then we write  $F \succ F'$  and  $FF' \in \mathbb{F}(\Pi_1, \Pi_3)$  for the concatenation of F and F'. If  $F \succ F'$ ,

 $\tau \in \operatorname{tm}(F)$  and  $\tau' \in \operatorname{tm}(F')$ , and  $t > \tau_{\operatorname{rt}(F)}$ , write

$$(\tau/t/\tau')\colon (FF')^{\circ} \to (0,\infty); \quad \Pi \mapsto \begin{cases} \tau(\Pi), & \Pi \in F^{\circ}, \\ t + \tau'(\Pi), & \Pi \in (F')^{\circ}, \end{cases}$$

$$\xi\xi'\colon \operatorname{nd}(FF')^{\circ} \to E; \quad u \mapsto \begin{cases} \xi_{u}, & u \in \operatorname{nd}(F)^{\circ}, \\ \xi'_{u}, & u \in \operatorname{nd}(F')^{\circ}, \end{cases}$$

$$(3.1)$$

so that  $(\tau/t/\tau') \in \text{tm}(FF')$  and  $\xi\xi' \in \text{sp}(FF')$ . (The notation  $\xi\xi'$  is just a special case of the notation for joining maps that we introduced in the paragraph preceding (2.5).) If A is a statement, and x is an expression that evaluates to a real number if A is true (and may otherwise be ill-defined), then we use  $[x]_A$  as short-hand for x if A is true, and 1 otherwise.

Proof of Lemma 3.1. The possible values of Fr of a path in  $\mathcal{P}$  that starts at  $\Pi$  and passes through  $\Pi'$  are exactly FF' for  $F \in \mathbb{F}(\Pi, \Pi')$  and  $F' \in \mathbb{F}(\Pi')$ . If Fr = FF', then Tm has to be of the form  $(\tau/t/\tau')$  for  $\tau \in \text{tm}(F)$  with  $\tau_{\Pi'} < t$  and  $\tau' \in \text{tm}(F')$ , and Sp has to be of the form  $\xi\xi'$  for  $\xi \in \text{sp}(F)$  and  $\xi' \in \text{sp}(F')$ . Conditional on Fr = FF',  $Tm = (\tau/t/\tau')$ , and  $Sp = \xi\xi'$ , which happens with probability

$$\frac{1}{N^{\nu}(\boldsymbol{x})} f_{\nu}((\tau/t/\tau'), \xi \xi' \mid \boldsymbol{x}) \boldsymbol{\nu}_{F}(\mathrm{d}\xi) \boldsymbol{\nu}_{F'}(\mathrm{d}\xi') \,\mathrm{d}\tau \,\mathrm{d}\tau', \tag{3.2}$$

the probability density of  $X_t \in (\Pi', dy)$  is, by (2.12) and definition of  $K_x$ ,

$$K_{\boldsymbol{x}}\Big((FF', (\tau/t/\tau'), \xi\xi'), \{\boldsymbol{X}_t \in (\Pi', d\boldsymbol{y})\}\Big)$$

$$= \prod_{u \in \Pi'} p_{t-\tau_u}(y_u - (\boldsymbol{x}\xi)_u) \left[\frac{p_{\tau'_{\text{pr'}(u)}}(y_u - \xi'_{\text{pr'}(u)})}{p_{t+\tau'_{\text{pr'}(u)}} - \tau_u((\boldsymbol{x}\xi)_u - \xi'_{\text{pr'}(u)})}\right]_{\text{pr'}(u) \neq \emptyset} d\boldsymbol{y}, \quad (3.3)$$

where  $\operatorname{pr} := \operatorname{pr}_F$  and  $\operatorname{pr}' := \operatorname{pr}_{F'}$ . Multiplying (3.2) and (3.3) gives, after some careful cancellations,

$$\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = FF', \operatorname{Tm} \in \operatorname{d}(\tau/t/\tau'), \operatorname{Sp} \in \operatorname{d}(\xi\xi'), \boldsymbol{X}_{t} \in (\Pi', \operatorname{d}\boldsymbol{y}))$$

$$= \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} f_{\boldsymbol{\nu}}(\tau, \xi, t, \boldsymbol{y} \mid \boldsymbol{x}) f_{\boldsymbol{\nu}}(\tau', \xi' \mid \boldsymbol{y}) \boldsymbol{\nu}_{F}(\operatorname{d}\xi) \boldsymbol{\nu}_{F'}(\operatorname{d}\xi') \operatorname{d}\tau \operatorname{d}\tau' \operatorname{d}\boldsymbol{y}.$$

If F' is trivial and  $\lambda_{\Pi'} > 0$ , then the equality holds because both  $\mathbb{P}^{x}(\text{Fr} = FF')$  and thus the left-hand side (LHS), and  $f_{\nu}|_{\text{dec}(F')}$  and thus the right-hand side (RHS) are zero. Integrating over  $\tau, \xi, \tau', \xi'$  gives

$$\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = FF', \boldsymbol{X}_t \in (\Pi', d\boldsymbol{y})) = \frac{N_{\boldsymbol{\nu}}^{F'}(\boldsymbol{y})}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} \left( \int_{\operatorname{dec}(F)} f_{\boldsymbol{\nu}}(\tau, \xi, t, \boldsymbol{y} \,|\, \boldsymbol{x}) \boldsymbol{\nu}_F(d\xi) \, d\tau \right) d\boldsymbol{y},$$

and then summing over F and F' gives the claim.

# **Lemma 3.2.** If t, s > 0, then $S_t S_s = S_{t+s}$ .

A crucial ingredient in the proof is the following identity, which is an immediate consequence of the definitions. If F > F',  $(\tau, \xi) \in \text{dec}(F)$ ,  $(\tau', \xi') \in \text{dec}(F')$ , t, t' > 0 with

$$\tau_{\mathrm{rt}(F)} < t$$
, and  $\mathbf{y}' \in E_{\circ}^{\mathrm{lf}(F')}$ ,

$$\int_{E_{c}^{\mathrm{rt}(F)}} f_{\nu}(\tau, \xi, t, \boldsymbol{y} \mid \boldsymbol{x}) f_{\nu}(\tau', \xi', t', \boldsymbol{y}' \mid \boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} = f_{\nu}((\tau/t/\tau'), \xi \xi', t + t', \boldsymbol{y}' \mid \boldsymbol{x}). \tag{3.4}$$

Proof of Lemma 3.2. Fix  $(\Pi_1, \boldsymbol{x})$ , t, s > 0, and  $\Pi_2 \leq \Pi_1$ . Then by Lemma 3.1,

$$(S_{t}S_{s})((\Pi_{1},\boldsymbol{x}),(\Pi_{2},\mathrm{d}\boldsymbol{y}))/\mathrm{d}\boldsymbol{y}$$

$$= \sum_{\Pi_{2} \leq \Pi' \leq \Pi_{1}} \int_{\boldsymbol{z} \in E_{0}^{\Pi'}} S_{s}((\Pi',\boldsymbol{z}),(\Pi_{2},\mathrm{d}\boldsymbol{y}))S_{t}((\Pi_{1},\boldsymbol{x}),(\Pi',\mathrm{d}\boldsymbol{z}))/\mathrm{d}\boldsymbol{y}$$

$$= \sum_{\Pi_{2} \leq \Pi' \leq \Pi_{1}} \int \mathrm{d}\boldsymbol{z} \left( \frac{N^{\boldsymbol{\nu}}(\boldsymbol{y})}{N^{\boldsymbol{\nu}}(\boldsymbol{z})} \sum_{F_{2} \in \mathbb{F}(\Pi',\Pi_{2})} \int_{\mathrm{dec}(F_{2})} f_{\boldsymbol{\nu}}(\tau_{2},\xi_{2},s,\boldsymbol{y} \mid \boldsymbol{z}) \boldsymbol{\nu}_{F_{2}}(\mathrm{d}\xi_{2}) \,\mathrm{d}\tau_{2}$$

$$\times \frac{N^{\boldsymbol{\nu}}(\boldsymbol{z})}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} \sum_{F_{1} \in \mathbb{F}(\Pi_{1},\Pi')} \int_{\mathrm{dec}(F_{1})} f_{\boldsymbol{\nu}}(\tau_{1},\xi_{1},t,\boldsymbol{z} \mid \boldsymbol{x}) \boldsymbol{\nu}_{F_{1}}(\mathrm{d}\xi_{1}) \,\mathrm{d}\tau_{1} \right)$$

$$\stackrel{(3.4)}{=} \frac{N^{\boldsymbol{\nu}}(\boldsymbol{y})}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} \sum_{\substack{\Pi_{2} \leq \Pi' \leq \Pi_{1} \text{ dec}(F_{1}) \text{ dec}(F_{2})}} \int_{F_{1} \in \mathbb{F}(\Pi'_{1},\Pi')} \int_{F_{2} \in \mathbb{F}(\Pi'_{1},\Pi_{2})} f_{\boldsymbol{\nu}}((\tau_{1}/t/\tau_{2}),\xi_{1}\xi_{2},t+s,\boldsymbol{y} \mid \boldsymbol{x}) \boldsymbol{\nu}_{F_{1}}(\mathrm{d}\xi_{1}) \,\mathrm{d}\tau_{1}}{\boldsymbol{\nu}_{F_{2}}(\mathrm{d}\xi_{2}) \,\mathrm{d}\tau_{2}}.$$

$$(3.5)$$

If  $\Pi_2 \leq \Pi' \leq \Pi_1$  and  $F_1 \in \mathbb{F}(\Pi_1, \Pi')$ ,  $F_2 \in \mathbb{F}(\Pi', \Pi_2)$ , then  $\xi \in \operatorname{sp}(F_1 F_2)$  is always of the form  $\xi = \xi_1 \xi_2$  for  $\xi_{1,2} \in \operatorname{sp}(F_{1,2})$ , and  $\tau \in \operatorname{tm}(F_1 F_2)$  is of the form  $\tau = (\tau_1/t/\tau_2)$  for  $\tau_{1,2} \in \operatorname{tm}(F_{1,2})$  if and only if  $\tau_{\Pi'} < t < \tau_{\Pi'}^+$ . Thus the final integral on the RHS of (3.5) is equal to

$$\int_{\operatorname{dec}(F_1F_2)} \mathbb{1}_{\left\{\tau_{\Pi'} < t < \tau_{\Pi'}^+\right\}} f_{\boldsymbol{\nu}}(\tau, \xi, t + s, \boldsymbol{y} \,|\, \boldsymbol{x}) \, \mathrm{d}\tau \, \mathrm{d}\xi,$$

so

$$(S_{t}S_{s})((\Pi_{1}, \boldsymbol{x}), (\Pi_{2}, d\boldsymbol{y}))/d\boldsymbol{y}$$

$$= \frac{N^{\boldsymbol{\nu}}(\boldsymbol{y})}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} \sum_{F \in \mathbb{F}(\Pi_{1}, \Pi_{2})} \sum_{\Pi' \in F} \int_{\operatorname{dec}(F)} \mathbb{1}_{\{\tau_{\Pi'} < t < \tau_{\Pi'}^{+}\}} f_{\boldsymbol{\nu}}(\tau, \xi, t + s, \boldsymbol{y} \mid \boldsymbol{x}) \boldsymbol{\nu}_{F}(d\xi) d\tau$$

$$= \frac{N^{\boldsymbol{\nu}}(\boldsymbol{y})}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} \sum_{F \in \mathbb{F}(\Pi_{1}, \Pi_{2})} \int_{\operatorname{dec}(F)} f_{\boldsymbol{\nu}}(\tau, \xi, t + s, \boldsymbol{y} \mid \boldsymbol{x}) \boldsymbol{\nu}_{F}(d\xi) d\tau$$

$$= S_{t+s}((\Pi_{1}, \boldsymbol{x}), (\Pi_{2}, d\boldsymbol{y}))/d\boldsymbol{y}.$$

This proves that the Brownian spatial coalescent with transition measures  $\nu$  is a Markov process.

**Lemma 3.3.** The map  $x \mapsto \mathbb{P}^x$  is continuous w.r.t. the topology of weak convergence of probability measures. In particular,  $(\mathbb{P}^x)_{x \in \mathcal{X}}$  has the Feller property.

*Proof.* Suppose  $O \subset \Omega$  is open, and  $(\Pi_n, \boldsymbol{x}_n) \to (\Pi, \boldsymbol{x})$  in  $\mathcal{X}$ , without loss of generality  $\Pi_n = \Pi$  for all  $n \in \mathbb{N}$ . Then,

$$\mathbb{P}^{\boldsymbol{x}_n}(O) = \int_{\operatorname{dec}(\mathbb{F})} K_{\boldsymbol{x}_n}(F^*, O) P^{\boldsymbol{x}_n}(\mathrm{d}F^*)$$

$$= \sum_{F \in \mathbb{F}(\Pi)} \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x}_n)} \int_{\operatorname{dec}(F)} K_{\boldsymbol{x}_n}((F, \tau, \xi), O) f_{\boldsymbol{\nu}}(\tau, \xi \mid \boldsymbol{x}_n) \boldsymbol{\nu}_F(\mathrm{d}\xi) \, \mathrm{d}\tau.$$

By Lemmas 2.9 and 2.12, and definition of  $f_{\nu}$ , we have  $N^{\nu}(\boldsymbol{x}_n) \to N^{\nu}(\boldsymbol{x})$ ,  $f_{\nu}(\tau, \xi \mid \boldsymbol{x}_n) \to f_{\nu}(\tau, \xi \mid \boldsymbol{x})$ , and  $\underline{\lim}_{n \to \infty} K_{\boldsymbol{x}_n}(F^{\star}, O) \geq K_{\boldsymbol{x}}(F^{\star}, O)$  for all  $F^{\star} = (F, \tau, \xi) \in \operatorname{dec}(\mathbb{F})$ . Thus by Fatou's lemma,

$$\underline{\lim}_{n\to\infty}\mathbb{P}^{\boldsymbol{x}_n}(O)\geq \sum_{F\in\mathbb{F}(\Pi)}\frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x})}\int_{\mathrm{dec}(F)}K_{\boldsymbol{x}}((F,\tau,\xi),O)f_{\boldsymbol{\nu}}(\tau,\xi\,|\,\boldsymbol{x}_n)\boldsymbol{\nu}_F(\mathrm{d}\xi)\,\mathrm{d}\tau=\mathbb{P}^{\boldsymbol{x}}(O).$$

This proves that  $x \mapsto \mathbb{P}^x$  is continuous by the Portmanteau theorem.

3.1.2. "Only if" Direction of Theorem 2.14. Fix a Brownian spatial coalescent process, that is a spatial coalescent whose laws can be written as  $\mathbb{P}^{x} = P^{x} \otimes K_{x}$  where  $P^{x} = \text{Dec } \#\mathbb{P}^{x}$  for  $x \in \mathcal{X}$  (recall Definition 2.10). We have to prove that the laws  $(P^{x})$  are of the form (2.12).

The first step revolves around the following idea. Suppose  $(\Pi, \boldsymbol{x}) \in \mathcal{X}$ , s > 0, and  $F_0 \in \mathbb{F}(\Pi)$  is the trivial forest. Then we can evaluate  $\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F_0, \boldsymbol{X}_s \in \mathrm{d}\boldsymbol{y})$ , a subprobability measure on  $E_{\circ}^{\Pi}$ , in two ways: by applying the Markov property at time s it equals  $S_s(\boldsymbol{x}, \mathrm{d}\boldsymbol{y})\mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr} = F_0)$ , and by (2.12) it equals  $\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F_0)K_{\boldsymbol{x}}(F_0, \boldsymbol{X}_s \in \mathrm{d}\boldsymbol{y})$ , giving

$$S_s(\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) \mathbb{P}^{\boldsymbol{y}}(\mathrm{Fr} = F_0) = \mathbb{P}^{\boldsymbol{x}}(\mathrm{Fr} = F_0) \prod_{u \in \Pi} p_s(\boldsymbol{x}_u - \boldsymbol{y}_u) \, \mathrm{d}\boldsymbol{y}$$

as measures on  $E_{\circ}^{\Pi}$ , where we expanded  $K_{\boldsymbol{x}}(F_0, \boldsymbol{X}_s \in \mathrm{d}\boldsymbol{y})$ . This already implies the main claim of the following proposition in the case where  $\mathbb{P}^{\boldsymbol{x}}(\mathrm{Fr}=F_0)>0$  for all  $\boldsymbol{x}\in E_{\circ}^{\Pi}$ . For  $F\in\mathbb{F},\ \tau\in\mathrm{tm}(F)$  and  $s\geq 0$  we write  $\tau+s$  for the map  $u\mapsto \tau_u+s$ , and  $\tau\geq s$  if  $\tau=\tau'+s$  for some  $\tau'\in\mathrm{tm}(F)$ .

**Proposition 3.4.** For every  $\Pi \in \mathcal{P}$ , there is a unique constant  $\beta_{\Pi} \geq 0$ , and a continuous function  $\widetilde{N}_{\Pi} \colon E_{\circ}^{\Pi} \to (0, \infty)$  unique up to a constant positive multiple such that for all s > 0,

$$S_s((\Pi, \boldsymbol{x}), (\Pi, d\boldsymbol{y})) = e^{-\beta_\Pi s} \frac{\widetilde{N}_\Pi(\boldsymbol{y})}{\widetilde{N}_\Pi(\boldsymbol{x})} \prod_{u \in \Pi} p_s(\boldsymbol{x}_u - \boldsymbol{y}_u) d\boldsymbol{y}.$$
 (3.6)

Furthermore, for any  $F \in \mathbb{F}(\Pi)$ , exactly one of the following hold

- (i)  $\mathbb{P}^{(\Pi, \boldsymbol{x})}(\operatorname{Fr} = F) > 0$  for all  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ , in which case we call F possible,
- (ii)  $\mathbb{P}^{(\Pi, \boldsymbol{x})}(\operatorname{Fr} = F) = 0$  for all  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ , in which case we call F impossible.

If  $\beta_{\Pi} > 0$  then the trivial forest is impossible, and if  $\beta_{\Pi} = 0$  then the trivial forest is the only possible forest and  $\widetilde{N}_{\Pi}$  is constant.

It will turn out that  $\beta_{\Pi} = |\nu_{\Pi}|$ , and that  $\widetilde{N}_{\Pi}$  is (a constant multiple of) the normalisation  $N^{\nu}$  for the transition measures  $\nu \in R$  that define this Brownian spatial coalescent, see Theorem 3.9. Proposition 3.4 is proved in a sequence of lemmas.

**Lemma 3.5.** If  $(\Pi, \boldsymbol{x}) \in \mathcal{X}$ , t > 0, then  $S_t((\Pi, \boldsymbol{x}), (\Pi, d\boldsymbol{y}))$  is not the zero-measure on  $E_{\circ}^{\Pi}$ .

*Proof.* Fix  $(\Pi, \boldsymbol{x}) \in \mathcal{X}$ . Since  $(\Pi_t, \boldsymbol{X}_t)_{t \geq 0}$  is right-continuous, we have  $\Pi_t \to \Pi_0 = \Pi$  a.s. as  $t \to 0$ , in particular in distribution, so  $S_t(\boldsymbol{x}, E_\circ^\Pi) = \mathbb{P}^{\boldsymbol{x}}(\Pi_t = \Pi) \to 1$  as  $t \to 0$ . This implies the claim for sufficiently small s > 0, and repeated applications of  $S_{2s} = S_s S_s$  finish the proof.

**Lemma 3.6.** For every  $\Pi \in \mathcal{P}$  there are a unique  $\beta_{\Pi} \geq 0$  and a function  $\widetilde{N}_{\Pi} : E_{\circ}^{\Pi} \to (0, \infty)$  unique up to a constant positive multiple such that (3.6) holds. Every  $F \in \mathbb{F}(\Pi)$  is either possible or impossible, and if the trivial forest  $F_0 \in \mathbb{F}(\Pi)$  is possible, then  $\widetilde{N}_{\Pi}(\mathbf{x})\mathbb{P}^{\mathbf{x}}(\mathrm{Fr} = F_0)$  is constant in  $\mathbf{x} \in E_{\circ}^{\Pi}$  and positive. Furthermore, for every possible  $F \in \mathbb{F}(\Pi)$ , every  $\mathbf{x}, \mathbf{y} \in E_{\circ}^{\Pi}$ , and  $s \geq 0$ ,

$$e^{\beta_{\Pi}s} \frac{\widetilde{N}_{\Pi}(\boldsymbol{x})P^{\boldsymbol{x}}(F, d(\tau+s), d\xi)}{\prod_{\substack{u \in \Pi \\ \text{pr}_{F}(u) \neq \emptyset}} p(\tau_{\text{pr}_{F}(u)} + s, \xi_{\text{pr}_{f}(u)} - \boldsymbol{x}_{u})} = \frac{\widetilde{N}_{\Pi}(\boldsymbol{y})P^{\boldsymbol{y}}(F, d\tau, d\xi)}{\prod_{\substack{u \in \Pi \\ \text{pr}_{F}(u) \neq \emptyset}} p(\tau_{\text{pr}_{F}(u)}, \xi_{\text{pr}_{F}(u)} - \boldsymbol{y}_{u})}$$
(3.7)

as measures on dec(F).

*Proof.* Fix  $\Pi \in \mathcal{P}$  throughout the proof. Uniqueness is straightforward: if (3.6) holds with  $\beta$  and  $\widetilde{N}$ , but also with  $\beta'$  and  $\widetilde{N}'$ , then for every s > 0,  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ , and Lebesgue-almost all  $\boldsymbol{y} \in E_{\circ}^{\Pi}$  (with the exceptional null-set depending on  $\boldsymbol{x}$  and s),

$$e^{-\beta s} \frac{\widetilde{N}(\mathbf{y})}{\widetilde{N}(\mathbf{x})} = e^{-\beta' s} \frac{\widetilde{N}'(\mathbf{y})}{\widetilde{N}'(\mathbf{x})}.$$
 (3.8)

Letting  $s \to 0$  in  $\mathbb{Q}$ , it follows that for every  $\boldsymbol{x}$ , and all  $\boldsymbol{y} \notin N_{\boldsymbol{x}}$  for a null-set  $N_{\boldsymbol{x}} \subset E_{\circ}^{\Pi}$ , we have  $\widetilde{N}(\boldsymbol{y})/\widetilde{N}(\boldsymbol{x}) = \widetilde{N}'(\boldsymbol{y})/\widetilde{N}'(\boldsymbol{x})$ . For a fixed choice  $\boldsymbol{x}_0$  and  $a := \widetilde{N}(\boldsymbol{x}_0)/\widetilde{N}'(\boldsymbol{x}_0) > 0$ , this implies  $\widetilde{N}(\boldsymbol{y}) = a\widetilde{N}'(\boldsymbol{y})$  for all  $\boldsymbol{y} \in E_{\circ}^{\Pi} \setminus N_{\boldsymbol{x}_0}$ . Then for any  $\boldsymbol{x} \in E_{\circ}^{\Pi}$  we can choose  $\boldsymbol{y} \in E_{\circ}^{\Pi} \setminus (N_{\boldsymbol{x}_0} \cup N_{\boldsymbol{x}})$  to obtain  $\widetilde{N}(\boldsymbol{x}) = \frac{\widetilde{N}(\boldsymbol{y})}{\widetilde{N}'(\boldsymbol{y})}\widetilde{N}'(\boldsymbol{x}) = a\widetilde{N}'(\boldsymbol{x})$ . Back into (3.8) this also implies  $\beta = \beta'$ . Conversely, if (3.7) holds for some function  $\widetilde{N}_{\Pi}$ , then it also holds for  $c\widetilde{N}_{\Pi}$  for any c > 0.

If  $|\Pi| = 1$  then (3.6) holds for  $\beta_{\Pi} = 0$  and  $\widetilde{N}_{\Pi} \equiv 1$  (because a single lineage follows a Brownian motion), and the only element of  $\mathbb{F}(\Pi)$  is the trivial forest, which makes the remaining statements trivial. Assume  $|\Pi| > 1$  for the rest of the proof. Let  $F_0 \in \mathbb{F}(\Pi)$  be the trivial forest. For s > 0 and  $x \in E_{\circ}^{\Pi}$ , we evaluate  $\mathbb{P}^{x}(\operatorname{Fr} = F_0, X_s \in \mathrm{d}y)$ , which is a sub-probability measure on  $E_{\circ}^{\Pi}$ , in two ways. On the one hand, by applying the Markov property at time s it equals  $S_s(x, \mathrm{d}y)\mathbb{P}^y(\operatorname{Fr} = F_0)$ . On the other hand, by (2.12) it equals  $\mathbb{P}^{x}(\operatorname{Fr} = F_0)K_x(F_0, X_s \in \mathrm{d}y)$ . This gives

$$S_s(\boldsymbol{x}, d\boldsymbol{y}) \mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr} = F_0) = \mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F_0) \prod_{u \in \Pi} p_s(\boldsymbol{x}_u - \boldsymbol{y}_u) d\boldsymbol{y}$$
 (3.9)

as measures on  $E_{\circ}^{\Pi}$ , where we expanded  $K_{\boldsymbol{x}}(F_0, \boldsymbol{X}_s \in \mathrm{d}\boldsymbol{y})$ . We now explain why  $F_0$  is either possible or impossible. If  $\mathbb{P}^{\boldsymbol{y}}(\mathrm{Fr} = F_0) = 0$  on a set of positive Lebesgue measure, then integrating (3.9) over  $\boldsymbol{y}$  in this set gives zero on the LHS, and the integral on the RHS is zero only if  $\mathbb{P}^{\boldsymbol{x}}(\mathrm{Fr} = F_0) = 0$ , so we must have  $\mathbb{P}^{\boldsymbol{x}}(\mathrm{Fr} = F_0) = 0$  for all  $\boldsymbol{x} \in E_{\circ}^{\Pi}$  and  $F_0$  is impossible. Now assume that  $\mathbb{P}^{\boldsymbol{y}}(\mathrm{Fr} = F_0) > 0$  Lebesgue-almost everywhere. Then

integrating (3.9) over  $\boldsymbol{y} \in E_0^{\Pi}$  gives a positive number on the LHS by Lemma 3.5, and  $\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F_0)$  on the RHS, so  $\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F_0) > 0$  for all  $\boldsymbol{x} \in E_0^{\Pi}$  and  $F_0$  is possible. In that case, (3.9) implies (3.6) with  $\beta_{\Pi} = 0$  and  $\widetilde{N}_{\Pi}(\boldsymbol{x}) = 1/\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F_0)$ . Eq. (3.7) for the trivial forest  $F_0$  is just  $\widetilde{N}_{\Pi}(\boldsymbol{x})\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F_0) = \widetilde{N}_{\Pi}(\boldsymbol{y})\mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr} = F_0)$ , which holds by definition of  $\widetilde{N}_{\Pi}$ . Let now  $F \in \mathbb{F}(\Pi)$  be non-trivial. For s > 0 and  $\boldsymbol{x} \in E_0^{\Pi}$ , we evaluate

$$\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F, \operatorname{Tm} \in \operatorname{d}(\tau + s), \operatorname{Sp} \in \operatorname{d}\xi, \boldsymbol{X}_s \in \operatorname{d}\boldsymbol{y}), \tag{3.10}$$

which is a sub-probability measure on  $\operatorname{dec}(F) \times E_{\circ}^{\Pi}$ , in two ways. On the one hand, by applying the Markov property at time s it equals  $S_s(\boldsymbol{x}, \mathrm{d}\boldsymbol{y})P^{\boldsymbol{y}}(F, \mathrm{d}\tau, \mathrm{d}\xi)$ . On the other hand, by conditioning on  $\operatorname{Dec} = (F, \tau, \xi)$  and (2.12), the expression (3.10) equals

$$P^{\boldsymbol{x}}(F, d(\tau + s), d\xi)K_{\boldsymbol{x}}((F, \tau + s, \xi), \{\boldsymbol{X}_s \in d\boldsymbol{y}\}).$$

Thus we obtain

 $S_s(\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) P^{\boldsymbol{y}}(F, \mathrm{d}\tau, \mathrm{d}\xi)$ 

$$= P^{\boldsymbol{x}}(F, d(\tau+s), d\xi) \prod_{u \in \Pi} p_s(\boldsymbol{x}_u - \boldsymbol{y}_u) \left[ \frac{p(\tau_{\text{pr}(u)}, \xi_{\text{pr}(u)} - \boldsymbol{y}_u)}{p(\tau_{\text{pr}(u)} + s, \xi_{\text{pr}(u)} - \boldsymbol{x}_u)} \right]_{\text{pr}(u) \neq \emptyset} d\boldsymbol{y},$$
(3.11)

as an equality of measures on  $\operatorname{tm}(F) \times \operatorname{sp}(F) \times E_{\circ}^{\Pi}$ , where we expanded  $K_{\boldsymbol{x}}((F, \tau + s, \xi), \boldsymbol{X}_{s} \in d\boldsymbol{y})$  on the RHS, and abbreviated  $\operatorname{pr} = \operatorname{pr}_{F}$ .

We now explain why F is either possible or impossible. Suppose that  $\mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr}=F)=0$  for  $\boldsymbol{y}$  in a set of positive Lebesgue measure. Integrating over this set and  $(\tau,\xi)\in\operatorname{Dec}(F)$  gives zero on the LHS, and on the RHS a quantity that is zero if and only if  $P^{\boldsymbol{x}}(F,\{\tau\geq s\})=0$ , which converges to  $\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr}=F)$  as  $s\to 0$  by continuity from below, so  $\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr}=F)=0$  for all  $\boldsymbol{x}\in E_o^\Pi$  and F is impossible. Now suppose that  $\mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr}=F)>0$  for Lebesgue almost-all  $\boldsymbol{y}\in E_o^\Pi$ . Then  $S_s(\boldsymbol{x},\mathrm{d}\boldsymbol{y})$  has a Lebesgue density. Indeed, integrating (3.11) over  $\boldsymbol{y}\in N$  for a Lebesgue null set N, and  $(\tau,\xi)\in\operatorname{dec}(F)$ , gives zero on the RHS, and on the LHS it gives  $\int_N \mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr}=F)S_s(\boldsymbol{x},\mathrm{d}\boldsymbol{y})$ , which was positive if  $S_s(\boldsymbol{x},N)>0$ . Then, for an arbitrary s>0,

$$\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F) \ge \mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F, \tau \ge s) = \int_{E^{\Pi}} \mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr} = F) S_s(\boldsymbol{x}, d\boldsymbol{y}) > 0, \tag{3.12}$$

so F is possible. In the second step, we applied the Markov property at time s, and the last expression is positive because  $\mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr}=F)>0$  for Lebesgue-a.e.  $\boldsymbol{y}\in E_{\circ}^{\Pi}$ , and  $S_{s}(\boldsymbol{x},\cdot)$  is non-zero (Lemma 3.5) and has a Lebesgue density. In fact, (3.12) shows that if F is possible, then  $\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr}=F,\tau\geq s)>0$  for all  $\boldsymbol{x}\in E_{\circ}^{\Pi}$  and  $s\geq 0$ .

We now prove (3.6) in case the trivial forest is impossible, as well as (3.7) for non-trivial forests. Thus we may assume that there exists a possible non-trivial forest  $F \in \mathbb{F}(\Pi)$ , for otherwise we are already done. Let such an F be given, and define for every  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ ,

$$G(\boldsymbol{x}, F, d\tau, d\xi) := \frac{P^{\boldsymbol{x}}(F, d\tau, d\xi)}{\prod_{\substack{u \in \Pi \\ \operatorname{pr}_{F}(u) \neq \emptyset}} p(\tau_{\operatorname{pr}_{F}(u)}, \xi_{\operatorname{pr}_{F}(u)} - \boldsymbol{x}_{u})},$$
(3.13)

which is a non-zero measure on  $\operatorname{dec}(F)$  that is finite on  $\{\tau \geq a\} \times \operatorname{sp}(F) \subset \operatorname{dec}(F)$  for every a > 0. Recall from the paragraph following (3.12) that  $P^{x}(F, \{\tau \geq a\}) > 0$  for every  $a \geq 0$ 

and  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ , so

$$G(\boldsymbol{x},s) \coloneqq G(\boldsymbol{x},F,\{\tau>1+s\}\times \operatorname{sp}(F)) = \int\limits_{\operatorname{tm}(F)} \mathbb{1}_{\{\tau>1+s\}} \int\limits_{\operatorname{sp}(F)} G(\boldsymbol{x},F,\mathrm{d}\tau,\mathrm{d}\xi) \in (0,\infty)$$

for all  $\boldsymbol{x} \in E_{\circ}^{\Pi}$  and  $s \geq 0$ . By continuity from below,  $G(\boldsymbol{x}, \cdot)$  is right-continuous on  $[0, \infty)$ . Rearranging (3.11) gives

$$G(\boldsymbol{x}, F, d(\tau + s), d\xi) d\boldsymbol{y} = \frac{S_s(\boldsymbol{x}, d\boldsymbol{y})}{\prod_{\boldsymbol{u} \in \Pi} p_s(\boldsymbol{x}_{\boldsymbol{u}} - \boldsymbol{y}_{\boldsymbol{u}})} G(\boldsymbol{y}, F, d\tau, d\xi).$$

If we integrate this over  $\{\tau > 1 + t\} \times \operatorname{sp}(F) \subset \operatorname{dec}(F)$  for some  $t \ge 0$ , we obtain

$$\frac{S_s(\boldsymbol{x}, d\boldsymbol{y})}{\prod_{u \in \Pi} p_s(\boldsymbol{x}_u - \boldsymbol{y}_u)} = \frac{G(\boldsymbol{x}, t + s)}{G(\boldsymbol{y}, t)} d\boldsymbol{y}$$
(3.14)

for s>0. The LHS is independent of t, so we can equate the RHS for t=0 and  $t\geq 0$  to obtain that for every  $s>0, t\geq 0, x\in E_{\circ}^{\Pi}$ ,

$$g(s, \boldsymbol{x}, \boldsymbol{y}) \coloneqq \frac{G(\boldsymbol{x}, s)}{G(\boldsymbol{y}, 0)} = \frac{G(\boldsymbol{x}, t + s)}{G(\boldsymbol{y}, t)},$$

for Lebesgue-almost all  $\boldsymbol{y} \in E_{\circ}^{\Pi}$ . By right-continuity of  $G(\boldsymbol{x},\cdot)$ , the same is true for s=0 (and any  $t\geq 0$ ). Thus for  $t,s\geq 0$  and  $\boldsymbol{x}\in E_{\circ}^{\Pi}$ , there is a Lebesgue-null set  $N_{\boldsymbol{x},s,t}\subset E_{\circ}^{\Pi}$  such that for all  $\boldsymbol{z}\not\in N_{\boldsymbol{x},s,t}$ , and all  $\boldsymbol{y}\in E_{\circ}^{\Pi}$ ,

$$g(s,\boldsymbol{x},\boldsymbol{z})g(t,\boldsymbol{z},\boldsymbol{y}) = \frac{G(\boldsymbol{x},s)}{G(\boldsymbol{z},0)}\frac{G(\boldsymbol{z},t)}{G(\boldsymbol{y},0)} = \frac{G(\boldsymbol{x},s+t)}{G(\boldsymbol{z},t)}\frac{G(\boldsymbol{z},t)}{G(\boldsymbol{y},0)} = g(s+t,\boldsymbol{x},\boldsymbol{y}).$$

Now apply Lemma A.10 to  $\log g(s, \boldsymbol{x}, \boldsymbol{y})$ , which is well-defined because  $g(s, \boldsymbol{x}, \boldsymbol{y}) > 0$ , and the assumptions of Lemma A.10 are satisfied with  $b(\boldsymbol{x}) = \log G(\boldsymbol{x}, 0)$ , and  $A_{\boldsymbol{x}} := E_{\circ}^{\Pi} \setminus \bigcup_{s,t \in \mathbb{Q} \cap [0,\infty)} N_{\boldsymbol{x},s,t}$ . Then there exists  $\beta = \beta_F \in \mathbb{R}$  (since g depends by definition on F) such that for  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ ,  $s \in \mathbb{Q} \cap [0,\infty)$ , and Lebesgue-almost all  $\boldsymbol{y} \in E_{\circ}^{\Pi}$ ,

$$g(s, \boldsymbol{x}, \boldsymbol{y}) = e^{-\beta_F s} \frac{\widetilde{N}_F(\boldsymbol{y})}{\widetilde{N}_F(\boldsymbol{x})},$$
(3.15)

where  $\widetilde{N}_F(\boldsymbol{x}) = G(\boldsymbol{x},0)^{-1}$ . If the trivial forest is impossible, in which case we have not yet found  $\beta_{\Pi}$  and  $\widetilde{N}_{\Pi}$ , (3.14) and (3.15) imply (3.6) for  $s \in \mathbb{Q} \cap (0,\infty)$  with  $\beta_{\Pi} := \beta_F$  and  $\widetilde{N}_{\Pi} := \widetilde{N}_F$ . In that case, if we assume  $\beta_{\Pi} < 0$ , then integrating (3.6) over  $\boldsymbol{y} \in E_0^{\Pi}$  gives

$$1 \ge \int S_s(\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) \ge \mathrm{e}^{|\beta_\Pi|s} \frac{1}{\widetilde{N}_\Pi(\boldsymbol{x})} (\min_E p_s)^{|\Pi|} \int \widetilde{N}_\Pi(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \stackrel{s \to \infty}{\longrightarrow} \infty,$$

a contradiction. We now explain why it sufficed to show (3.6) for  $s \in \mathbb{Q}_+ := \mathbb{Q} \cap (0, \infty)$ . Denote the RHS in (3.6) for fixed  $\boldsymbol{x} \in E_{\circ}^{\Pi}$  temporarily by  $h(s, \boldsymbol{y}) \, \mathrm{d} \boldsymbol{y}$ , which has now been shown to be a finite measure on  $E_{\circ}^{\Pi}$  for all  $s \in \mathbb{Q}_+$  (because the LHS in (3.6) is a finite measure). Then  $\widetilde{N}_{\Pi}$  is integrable because  $\inf p_s > 0$  for any fixed  $s \in \mathbb{Q}_+$ , which implies that  $h(s, \boldsymbol{y}) \, \mathrm{d} \boldsymbol{y}$  is a finite measure for all s > 0, and enables an application of dominated convergence to show that  $s \mapsto h(s, \boldsymbol{y}) \, \mathrm{d} \boldsymbol{y}$  is continuous in the weak topology on  $\mathcal{M}_F(E_{\circ}^{\Pi})$ . The LHS of (3.6) is right-continuous in s w.r.t. the same topology because  $(\Pi_t, \boldsymbol{X}_t)$  is almost-surely right-continuous, which implies that (3.6) holds for all s > 0.

We prove (3.7) for every non-trivial, possible forest  $F \in \mathbb{F}(\Pi)$ . Fix one, if it exists, abbreviate  $\beta = \beta_{\Pi}$ ,  $\widetilde{N} = \widetilde{N}_{\Pi}$ , and recall the definition of  $G(\boldsymbol{x}, F, d\tau, d\xi)$  from (3.13). Then (3.7) is equivalent to  $e^{\beta s}\widetilde{N}(\boldsymbol{x})G(\boldsymbol{x}, F, d(\tau + s), d\xi) = \widetilde{N}(\boldsymbol{y})G(\boldsymbol{y}, F, d\tau, d\xi)$ . Plugging (3.6) into (3.11) and rearranging shows that for every  $\boldsymbol{x} \in E_{\circ}^{\Pi}$  and s > 0,

$$e^{\beta s} \widetilde{N}(\boldsymbol{x}) G(\boldsymbol{x}, F, d(\tau + s), d\xi) d\boldsymbol{y} = \widetilde{N}(\boldsymbol{y}) G(\boldsymbol{y}, F, d\tau, d\xi) d\boldsymbol{y}$$
(3.16)

as measures on  $\operatorname{dec}(F) \times E_{\circ}^{\Pi}$ . Since  $G(\boldsymbol{x}, F, \operatorname{d}\tau, \operatorname{d}\xi)$  is a locally finite measure on  $\operatorname{dec}(F)$ , the LHS in (3.16) is continuous in  $s \in [0, \infty)$  w.r.t. the topology of vague convergence on the space of locally finite measures on  $\operatorname{dec}(F) \times E_{\circ}^{\Pi}$ . Thus we can let  $s \to 0$  in (3.16) and obtain that it also holds for s = 0. Since the  $\sigma$ -algebra on  $\operatorname{dec}(F)$  is countably generated, we can further find for every  $\boldsymbol{x} \in E_{\circ}^{\Pi}$  and  $s \ge 0$  a single null set  $N_{\boldsymbol{x},s} \subset E_{\circ}^{\Pi}$  such that for all  $\boldsymbol{y} \in E_{\circ}^{\Pi} \setminus N_{\boldsymbol{x},s}$ ,

$$e^{\beta s}\widetilde{N}(\boldsymbol{x})G(\boldsymbol{x},F,d(\tau+s),d\xi) = \widetilde{N}(\boldsymbol{y})G(\boldsymbol{y},F,d\tau,d\xi)$$

as measures on  $\operatorname{dec}(F)$ , which is (3.7). If s=0 then this equality is symmetric in  $\boldsymbol{x}$  and  $\boldsymbol{y}$  and must thus already hold for all pairs  $\boldsymbol{x},\boldsymbol{y}\in E_{\circ}^{\Pi}$ . Then for  $s>0,\ \boldsymbol{x},\boldsymbol{y}\in E_{\circ}^{\Pi}$ , and  $\boldsymbol{y}'\notin N_{\boldsymbol{x},s}$ ,

$$e^{\beta s}\widetilde{N}(\boldsymbol{x})G(\boldsymbol{x},F,\mathrm{d}(\tau+s),\mathrm{d}\xi) = \widetilde{N}(\boldsymbol{y}')G(\boldsymbol{y}',F,\mathrm{d}\tau,\mathrm{d}\xi) = \widetilde{N}(\boldsymbol{y})G(\boldsymbol{y},F,\mathrm{d}\tau,\mathrm{d}\xi).$$

For the rest of this section, we fix an arbitrary choice for  $\widetilde{N}_{\Pi}$ ,  $\Pi \in \mathcal{P}$ .

**Lemma 3.7.**  $\widetilde{N}_{\Pi}$  is continuous for every  $\Pi \in \mathcal{P}$ .

*Proof.* Fix  $\Pi \in \mathcal{P}$ . First note that (3.6) implies integrability of  $\widetilde{N}_{\Pi}$  over  $E_{\circ}^{\Pi}$ , because the LHS is integrable, and  $\prod_{u \in \Pi} p_s(\boldsymbol{x}_u - \boldsymbol{y}_u)$  can be uniformly bounded below by a positive constant for fixed s > 0. Now suppose that  $\boldsymbol{x}_n \to \boldsymbol{x}$  in  $E_{\circ}^{\Pi}$ , and define  $f \in C_b(\mathcal{X})$  by  $f(\Pi', \boldsymbol{x}) := \mathbb{1}_{\{\Pi' = \Pi\}}$ . Then  $S_1 f \in C_b(\mathcal{X})$ , and

$$(S_1 f)(\boldsymbol{x}_n) = \int_{E_0^{\Pi}} S_t((\Pi, \boldsymbol{x}_n), (\Pi, d\boldsymbol{y})) = e^{-\beta_{\Pi}} \frac{1}{\widetilde{N}_{\Pi}(\boldsymbol{x}_n)} \int \widetilde{N}_{\Pi}(\boldsymbol{y}) \prod_{u \in \Pi} p_1(\boldsymbol{y}_u - \boldsymbol{x}_n(u)) d\boldsymbol{y}.$$

By dominated convergence and integrability of  $\widetilde{N}_{\Pi}$ , we can pull the limit  $n \to \infty$  into the integral on the RHS, the LHS converges to  $(S_1f)(\boldsymbol{x})$  because  $S_1f$  is continuous, and all involved quantities are positive, so we must have  $\widetilde{N}_{\Pi}(\boldsymbol{x}_n) \to \widetilde{N}_{\Pi}(\boldsymbol{x})$ .

**Lemma 3.8.** Let  $\Pi \in \mathcal{P}$ . If  $\beta_{\Pi} > 0$  then the trivial forest is impossible, and if  $\beta_{\Pi} = 0$  then the trivial forest is the only possible forest and  $\widetilde{N}_{\Pi}$  is constant.

*Proof.* Fix  $\Pi \in \mathcal{P}$  and  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ , and let  $F_0 \in \mathbb{F}(\Pi)$  be the trivial forest. If we integrate (3.6) over  $\boldsymbol{y} \in E_{\circ}^{\Pi}$  and let  $s \to \infty$ , we obtain

$$\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F_0) = \lim_{s \to \infty} \frac{e^{-\beta_{\Pi} s}}{\widetilde{N}_{\Pi}(\boldsymbol{x})} \int \widetilde{N}_{\Pi}(\boldsymbol{y}) \prod_{u \in \Pi} p_s(\boldsymbol{x}_u - \boldsymbol{y}_u) \, d\boldsymbol{y} = \frac{\mathbb{1}_{\{\beta_{\Pi} = 0\}}}{\widetilde{N}_{\Pi}(\boldsymbol{x})} \int \widetilde{N}_{\Pi}(\boldsymbol{y}) \, d\boldsymbol{y}, \quad (3.17)$$

where we used that  $p_s \to 1$  uniformly as  $s \to \infty$ . If  $\beta_{\Pi} > 0$  then (3.17) implies  $\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F_0) = 0$  for all  $\boldsymbol{x} \in E_0^{\Pi}$ , so  $F_0$  is impossible. Otherwise,  $\boldsymbol{z} \mapsto \widetilde{N}_{\Pi}(\boldsymbol{z})\mathbb{P}^{\boldsymbol{z}}(\operatorname{Fr} = F_0)$  is constant

and positive by Lemma 3.6, so (3.17) turns into

$$1 = \int \mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr} = F_0)^{-1} \, \mathrm{d}\boldsymbol{y},$$

which implies that  $\mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr}=F_0)=1$  for almost-all  $\boldsymbol{y}\in E_{\circ}^{\Pi}$ , in particular at least one  $\boldsymbol{y}_0\in E_{\circ}^{\Pi}$ . Then  $\mathbb{P}^{\boldsymbol{y}_0}(\operatorname{Fr}=F)=0$  for every  $F\in\mathbb{F}(\Pi)\setminus\{F_0\}$ , so all non-trivial forests are impossible. Thus  $\mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr}=F_0)=1$  for all  $\boldsymbol{y}\in E_{\circ}^{\Pi}$ , so  $\widetilde{N}_{\Pi}(\boldsymbol{y})=\widetilde{N}_{\Pi}(\boldsymbol{y})\mathbb{P}^{\boldsymbol{y}}(\operatorname{Fr}=F_0)$  is constant in  $\boldsymbol{y}\in E_{\circ}^{\Pi}$ .

Lemmas 3.6 to 3.8 together imply Proposition 3.4.

**Theorem 3.9.** There exists a unique family  $\nu \in R$  such that (2.12) holds for every  $F \in \mathbb{F}$  and  $x \in E_{\circ}^{\mathrm{lf}(F)}$ , that is

$$P^{\boldsymbol{x}}(F, d\tau, d\xi) = \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} f_{\boldsymbol{\nu}}(\tau, \xi \mid \boldsymbol{x}) \boldsymbol{\nu}_F(d\xi) d\tau$$

as measures on  $\operatorname{dec}(F)$ . For all  $\Pi \in \mathcal{P}$ ,  $\beta_{\Pi} = |\nu_{\Pi}|$ , and  $\widetilde{N}_{\Pi}/N^{\nu}$  is constant on  $E_{\circ}^{\Pi}$ .

That is, there are choices for  $\widetilde{N}_{\Pi}$ ,  $\Pi \in \mathcal{P}$  such that  $\widetilde{N}_{\Pi} = N^{\nu}|_{E_{\circ}^{\Pi}}$ . We first prove a helpful consequence of Proposition 3.4. If  $F = (\Pi_0^F, \Pi_1^F, \dots, \Pi_m^F) \in \mathbb{F}$  with m > 1,  $G := (\Pi_1^F, \dots, \Pi_m^F)$ , and  $(\tau, \xi) \in \text{dec}(G)$ , s > 0,  $\mathbf{z} \in E_{\circ}^{\Pi_1 \setminus \Pi}$ , then we define  $s\tau \in \text{tm}(F)$  by  $\Pi_0 \mapsto s$  and  $\Pi_i \mapsto s + \tau(\Pi_i)$  for  $i \in [m]$ , and  $\mathbf{z}\xi \in \text{sp}(F)$  in the sense of (2.4). This defines a bijection

$$(0,\infty) \times E_{\circ}^{\Pi_1 \backslash \Pi} \times \operatorname{dec}(G) \quad \to \quad \operatorname{dec}(F),$$
$$(s, \boldsymbol{z}, \tau, \boldsymbol{\xi}) \quad \mapsto \quad (s\tau, \boldsymbol{z}\boldsymbol{\xi}).$$

**Lemma 3.10.** Let  $F = (\Pi, \Pi_1^F, \dots, \Pi_m^F) \in \mathbb{F}$  be non-trivial, then

$$H_F(d\tau, d\xi) := e^{\beta_{\Pi} \min \tau} \frac{\widetilde{N}_{\Pi}(\boldsymbol{x}) P^{\boldsymbol{x}}(F, d\tau, d\xi)}{\prod_{\substack{u \in \Pi \\ \operatorname{pr}_F(u) \neq \emptyset}} p(\tau_{\operatorname{pr}_F(u)}, \xi_{\operatorname{pr}_F(u)} - \boldsymbol{x}_u)},$$

defines a finite measure on  $\operatorname{dec}(F)$  whose definition does not depend on  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ . If m=1, then there is a finite measure  $H_{\Pi,\Pi_{1}^{F}}^{\circ}$  on  $E_{\circ}^{\Pi_{1}^{F} \setminus \Pi}$  such that

$$H_F(\mathrm{d}\tau,\mathrm{d}\xi) = H_{\Pi,\Pi_1^F}^{\circ}(\mathrm{d}\xi)\,\mathrm{d}\tau.$$

If m > 1, so  $G := (\Pi_1^F, \dots, \Pi_m^F)$  is non-trivial, then there exists a finite measure  $H_F^{\circ}$  on  $E_{\circ}^{\Pi_1^F \setminus \Pi} \times \operatorname{dec}(G)$  such that

$$H_F(d(s\tau), d(z\xi)) = H_F^{\circ}(dz, d\tau, d\xi) ds.$$

*Proof.* Abbreviate  $pr = pr_F$  and  $ch = ch_F$ . If we multiply (3.7) with  $e^{\beta_{\Pi} \min \tau}$ , we get

$$e^{\beta_{\Pi} \min(\tau+s)} \frac{\widetilde{N}_{\Pi}(\boldsymbol{x}) P^{\boldsymbol{x}}(F, d(\tau+s), d\xi)}{\prod_{\substack{u \in \Pi \\ \operatorname{pr}(u) \neq \emptyset}} p(\tau_{\operatorname{pr}(u)} + s, \xi_{\operatorname{pr}(u)} - \boldsymbol{x}_u)} = e^{\beta_{\Pi} \min \tau} \frac{\widetilde{N}_{\Pi}(\boldsymbol{y}) P^{\boldsymbol{y}}(F, d\tau, d\xi)}{\prod_{\substack{u \in \Pi \\ \operatorname{pr}(u) \neq \emptyset}} p(\tau_{\operatorname{pr}(u)}, \xi_{\operatorname{pr}(u)} - \boldsymbol{y}_u)}$$
(3.18)

for every  $x, y \in E_{\circ}^{\Pi}$  and  $s \geq 0$  as measures on dec(F). Putting s = 0 shows that the definition of  $H_F$  does not depend on x, and in general we can rewrite (3.18) as

$$H_F(d(\tau + s), d\xi) = H_F(d\tau, d\xi), \qquad s \ge 0, \tag{3.19}$$

as measures on dec(F).

Now assume that m=1, then  $F=(\Pi,\Pi_1^F)$ , and  $H_F$  is a finite measure on  $(0,\infty)\times E_\circ^{\Pi_1^F\setminus\Pi}$  which is translation-invariant in its first component. For  $A\subset E_\circ^{\Pi_1^F\setminus\Pi}$  measurable,  $H_F(\cdot,A)$  is a translation-invariant measure on  $(0,\infty)$ , thus a multiple  $H_{\Pi,\Pi_1^F}^\circ(A)\in[0,\infty]$  of Lebesgue measure. Then

$$H_{\Pi,\Pi_1^F}^{\circ}(A) = H_F((1,2) \times A), \qquad A \subset E^{\Pi_1^F \setminus \Pi}$$
 measurable,

which implies that  $H_{\Pi,\Pi_f^F}^{\circ}$  is a finite measure. Then  $H_F(\mathrm{d} s,\mathrm{d} \xi)=H_{\Pi,\Pi_f^F}^{\circ}(\mathrm{d} \xi)\,\mathrm{d} s.$ 

Now suppose that m > 1, then (3.19) implies that, for  $A \subset E_{\circ}^{\Pi_1^F \setminus \Pi} \times \operatorname{dec}(G)$  measurable,

$$H_F\left(\left\{(s\tau, \mathbf{z}\xi) \in \operatorname{dec}(F) \colon s \in \cdot, (\mathbf{z}, (\tau, \xi)) \in A\right\}\right)$$

is a translation-invariant measure on  $(0, \infty)$  and thus a multiple of Lebesgue measure, which implies with a similar argument that

$$H_F(d(s\tau), d(z\xi)) = H_F^{\circ}(dz, d\tau, d\xi) ds$$

for a finite measure  $H_F^{\circ}$  on  $E_{\circ}^{\Pi_1^F \backslash \Pi} \times \operatorname{dec}(G)$ .

Proof of Theorem 3.9. We use an induction over  $|\operatorname{lf}(F)|$  to prove existence of  $\boldsymbol{\nu}$ , and comment on uniqueness at the end of the proof. If  $\Pi \in \mathcal{P}$  with  $|\Pi| = 1$  then the only  $F \in \mathbb{F}(\Pi)$  is trivial and (2.12) holds automatically. Furthermore  $\beta_{\Pi} = |\nu_{\Pi}| = 0$  by definition, and both  $N^{\boldsymbol{\nu}}|_{E_{\circ}^{\Pi}}$  (for any  $\boldsymbol{\nu} \in R$  by definition) and  $\widetilde{N}_{\Pi}$  are constant (Lemma 3.8). Let  $\Pi \in \mathcal{P}$  with  $|\Pi| \geq 2$  and suppose that  $\nu_{\Pi',\Pi''}$  for  $\Pi'' < \Pi'$  with  $|\Pi'| < |\Pi|$  have already been found such that (2.12) holds for all  $F \in \mathbb{F}$  with  $|\operatorname{lf}(F)| < |\Pi|$ , and such that for all  $\Pi' \in \mathcal{P}$  with  $|\Pi'| < |\Pi|$ ,  $\beta_{\Pi'} = |\nu_{\Pi'}|$ , and  $\widetilde{N}_{\Pi'}/N^{\boldsymbol{\nu}}$  is constant on  $E_{\circ}^{\Pi'}$  (this is a well-defined statement because  $N^{\boldsymbol{\nu}}$  on  $E_{\circ}^{\Pi'}$  only depends on transition measures  $\nu_{\Pi_1,\Pi_2}$  with  $\Pi_1 \leq \Pi'$ ).

Note that it is now enough to find finite measures  $\nu_{\Pi,\Pi'}$  for  $\Pi' \in \mathcal{P}$  with  $\Pi' < \Pi$  such that  $|\nu_{\Pi}| = \beta_{\Pi}$  and, for some  $c_{\Pi} > 0$ ,

$$P^{\boldsymbol{x}}(F, d\tau, d\xi) = \frac{c_{\Pi}}{\widetilde{N}_{\Pi}(\boldsymbol{x})} f_{\boldsymbol{\nu}}(\tau, \xi \mid \boldsymbol{x}) \boldsymbol{\nu}_{F}(d\xi) d\tau$$
 (3.20)

as measures on  $\operatorname{dec}(F)$  for every  $F \in \mathbb{F}(\Pi)$  and  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ . Indeed, the left-hand side defines a probability measure on  $\operatorname{dec}(\mathbb{F}(\Pi))$ , and  $N^{\boldsymbol{\nu}}$  is, by definition, the correct normalisation, so  $c_{\Pi}/\widetilde{N}_{\Pi} = 1/N^{\boldsymbol{\nu}}$  on  $E_{\circ}^{\Pi}$ .

If  $\beta_{\Pi}=0$ , then all  $F\in\mathbb{F}(\Pi)$  except for the trivial forest are impossible, and (2.12) holds for all  $F\in\mathbb{F}(\Pi)$  by putting  $\nu_{\Pi,\Pi'}=0$  for all  $\Pi'<\Pi$ . Then  $|\nu_{\Pi}|=0=\beta_{\Pi}$ , and  $\widetilde{N}_{\Pi}$  is constant (Lemma 3.8). Assume for the remainder of the inductive step that  $\beta_{\Pi}>0$ , and let  $F=(\Pi,\Pi_1^F,\ldots,\Pi_m^F)\in\mathbb{F}$  with  $m\geq 1$ , for if F is trivial then both sides of (2.12) vanish automatically (given the success of the remaining inductive step, so that  $|\nu_{\Pi}|=\beta_{\Pi}>0$ ). If m=1 and  $\beta_{\Pi_1^F}>0$ , then  $F=(\Pi,\Pi_1^F)$  is impossible, and both left- and right-hand side of (2.12) are zero (the latter because  $|\nu_{\Pi_1^F}|=\beta_{\Pi_1^F}>0$  by induction hypothesis). If  $\beta_{\Pi_1^F}=0$ ,

we let  $\nu_{\Pi,\Pi_1^F} = c_{\Pi}^{-1} H_{\Pi,\Pi_1^F}^{\circ}$  for a  $c_{\Pi} > 0$  that we specify later. Then rearranging the definition of  $H_F$  in Lemma 3.10 implies

$$P^{\boldsymbol{y}}(F, d\tau, d\xi) = e^{-\beta_{\Pi}\tau} \frac{c_{\Pi}}{\widetilde{N}_{\Pi}(\boldsymbol{y})} \prod_{\substack{u \in \Pi \\ \operatorname{pr}(u) \neq \emptyset}} p(\tau_{\operatorname{pr}(u)}, \xi_{\operatorname{pr}(u)} - \boldsymbol{y}_u) \nu_{\Pi, \Pi_1^F}(d\xi) d\tau,$$

where we abbreviated pr = pr<sub>F</sub> and ch = ch<sub>F</sub>. Assuming that  $|\nu_{\Pi}| = \beta_{\Pi}$  after choosing the remaining transition measures and  $c_{\Pi}$ , this is equivalent to (3.20) (recall the relevant definitions from (2.9) to (2.11)).

Now suppose m>1. The claim has already been proved for  $G=(\Pi_1^F,\ldots,\Pi_m^F)$  by induction hypothesis because  $|\Pi_1^F|<|\Pi|$ . If G is impossible then by (2.12) for G we must have  $\beta_{\mathrm{rt}(G)}=\beta_{\mathrm{rt}(F)}>0$  or  $\nu_{\Pi_i^F,\Pi_{i+1}^F}=0$  for some  $i\in[m-1]$ , and since F is also impossible, both left- and right-hand sides of (2.12) for F are zero. Suppose thus that G is possible, so  $\beta_{\mathrm{rt}(F)}=0$  and  $\nu_{\Pi_i^F,\Pi_{i+1}^F}\neq0$  for all  $i\in[m-1]$ . Let  $\Pi_{\mathrm{m}}=\Pi\setminus\Pi_1^F$  and  $\Pi_{\mathrm{n}}=\Pi_1^F\cap\Pi$ , the leaf nodes that did and did not merge in the first transition, respectively. Rearranging the definition of  $H_F$  in Lemma 3.10 gives

$$P^{\boldsymbol{x}}(F, d(s\tau), d(\boldsymbol{z}\xi))$$

$$= \frac{e^{-\beta_{\Pi}s}}{\widetilde{N}_{\Pi}(\boldsymbol{x})} H_{F}(d(s\tau), d(\boldsymbol{z}\xi)) \left[ \prod_{\substack{u \in \Pi \\ \operatorname{pr}(u) \neq \emptyset}} p((s\tau)_{\operatorname{pr}(u)}, (\boldsymbol{z}\xi)_{\operatorname{pr}(u)} - \boldsymbol{x}_{u}) \right]$$

$$= \frac{e^{-\beta_{\Pi}s}}{\widetilde{N}_{\Pi}(\boldsymbol{x})} H_{F}^{\circ}(d\boldsymbol{z}, d\tau, d\xi) ds \left[ \prod_{\substack{u \in \Pi_{n} \\ \operatorname{pr}(u) \neq \emptyset}} p(\tau_{\operatorname{pr}(u)} + s, \xi_{\operatorname{pr}(u)} - \boldsymbol{x}_{u}) \prod_{u \in \Pi_{m}} p_{s}(\boldsymbol{x}_{u} - \boldsymbol{z}_{\operatorname{pr}(u)}) \right].$$
(3.21)

We now evaluate

$$\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F, \operatorname{Tm} \in d(s\tau), \operatorname{Sp} \in d(\boldsymbol{z}\xi), \boldsymbol{X}_{s}|_{\Pi_{n}} \in d\boldsymbol{y}), \tag{3.22}$$

a sub-probability measure on  $\operatorname{dec}(F) \times E_{\circ}^{\Pi_n}$ , in two ways. On the one hand, by (2.12) it equals

$$P^{\boldsymbol{x}}(F, d(s\tau), d(\boldsymbol{z}\xi))K_{\boldsymbol{x}}((F, s\tau, z\xi), \{\boldsymbol{X}_s|_{\Pi_n} \in d\boldsymbol{y}\})$$

$$= P^{\boldsymbol{x}}(F, d(s\tau), d(\boldsymbol{z}\xi)) \prod_{u \in \Pi_n} p_s(\boldsymbol{x}_u - \boldsymbol{y}_u) \left[ \frac{p(\tau_{\text{pr}(u)}, \boldsymbol{y}_u - \xi_{\text{pr}(u)})}{p(\tau_{\text{pr}(u)} + s, \boldsymbol{x}_u - \xi_{\text{pr}(u)})} \right]_{\text{pr}(u) \neq \emptyset} d\boldsymbol{y}. \quad (3.23)$$

Let  $\tau_0 := \inf\{t > 0 : \Pi_t \neq \Pi_0\}$ , and  $\tau_{\Pi_1^F} := \inf\{t > 0 : \Pi_t = \Pi_1^F\}$ , which are both  $(\mathcal{F}_t^{\boldsymbol{X}})$ -stopping times, and write

$$P_{\Pi,\Pi_1^F}^{\boldsymbol{x}}(\mathrm{d} s,\mathrm{d} z,\mathrm{d} \boldsymbol{y}) \coloneqq \mathbb{P}^{\boldsymbol{x}}(\tau_0 = \tau_{\Pi_1^F} \in \mathrm{d} s,\boldsymbol{X}_s \in \mathrm{d}(\boldsymbol{y}\boldsymbol{z})),$$

a sub-probability measure on  $(0, \infty) \times E_{\circ}^{\Pi}$  whose total mass is the probability that the first jump is  $\Pi \to \Pi_1^F$  when starting at  $\boldsymbol{x}$ . For the second evaluation, by the strong Markov property at time  $\tau_{\Pi_1^F}$ , (3.22) equals  $P_{\Pi,\Pi_1^F}^{\boldsymbol{x}}(\mathrm{d}\boldsymbol{s},\mathrm{d}\boldsymbol{z},\mathrm{d}\boldsymbol{y})P^{\boldsymbol{y}\boldsymbol{z}}(G,\mathrm{d}\tau,\mathrm{d}\xi)$ , and equating with

(3.23) gives

$$P^{\boldsymbol{x}}(F, d(s\tau), d(\boldsymbol{z}\xi)) d\boldsymbol{y} = \frac{P_{\Pi, \Pi_{1}^{F}}^{\boldsymbol{x}}(ds, d\boldsymbol{z}, d\boldsymbol{y})}{\prod_{u \in \Pi_{n}} p_{s}(\boldsymbol{x}_{u} - \boldsymbol{y}_{u})} P^{\boldsymbol{y}\boldsymbol{z}}(G, d\tau, d\xi) \prod_{\substack{u \in \Pi_{n} \\ \operatorname{pr}(u) \neq \emptyset}} \frac{p(\tau_{\operatorname{pr}(u)} + s, \boldsymbol{x}_{u} - \xi_{\operatorname{pr}(u)})}{p(\tau_{\operatorname{pr}(u)}, \boldsymbol{y}_{u} - \xi_{\operatorname{pr}(u)})}.$$

Combining with (3.21) gives, after some cancellation,

$$P_{\Pi,\Pi_{1}^{F}}^{\boldsymbol{x}}(\mathrm{d}s,\mathrm{d}\boldsymbol{z},\mathrm{d}\boldsymbol{y})P^{\boldsymbol{y}\boldsymbol{z}}(G,\mathrm{d}\tau,\mathrm{d}\xi) = \frac{\mathrm{e}^{-\beta_{\Pi}s}}{\widetilde{N}_{\Pi}(\boldsymbol{x})}H_{F}^{\circ}(\mathrm{d}\boldsymbol{z},\mathrm{d}\tau,\mathrm{d}\xi)\,\mathrm{d}s$$

$$\times \prod_{u\in\Pi_{n}} p_{s}(\boldsymbol{x}_{u}-\boldsymbol{y}_{u})\prod_{u\in\Pi_{m}} p_{s}(\boldsymbol{x}_{u}-\boldsymbol{z}_{\mathrm{pr}(u)})\prod_{\substack{u\in\Pi_{n}\\\mathrm{pr}(u)\neq\emptyset}} p(\tau_{\mathrm{pr}(u)},\boldsymbol{y}_{u}-\xi_{\mathrm{pr}(u)})\,\mathrm{d}\boldsymbol{y}.$$

for fixed  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ . By (2.12) which already holds for G,  $P^{\boldsymbol{y}\boldsymbol{z}}(G, d\tau, d\xi)$  has a strictly positive density  $\frac{1}{N^{\nu}(\boldsymbol{y}\boldsymbol{z})}f_{\nu}(\cdot | \boldsymbol{y}\boldsymbol{z})|_{\text{dec}(G)}$  w.r.t. the finite measure  $\boldsymbol{\nu}_{G}(d\xi)d\tau$ , so we can apply Lemma 3.11 below with  $\boldsymbol{x}=(\boldsymbol{z},\boldsymbol{y})$  and  $\boldsymbol{\zeta}=(\tau,\xi)$ , to obtain existence of a measure  $K(d\boldsymbol{y},d\boldsymbol{z})$  on  $E_{\circ}^{\Pi_{1}^{F}}$  (which may depend on  $\boldsymbol{x}$  and F) with

$$K(d\mathbf{y}, d\mathbf{z}) ds = e^{\beta_{\Pi} s} \widetilde{N}_{\Pi}(\mathbf{z}) \frac{P_{\Pi, \Pi_{1}^{F}}^{\mathbf{z}}(ds, d\mathbf{y}, d\mathbf{z})}{\prod_{u \in \Pi_{n}} p_{s}(\mathbf{z}_{u} - \mathbf{y}_{u}) \prod_{u \in \Pi_{m}} p_{s}(\mathbf{z}_{u} - \mathbf{z}_{\text{pr}(u)})}$$
(3.24)

as measures on  $E_{\circ}^{\Pi_{1}^{F}} \times (0, \infty)$ , and

$$K(\mathrm{d}\boldsymbol{y},\mathrm{d}\boldsymbol{z})\boldsymbol{\nu}_{G}(\mathrm{d}\boldsymbol{\xi})\,\mathrm{d}\tau = H_{F}^{\circ}(\mathrm{d}\boldsymbol{z},\mathrm{d}\tau,\mathrm{d}\boldsymbol{\xi})\frac{N^{\boldsymbol{\nu}}(\boldsymbol{y}\boldsymbol{z})}{f_{\boldsymbol{\nu}}(\tau,\boldsymbol{\xi}\mid\boldsymbol{y}\boldsymbol{z})}\prod_{\substack{u\in\Pi_{\mathrm{n}}\\\mathrm{pr}(u)\neq\emptyset}}p(\tau_{\mathrm{pr}(u)},\boldsymbol{y}_{u}-\xi_{\mathrm{pr}(u)})\,\mathrm{d}\boldsymbol{y},\quad(3.25)$$

as measures on  $E_{\circ}^{\Pi_{1}^{F}} \times \operatorname{dec}(G)$ . From (3.24) it follows that K depends on F only through  $(\Pi, \Pi_{1}^{F})$ , and from (3.25) it follows that K does not depend on  $\boldsymbol{x}$ . In other words, had we chosen another  $\boldsymbol{x} \in E_{\circ}^{\Pi}$  and another possible forest  $F \in \mathbb{F}(\Pi)$  whose first transition is  $\Pi \to \Pi_{1}^{F}$ , we would have obtained the same measure  $K = K_{\Pi,\Pi_{1}^{F}}$ . It follows from (3.25) that K is absolutely continuous in its first component, that is there exist measures  $K(\boldsymbol{y}, \mathrm{d}\boldsymbol{z})$  on  $E_{\boldsymbol{y}}^{\Pi_{1}^{F} \setminus \Pi} := \{\boldsymbol{z} \in E_{\circ}^{\Pi_{1}^{F} \setminus \Pi} : \boldsymbol{y}\boldsymbol{z} \in E_{\circ}^{\Pi_{1}^{F}}\}$  such that  $\boldsymbol{y} \mapsto K(\boldsymbol{y}, A)$  is measurable for measurable A, and such that

$$K(\boldsymbol{y}, \mathrm{d}\boldsymbol{z})\boldsymbol{\nu}_{G}(\mathrm{d}\boldsymbol{\xi})\,\mathrm{d}\tau = H_{F}^{\circ}(\mathrm{d}\boldsymbol{z}, \mathrm{d}\tau, \mathrm{d}\boldsymbol{\xi})\frac{N^{\boldsymbol{\nu}}(\boldsymbol{y}\boldsymbol{z})}{f_{\boldsymbol{\nu}}(\tau, \boldsymbol{\xi} \mid \boldsymbol{y}\boldsymbol{z})} \prod_{\substack{u \in \Pi_{\mathrm{n}} \\ \mathrm{pr}(u) \neq \emptyset}} p(\tau_{\mathrm{pr}(u)}, \boldsymbol{y}_{u} - \xi_{\mathrm{pr}(u)})$$
(3.26)

as measures on  $E_{\boldsymbol{y}}^{\Pi_1^F\backslash\Pi} \times \operatorname{dec}(G)$  for Lebesgue-almost all  $\boldsymbol{y} \in E_{\circ}^{\Pi_n}$ . (The fact that we can choose a single null-set outside of which equality holds as measures is because the  $\sigma$ -algebra of  $E_{\boldsymbol{y}}^{\Pi_1^F\backslash\Pi} \times \operatorname{dec}(G)$  is countably generated.) If we rearrange (3.26) for  $H_F^{\circ}$  and plug it into

(3.21), we obtain

 $P^{\boldsymbol{x}}(F, d(s\tau), d(\boldsymbol{z}\xi))$ 

$$= \frac{e^{-\beta_{\Pi} s}}{\widetilde{N}_{\Pi}(\boldsymbol{x})} \left[ \prod_{\substack{u \in \Pi_{\mathrm{n}} \\ \mathrm{pr}(u) \neq \emptyset}} \frac{p(\tau_{\mathrm{pr}(u)}, \xi_{\mathrm{pr}(u)} - \boldsymbol{x}_{u})}{p(\tau_{\mathrm{pr}(u)}, \xi_{\mathrm{pr}(u)} - \boldsymbol{y}_{u})} \prod_{u \in \Pi_{\mathrm{m}}} p_{s}(\boldsymbol{x}_{u} - \boldsymbol{z}_{\mathrm{pr}(u)}) \right] f_{\boldsymbol{\nu}}(\tau, \xi \mid \boldsymbol{y}\boldsymbol{z})$$

$$\times \frac{K(\boldsymbol{y}, \mathrm{d}\boldsymbol{z})}{N^{\boldsymbol{\nu}}(\boldsymbol{y}\boldsymbol{z})} \boldsymbol{\nu}_{G}(\mathrm{d}\xi) \,\mathrm{d}\tau \,\mathrm{d}s$$

for Lebesgue-almost all y. By the definition of  $f_{\nu}$  (2.10), this is equal to

$$P^{\boldsymbol{x}}(F, d(s\tau), d(\boldsymbol{z}\xi)) = \frac{1}{\widetilde{N}_{\Pi}(\boldsymbol{x})} f_{\boldsymbol{\nu}}(s\tau, \boldsymbol{z}\xi \mid \boldsymbol{x}) \frac{K(\boldsymbol{y}, d\boldsymbol{z})}{N^{\boldsymbol{\nu}}(\boldsymbol{y}\boldsymbol{z})} \boldsymbol{\nu}_{G}(d\xi) d(s\tau). \tag{3.27}$$

Fix an arbitrary  $\mathbf{y}_0 \in E_{\circ}^{\Pi_n}$  for which this holds and put  $\nu_{\Pi,\Pi_1^F}(\mathrm{d}\mathbf{z}) \coloneqq c_{\Pi}^{-1} \frac{K(\mathbf{y}_0,\mathrm{d}\mathbf{z})}{N^{\nu}(\mathbf{y}_0\mathbf{z})}$  for the same  $c_{\Pi} > 0$  we introduced before. If we divide by  $N^{\nu}(\mathbf{y}_0\mathbf{z})$  in (3.26) and integrate over  $\xi \in \mathrm{sp}(G)$  and  $\tau \in \{1 \le \tau \le 2\} \subset \mathrm{tm}(G)$  say it follows that  $\nu_{\Pi,\Pi_1^F}$  is a finite measure. Now (3.27) evaluated for  $\mathbf{y} = \mathbf{y}_0$  becomes

$$P^{\boldsymbol{x}}(F, d(s\tau), d(\boldsymbol{z}\xi)) = \frac{c_{\Pi}}{\widetilde{N}_{\Pi}(\boldsymbol{x})} f_{\boldsymbol{\nu}}(s\tau, \boldsymbol{z}\xi \mid \boldsymbol{x}) \boldsymbol{\nu}_{F}(d(\boldsymbol{z}\xi)) d(s\tau),$$

which implies (3.20).

We have now defined a finite measure  $\nu_{\Pi,\Pi'}$  for every  $\Pi' \in \mathcal{P}$  with  $\Pi' < \Pi$ : if  $\beta_{\Pi'} = 0$ , then  $\nu_{\Pi,\Pi'}$  is defined in the "m = 1 case". Otherwise, there exists a possible, non-trivial forest  $G \in \mathbb{F}(\Pi')$  so that  $\nu_{\Pi,\Pi'}$  is defined in the "m > 1 case", in which we explained why the definition of  $\nu_{\Pi,\Pi'}$  in that case does not depend on the choice of  $G \in \mathbb{F}(\Pi')$ .

It remains to show that we can choose  $c_{\Pi} > 0$  such that  $|\nu_{\Pi}| = \beta_{\Pi}$ . Since  $\beta_{\Pi} > 0$  there must be at least one non-trivial, possible forest  $F \in \mathbb{F}(\Pi)$ , in which case (3.27) necessitates that the associated  $\nu_{\Pi,\Pi_1^F}$  is non-zero, so  $|\nu_{\Pi}| > 0$ . Thus there exists a unique choice for  $c_{\Pi} > 0$  (which by definition is a linear factor in every non-zero  $\nu_{\Pi,\Pi'}$ ) for which  $|\nu_{\Pi}| = \beta_{\Pi}$ .

Finally, the construction of  $\nu$  satisfying (2.12) in this proof was unique. The sceptical reader can prove uniqueness of the transition measures  $\nu$  associated with a Brownian spatial coalescent directly with an induction of a similar structure to this proof.

**Lemma 3.11.** Suppose  $\Omega_1, \Omega_2, \Omega_3$  are measurable spaces, and

$$G(ds, dx) f(x, \zeta) \nu(d\zeta) = g(s, x) F(dx, d\zeta) \mu(ds),$$

where G and F are measures on  $\Omega_1 \times \Omega_2$  and  $\Omega_2 \times \Omega_3$  respectively,  $f : \Omega_2 \times \Omega_3 \to \mathbb{R}$  and  $g : \Omega_1 \times \Omega_2 \to \mathbb{R}$  are strictly positive measurable functions, and  $\nu$  and  $\mu$  are measures on  $\Omega_3$  and  $\Omega_1$  respectively, such that there exist  $A \subset \Omega_3$  and  $B \subset \Omega_1$  with  $\mu(A), \nu(B) \in (0, \infty)$ . Then there is a measure H on  $\Omega_2$  such that

$$\frac{G(\mathrm{d}s,\mathrm{d}x)}{g(s,x)} = H(\mathrm{d}x)\mu(\mathrm{d}s), \qquad \frac{F(\mathrm{d}\zeta,\mathrm{d}x)}{f(\zeta,x)}\nu(\mathrm{d}\zeta) = H(\mathrm{d}x)\nu(\mathrm{d}\zeta),$$

respectively as measures on  $\Omega_1 \times \Omega_2$  and  $\Omega_2 \times \Omega_3$ .

*Proof.* Rewrite the equality as an equality of measures

$$\frac{G(\mathrm{d}s,\mathrm{d}x)}{g(s,x)}\nu(\mathrm{d}\zeta) = \frac{F(\mathrm{d}x,\mathrm{d}\zeta)}{f(x,\zeta)}\mu(\mathrm{d}s). \tag{3.28}$$

Let A, B be sets of positive and finite measure w.r.t.  $\mu$  and  $\nu$  respectively (by assumption there exists at least one each), and define H(dx) by integrating (3.28) over  $s \in A, \zeta \in B$  and multiplying with  $\frac{1}{\mu(A)\nu(B)}$ , that is

$$H(\mathrm{d}x) := \frac{1}{\mu(A)} \int_{s \in A} \frac{G(\mathrm{d}s, \mathrm{d}x)}{g(s, x)} = \frac{1}{\nu(B)} \int_{\zeta \in B} \frac{F(\mathrm{d}x, \mathrm{d}\zeta)}{f(x, \zeta)}.$$

Note H is independent of B by the first and A by the second equality. To prove the first equality of measures in the claim, take a set A with  $\mu(A) \in (0, \infty)$  and  $C \subset \Omega_2$ . Then,

$$\int_{x \in C} \int_{s \in A} H(\mathrm{d}x) \mu(\mathrm{d}s) = H(C) \mu(A) = \mu(A) \int_{C} H(\mathrm{d}x) = \int_{x \in C} \int_{s \in A} \frac{G(\mathrm{d}s, \mathrm{d}x)}{g(s, x)}.$$

If A is a  $\mu$ -null set, then by integrating (3.28) over it and some set of positive  $\nu$ -measure we find that  $\int_A \frac{G(\mathrm{d} s, \mathrm{d} x)}{g(s, x)} = 0$ , so equality holds in this case as well. Since  $\mu$  is  $\sigma$ -finite, we find equality for all measurable sets A. The second equality follows similarly.

3.2. Characterisation of Label Invariance. In preparation for the proof of Theorem 1.12, this section is devoted to a characterisation of label invariance for Brownian spatial coalescents.

For  $(n, \vec{k}) \in M$  and  $\Pi, \Pi' \in \mathcal{P}$  with  $n = |\Pi|$  and  $\Pi' < \Pi$ , we call  $(\Pi, \Pi')$  an  $(n, \vec{k})$ -merger if  $\Pi'$  can be obtained from  $\Pi$  by coalescing m disjoint sets  $I_1, \ldots, I_m \subset \Pi$  of sizes  $k_1$  to  $k_m$  into one each. A non-spatial coalescent with transition rates  $\lambda \in R^\circ$  is label invariant if and only if there are numbers  $\lambda_{n,\vec{k}} \geq 0$  for  $(n,\vec{k}) \in M$  such that  $\lambda_{\Pi,\Pi'} = \lambda_{n,\vec{k}}$  whenever  $(\Pi,\Pi')$  is an  $(n,\vec{k})$  merger (recall Definition 1.1 and the following paragraph). We show that a similar characterisation holds for Brownian spatial coalescents. For  $m \in \mathbb{N}$  define

$$E_{\circ}^{m} = \left\{ \boldsymbol{x} \in E^{m} \colon \left[ \{i\} \mapsto \boldsymbol{x}_{i} \right] \in E_{\circ}^{\left\{ \{1\}, \dots, \{m\} \right\}} \right\}.$$

In particular,  $E_{\circ}^{1} = E$ . If  $(n, k_{1}, \ldots, k_{m}) \in M$ , choose a bijection  $\ell : [m] \to \Pi' \setminus \Pi$  such that  $\ell(i)$  for  $i \in [m]$  is the union of  $k_{i}$  distinct elements of  $\Pi$ , which is unique up to a permutation of pairs (i, j) with  $k_{i} = k_{j}$ . Define  $\kappa_{\Pi,\Pi'} : E_{\circ}^{\Pi' \setminus \Pi} \to E_{\circ}^{m}$  by  $\boldsymbol{x} \mapsto \boldsymbol{x} \circ \ell$  (here and in the following, we occasionally identify a vector  $\boldsymbol{x} \in E^{m}$  with the map  $[i \mapsto \boldsymbol{x}_{i}] \in E^{[m]}$ ). Recall the definition of  $\mathcal{I}(\Pi_{0},\Pi_{1})$  from the paragraph preceding Definition 2.4. We further extend a map  $\iota \in \mathcal{I}(\Pi_{0},\Pi_{1})$  to

$$\mathbb{F}(\Pi_0) \to \mathbb{F}(\Pi_1); \quad F = (\Pi_0^F, \dots, \Pi_m^F) \mapsto (\iota(\Pi_0^F), \dots, \iota(\Pi_m^F)),$$

and, for every  $F \in \mathbb{F}(\Pi_0)$ , to

$$\operatorname{sp}(F) \to \operatorname{sp}(\iota(F)); \quad \xi \mapsto \xi \circ \iota^{-1},$$
  
 $\operatorname{tm}(F) \to \operatorname{tm}(\iota(F)); \quad \tau \mapsto \tau \circ \iota^{-1}.$ 

Finally, this lets us define  $\iota$  on  $\operatorname{dec}(\mathbb{F}(\Pi_0))$  by  $(F, \tau, \xi) \mapsto (\iota(F), \iota(\tau), \iota(\xi))$ .

**Lemma 3.12.** The Brownian spatial coalescent with transition measures  $\nu \in R^{\circ}$  is label invariant if and only if for all  $\Pi_0, \Pi_1 \in \mathcal{P}$  of equal size and  $\iota \in \mathcal{I}(\Pi_0, \Pi_1)$ ,

$$\forall \Pi' < \Pi_0 \colon \iota \# \nu_{\Pi_0,\Pi'} = \nu_{\iota(\Pi_0),\iota(\Pi')}.$$

In that case, there exists for every  $(n, \vec{k}) = (n, k_1, \dots, k_m) \in M$  a finite measure  $\nu_{n,\vec{k}}$  on  $E_{\circ}^m$  such that for every  $(n, \vec{k})$ -merger  $(\Pi, \Pi')$ ,

$$\kappa_{\Pi,\Pi'} \# \nu_{\Pi,\Pi'} = \nu_{n,\vec{k}}.$$

If  $k_i = k_i$ , then  $\nu_{n,\vec{k}}$  is symmetric in the ith and jth coordinate.

Lemma 3.12 justifies that, for a label invariant Brownian spatial coalescent, we identify  $\nu_{\Pi,\Pi'}$  for a  $(n,\vec{k})$ -merger  $(\Pi,\Pi')$  with  $\nu_{n,\vec{k}}$ , and by the symmetry of  $\nu_{n,\vec{k}}$  the choice of the underlying map  $\kappa_{\Pi,\Pi'}$  is irrelevant.

Proof of Lemma 3.12. Let  $\nu \in R$  be the transition measures of a Brownian spatial coalescent. We assume that  $|\nu_{\Pi}| > 0$  for all  $\Pi \in \mathcal{P}$  with  $|\Pi| \geq 2$  to slightly simplify the proof. Let  $\Pi_0, \Pi_1 \in \mathcal{P}$  be of equal size. Note that the right-hand condition is equivalent by a simple induction to  $\iota \# \nu_F = \nu_{\iota(F)}$  for all  $\iota \in \mathcal{I}(\Pi_0, \Pi_1)$  and  $F \in \mathbb{F}(\Pi_0)$ . Let  $\iota \in \mathcal{I}(\Pi_0, \Pi_1)$ , then (2.3) for  $\boldsymbol{x} \in E_0^{\Pi_0}$  is equivalent to  $P^{\iota(\boldsymbol{x})}(F^* \in \cdot) = P^{\boldsymbol{x}}(\iota(F^*) \in \cdot)$ , that is

$$f_{\boldsymbol{\nu}}(\tau,\xi \mid \iota(\boldsymbol{x}))\boldsymbol{\nu}_{F}(\mathrm{d}\xi)\,\mathrm{d}\tau = f_{\boldsymbol{\nu}}(\iota^{-1}(\tau),\iota^{-1}(\xi) \mid \boldsymbol{x})(\iota\#\boldsymbol{\nu}_{\iota^{-1}(F)})(\mathrm{d}\xi)\,\mathrm{d}\tau$$

as measures on dec(F) for every  $F \in \mathbb{F}(\Pi_1)$ , which in turn is equivalent to

$$\prod_{(\Pi,\Pi')\in F} e^{-|\nu_{\Pi}|(\tau_{\Pi'}-\tau_{\Pi})} \nu_{F}(d\xi) d\tau = \prod_{(\Pi,\Pi')\in F} e^{-|\nu_{\iota^{-1}(\Pi)}|(\tau_{\Pi'}-\tau_{\Pi})} (\iota \# \nu_{\iota^{-1}(F)}) (d\xi) d\tau$$
(3.29)

for all  $F \in \mathbb{F}(\Pi_1)$ . Clearly  $\nu_F = \iota \# \nu_{\iota^{-1}(F)}$  for all  $F \in \mathbb{F}(\Pi_1)$  implies (3.29). If label invariance holds, then integrating out  $\xi \in \operatorname{sp}(F)$  and varying  $\tau$  in (3.29) implies  $|\nu_{\Pi}| = |\nu_{\iota^{-1}(\Pi)}|$  for all  $(\Pi, \Pi') \in F$ , which back into (3.29) implies  $\iota \# \nu_{\iota^{-1}(F)} = \nu_F$ .

Now suppose label invariance holds, and let  $n \geq 2$  and  $k_1 \geq \ldots \geq k_m \geq 2$  with  $\sum_i k_i \leq n$ . For existence of  $\nu_{n,\vec{k}}$  it suffices to show that  $\kappa_{\Pi,\Pi'} \# \nu_{\Pi,\Pi'}$  is the same for every  $(n,\vec{k})$ -merger  $(\Pi,\Pi')$ . Let  $(\Pi_0,\Pi'_0)$  and  $(\Pi_1,\Pi'_1)$  be  $(n,\vec{k})$ -mergers, and abbreviate  $\kappa_i := \kappa_{\Pi_i,\Pi'_i}$  and  $\nu_i := \nu_{\Pi_i,\Pi'_i}$  for i = 1, 2. Fix some  $\iota \in \mathcal{I}(\Pi_0,\Pi_1)$  for which  $\iota(\Pi'_0) = \Pi'_1$ , then

$$\kappa_1 \# \nu_1 = \kappa_1 \# (\iota \# \nu_0) = (\kappa_1 \circ \iota) \# \nu_0. \tag{3.30}$$

Here  $\kappa_1 \circ \iota \colon E_\circ^{\Pi_0' \setminus \Pi_0} \to E_\circ^m$  depends on the choices made for  $\kappa_0$  and  $\iota$ , but is always of the same form as  $\kappa_0$ , that is  $\boldsymbol{x} \mapsto \boldsymbol{x} \circ \ell$  for some bijection  $\ell \colon [m] \to \Pi' \setminus \Pi$  such that  $\ell(i)$  is the union of  $k_i$  disjoint elements of  $\Pi_0$ . If we show that  $(\cdot \circ \ell) \# \nu_0$  is the same for each such choice for  $\ell$ , then both equality of (3.30) to  $\kappa_0 \# \nu_0$ , and the claimed symmetry of  $\nu_{n,\vec{k}}$  follow. If two such bijections  $\ell,\ell' \colon [m] \to \Pi' \setminus \Pi$  are given, then there is a permutation  $\pi$  of  $\Pi' \setminus \Pi$  such that  $\pi \circ \ell = \ell'$ , and  $\pi(u)$  for  $u \in \Pi' \setminus \Pi$  is the result of merging as many blocks from  $\Pi$  as u. Thus we can construct a bijection  $\iota_0 \colon \Pi_0 \to \Pi_0$  such that  $\iota_0(u) = u$  for  $u \in \Pi_0 \setminus \Pi'_0$ , and  $\iota_0$  maps  $\{u \in \Pi_0 \colon u \subset \pi(v)\}$  to  $\{u \in \Pi_0 \colon u \subset v\}$  for each  $v \in \Pi'_0 \setminus \Pi_0$ . Then after extending  $\iota_0$  to  $\Pi'_0$  and  $E_\circ^{\Pi'_0 \setminus \Pi_0}$  as usual,  $\iota_0|_{\Pi'_0 \setminus \Pi_0} = \pi^{-1}$  and thus  $\iota_0|_{E_\circ^{\Pi'_0 \setminus \Pi_0}} = (\cdot \circ \pi)$ . In particular,  $\iota_0(\Pi_0) = \Pi_0$  and  $\iota_0(\Pi'_0) = \Pi'_0$ , so  $(\cdot \circ \pi) \# \nu_0 = \iota_0 \# \nu_0 = \nu_0$ , which implies  $(\cdot \circ \ell) \# \nu_0 = (\cdot \circ \ell') \# \nu_0$ .

Given a Brownian spatial coalescent, say with transition measures  $\boldsymbol{\nu} \in R^{\circ}$ , recall that a forest  $F \in \mathbb{F}$  is either possible, that is  $\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F) > 0$  for all  $\boldsymbol{x} \in E^{\operatorname{lf}(F)}_{\circ}$ , or impossible, that is  $\mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F) = 0$  for all such  $\boldsymbol{x}$ . We call a merge event  $(\Pi, \Pi')$  possible if there exists a

possible forest  $F = (\Pi, \Pi', ...) \in \mathbb{F}$ , that is if for every  $\mathbf{x} \in E_{\circ}^{\Pi}$  there is a positive probability that the first merge event is  $(\Pi, \Pi')$ . We collect the following obvious but useful statement.

**Lemma 3.13.** A forest  $F \in \mathbb{F}$  is possible if and only if  $\nu_{\Pi,\Pi'} \neq 0$  for all  $(\Pi,\Pi') \in F$  and  $|\nu_{\mathrm{rt}(F)}| = 0$ . An  $(\Pi,\Pi')$ -merger is possible if and only if  $\nu_{\Pi,\Pi'} \neq 0$ .

In particular, if  $|\nu_{\Pi}| > 0$  for every  $\Pi \in \mathcal{P}$  with  $|\Pi| \geq 2$ , then the only possible forests are trees. This will be the case for sampling consistent Brownian spatial coalescents, as we will see in the following section.

3.3. Characterisation of Sampling Consistency. In this section we prove Theorem 1.12. Throughout this section, fix a label invariant Brownian spatial coalescent, say with transition measures  $\boldsymbol{\nu} \in R$ . Recall in this context the measures  $\nu_{n,\vec{k}}$ , and write  $|\nu_n| := |\nu_{\Pi}|$  for any  $\Pi \in \mathcal{P}$  with  $|\Pi| = n$ , and  $|\nu_1| := 0$ . The following lemma is a core part of the proof. For two finite measures write  $m_1 \sim m_2$  if  $m_1 = cm_2$  for some c > 0.

**Lemma 3.14.** Let  $\nu(d\xi) = \lambda d\xi$  for some  $\lambda \in \mathbb{R}^{\circ}$ . Then the following are equivalent

- (i) The non-spatial coalescent with transition rates  $\lambda$  is sampling consistent.
- (ii)  $\int_{E_{\boldsymbol{x}}} N^{\boldsymbol{\nu}}(\boldsymbol{x}y) \, \mathrm{d}y < \infty$  for every  $\boldsymbol{x} \in \mathcal{X}$ , and the Brownian spatial coalescent with transition measures  $\boldsymbol{\nu}$  is sampling consistent w.r.t. the probability measures  $(\mu_{\boldsymbol{x}})$  satisfying  $\mu_{\boldsymbol{x}}(\mathrm{d}y) \sim N^{\boldsymbol{\nu}}(\boldsymbol{x}y) \, \mathrm{d}y$ .

In that case,  $\int N^{\nu}(xy) dy = N^{\nu}(x)$ .

Lemma 3.14 is proved in Section 3.3.1, and implies the "if" direction of Theorem 1.12. For the "only if" direction it remains to show that sampling consistency of a Brownian spatial coalescent w.r.t. some family  $(\mu_x)$  of probability measures implies  $\nu(d\xi) \sim d\xi$ , and that  $\mu_x(dy) \sim N^{\nu}(xy) dy$ . That is the content of Section 3.3.2. In the remainder of this section, we set up the notation used in the proof and make some technical observations related to sampling consistency.

Excluding from now the trivial case where  $|\nu_n| = 0$  for all  $n \in \mathbb{N}$  (so  $\nu_{n,\vec{k}} = 0$  for all  $(n,\vec{k})$ ), sampling consistency necessitates  $|\nu_n| > 0$  for all  $n \geq 2$ , in particular that only trees are possible. This is part of Lemma 3.20 below. We now present a characterisation of all trees G from which a fixed tree F can be "subsampled" by removing one leaf. This characterisation will be at the core of our main arguments.

For every  $\Pi_0 \in \mathcal{P}$  we fix some  $v_{\oplus} \subset \mathbb{N}$  disjoint from all members of  $\Pi_0$  and put  $\Pi_0^{\oplus} \coloneqq \Pi_0 \cup \{v_{\oplus}\} \in \mathcal{P}$ . For  $F \in \mathbb{T}(\Pi_0)$  and  $G \in \mathbb{T}(\Pi_0^{\oplus})$  we say that G extends F if  $F = G \downarrow_{\Pi_0} \coloneqq \{\Pi \setminus v_{\oplus} \colon \Pi \in G\}$  (using the set-representation  $F = \{\Pi_0^F, \dots, \Pi_m^F\}$  of a forest), and write  $\mathbb{T}^{\oplus}(F)$  for the set of such G. In that case, if  $(\tau, \xi) \in \operatorname{dec}(G)$  then  $\tau \downarrow_F \in \operatorname{tm}(F)$  and  $\xi \downarrow_F \in \operatorname{sp}(F)$  denote the decorations induced on F, respectively defined by  $\Pi \setminus v_{\oplus} \mapsto \operatorname{min}\{\tau(\Pi')\colon \Pi' \setminus v_{\oplus} = \Pi \setminus v_{\oplus}\}$  and  $u \setminus v_{\oplus} \mapsto \xi_u$ ; write  $\tau_{\oplus} \coloneqq \tau_{\operatorname{pr}_G(v_{\oplus})}$  and  $\xi_{\oplus} \coloneqq \xi_{\operatorname{pr}_G(v_{\oplus})}$  for time and location of the first merge of  $v_{\oplus}$ , see Fig. 5.

If  $G^{\star} = (G, \tau, \xi) \in \operatorname{dec}(\mathbb{T})$  and  $G\downarrow_{\Pi_0} = F$  then we denote  $G^{\star}\downarrow_{\Pi_0} = (G\downarrow_{\Pi_0}, \tau\downarrow_F, \xi\downarrow_F)$ . For  $F^{\star} \in \operatorname{dec}(\mathbb{T})$  write  $\operatorname{dec}^{\oplus}(F^{\star}) = \{G^{\star} \colon G^{\star}\downarrow_{\Pi_0} = F^{\star}\}$ , and for fixed G extending F,  $\operatorname{dec}^{\oplus}(F^{\star} \mid G) = \{(G, \tau, \xi) \in \operatorname{dec}^{\oplus}(F^{\star})\}$ . Write  $\Pi_G^{\oplus} = \max\{\Pi \in G \colon v_{\oplus} \notin \Pi\}$  for the state of the tree immediately after the first merge involving  $v_{\oplus}$ , and  $\Pi_F^{\oplus} \coloneqq \Pi_G^{\oplus} \setminus v_{\oplus}$  for the same

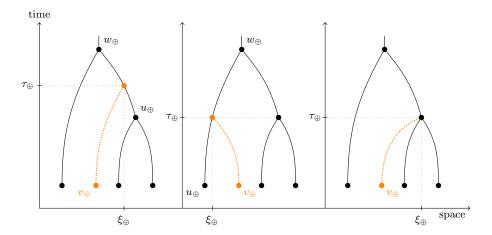


FIGURE 5. There are three ways to extend a fixed tree to an additional leaf. Either the leaf  $v_{\oplus}$  merges binary with another node  $u_{\oplus}$ , at a time at which no merge happens in the original tree (left), or simultaneously with an existing merge event (middle); or  $v_{\oplus}$  joins an existing merge event (right). The additional degrees of freedom in the tree decoration given that of the underlying tree are, from right to left, none, the location  $\xi_{\oplus}$ , and the time  $\tau_{\oplus}$  and location  $\xi_{\oplus}$  of the additional merge event.

state projected on  $\Pi_0$ . If G extends F, it has to fall within exactly one of the following three classes (see also Fig. 5).

- (i) *Multiple merge*:  $v_{\oplus}$  is part of a multiple merger (Figure 5 right). In this case we can identify  $\operatorname{dec}(G)$  and  $\operatorname{dec}(F)$ , so  $\operatorname{dec}^{\oplus}(F^{\star} | G)$  is a singleton.
- (ii) Binary merge:  $v_{\oplus}$  merges binary with a node  $u_{\oplus} \in \operatorname{nd}(G)$ , and no other merge takes place simultaneously (Figure 5 left). In this case, for  $F^* = (F, \tau_0, \xi_0) \in \operatorname{dec}(\mathbb{F})$  there is a bijection

$$\det^{\oplus}(F^{\star} \mid G) \quad \longleftrightarrow \quad (\tau_{0}(\Pi_{F}^{\oplus}), \tau_{0}(\Pi_{F}^{\oplus})^{+}) \times E,$$

$$(\tau, \xi) \quad \mapsto \quad (\tau_{\oplus}, \xi_{\oplus}),$$

$$(3.31)$$

whose inverse we denote  $(s, z) \mapsto (s^{\oplus} \tau_0, z^{\oplus} \xi_0)$ . In this case  $w_{\oplus} := \operatorname{pr}_G(u_{\oplus} \cup v_{\oplus}) \neq \emptyset$  except if the merge event involving  $v_{\oplus}$  is the final one, that is  $|\Pi_G^{\oplus}| = 1$ .

(iii) Simultaneous binary merge:  $v_{\oplus}$  merges binary with a node  $u_{\oplus} \in \operatorname{nd}(G)$  as part of a simultaneous merge event (Figure 5 middle). Then  $w_{\oplus} := \operatorname{pr}_{G}(u_{\oplus} \cup v_{\oplus}) \neq \emptyset$ , and we can identify  $\operatorname{tm}(G)$  and  $\operatorname{tm}(F)$ , and for  $F^{\star} = (F, \tau_{0}, \xi_{0}) \in \operatorname{dec}(\mathbb{F})$  there is a bijection

$$\operatorname{dec}^{\oplus}(F^{\star} \mid G) \longleftrightarrow E_{\xi_{0}|_{\Pi_{F}^{\oplus}}}; \quad (\tau, \xi) \mapsto \xi_{\oplus}, \tag{3.32}$$

(recall (2.5),) the inverse of which we denote  $z \mapsto (\tau_0, z^{\oplus} \xi_0)$ .

Denote the sets of such trees by  $\mathbb{T}_{\mathrm{m}}^{\oplus}(F)$ ,  $\mathbb{T}_{\mathrm{b}}^{\oplus}(F)$ , and  $\mathbb{T}_{\mathrm{sb}}^{\oplus}(F)$ , respectively. We do not explicitly denote dependence of  $u_{\oplus}$  and  $w_{\oplus}$  etc. on G, but it will always be clear from context.

This allows us to formulate sampling consistency on a more precise technical level. If all transition measures are absolutely continuous (which we will prove is a consequence of sampling consistency in Lemma 3.23), then with a slight abuse of notation, we write  $\nu_{n,\vec{k}}(\mathrm{d}z) = \nu_{n,\vec{k}}(z)\,\mathrm{d}z$  and  $\nu_F(\mathrm{d}\xi) = \nu_F(\xi)\,\mathrm{d}\xi$  etc., as well as  $P^x(F^*) \coloneqq P^x(F,\tau,\xi) \coloneqq \frac{1}{N^\nu(x)}f_\nu(\tau,\xi\,|\,x)\nu_F(\xi)$ , so that  $P^x(F,\mathrm{d}\tau,\mathrm{d}\xi) = P^x(F,\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi$ .

**Lemma 3.15.** Let  $\boldsymbol{\nu} \in R$  be a family of transition measures that are absolutely continuous w.r.t. Lebesgue measure. Then the associated coalescent process is sampling consistent w.r.t. a family of probability measures  $(\mu_{\boldsymbol{x}})_{\boldsymbol{x} \in \mathcal{X}}$  if and only if  $|\nu_n| > 0$  for all  $n \geq 2$ , and for every  $\Pi_0 \in \mathcal{P}$ ,  $F \in \mathbb{T}(\Pi_0)$ ,  $\boldsymbol{x} \in E_0^{\Pi_0}$ , and Lebesgue-a.e.  $F^* = (F, \tau, \xi) \in \operatorname{dec}(F)$ ,

$$K_{\boldsymbol{x}}(F^{\star},\cdot)P^{\boldsymbol{x}}(F,\tau,\xi) = K_{\boldsymbol{x}}(F^{\star},\cdot)\sum_{G\in\mathbb{T}_{m}^{\oplus}(F)} \int P^{\boldsymbol{x}y}(G,\tau,\xi)\mu_{\boldsymbol{x}}(\mathrm{d}y)$$

$$+\sum_{G\in\mathbb{T}_{b}^{\oplus}(F)} \iiint K_{\boldsymbol{x}y}((G,s^{\oplus}\tau,z^{\oplus}\xi),\{\mathring{\boldsymbol{X}}\in\cdot\})P^{\boldsymbol{x}y}(G,s^{\oplus}\tau,z^{\oplus}\xi)\,\mathrm{d}s\,\mathrm{d}z\mu_{\boldsymbol{x}}(\mathrm{d}y) \quad (3.33)$$

$$+\sum_{G\in\mathbb{T}_{b}^{\oplus}(F)} \iint K_{\boldsymbol{x}y}((G,\tau,z^{\oplus}\xi),\{\mathring{\boldsymbol{X}}\in\cdot\})P^{\boldsymbol{x}y}(G,\tau,z^{\oplus}\xi)\,\mathrm{d}z\mu_{\boldsymbol{x}}(\mathrm{d}y),$$

where  $(\mathring{X}_t) := (X_t \setminus v_{\oplus})$ , and domains of integration for s and z are as indicated by the bijections (3.31) and (3.32).

Proof. Eq. (3.33) is a direct reformulation of (2.6) using (2.8), the characterisation of  $\mathbb{T}^{\oplus}(F)$  for  $F \in \mathbb{T}$ , and in order to get an equation pointwise for a.e.  $F^{\star}$ , the fact that  $K_{\boldsymbol{x}}(F^{\star}, \{\text{Dec} \in A\} \cap (\cdot)) = \mathbb{1}_{\{F^{\star} \in A\}} K_{\boldsymbol{x}}(F^{\star}, \cdot)$  for all  $F^{\star} \in \text{dec}(\mathbb{T})$  and measurable  $A \subset \text{dec}(F)$ . That it suffices to consider trees is because  $|\nu_n| > 0$  for  $n \geq 2$  makes all forests that are not trees impossible.

Before moving on to the proof of Lemma 3.14, we give an explicit description of the law of a non-spatial coalescent when regarded as a random, time decorated forest, which will make it easier to connect with the laws  $(P^x)$  of a Brownian spatial coalescent. It follows from a simple argument about competing exponential clocks.

**Lemma 3.16.** Let  $\lambda \in R^{\circ}$  be the transition rates of a non-spatial coalescent  $(\mathbb{P}^{\Pi})_{\Pi \in \mathcal{P}}$ , and denote the total jump rate while at  $\Pi$  by  $\lambda_{\Pi} \geq 0$ . We define  $f_{\mathrm{tm}}^{\lambda} \colon \mathrm{tm}(\mathbb{F}) \to (0, \infty)$  by  $f_{\mathrm{tm}}^{\lambda}|_{\mathrm{tm}(F)} \equiv 0$  if  $\lambda_{\mathrm{rt}(F)} > 0$ , otherwise

$$f_{\mathrm{tm}}^{\boldsymbol{\lambda}}(F,\tau) := \prod_{(\Pi,\Pi')\in F} \lambda_{\Pi,\Pi'} \mathrm{e}^{-\lambda_{\Pi}(\tau_{\Pi'}-\tau_{\Pi})}, \qquad \tau \in \mathrm{tm}(F),$$

which is 1 if F is trivial. Then for  $\Pi \in \mathcal{P}$  and  $F \in \mathbb{F}(\Pi)$ ,

$$(\operatorname{Tm} \# \mathbb{P}^{\Pi})(F, d\tau) = f_{\operatorname{tm}}^{\lambda}(F, \tau) d\tau.$$

3.3.1. Proof of Lemma 3.14. In this section we fix label invariant rates  $\lambda \in R^{\circ}$  and define  $\nu \in R$  by  $\nu_{n,\vec{k}}(\mathrm{d}z) := \lambda_{n,\vec{k}} \,\mathrm{d}z$ . Then the laws  $(P^x)$  that characterise the Brownian spatial coalescent process with transition measures  $\nu$  (recall (2.12)) take the simpler form

$$P^{\boldsymbol{x}}(F,\tau,\xi) = \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} f_{\text{tm}}^{\boldsymbol{\lambda}}(F,\tau) f_{\text{sp}}^{F}(\xi \mid \tau, \boldsymbol{x}), \tag{3.34}$$

where in this chapter we will explicitly denote dependence of  $f_{\rm tm}^{\lambda}$  and  $f_{\rm sp}$  on F. We further remark that a characterisation similar to Lemma 3.15 holds for non-spatial coalescent processes, which are sampling consistent with transition rates  $\lambda$  if and only if  $\lambda_n > 0$  for all  $n \geq 2$  and

$$f_{\text{tm}}^{\lambda}(F,\tau) = \sum_{G \in \mathbb{T}_{b}^{\oplus}(F)} \int f_{\text{tm}}^{\lambda}(G,s^{\oplus}\tau) \, \mathrm{d}s + \sum_{G \in \mathbb{T}_{sb}^{\oplus}(F)} f_{\text{tm}}^{\lambda}(G,\tau) + \sum_{G \in \mathbb{T}_{m}^{\oplus}(F)} f_{\text{tm}}^{\lambda}(G,\tau)$$
(3.35)

for every  $\Pi_0 \in \mathcal{P}$ ,  $F \in \mathbb{T}(\Pi_0)$ , and Lebesgue-a.e.  $\tau \in \text{tm}(F)$ , where the RHS is simply a density of  $\mathbb{P}^{\Pi_0^{\oplus}}(\{(G,\tau'): G\downarrow_{\Pi_0} = F, \tau'\downarrow_F \in \cdot\})$  on tm(F).

**Lemma 3.17.** If the non-spatial coalescent with transition rates  $\lambda$  is sampling consistent, then  $\int N^{\nu}(\mathbf{x}y) \, \mathrm{d}y = N^{\nu}(\mathbf{x})$  for all  $\mathbf{x} \in \mathcal{X}$ , and the Brownian spatial coalescent with transition measures  $\nu$  is sampling consistent w.r.t. the probability measures defined by  $\mu_{\mathbf{x}} = \frac{N^{\nu}(\mathbf{x}y)}{N^{\nu}(\mathbf{x})} \, \mathrm{d}y$ .

Proof. In that case  $|\nu_n| = \lambda_n > 0$  for  $n \geq 2$ , so the assumptions of Lemma 3.15 are satisfied and we have to confirm (3.33). Note that passing to the total mass in (3.33) and then integrating over  $(\tau, \xi) \in \text{dec}(F)$  gives 1 on both sides (it equates to taking the total mass in (2.6)). In particular, if we prove (3.33) for any family of measures, in this case  $\mu_{\boldsymbol{x}}(\mathrm{d}y) = \frac{N^{\nu}(\boldsymbol{x}y)}{N^{\nu}(\boldsymbol{x})}\,\mathrm{d}y$ , then they must already be probability measures (in particular finite), that is  $\int N^{\nu}(\boldsymbol{x}y)\,\mathrm{d}y = N^{\nu}(\boldsymbol{x})$ . Let  $\Pi_0 \in \mathcal{P}$ ,  $F^{\star} = (F, \tau, \xi) \in \mathrm{dec}(\mathbb{T}(\Pi_0))$  and  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ . The laws on both sides of (3.33) are determined by their pushforward under the collection of maps  $(\mathrm{Path}_u^{F^{\star}})_{u \in \mathrm{nd}(F)}$ , which are an independent family with respect to both. Hence it suffices to show that (3.33) holds when applying a pushforward with  $\mathrm{Path}_u \coloneqq \mathrm{Path}_u^{F^{\star}}$  for a fixed but arbitrary  $u \in \mathrm{nd}(F)$ . Assume u is not the root, so  $w \coloneqq \mathrm{pr}_F(u) \neq \emptyset$ , otherwise the proof is similar. Then  $\mathrm{Path}_u \# K_{\boldsymbol{x}}(F^{\star},\cdot) = B^{(\tau_u,(\boldsymbol{x}\xi)_u) \to (\tau_w,\xi_w)}$ . If  $G^{\star} = (G,s^{\oplus}\tau,z^{\oplus}\xi) \in \mathbb{T}_b^{\oplus}(F)$  with  $u_{\oplus} = u$  then

$$\operatorname{Path}_{u} \# K_{\boldsymbol{x}y}(G^{\star}, \{\mathring{\boldsymbol{X}} \in \cdot\}) = B^{(\tau_{u}, (\boldsymbol{x}\xi)_{u}) \to (s, z) \to (\tau_{w}, \xi_{w})},$$

and

$$\iiint B^{(\tau_{u},(\boldsymbol{x}\xi)_{u})\to(s,z)\to(\tau_{w},\xi_{w})} P^{\boldsymbol{x}y}(G,s^{\oplus}\tau,z^{\oplus}\xi) \,\mathrm{d}s \,\mathrm{d}z \mu_{\boldsymbol{x}}(\mathrm{d}y)$$

$$= \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} \int f_{\mathrm{tm}}^{\boldsymbol{\lambda}}(G,s^{\oplus}\tau) \left( \iint B^{(\tau_{u},(\boldsymbol{x}\xi)_{u})\to(s,z)\to(\tau_{w},\xi_{w})}(\cdot) f_{\mathrm{sp}}^{G}(z^{\oplus}\xi \mid s^{\oplus}\tau,\boldsymbol{x}y) \,\mathrm{d}z \,\mathrm{d}y \right) \,\mathrm{d}s$$

$$= \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} \left( \int f_{\mathrm{tm}}^{\boldsymbol{\lambda}}(G,s^{\oplus}\tau) \,\mathrm{d}s \right) B^{(\tau_{u},(\boldsymbol{x}\xi)_{u})\to(\tau_{w},\xi_{w})}(\cdot) f_{\mathrm{sp}}^{F}(\xi \mid \tau,\boldsymbol{x}), \tag{3.36}$$

where the second equality followed directly from Lemma A.1 because

$$f_{\rm sp}^G(z^{\oplus}\xi \mid s^{\oplus}\tau, \boldsymbol{x}y) = f_{\rm sp}^F(\xi \mid \tau, \boldsymbol{x}) \frac{p_s(y-z)p_{s-\tau_u}(z-(\boldsymbol{x}\xi)_u)p_{\tau_w-s}(\xi_w-z)}{p_{\tau_w-\tau_u}(\xi_w-(\boldsymbol{x}\xi)_u)}.$$
 (3.37)

If  $u_{\oplus} \neq u$  then the pushforward of  $K_{xy}(G^{\star}, \{\mathring{X} \in \cdot\})$  under  $\mathrm{Path}_{u}$  is the same as that of  $K_{x}(F^{\star}, \cdot)$ , and then

$$\iiint P^{\boldsymbol{x}y}(G,s^{\oplus}\tau,z^{\oplus}\xi)\,\mathrm{d}s\,\mathrm{d}z\mu_{\boldsymbol{x}}(\mathrm{d}y) = \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x})}f^F_{\mathrm{sp}}(\xi\,|\,\tau,\boldsymbol{x})\int f^{\boldsymbol{\lambda}}_{\mathrm{tm}}(G,s^{\oplus}\tau)\,\mathrm{d}s$$

follows from (3.36). Similar calculations can be made for  $G \in \mathbb{T}^{\oplus}_{\mathrm{sb}}(F)$ , and if  $G \in \mathbb{T}^{\oplus}_{\mathrm{m}}(F)$  then  $f_{\mathrm{sp}}^G(\xi \mid \tau, \boldsymbol{x}y) = f_{\mathrm{sp}}^F(\xi \mid \tau, \boldsymbol{x})p_{\tau_{\oplus}}(\xi_{\oplus} - y)$  so

$$\int P^{xy}(G,\tau,\xi)\mu_{x}(\mathrm{d}y) = \frac{1}{N^{\nu}(x)} f_{\mathrm{tm}}^{\lambda}(G,\tau) f_{\mathrm{sp}}^{F}(\xi \mid \tau, x).$$

We can thus cancel  $\frac{1}{N^{\nu}(\boldsymbol{x})}f_{\mathrm{sp}}^{F}(\xi \mid \tau, \boldsymbol{x})B^{(\tau_{u},(\boldsymbol{x}\xi)_{u}\to(\tau_{w},\xi_{w})}$  to conclude that the pushforward of (3.33) under any  $u \in \mathrm{nd}(F)$  is equivalent exactly to (3.35) and thus implied by sampling consistency of the non-spatial coalescent with transition rates  $\boldsymbol{\lambda}$ .

**Lemma 3.18.** If  $\int N^{\nu}(xy) \, dy < \infty$  for all  $x \in \mathcal{X}$ , and the Brownian spatial coalescent with transition measures  $\nu(d\xi) = \lambda \, d\xi$  is sampling consistent w.r.t. the probability measures  $(\mu_x)$  defined by  $\mu_x(dy) \sim N^{\nu}(xy) \, dy$ , then the non-spatial coalescent with transition rates  $\lambda$  is sampling consistent and  $\int N^{\nu}(xy) \, dy = N^{\nu}(x)$ .

*Proof.* The assumptions of Lemma 3.15 are satisfied, so (3.33) holds with

$$\mu_{\boldsymbol{x}}(\mathrm{d}y) = \frac{N^{\boldsymbol{\nu}}(\boldsymbol{x}y)}{\int N^{\boldsymbol{\nu}}(\boldsymbol{x}y')\,\mathrm{d}y'}\,\mathrm{d}y, \quad \boldsymbol{x} \in \mathcal{X}.$$

By the same calculations made in the proof of Lemma 3.17, this implies

$$\frac{1}{N^{\nu}(\boldsymbol{x})} f_{\text{tm}}^{\lambda}(F,\tau) \\
= \frac{1}{\int N^{\nu}(\boldsymbol{x}y) \, \mathrm{d}y} \left( \sum_{G \in \mathbb{T}_{b}^{\oplus}(F)} \int f_{\text{tm}}^{\lambda}(G,s^{\oplus}\tau) \, \mathrm{d}s + \sum_{G \in \mathbb{T}_{sb}^{\oplus}(F)} f_{\text{tm}}^{\lambda}(G,\tau) + \sum_{G \in \mathbb{T}_{m}^{\oplus}(F)} f_{\text{tm}}^{\lambda}(G,\tau) \right) \tag{3.38}$$

for every  $\Pi_0 \in \mathcal{P}$ ,  $F \in \mathbb{T}(\Pi_0)$ , and  $\tau \in \text{tm}(F)$ . Integrating out  $\tau \in \text{tm}(F)$  gives  $N^{\boldsymbol{\nu}}(\boldsymbol{x}) = \int N^{\boldsymbol{\nu}}(\boldsymbol{x}y) \, \mathrm{d}y$ , and then (3.38) turns into (3.35) which implies sampling consistency of the non-spatial coalescent with transition rates  $\boldsymbol{\lambda}$ .

Lemmas 3.17 and 3.18 together imply Lemma 3.14.

3.3.2. Consequences of Sampling Consistency. In this section we fix some label invariant Brownian spatial coalescent, say with transition measures  $\boldsymbol{\nu} \in R$ , and assume it is sampling consistent w.r.t. some family of probability measures  $(\mu_{\boldsymbol{x}})_{\boldsymbol{x} \in \mathcal{X}}$ . The goal is to prove  $\boldsymbol{\nu}(\mathrm{d}\xi) \sim \mathrm{d}\xi$ , and that  $\mu_{\boldsymbol{x}}(\mathrm{d}y) \sim N^{\boldsymbol{\nu}}(\boldsymbol{x}y)\,\mathrm{d}y$  for every  $\boldsymbol{x} \in \mathcal{X}$ .

We write  $(n, k_1, \ldots, k_m) \leq (n', k'_1, \ldots, k'_{m'})$  if there is a way to sample n particles out of the n' particles in an  $(n', \vec{k}')$ -merger restricted to which we observe an  $(n, \vec{k})$ -merger. That is, if  $n \leq n'$  and  $m \leq m'$  and there exists an assignment of n particles to buckets of sizes  $k'_0, k'_1, \ldots, k_{m'}$  (where  $k'_0 = n' - \sum k'_i$ ) such that the numbers of particles in buckets 1 to m' have sizes  $k_1$  to  $k_m$  (ignoring empty buckets, and without maintaining order).

**Lemma 3.19.** (i) If  $(n, \vec{k}) \leq (n', \vec{k'})$  and  $(n', \vec{k'})$  is possible, then so is  $(n, \vec{k})$ .

(ii) If  $(n, \vec{k})$  is possible and n' > n, then there exists  $\vec{k}'$  such that  $(n', \vec{k}') \ge (n, \vec{k})$  and  $(n', \vec{k}')$  is possible.

Proof. Suppose  $(n, \vec{k}) \leq (n', \vec{k'})$  and the latter is possible. Let  $\Pi, \Pi' \in \mathcal{P}$  with  $|\Pi| = n$ ,  $|\Pi'| = n'$  and  $\Pi \subset \Pi'$ . There exists a possible forest  $F' \in \mathbb{F}(\Pi')$  that starts with an  $(n', \vec{k'})$ -merger such that the forest  $F \coloneqq F' \downarrow_{\Pi}$  induced on  $\Pi$  starts with an  $(n, \vec{k})$ -merger. It suffices to show that F is possible. By sampling consistency, for some fixed  $\boldsymbol{x} \in E_{\circ}^{\Pi}$  there is a probability measure  $\mu$  on  $E_{\circ}^{\Pi' \setminus \Pi}$  such that

$$\mathbb{P}^{\boldsymbol{x}}(\mathrm{Fr}=F) = \int \mathbb{P}^{\boldsymbol{x}\boldsymbol{z}}(\mathrm{Fr}|_{\Pi}=F)\mu(\mathrm{d}\boldsymbol{z}) \geq \int \mathbb{P}^{\boldsymbol{x}\boldsymbol{z}}(\mathrm{Fr}=F')\mu(\mathrm{d}\boldsymbol{z}) > 0,$$

so F is possible. Now suppose  $(n, \vec{k})$  is possible and n' > n. Let  $\Pi, \Pi' \in \mathcal{P}$  with  $|\Pi| = n$ ,  $|\Pi'| = n'$  and  $\Pi \subset \Pi'$ , and let  $F \in \mathbb{F}(\Pi)$  such that F is possible and it starts with an  $(n, \vec{k})$ -merger. Then

$$0 < \mathbb{P}^{\boldsymbol{x}}(\operatorname{Fr} = F) = \int \mathbb{P}^{\boldsymbol{x}\boldsymbol{z}}(\operatorname{Fr}|_{\Pi} = F)\mu(\mathrm{d}\boldsymbol{z}) \leq \sum_{\substack{F' \in \mathbb{F}(\Pi') \\ F'|_{\Pi} = F}} \int \mathbb{P}^{\boldsymbol{x}\boldsymbol{z}}(\operatorname{Fr} = F')\mu(\mathrm{d}\boldsymbol{z}).$$

Thus there exists a possible forest  $F' \in \mathbb{F}(\Pi')$  with  $F'|_{\Pi} = F$ , whose first merge event satisfies  $(n', \vec{k}') \geq (n, \vec{k})$ .

**Lemma 3.20.** Exactly one of the following hold.

- (i) All mergers are impossible.
- (ii) All of the mergers (n,n) for  $n \geq 2$  are possible and all other mergers are impossible.
- (iii) For every  $n \geq 3$  there exists a possible  $(n, \vec{k})$ -merger with  $\sum_i k_i < n$ , and (2, 2) is possible.

In cases (ii) and (iii),  $|\nu_n| > 0$  for all  $n \geq 2$ , in particular the only possible forests are trees.

The extremal cases (i) and (ii) correspond in the non-spatial setting to the  $\Lambda$ -coalescent with  $\Lambda$  the zero-measure, and a Dirac mass at 1, respectively. Lemma 3.20 states that if neither (i) nor (ii) hold, then for every initial condition with at least three particles there is a positive probability that the first merge event leaves at least one particle untouched.

Proof of Lemma 3.20. Suppose that every merger that isn't of the form (n,n) for  $n \geq 2$  is impossible. Then either (i) holds, or there exists an  $n \geq 2$  such that (n,n) is possible. Then  $(k,k) \leq (n,n)$  is possible for all  $2 \leq k < n$ . If n' > n, then  $(n',\vec{k}')$  is possible for some  $\vec{k}'$ , but then it must be of the form (n',n'), so in fact (n,n) is possible for all  $n \geq 2$  and all other merge events are impossible, so (ii) holds. In particular  $|\nu_n| > 0$  for all  $n \geq 2$ .

Now assume that there exists  $n_0 \geq 3$  such that a merge event  $(n_0, \vec{k}_0) \neq (n_0, n_0)$  is possible, and we want to show (iii) holds. Then  $(2,2) \leq (n_0, \vec{k}_0)$  is possible, and for any  $n > n_0$ , by Lemma 3.19(ii) there exists a possible merge event  $(n, \vec{k}) \geq (n_0, \vec{k}_0)$ , which cannot be (n, n) because otherwise  $(n_0, \vec{k}_0) = (n_0, n_0)$ . Thus if  $n \geq 3$  arbitrary, we can

take  $N \coloneqq (3n) \vee n_0$  and a possible merge event  $(N, k_1, \ldots, k_m) \neq (N, N)$ . If m = 1, then  $k_1 < N$  so  $(n, n - 1) \le (N, \vec{k})$  is possible by Lemma 3.19(i). If  $m \ge 2$ , then we construct a merger  $(n, \vec{k}') \le (N, \vec{k})$  with  $\sum_i k_i' < n$  by assigning one particle to the smallest bucket of size  $k_m$ , and spread all other particles over the remaining buckets, which works as long as  $n - 1 \le N - k_1$ , but if this wasn't true then  $N \ge mk_1 \ge 2(N - n + 1)$  and thus  $n \ge N/2$ , a contradiction.

We assume from now that we are in case (iii). In case (i) there is nothing to show, and in case (ii) proofs are easier than and can directly be adapted from those that follow. We will from now explicitly denote dependence of  $f_{\nu}$  on F. The following lemma serves as a start to an inductive argument, and gently introduces the general proof idea.

**Lemma 3.21.** There is  $\lambda_{2,2} > 0$  such that  $\nu_{2,2}(\mathrm{d}z) = \lambda_{2,2} \, \mathrm{d}z$ , and for every  $x \in E$ ,  $\mu_x(\mathrm{d}y) \sim N^{\nu}(xy) \, \mathrm{d}y$ .

Proof. Let  $\Pi_0 = \{u, v\} \in \mathcal{P}$  and  $x \in E^{\{u\}}$ . Abbreviate  $\mu(\mathrm{d}y) := N^{\boldsymbol{\nu}}(xy)^{-1}\mu_x(\mathrm{d}y)$  and  $\nu := \nu_{2,2}$ , so that  $\mu(\mathrm{d}y), \nu(\mathrm{d}y) \sim \mathrm{d}y$  are to be shown. The law of  $\boldsymbol{X}_t(u)$  under  $\mathbb{P}^x$  is  $B^{(0,x)+}$ . If  $y \in E^{\{v\}}$ , then started from xy the only possible tree is  $F = \{\Pi_0, \{uv\}\}$ , and conditional on  $\mathrm{Dec} = (\tau, \xi) \in \mathrm{dec}(F)$ , the law of  $\boldsymbol{X}_t(u)$  (recall  $\boldsymbol{X}$  from Lemma 3.15) under  $\mathbb{P}^{xy}$  is  $B^{(0,x)\to(\tau,\xi)+}$ . Thus by sampling consistency,

$$B^{(0,x)+}(\cdot) = \int_{E} \int_{\operatorname{dec}(F)} B^{(0,x)\to(\tau,\xi)+}(\cdot) P^{xy}(F, d\tau, d\xi) \mu_{x}(dy),$$
  
$$= \int_{E} \int_{E} \int_{0}^{\infty} B^{(0,x)\to(\tau,\xi)+}(\cdot) f_{\boldsymbol{\nu}}^{F}(\tau,\xi \mid xy) d\tau \nu(d\xi) \mu(dy),$$

which by Lemma A.1 and definition of  $f_{\nu}^{F}(\cdot \mid xy)$  implies

$$\left(\int_{E} p_{s}(x-z)p_{s}(y-z)\mu(\mathrm{d}y)\right)\nu(\mathrm{d}z) \stackrel{!}{\sim} p_{s}(x-z)\,\mathrm{d}z$$

for Lebesgue-a.a. s>0. That is, for such s>0 there is a constant c(s)>0 such that  $\int p_s(y-z)\mu(\mathrm{d}y) \nu(\mathrm{d}z) = c(s)\,\mathrm{d}z$ . Then  $\nu$  must have a positive Lebesgue-density which we can write as 1/g for some measurable  $g\colon E\to (0,\infty)$ , and then  $T_s\mu=c(s)g$  Lebesgue-a.e. for Lebesgue-a.a. s>0. By Lemma 3.22 below there is a constant c>0 such that  $g\equiv c$  a.e. and  $\mu(\mathrm{d}z)=c\,\mathrm{d}z$ .

**Lemma 3.22.** Suppose that  $\mu$  is a finite, non-zero measure on E,  $g: E \to (0, \infty)$  is measurable,  $N \subset (0, \infty)$  is a Lebesgue-null set and for  $s \in (0, \infty) \setminus N$  there is c(s) > 0 such that  $T_s \mu = c(s)g$  Lebesgue-a.e. Then there is a constant c > 0 such that g(z) = c a.e. and  $\mu(\mathrm{d}z) = c\,\mathrm{d}z$ .

Proof. All equalities of functions  $E \to \mathbb{R}$  in this proof are meant Lebesgue-a.e. First note that g is bounded because  $T_s\mu$  is bounded for any s>0. Let  $s_0\in S$ , put  $f:=T_{s_0}\mu=c(s_0)g$ , and  $\widetilde{c}(s):=\frac{c(s+s_0)}{c(s_0)}$  whenever  $s\in S:=(0,\infty)\setminus (N-s_0)$ . Then  $T_sf=c(s+s_0)g=\widetilde{c}(s)f$  for all  $s\in S$ , which by the semigroup property implies  $\widetilde{c}(t+s)=\widetilde{c}(s)\widetilde{c}(t)$  whenever  $s,t,s+t\in S$ . This implies that there exists  $\alpha\in\mathbb{R}$  with  $\widetilde{c}(s)=\mathrm{e}^{\alpha s}$  for all s in some dense subset A of S. Indeed, the set  $N':=\bigcup_{r\in\mathbb{Q}_+}\frac{N-s_0}{r}$  is null and its complement  $S'\subset S$  has the property that

for all  $s \in S'$  and  $r \in \mathbb{Q}_+$  also  $rs \in S'$ . Fix some  $s_1 \in S'$ , then  $\widetilde{c}(\frac{p}{q}s_1)^q = \widetilde{c}(ps_1) = \widetilde{c}(s_1)^p$  for all  $p,q \in \mathbb{N}$ , that is  $\widetilde{c}(rs_1) = \widetilde{c}(s_1)^r$  for all  $r \in \mathbb{Q}_+$ , which implies the claim with  $\alpha = \log c(s_1)/s_1$  and  $A = s_1\mathbb{Q}_+$ . We have proved that  $T_s g = \mathrm{e}^{\alpha s} g$  for all  $s \in A$ . The LHS tends to the constant  $c := \int g(x) \, \mathrm{d}x$  uniformly on E as  $s \to \infty$ , which necessitates that  $\alpha = 0$  and that  $g \equiv c$  a.e. Then  $(T_s \mu)(\mathrm{d}z) = c \, \mathrm{d}z$  as measures for all  $s \in (0, \infty) \setminus N$ , and the LHS tends to  $\mu$  weakly as  $s \to \infty$ , so  $\mu(\mathrm{d}z) = c \, \mathrm{d}z$ .

**Lemma 3.23.** All measures  $\nu_{n,\vec{k}}$  are absolutely continuous w.r.t. Lebesgue measure.

Proof. We prove the claim with an induction over  $n \geq 2$ , the start of which is due to Lemma 3.21. Let  $n \geq 3$ , assume that the claim is proved for  $(n', \vec{k}')$  for all n' < n, and take  $(n, k_1, \ldots, k_m)$  possible. Let  $\Pi_0 \in \mathcal{P}$  with  $|\Pi_0| = n$ , and  $\Pi_0^{\oplus}$  etc. as above. If  $k_1 \geq 3$  then  $(n-1, \vec{k}') \coloneqq (n-1, k_1-1, k_2, \ldots, k_m) \leq (n, \vec{k})$  is possible and there are possible trees  $F \in \mathbb{T}(\Pi_0)$ ,  $G \in \mathbb{T}(\Pi^{\oplus})$  with  $G \downarrow_{\Pi_0} = F$  whose first merge events are respectively  $(n-1, \vec{k}')$  and  $(n, \vec{k})$ . If  $\xi \in \operatorname{sp}(F)$  write  $\xi^{(0)} \coloneqq \xi|_{\Pi_F^1 \setminus \operatorname{lf}(F)}$  for the location(s) of the first merge event, and similarly for  $\xi \in \operatorname{sp}(G)$ . Then, for fixed  $x \in E_0^{\Pi_0}$  and  $y \in E_x$ ,  $P^x(F, \xi^{(0)} \in \cdot)$  and  $P^{xy}(G, \xi^{(0)} \in \cdot)$  have positive densities w.r.t.  $\nu_{n-1, \vec{k}'}$  and  $\nu_{n, \vec{k}}$ , respectively. Indeed,

$$P^{\boldsymbol{x}}(F,\xi^{(0)} \in \mathrm{d}\boldsymbol{z}) = \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} \left( \int_{\mathrm{tm}(F)} \int_{\mathrm{sp}(F \setminus \Pi_0)} f_{\boldsymbol{\nu}}^F(\tau,\xi'\boldsymbol{z} \,|\, \boldsymbol{x}) \, \mathrm{d}\tau \, \boldsymbol{\nu}_{F \setminus \Pi_0}(\mathrm{d}\xi') \right) \nu_{n-1,\vec{k}'}(\mathrm{d}\boldsymbol{z}),$$

and similarly for  $P^{xy}(G, \xi^{(0)} \in \mathrm{d}\boldsymbol{z})$ . Since  $P^{x}(F, \xi^{(0)} \in \cdot) \geq \int P^{xy}(G, \xi^{(0)} \in \cdot) \mu_{x}(\mathrm{d}y)$  by sampling consistency, this implies that  $\nu_{n,\vec{k}}$  is absolutely continuous w.r.t.  $\nu_{n-1,\vec{k}'}$  and thus has a Lebesgue density. If  $k_1 = \ldots = k_m = 2$ , then  $(n-1,\vec{k}') \coloneqq (n-1,k_1,\ldots,k_{m-1}) \leq (n,\vec{k})$  is possible, and by an analogous argument we obtain that  $\nu_{n,\vec{k}}$  is absolutely continuous w.r.t. Lebesgue measure in the first m-1 arguments, and thus by symmetry (Lemma 3.12) in all arguments.

We have now proved the assumptions of Lemma 3.15, of which we use the following corollary.

**Lemma 3.24.** Let  $\Pi_0 \in \mathcal{P}$ ,  $F \in \mathbb{T}(\Pi_0)$ , and  $\mathbf{x} \in E_0^{\Pi_0}$ , then

$$K_{\boldsymbol{x}}(F^{\star},\cdot) \sim \sum_{G \in \mathbb{T}_{b}^{\oplus}(F)} \iiint K_{\boldsymbol{x}y}((G,s^{\oplus}\tau,z^{\oplus}\xi),\{\mathring{\boldsymbol{X}} \in \cdot\})P^{\boldsymbol{x}y}((G,s^{\oplus}\tau,z^{\oplus}\xi)) \,\mathrm{d}s \,\mathrm{d}z\mu_{\boldsymbol{x}}(\mathrm{d}y)$$
$$+ \sum_{G \in \mathbb{T}_{a}^{\oplus}(F)} \iint K_{\boldsymbol{x}y}((G,\tau,z^{\oplus}\xi),\{\mathring{\boldsymbol{X}} \in \cdot\})P^{\boldsymbol{x}y}((G,\tau,z^{\oplus}\xi)) \,\mathrm{d}z\mu_{\boldsymbol{x}}(\mathrm{d}y) \qquad (3.39)$$

for Lebesgue-a.e.  $F^* = (F, \tau, \xi) \in \operatorname{dec}(F)$ , where domains of integration for s and z are as indicated by the bijections (3.31) and (3.32). For fixed  $F^*$ , if  $G \in \mathbb{T}_b^{\oplus}(F)$  and  $y \in E_x$ ,

$$P^{xy}((G, s^{\oplus}\tau, z^{\oplus}\xi)) \sim \frac{\mathrm{e}^{-\beta_G^{\oplus}s}}{N^{\nu}(xy)} p_s(z-y) p_{s-\tau_{u_{\oplus}}}(z-(x\xi)_{u_{\oplus}}) \left[ p_{\tau_{w_{\oplus}}-s}(\xi_{w_{\oplus}}-z) \right]_{w_{\oplus}\neq\emptyset} \nu_G(z^{\oplus}\xi)$$

$$(3.40)$$

where  $\beta_G^{\oplus} = |\nu_{|\Pi_G^{\oplus}|}| - |\nu_{|\Pi_G^{\oplus}|+1}|$ , and if  $G \in \mathbb{T}_{sb}^{\oplus}(F)$ ,

$$P^{xy}((G,\tau,z^{\oplus}\xi)) \sim \frac{1}{N^{\nu}(xy)} p_{\tau_{\oplus}}(z-y) p_{\tau_{\oplus}-\tau_{u_{\oplus}}}(z-(x\xi)_{u_{\oplus}}) p_{\tau_{w_{\oplus}}-\tau_{\oplus}}(\xi_{w_{\oplus}}-z) \nu_{G}(z^{\oplus}\xi).$$
(3.41)

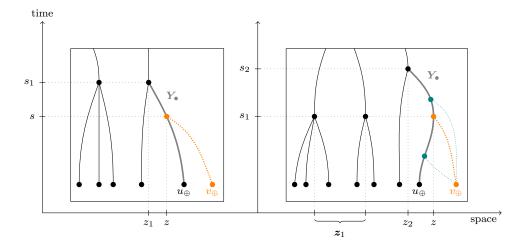


Figure 6. Illustrations for the proof of Lemma 3.26.

The constants in both cases only depend on  $F^*$  and x.

*Proof.* The first claim follows directly from (3.33), and (3.40) and (3.41) follow from the definition of  $f_{\nu}$  by discarding terms that don't depend on s, z, y, and z, y, respectively.  $\square$ 

**Lemma 3.25.** For every  $\mathbf{x} \in \mathcal{X}$ ,  $\mu_{\mathbf{x}}(\mathrm{d}y) \sim N^{\nu}(\mathbf{x}y) \,\mathrm{d}y$ .

Proof. Let  $n \geq 3$ ,  $\Pi_0 \in \mathcal{P}$  with  $|\Pi_0| = n$ , and  $\boldsymbol{x} \in E_{\circ}^{\Pi_0}$ . By Lemma 3.20 there exists a possible tree  $F = (\Pi_0, \dots, \Pi_m = \{u_{\oplus}\})$  such that  $G \coloneqq (\Pi_0 \cup \{v_{\oplus}\}, \dots, \Pi_{m-1} \cup \{v_{\oplus}\}, \{u_{\oplus}, v_{\oplus}\}, \{u_{\oplus} \cup v_{\oplus}\})$  is possible. Note  $G \downarrow_{\Pi_0} = F$ . In words, in G the first n-1 particles merge according to F and then the root of F merges with the nth particle  $v_{\oplus}$ . Fix  $F^* = (F, \tau, \xi) \in \text{dec}(F)$  for which (3.39) holds, and denote by  $s_0 \coloneqq \tau(u_{\oplus}), z_0 \coloneqq \xi(u_{\oplus})$  time and place of the birth of F's root. Then the law of  $\mathbf{Y}_{\bullet} \coloneqq \mathbf{X}_{s_0+\bullet}(u_{\oplus})$ , the motion of F's root, under  $K_{\mathbf{x}}(F^*, \cdot)$  is  $B^{(s_0, z_0)+}$ . For any  $G'^* \in \text{dec}^{\oplus}(F^*)$  with  $G' \neq G$  the law of  $\mathbf{Y}$  under  $K_{\mathbf{x}}(G'^*, \{\mathring{\mathbf{X}} \in \cdot\})$  is the same, because  $v_{\oplus}$  merges into F before  $s_0$ . If  $G^* = (G, s^{\oplus}\tau, z^{\oplus}\xi) \in \text{dec}^{\oplus}(F^* \mid G)$ , then the law of  $\mathbf{Y}$  under  $K_{\mathbf{x}}(G^*, \{\mathring{\mathbf{X}} \in \cdot\})$  is  $B^{(s_0, z_0) \to (s, z)+}$ , so by Lemma 3.24,

$$B^{(s_0,z_0)+}(\cdot) \sim \int_{E_{x}} \int_{E} \int_{s_0}^{\infty} B^{(s_0,z_0)\to(s,z)+}(\cdot) P^{xy}((G,s^{\oplus}\tau,z^{\oplus}\xi)) \,\mathrm{d}s \,\mathrm{d}z \mu_{x}(\mathrm{d}y)$$
$$\sim \iiint B^{(s_0,z_0)\to(s,z)+}(\cdot) p_{s}(z-y) p_{s-s_0}(z-z_0) \mathrm{e}^{-|\nu_2|s} \,\mathrm{d}s \,\mathrm{d}z \mu(\mathrm{d}y),$$

where we put  $\mu(dy) := \frac{1}{N^{\nu}(xy)}\mu_{x}(dy)$  and used that  $\nu_{2,2}(dz) \sim dz$  is already known (that is,  $\nu_{2,2}(z)$  is constant in z). By Lemma A.1, this implies that, for Lebesgue-a.a.  $s \in (s_0, \infty)$ ,

$$\left( \int_{E_{x}} p_{s}(z-y) p_{s-s_{0}}(z-z_{0}) \mu(\mathrm{d}y) \right) \mathrm{d}z \sim p_{s-s_{0}}(z-z_{0}) \, \mathrm{d}z,$$

that is  $T_s\mu$  is a.e. constant for a.a. s>0, and thus  $\mu(\mathrm{d}z)\sim\mathrm{d}z$  by Lemma 3.22.

**Lemma 3.26.** For every  $(n, \vec{k}) = (n, k_1, \dots, k_m) \in M$  and  $i \in [m]$ ,  $\nu_{n,\vec{k}}$  is constant in the ith argument if  $k_i = 2$ . In particular,  $\nu_{n,2}(dz) = \lambda_{n,2} dz$  for some  $\lambda_{n,2} \ge 0$  for each  $n \ge 2$ .

Proof. We first prove  $\nu_{n,2}(\mathrm{d}z) \sim \mathrm{d}z$  for all  $n \geq 2$ . This is already proved for n = 2, so let  $n \geq 3$ ,  $(\Pi_0, \boldsymbol{x}) \in \mathcal{X}$  with  $|\Pi_0| = n - 1$ , and  $F \in \mathbb{T}(\Pi_0)$  a possible tree. Let  $u_{\oplus} \in \Pi_0$  be one of the leaves involved in the first merge event, and denote by  $G \in \mathbb{T}(\Pi_0^{\oplus})$  the tree in which an additional leaf  $v_{\oplus}$  merges with  $u_{\oplus}$  before the first merge of F, so  $G \downarrow_{\Pi_0} = F$ . Fix  $F^* = (F, \tau, \xi) \in \mathrm{dec}(F)$  for which (3.39) holds, and put  $s_1 \coloneqq \tau(\mathrm{pr}_F(u_{\oplus}))$ ,  $z_1 \coloneqq \tau(\mathrm{pr}_F(u_{\oplus}))$ , the time and place of the first merge of  $u_{\oplus}$  in F. See Fig. 6 (left). Then the law of  $(\mathbf{Y}_r \coloneqq \mathbf{X}_r(u_{\oplus}))_{0 \leq r \leq s_1}$  under  $K_{\mathbf{x}}(F^*, \cdot)$  is  $B^{(0,x_0) \to (s_1,z_1)}$  where  $x_0 \coloneqq \mathbf{x}(u_{\oplus})$ . For any  $G'^* \in \mathrm{dec}^{\oplus}(F^*)$  with  $G' \neq G$  the law of  $\mathbf{Y}$  under  $K_{\mathbf{x}}(G'^*, \{\mathring{\mathbf{X}} \in \cdot\})$  is the same, but if  $G^* = (G, s^{\oplus}\tau, z^{\oplus}\xi) \in \mathrm{dec}^{\oplus}(F^* \mid G)$ , then the law of  $\mathbf{Y}$  under  $K_{\mathbf{x}}(G^*, \{\mathring{\mathbf{X}} \in \cdot\})$  is  $B^{(0,x_0) \to (s_1,z_1)}$ . Thus by Lemma 3.24, for some  $\beta \in \mathbb{R}$ ,

$$B^{(0,x_0)\to(s_1,z_1)}(\cdot) \sim \int_{E_{\boldsymbol{x}}} \int_{E} \int_{0}^{s_1} B^{(0,x_0)\to(s,z)\to(s_1,z_1)}(\cdot) P^{\boldsymbol{x}y}((G,s^{\oplus}\tau,z^{\oplus}\xi)) \,\mathrm{d}s \,\mathrm{d}z \mu_{\boldsymbol{x}}(\mathrm{d}y)$$

$$\sim \iiint B^{(0,x_0)\to(s,z)\to(s_1,z_1)}(\cdot) \mathrm{e}^{-\beta s} p_s(z-y) p_s(z-x_0)$$

$$p_{s_1-s}(z_1-z) \nu_{n,2}(z) \,\mathrm{d}s \,\mathrm{d}z \,\mathrm{d}y$$

$$= \iint B^{(0,x_0)\to(s,z)\to(s_1,z_1)}(\cdot) \mathrm{e}^{-\beta s} p_s(z-x_0) p_{s_1-s}(z_1-z) \nu_{n,2}(\mathrm{d}z) \,\mathrm{d}s,$$

which by Lemma A.1 implies  $\nu_{n,2}(\mathrm{d}z) \sim \mathrm{d}z$ .

Now let  $n \geq 3$  and  $(n, k_1, \ldots, k_m)$  possible with  $m \geq 2$  and  $k_m = 2$ . Take  $\Pi_0 \in \mathcal{P}$  with  $|\Pi_0| = n - 1$ , and let  $G \in \mathbb{T}(\Pi_0^{\oplus})$  be a tree in which the first merge event is  $(n, \vec{k})$ , and in which the parent of the two leaves  $u_{\oplus}, v_{\oplus}$  comprising the merge associated to  $k_m$ , is involved in the merge event following the first. See Fig. 6 (right). Let  $F := G \setminus v_{\oplus} \in \mathbb{T}(\Pi_0)$ , so  $G\downarrow_{\Pi_0} = F$ . Fix  $F^* = (F, \tau, \xi) \in \text{dec}(F)$  for which (3.39) holds, and denote  $s_1 := \tau(\Pi_1^F)$ ,  $z_1 := \xi|_{\Pi_1^F\backslash\Pi_0}, s_2 := \tau(\text{pr}_F(u_{\oplus})), z_2 := \xi(\text{pr}_F(u_{\oplus}))$ , then the law of  $(Y_r := X_r(u_{\oplus}))_{0 \leq r \leq s_2}$  under  $K_{\boldsymbol{x}}(F^*, \cdot)$  is  $B^{(0,x_0)\to(s_2,z_2)}$ , where  $x_0 = \boldsymbol{x}(u_{\oplus})$ . If  $G'^* \in \text{dec}^{\oplus}(F^*)$  with  $G' \neq G$ , then the law of Y under  $K_{\boldsymbol{x}y}(G'^*, \{\mathring{\boldsymbol{X}} \in \cdot\})$  is the same, except in the two cases where G' merges  $v_{\oplus}$  with  $u_{\oplus}$  in  $(0,s_1)$  or in  $(s_1,s_2)$  (see teal coloured parts of Fig. 6). In the former case,

$$\iiint Y \# K_{xy}(G'^{\star}, \{\mathring{X} \in \cdot\}) P^{xy}((G, s^{\oplus}\tau, z^{\oplus}\xi)) \, ds \, dz \mu_{x}(dy) 
\sim \iiint B^{(0,x_{0}) \to (s,z) \to (s_{2},z_{2})}(\cdot) e^{-\beta s} p_{s}(z-y) p_{s}(z-x_{0}) p_{s_{2}-s}(z_{2}-z) \, ds \, dz \, dy 
= \int_{0}^{s_{1}} B^{(0,x_{0}) \to (s,z) \to (s_{2},z_{2})}(\cdot) e^{-\beta s} p_{s}(x_{0}-z) p_{s_{2}-s}(z-z_{2}) \, dz \, ds 
\sim B^{(0,x_{0}) \to (s_{2},z_{2})}(\cdot),$$

where we used Lemma A.1, and similarly if the merge is in  $(s_1, s_2)$ . Now, if  $G^* = (G, \tau, z^{\oplus} \xi) \in dec^{\oplus}(F^* \mid G)$  (see the orange coloured part of Fig. 6, right), then the law of Y under

 $K_{xy}(G^*, \{\mathring{X} \in \cdot\})$  is  $B^{(0,x_0)\to(s_1,z)\to(s_2,z_2)}$ , so by Lemma 3.24,

$$\begin{split} B^{(0,x_0)\to(s_2,z_2)}(\cdot) &\sim \int_{E_{\boldsymbol{x}}} \int_{E_{\boldsymbol{z}_1}} B^{(0,x_0)\to(s_1,z)\to(s_2,z_2)}(\cdot) P^{\boldsymbol{x}y}((G,\tau,z^{\oplus}\xi)) \,\mathrm{d}z \mu_{\boldsymbol{x}}(\mathrm{d}y) \\ &\sim \iint B^{(0,x_0)\to(s_1,z)\to(s_2,z_2)}(\cdot) p_{s_1}(z-x_0) p_{s_1}(z-y) \\ &\qquad \qquad p_{s_2-s_1}(z_2-z) \nu(\boldsymbol{z}_1,z) \,\mathrm{d}z \,\mathrm{d}y \\ &\sim \iint B^{(0,x_0)\to(s_1,z)\to(s_2,z_2)}(\cdot) p_{s_1}(z-x_0) p_{s_2-s_1}(z_2-z) \nu(\boldsymbol{z}_1,z) \,\mathrm{d}z. \end{split}$$

Evaluating both laws at time  $s_1$  gives that  $\nu_{n,\vec{k}}(z_1,\cdot)$  is (a.e.) constant. By symmetry of  $\nu_{n,\vec{k}}$  it must then be constant in all arguments i with  $k_i=2$ .

This means we can write  $\nu_{n,\vec{k}}(z) = \nu_{n,\vec{k}}(z_1,\ldots,z_j)$  if  $(n,k_1,\ldots,k_m) \in M$  with  $m \geq 2$  and  $k_j = \ldots = k_m = 2$ . The following lemma finishes this section.

# Lemma 3.27. $\nu(d\xi) \sim d\xi$ .

*Proof.* We first prove that for every  $(n, \vec{k}) \in M$  there exist non-negative constants  $(c^{(j)})_{0 \le j \le m}$  and  $c^{\text{sb}}$  (whose dependence on  $(n, \vec{k})$  we suppress) such that

$$\nu_{n,\vec{k}}(z) = c^{(0)}\nu_{n+1,\vec{k}}(z) + c^{\text{sb}}\nu_{n+1,k_1,\dots,k_m,2}(z) + \sum_{i=1}^m c^{(i)}\nu_{n+1,k_1,\dots,k_j+1,\dots,k_m}(z), \quad (3.42)$$

for a.e. z, and  $c^{(j)} > 0$  for  $j \in [m]$ . Fix  $(n, \vec{k})$  and let F be a possible tree that starts with an  $(n, \vec{k})$ -merger, and put  $F^{\circ} := F \setminus lf(F)$ . By passing to the total mass in (3.33), for a.e.  $z \in E_0^{\Pi_1^F \setminus \Pi_0^F}$ ,  $\xi \in sp(F^{\circ})$ , and  $\tau \in tm(F)$ ,

$$P^{\boldsymbol{x}}((F,\tau,\boldsymbol{z}\xi)) = \sum_{G \in \mathbb{T}_{b}^{\oplus}(F)} \iiint P^{\boldsymbol{x}y}((G,s^{\oplus}\tau,z^{\oplus}\boldsymbol{z}\xi)) \, \mathrm{d}s \, \mathrm{d}z\mu_{\boldsymbol{x}}(\mathrm{d}y)$$

$$+ \sum_{G \in \mathbb{T}_{b}^{\oplus}(F)} \iint P^{\boldsymbol{x}y}((G,\tau,z^{\oplus}\boldsymbol{z}\xi)) \, \mathrm{d}z\mu_{\boldsymbol{x}}(\mathrm{d}y) + \sum_{G \in \mathbb{T}_{m}^{\oplus}(F)} \int P^{\boldsymbol{x}y}((G,\tau,\boldsymbol{z}\xi))\mu_{\boldsymbol{x}}(\mathrm{d}y).$$

$$(3.43)$$

For the remainder of this proof put  $f_F(\tau) := \prod_{(\Pi,\Pi')\in F} e^{-|\nu_{\Pi}|(\tau_{\Pi'}-\tau_{\Pi})}$  and analogously for other trees, so that

$$f_{m{
u}}^F( au, m{z}\xi \mid m{x}) = rac{1}{N^{m{
u}}(m{x})} f_F( au) f_{
m sp}^F(m{z}\xi \mid au, m{x}).$$

Further recall that  $P^{\boldsymbol{x}}((F,\tau,\boldsymbol{z}\xi)) = f_{\boldsymbol{\nu}}^F(\tau,\boldsymbol{z}\xi\,|\,\boldsymbol{x})\boldsymbol{\nu}_F(\mathrm{d}(\boldsymbol{z}\xi))$ . We will now divide (3.43) by  $f_{\mathrm{sp}}^F(\boldsymbol{z}\xi\,|\,\tau,\boldsymbol{x})$  and then integrate it over  $\xi$  and  $\tau$ . On the left we get  $\frac{|\boldsymbol{\nu}_F\circ|}{N^{\boldsymbol{\nu}}(\boldsymbol{x})}\left(\int f_F(\tau)\,\mathrm{d}\tau\right)\boldsymbol{\nu}_{n,\vec{k}}(\boldsymbol{z})$ . Now let  $G\in\mathbb{T}_\mathrm{b}^\oplus(F)$  such that the binary merge is before the first merge of F, then the corresponding summand in (3.43) is  $\lambda_{2,2}/\int N^{\boldsymbol{\nu}}(\boldsymbol{x}y)\,\mathrm{d}y$  times

$$\int \left( \iint f_{\mathrm{sp}}^{G}(z^{\oplus} \boldsymbol{z} \boldsymbol{\xi} \mid s^{\oplus} \tau, \boldsymbol{x} \boldsymbol{y}) \, \mathrm{d} \boldsymbol{y} \, \mathrm{d} \boldsymbol{z} \right) f_{G}(s^{\oplus} \tau) \nu_{n, \vec{k}}(\boldsymbol{z}) \boldsymbol{\nu}_{F^{\circ}}(\boldsymbol{\xi}) \, \mathrm{d} s$$

$$= f_{\mathrm{sp}}^{F}(\boldsymbol{z} \boldsymbol{\xi} \mid \tau, \boldsymbol{x}) \left( \int f_{G}(s^{\oplus} \tau) \, \mathrm{d} s \right) \nu_{n, \vec{k}}(\boldsymbol{z}) \boldsymbol{\nu}_{F^{\circ}}(\boldsymbol{\xi}),$$

an equality we've used before, cmp. (3.37). Dividing by  $f_{\mathrm{sp}}^F(z\xi \mid \tau, x)$  and integrating over  $\xi$  and  $\tau$  gives a positive (because  $F^{\circ}$  is possible) multiple of  $\nu_{n,\vec{k}}(z)$ . Similarly, if the merge is after the first of F, we obtain a non-negative (because then G may be impossible) multiple of  $\nu_{n+1,\vec{k}}(z)$ . If  $G \in \mathbb{T}_{\mathrm{sb}}^{\oplus}(F)$  then we get non-negative multiples of either  $\nu_{n,\vec{k}}(z)$  or  $\nu_{n,k_1,\ldots,k_m,2}(z)$ , and if  $G \in \mathbb{T}_{\mathrm{m}}^{\oplus}(F)$  then we get either a non-negative multiple of  $\nu_{n+1,\vec{k}}(z)$ , or a positive multiple of  $\nu_{n+1,k_1,\ldots k_j+1,\ldots k_m}(z)$  for some  $j \in [m]$ , and for each such j there is a corresponding  $G \in \mathbb{T}_{\mathrm{m}}^{\oplus}(F)$ .

This proves the claim surrounding (3.42), from which we conclude with an induction over k in the statement "Every  $\nu_{n,\vec{k}}$  is constant in the ith argument if  $k_i \leq k$ ". For k=2 this is Lemma 3.26, now suppose it has been proved for some fixed  $k \geq 2$ . Take some  $(n,\vec{k})$  for which there is  $j \in [m]$  with  $k_j = k+1$  and  $k_i \leq k$  for i > j. Put  $\vec{k'} := (k_1, \ldots k_{j-1}, k_j - 1, k_{j+1}, \ldots, k_m)$ , and invoke (3.42) with  $\nu_{n,\vec{k'}}$  on the LHS. Then  $\nu_{n,\vec{k}}$  appears as a summand on the RHS (perhaps multiple times), and all other summands and the LHS are, by the induction hypothesis, constant in the jth to mth arguments. Hence also  $\nu_{n,\vec{k}}$  must be constant in the jth argument, and by symmetry in all arguments with  $k_i = k+1$ .

## 3.4. Drift Representation.

Proof of Theorem 1.15. Denote by  $(\Pi_t, \mathbf{X}_t)_{t\geq 0}$  the Brownian spatial coalescent with transition rates  $\boldsymbol{\nu}(\mathrm{d}\boldsymbol{z}) = \boldsymbol{\lambda}\,\mathrm{d}\boldsymbol{z}$  for some  $\boldsymbol{\lambda} \in R^\circ$ , started from  $\Pi_0 = \{\{1\}, \ldots, \{n\}\}$  and  $\mathbf{X}_0 = \boldsymbol{x} \in E_0^{\Pi_0}$ . Denote  $\mathbf{Z}_t = (Z_t^1, \ldots, Z_t^n) = (\mathbf{X}_t(\{1\}), \ldots, \mathbf{X}_t(\{n\}))$  as long as t is smaller than the random time of the first merge event, that is for  $t < \inf\{s > 0 \colon \Pi_s = \Pi_0\}$ .

We perform a slightly informal generator calculation, and leave technical details to the interested reader. Recall that the (time-dependent) generator of a Brownian bridge on the torus E, starting at  $x_0$  at time  $t_0$ , going to  $x_1$  at time  $t_1$ , is given by

$$A_s f(x) = \frac{1}{2} \Delta f(x) + \nabla_x \log p_{t_1 - s}(x_1 - x) \cdot \nabla f(x), \qquad s \in [t_0, t_1).$$

Then the generator of Z is

$$Af(z_{1},...,z_{n})$$

$$= \sum_{i=1}^{n} \frac{1}{N^{\nu}(\boldsymbol{z})} \int_{\operatorname{dec}(\mathbb{F}(\Pi_{0}))} \left(\frac{1}{2} \nabla_{z_{i}}^{2} f(\boldsymbol{z}) + \left[\nabla_{z_{i}} \log p(\tau_{\operatorname{pr}_{F}(\{i\})}, \xi_{\operatorname{pr}_{F}(\{i\})} - z_{i})\right]_{\operatorname{pr}_{F}(\{i\}) \neq \emptyset} \cdot \nabla_{z_{i}} f(\boldsymbol{z})\right) \times f_{\operatorname{tm}}^{\boldsymbol{\lambda}}(F, \tau) f_{\operatorname{sp}}(\xi \mid \tau, \boldsymbol{z}) \, d\tau \, d\xi$$

$$= \frac{1}{2}\Delta f(\boldsymbol{z}) + \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{z})} \sum_{i=1}^{n} \nabla_{z_{i}} f(\boldsymbol{z}) \cdot \int_{\text{dec}(\mathbb{F}(\Pi_{0}))} \left[ \nabla_{z_{i}} \log p(\tau_{\text{pr}_{F}(\{i\})}, \xi_{\text{pr}_{F}(\{i\})} - z_{i}) \right]_{\text{pr}_{F}(\{i\}) \neq \emptyset} \\ f_{\text{tm}}^{\boldsymbol{\lambda}}(F, \tau) f_{\text{sp}}(\boldsymbol{\xi} \mid \tau, \boldsymbol{z}) \, \mathrm{d}\tau \, \mathrm{d}\boldsymbol{\xi}.$$

If  $(F, \tau, \xi) \in \operatorname{dec}(\mathbb{F}(\Pi_0))$  with  $\operatorname{pr}_F(\{i\}) \neq \emptyset$ , then  $f_{\operatorname{sp}}(\xi \mid \tau, \mathbf{z})$  depends on  $z_i$  only through a factor  $p(\tau_{\operatorname{pr}_F(\{i\})}, \xi_{\operatorname{pr}_F(\{i\})} - z_i)$ , so

$$(\nabla_{z_i} \log p(\tau_{\operatorname{DF}_{\mathcal{D}}(\{i\})}, \xi_{\operatorname{DF}_{\mathcal{D}}(\{i\})} - z_i)) f_{\operatorname{SD}}(\xi \mid \tau, \boldsymbol{z}) = \nabla_{z_i} f_{\operatorname{SD}}(\xi \mid \tau, \boldsymbol{z}).$$

If  $\operatorname{pr}_F(\{i\}) = \emptyset$ , then  $f_{\operatorname{sp}}(\xi \mid \tau, \mathbf{z})$  is independent of  $z_i$ , so in any case

$$Af(z_1, \dots, z_n) = \frac{1}{2} \Delta f(\boldsymbol{z}) + \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{z})} \sum_{i=1}^n \nabla_{z_i} f(\boldsymbol{z}) \cdot \int_{\text{dec}(\mathbb{F}(\Pi_0))} f_{\text{tm}}^{\boldsymbol{\lambda}}(F, \tau) \nabla_{z_i} f_{\text{sp}}(\xi \mid \tau, \boldsymbol{z}) \, d\tau \, d\xi$$
$$= \frac{1}{2} \Delta f(\boldsymbol{z}) + \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{z})} \sum_{i=1}^n \nabla_{z_i} f(\boldsymbol{z}) \cdot \nabla_{z_i} N^{\boldsymbol{\nu}}(\boldsymbol{z})$$
$$= \frac{1}{2} \Delta f(\boldsymbol{z}) + \nabla f(\boldsymbol{z}) \cdot \nabla \log N^{\boldsymbol{\nu}}(\boldsymbol{z}).$$

To pull the derivative  $\nabla_{z_i}$  out of the integral in the second step (and thereby showing differentiability of  $N^{\nu}$ ), we require statements analogous to Lemmas B.1 and B.2, with  $f_{\rm sp}(\xi \mid \tau, \boldsymbol{x})$  replaced with  $\nabla_{x_i} f_{\rm sp}(\xi \mid \tau, \boldsymbol{x})$ . This is true using similar methods, with the modification that in d=1, points  $\boldsymbol{x}$  which have two identical coordinates have to be excluded (as is already the case in  $d \geq 2$ ); recall also Remark 1.16 (ii) on this issue.

### 3.5. Associated Population Models.

3.5.1.  $\Xi$ -Fleming-Viot Process. Fix a non-zero measure  $\Xi$  on  $\triangle$ , and recall Section 1.5. We begin by constructing a stationary, bi-infinite version of the particle representation (Y(t)) of the  $\Xi$ -Fleming-Viot process. Let  $\mathfrak{N}^{ij}$  for  $i,j\in\mathbb{N},\ i< j,$  and  $\mathfrak{M}$  be Poisson point processes as described in Section 1.5. Recall that for  $i,j\in\mathbb{N}$  and i< j, level j "looks down" to level i at points of  $\mathfrak{N}^{ij}$ , and at points of  $\mathfrak{M}$  if i and j are in the same basket, and that basket contains no levels smaller than i. In particular, the ordered set of times  $(\tau_l^j)_{l\in\mathbb{Z}}$  where level j looks down to some level i< j is itself a Poisson point process with finite intensity, so almost-surely countably infinite and without accumulation points. Denote by  $k_j^i \in \{1,\ldots,j-1\}$  for  $l\in\mathbb{Z}$  the level to which particle j looks down at time  $\tau_j^j$ .

$$(B_l^j(t): j \in \mathbb{N}, j \ge 2, l \in \mathbb{Z})_{t \ge 0}$$

be an independent family of Brownian motions on E, independent of  $B^1$  and the Poisson point processes, and all starting at zero. To construct the process  $(Y(t))_{t \in (-\infty,\infty)}$ , we put  $Y_1 = B^1$ , and then construct  $Y_j$  for  $j \geq 2$  inductively. Suppose  $(Y_1(t), \ldots, Y_{j-1}(t))_{t \in (-\infty,\infty)}$  has already been constructed for some  $j \geq 2$ . Then for every  $l \in \mathbb{Z}$  put

Let  $(B^1(t))_{t\in(-\infty,\infty)}$  be a Brownian motion running at stationarity on E, and

$$Y_j(t) = Y_{k,l}(\tau_l^j) + B_l^j(t - \tau_l^j), \qquad t \in [\tau_l^j, \tau_{l+1}^j).$$

In words, whenever j looks down to some level i < j at some time  $\tau$ , we let  $Y_j$  start at the location of  $Y_i$  at time  $\tau$  and evolve as a Brownian motion until the next time it looks down. Since these lookdown times are an infinite, discrete set, this defines  $Y_j(t)$  for all  $t \in (-\infty, \infty)$ . Clearly, for any fixed  $t_0 \in \mathbb{R}$ , the law of the evolution of the process  $(Y(t))_{t \geq t_0}$  is as described in Section 1.5, and since this entire construction is invariant under constant time shifts, the distribution of Y(t) and Y(s) is the same for any fixed  $s, t \in (-\infty, \infty)$ .

We will now prove Theorem 1.24, which will imply both Proposition 1.20 and Theorem 1.23. The statement is trivial for n=1, so fix  $n \geq 2$  for the rest of Section 3.5.1, and put  $\Pi_0 = \{\{1\}, \ldots, \{n\}\}$ . We present a formal construction of the process described in Theorem 1.24. Lemma A.2 can be used to obtain the following measurable maps:

- (i) a map  $h_0: E^n \to E^n$  such that the pushforward of Lebesgue measure  $d\mathbf{y}$  on  $E^n$  under  $h_0$  is  $N^{\Xi}(\mathbf{y}) d\mathbf{y}$ .
- (ii) for every  $m \in \{1, \ldots, n-1\}$  a map  $h_m \colon E^m \times E^{n-m} \to E^{n-m}$  such that for every  $\boldsymbol{x} \in E^m_{\circ}$ , the pushforward of Lebesgue measure  $d\boldsymbol{y}$  on  $E^{n-m}$  under  $h_m(\boldsymbol{x},\cdot)$  is  $\frac{N^{\Xi}(\boldsymbol{x}\boldsymbol{y})}{N^{\Xi}(\boldsymbol{x})} d\boldsymbol{y}$ , and  $h_m(\boldsymbol{x},\cdot) = \mathrm{id}_{E^{n-m}}$  if  $\boldsymbol{x} \in E^m \setminus E^m_{\circ}$ .
- (iii) for every  $F \in \mathbb{T}(\Pi)$  a map  $h_F : E^n \times \operatorname{dec}(F) \to \operatorname{dec}(F)$  such that for every  $\boldsymbol{x} \in E^n_{\circ}$ , the pushforward of  $f_{\operatorname{tm}}^{\boldsymbol{\lambda}}(F,\tau) \, \mathrm{d}\tau \, \mathrm{d}\xi$  under  $h_F(\boldsymbol{x},\cdot)$  is  $\frac{1}{N_F^{\boldsymbol{\lambda}}(\boldsymbol{x})} f_{\operatorname{tm}}^{\boldsymbol{\lambda}}(F,\tau) f_{\operatorname{sp}}(\xi \mid \tau,\boldsymbol{x}) \, \mathrm{d}\tau \, \mathrm{d}\xi$ , and  $h_F(\boldsymbol{x},\cdot) = \operatorname{id}_{\operatorname{dec}(F)}$  if  $\boldsymbol{x} \in E^n \setminus E^n_{\circ}$ .

The definitions of  $h_m(\boldsymbol{x},\cdot)$  and  $h_F(\boldsymbol{x},\cdot)$  if  $\boldsymbol{x} \in E^m \setminus E_{\circ}^m$  and  $\boldsymbol{x} \in E^n \setminus E_{\circ}^n$ , respectively, is irrelevant as long as all maps are measurable.

Now let  $(S, \mathcal{A}, \mathbb{P})$  be a probability space that supports the following random variables, all independent.

- (i) For every  $m \in \{0, ..., n-1\}$  an i.i.d. sequence  $(\omega_m^{(i)})_{i \in \mathbb{N}_0}$  of uniform  $E^{n-m}$ -valued random variables.
- (ii) For every  $F \in \mathbb{T}(\Pi_0)$  an i.i.d. sequence  $(\omega_F^{(i)})_{i \in \mathbb{N}_0}$  of random variables in  $\operatorname{dec}(F)$  with distribution  $f_{\operatorname{tm}}^{\lambda}(F,\tau) \, \mathrm{d}\tau \, \mathrm{d}\xi$ .
- (iii) For every non-empty  $u \subset [n]$  and  $i \in \mathbb{N}_0$ , a d-dimensional standard Brownian bridge  $\omega_B^{(i)} \in C([0,1],E)$  with  $\omega_B^{(i)}(0) = \omega_B^{(i)}(1) = 0$ .
- (iv) An i.i.d. sequence  $(\omega_U^{(i)})_{i\in\mathbb{N}_0}$  of uniform [0, 1)-random variables.

We further fix an ordering  $\mathbb{T}(\Pi_0) = \{F^{(1)}, \dots, F^{(|\mathbb{T}(\Pi_0)|)}\}\$ of  $\mathbb{T}(\Pi_0)$ .

Definition 3.28. Let  $\boldsymbol{\nu} \in R$ . The *n'th level Brownian spatial coalescent with resampling* associated with transition measures  $\boldsymbol{\nu}$  is a càglàd (left-continuous with right-limits)  $E^n$ -valued process  $\boldsymbol{Z}^n(t) = (Z_1^n(t), \dots, Z_n^n(t)), t \geq 0$ , defined on S as follows. Given a realisation  $\omega \in S$ :

- (i) Put  $\zeta^{(0)}:=h_0(\omega_0^{(0)})$ , and  $t^{(0)}:=0$ . (At the end of the construction,  $\zeta^{(i)}=Z^n(t^{(i)}+)$ .)
- (ii) For  $i \in \mathbb{N}_0$ , assuming  $0 = t^{(0)} < \ldots < t^{(i)}$  are defined and  $\mathbf{Z}^n(t)$  has been constructed for  $t \in [0, t^{(i)})$ :
  - (ii.1) If  $\zeta^{(i)} \in E^n \setminus E_o^n$  put j = 1, otherwise let  $j \in \{1, \ldots, |\mathbb{T}(\Pi_0)|\}$  be the unique number such that

$$\sum_{i' < i} N^\Xi_{F(i')}(\zeta^{(i)}) \leq \omega_U^{(i)} N^\Xi(\zeta^{(i)}) < \sum_{i' < i} N^\Xi_{F(i')}(\zeta^{(i)}).$$

Put  $F := F^{(j)}$ .

(ii.2) Let  $(\tau, \xi) = h_F(\zeta^{(i)}, \omega_F^{(i)}) \in \operatorname{dec}(F)$ , and  $t^{(i+1)} := \tau(\Pi_1^F)$  the time of the first merge event. For  $t \in (t^{(i)}, t^{(i+1)}]$ , and  $l \in [n]$ , put

$$Z_l^n(t) := \zeta_l^{(i)} + \frac{t - t^{(i)}}{\tau_{\operatorname{pr}_F(\{l\})}} (\xi_{\operatorname{pr}_F(\{l\})} - \zeta_l^{(i)}) + \sqrt{\tau_{\operatorname{pr}_F(\{l\})}} \omega_B^{(i)} \left( \frac{t - t^{(i)}}{\tau_{\operatorname{pr}_F(\{l\})}} \right),$$

which is a Brownian bridge started from  $\zeta_l^{(i)}$  at time  $t^{(i)}$ , going to  $\xi_{\operatorname{pr}_F(\{l\})}$  at time  $t^{(i)} + \tau_{\operatorname{pr}_F(\{l\})}$ , and stopped at time  $t^{(i+1)}$ .

(ii.3) Let  $I = \{\min u : u \in \Pi_1^F\}$  be the set of levels that were either not involved in the first merge event, or were minimal among the set of levels with which they merged. Let  $J = [n] \setminus I$ . For every  $l \in I$ , put  $\zeta_l^{(i+1)} = Z_l^n(t^{(i+1)})$ , and

$$\zeta_J^{(i)} := h_{|J|}(\zeta_I^{(i)}, \omega_{|J|}^{(i+1)}),$$

where 
$$\zeta_J^{(i+1)} = (\zeta_j^{(i+1)} : j \in J)$$
, and  $\zeta_J^{(i+1)}$  similarly.

This defines a measurable map from S into the space of càglàd paths  $[0,\infty) \to E^n$ .

All definitions dealing with cases where  $\mathbf{Z}^n(t+) \in E^n \setminus E_0^n$  for some  $t \geq 0$ , which happens with probability zero, are only in place for the map  $S \to D([0, \infty), E^n)$  to be well-defined, and do not affect the law of  $\mathbf{Z}^n$ .

Remark 3.29. If the Brownian spatial coalescent with transition measures  $\nu$  is sampling consistent (that is, associated to some finite measure  $\Xi$  on  $\Delta$ ), then for every m < n, the laws of  $(Z_1^m, \ldots, Z_m^m)$  and  $(Z_1^n, \ldots, Z_m^n)$  are the same, and we expect that a variant of the Kolmogorov extension theorem can be used to construct an  $E^{\infty}$ -valued process  $\mathbf{Z}(t) = (Z_1(t), Z_2(t), \ldots)$  whose first n levels have the same law as  $\mathbf{Z}^n$  for every  $n \in \mathbb{N}$ . Making this precise is not necessary to prove our results.

From now, let  $\mathbb{Z}^n$  be the n'th level Brownian spatial  $\Xi$ -coalescent with resampling.

**Theorem 3.30.** The laws of  $(Z_1^n(t), \ldots, Z_n^n(t))_{t\geq 0}$  and  $(Y_1(-t), \ldots, Y_n(-t))_{t\geq 0}$  are the same. In particular, the stationary distribution of  $\mathbf{Y}$  is an i.i.d. sample from a random realisation of  $\mu^{\Xi}$ .

*Proof.* It suffices to show that for every T>0, the laws of  $(Z_1^n(t),\ldots,Z_n^n(t))_{t\in[0,T]}$ , and  $(Y_1(-t),\ldots,Y_n(-t))_{t\in[0,T]}$ , which coincides with that of  $(Y_1(T-t),\ldots,Y_n(T-t))_{t\in[0,T]}$ , are the same. Denote by  $\mathbb{Q}^{\boldsymbol{y}}$  for  $\boldsymbol{y}\in E^n$  the probability measure on  $D([0,T],E^n)$  describing the law of  $(Y_1(t),\ldots,Y_n(t))_{t\in[0,T]}$  started from (or conditioned on)  $(Y_1(0),\ldots,Y_n(0))=\boldsymbol{y}$ . Then we show

$$\mathbb{P}((Z_1^n(t), \dots, Z_n^n(t))_{t \in [0,T]} \in \cdot) = \int_{E^n} \mathbb{Q}^{\mathbf{y}}((Y_1(T-t), \dots, Y_n(T-t))_{t \in [0,T]} \in \cdot) N^{\Xi}(\mathbf{y}) \, d\mathbf{y}.$$
(3.44)

In particular, this implies that

$$N^{\Xi}(\boldsymbol{z})\,\mathrm{d}\boldsymbol{z} = \mathbb{P}((Z_1^n(0),\ldots,Z_n^n(0))\in\mathrm{d}\boldsymbol{z}) = \int_{E^n}\mathbb{Q}^{\boldsymbol{y}}((Y_1(T),\ldots,Y_n(T))\in\mathrm{d}\boldsymbol{z})N^{\Xi}(\mathrm{d}\boldsymbol{y})\,\mathrm{d}\boldsymbol{y},$$

that is,  $N^{\Xi}(\boldsymbol{y}) d\boldsymbol{y}$  (which equals  $\mathbb{E}\left[\mu^{\Xi}(dy_1) \dots \mu^{\Xi}(dy_n)\right]$ , recall (1.5)) is the stationary distribution of  $(Y_1(t), \dots, Y_n(t))$ . Plugging this back into (3.44) gives

$$\mathbb{P}((Z_1^n(t), \dots, Z_n^n(t))_{t \in [0,T]} \in \cdot) = \mathbb{P}((Y_1(T-t), \dots, Y_n(T-t))_{t \in [0,T]} \in \cdot),$$

which is the claim.

We now prove (3.44). For simplicity we assume that  $\xi_j = 0$  for  $j \geq 2$  for  $\Xi$ -a.e.  $\boldsymbol{\xi} \in \Delta$ , in which case the evolution of  $\boldsymbol{Y}$  can be described in terms of a finite measure  $\Lambda$  on [0,1], see Remark 1.19. The general case requires more notation but no different ideas. Consider the event where  $(Y_1(t), \ldots, Y_n(t))$ , restricted to the first n levels, starts at  $\boldsymbol{y}^0 = (y_1^0, \ldots, y_n^0)$ , undergoes a branching event at time  $s \in (0,T)$  involving  $k \geq 2$  levels  $J = \{j_1 < \ldots < j_k\}$ , where  $Y^n(s-) = \boldsymbol{y}^s = (y_1^s, \ldots, y_n^s)$ , and has no further branching events until time T where it ends in  $Y^n(T) = \boldsymbol{y}^T = (y_1^T, \ldots, y_n^T)$ . The rate at which a branching event evolving exactly indices J happens is  $\lambda_{n,k}$ , the associated rate of the  $\Lambda$ -coalescent. Indeed, if k > 2 then the rate is

$$\int p^k (1-p)^{n-k} \frac{\Lambda(\mathrm{d}p)}{p^2} = \lambda_{n,k},$$

and if k=2 it is  $\Lambda(\{0\})=\int p^{k-2}(1-p)^{n-k}\Lambda(\mathrm{d}p)=\lambda_{n,2}$ . Thus the probability (density) of the entire event is

$$\underbrace{N^{\Lambda}(\boldsymbol{y}^{0})\,\mathrm{d}\boldsymbol{y}^{0}}_{\text{initial sample}} \times \underbrace{\prod_{j=1}^{n}p_{s}(y_{j}^{0}-y_{j}^{s})\,\mathrm{d}y_{j}^{s}}_{\text{spatial movement in }(0,s)} \times \underbrace{\lambda_{n,k}\mathrm{e}^{-\lambda_{n}s}}_{\text{branching event}} \times \underbrace{\prod_{j\in J}p_{T-s}(y_{j_{1}}^{s}-y_{j}^{T})\,\mathrm{d}y_{j}^{T}\prod_{j\not\in J}p_{T-s}(y_{j}^{s}-y_{j}^{T})\,\mathrm{d}y_{j}^{T}}_{\text{no branching events in }(s,T)} \times \underbrace{\underbrace{\mathrm{e}^{-\lambda_{n}(T-s)}}_{\text{no branching events in }(s,T)}. \quad (3.45)$$

The process  $((Z_1^n(T-t),\ldots,Z_n^n(T-t)))_{t\in[0,T]}$  lies in the same event if and only if

- (i) the initial state is  $\mathbf{Z}^n(0) = \mathbf{y}^T$ ,
- (ii) the first coalescence event of the Brownian spatial  $\Lambda$ -coalescent started from  $\mathbf{y}^T$  at time 0 is a multiple merger of the lineages with labels J at time T-s,
- (iii) the resampling at time T-s, given the locations  $\boldsymbol{y}^{s,1} \coloneqq (y_j^s)_{j \in [n] \setminus J \cup \{j_1\}}$  yields  $\boldsymbol{y}^{s,2} \coloneqq (y_j^s)_{j \in J \setminus \{j_1\}}$ ,
- (iv) the Brownian spatial  $\Lambda$ -coalescent started from  $y^s$  at time T-s has no coalescence events until time T, where it ends in state  $y^0$ .

The probability density for this is (recall Lemma 3.1)

$$\underbrace{N^{\Lambda}(\boldsymbol{y}^{T}) \, \mathrm{d}\boldsymbol{y}^{T}}_{(\mathrm{i})} \times \underbrace{\frac{N^{\Lambda}(\boldsymbol{y}^{s,1})}{N^{\Lambda}(\boldsymbol{y}^{T})} \lambda_{n,k} \mathrm{e}^{-\lambda_{n}(T-s)} \prod_{j \in J} p_{T-s}(y_{j_{1}}^{s} - y_{j}^{T}) \prod_{j \notin J} p_{T-s}(y_{j}^{s} - y_{j}^{T}) \, \mathrm{d}\boldsymbol{y}^{s,1}}_{(\mathrm{iii})} \times \underbrace{\frac{N^{\Lambda}(\boldsymbol{y}^{s}) \, \mathrm{d}\boldsymbol{y}^{s,2}}{N^{\Lambda}(\boldsymbol{y}^{s,1})}}_{(\mathrm{iii})} \times \underbrace{\frac{N^{\Lambda}(\boldsymbol{y}^{s}) \, \mathrm{d}\boldsymbol{y}^{s,2}}{N^{\Lambda}(\boldsymbol{y}^{s,1})}}_{(\mathrm{iii})} \times \underbrace{\frac{N^{\Lambda}(\boldsymbol{y}^{0})}{N^{\Lambda}(\boldsymbol{y}^{s})} \mathrm{e}^{-\lambda_{n}s} \prod_{j=1}^{n} p_{s}(y_{j}^{0} - y_{j}^{s}) \, \mathrm{d}y_{j}^{s}}_{(\mathrm{iv})},$$

which is equal to (3.45) after some cancellations. Conditional on this event, both the realisation of  $\mathbb{Z}^n$  and  $\mathbb{Y}$  are obtained by sampling Brownian bridges between  $\mathbb{y}^0$  and  $\mathbb{y}^s$ , and between  $\mathbb{y}^s$  and  $\mathbb{y}^T$ . The same argument applies (with no modifications but more notation) to all possible realisations.

This proves Proposition 1.20 and Theorem 1.24. To deduce Theorem 1.23, we need to show that the coalescent process obtained from  $\mathbb{Z}^n$  by projecting onto the trajectories of only the initial n particles and their ancestors (equivalently by forgetting about all resampled particles and their ancestors), has the law of a Brownian spatial coalescent. This essentially follows from the definitions, and the strong Markov property of the Brownian spatial coalescent. A formal argument first observes that the statement is true by definition if we stop at the time  $t^{(1)}$  of the first coalescence—resampling event, and using an inductive argument we can assume it is true started from the smaller number of particles remaining after the first merge event. Then the strong Markov property of the Brownian spatial coalescent applied at time  $t^{(1)}$ , and sampling consistency imply that the overall law is the same.

3.5.2. Spatial Cannings Models. Let  $N \in \mathbb{N}$ , and recall the definitions in Section 1.5.2. The construction of a bi-infinite, stationary version  $(\mathbf{Y}^N(t), \mathcal{Y}^N_t)_{t \in (-\infty, \infty)}$  is done in exactly the same way as the corresponding construction for the  $\Xi$ -Fleming-Viot process, presented at the beginning of Section 3.5.1.

Recall the definition of  $(\Pi_n^{N,n}, \boldsymbol{X}_t^{N,n})_{t\geq 0}$  from Section 1.5.2, which for  $N\in\mathbb{N}$  and  $n\in[N]$  is a random variable with values in  $\Omega$ .

**Lemma 3.31.** For any  $n, N \in \mathbb{N}$ ,  $n \leq N$ , the distribution of  $(\Pi_t^{N,n}, \mathbf{X}_t^{N,n})_{t\geq 0}$  is the Brownian spatial coalescent with transition measures

$$u_{n,\vec{k}}(\mathrm{d}z) = T_N p_{n,\vec{k}}^N \, \mathrm{d}z, \quad (n,\vec{k}) \in M,$$

started from  $\Pi_0^{N,n} = \{\{1\}, \dots, \{n\}\}$  and  $(\boldsymbol{X}_0^{N,n}(\{1\}), \dots, \boldsymbol{X}_0^{N,n}(\{n\}))$  a random sample from  $N^{\boldsymbol{\nu}}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$ . In particular, the stationary distribution of  $(Y_1^N, \dots, Y_n^N)$  is given by  $N^{\boldsymbol{\nu}}(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y}$ .

*Proof.* The distribution of  $(\Pi_t)_{t\geq 0}$  is a non-spatial coalescent with transition rates

$$\lambda_{n,\vec{k}} = T_N p_{n,\vec{k}}^N, \quad (n,\vec{k}) \in M.$$

This follows by a simple combinatorial argument explained in [29], see equation (3) in that paper. If  $(O_1, \ldots, O_N) = (1, \ldots, 1)$  a.s. then  $\lambda_{n,\vec{k}} = 0$  for all  $(n,\vec{k}) \in M$  and the statement is trivial. Otherwise  $\lambda_n \geq \lambda_{n,2} > 0$  for all  $n \geq 2$ , so that  $\lim_{t \to \infty} |\Pi_t| = 1$  almost-surely, and  $Z := \text{Dec}((\Pi^{N,n}, \boldsymbol{X}^{N,n}))$  takes values in  $\text{dec}(\mathbb{T})$  (recall (2.7)). By definition of the spatial

Cannings model, conditional on Z, we can obtain  $(\boldsymbol{X}^{N,n})$  by running independent Brownian bridges along the branches of the coalescence tree encoded in Z, and a Brownian motion after the final coalescence event (this is just the time reversal of the first level in the forwards process). Hence it suffices to show that the law of Z, conditional on  $\boldsymbol{X}_0^{N,n} = \boldsymbol{x}$ , is equal to  $\text{Dec } \#\mathbb{P}^{\boldsymbol{x}}$  where  $\mathbb{P}^{\boldsymbol{x}}$  is the law of a Brownian spatial coalescent with transition measures  $\boldsymbol{\nu}$ , started at  $\boldsymbol{x}$ . We do this by verifying (3.34).

Let  $(F, \tau, \xi) \in \operatorname{dec}(\mathbb{T})$  with  $\Pi_0^F = \Pi_0 := \{\{1\}, \dots, \{n\}\}$ , and put  $\pi = (\pi_t)_{t \geq 0} = \operatorname{Tm}^{-1}(F, \tau) \in \Omega_0$  (the unique path  $\pi$  of a non-spatial coalescent satisfying  $\operatorname{Tm}(\pi) = \tau$ ). Denote by  $u_0 = \{1, \dots, n\}$  the root of F (that is,  $\operatorname{rt}(F) = \{u_0\}$ ). Let  $\boldsymbol{x} \in E^{\Pi_0}$ . Then,

$$\begin{split} \mathbb{P}(Z \in \mathrm{d}(F, \tau, \xi), \boldsymbol{X}_0^{N,n} \in \mathrm{d}\boldsymbol{x}) \\ &= \mathbb{P}(\mathrm{Tm}((\Pi_t^{N,n})) \in \mathrm{d}\tau) \mathbb{P}\left(Z \in \mathrm{d}(F, \tau, \xi), \boldsymbol{X}_0^{N,n} \in \mathrm{d}\boldsymbol{x} \ \middle| \ \mathrm{Tm}((\Pi_t^{N,n})) \in \mathrm{d}\tau\right). \end{split}$$

By Lemma 3.16,

$$\mathbb{P}(\mathrm{Tm}((\Pi_t^{N,n})) \in \mathrm{d}\tau) = f_{\mathrm{tm}}^{\lambda}(F,\tau)\,\mathrm{d}\tau,$$

and because the movement of particles forwards in time, conditional on the branching, are independent Brownian motions,

$$\mathbb{P}(Z \in \mathrm{d}(F, \tau, \xi), \boldsymbol{X}_0^{N,n} \in \mathrm{d}\boldsymbol{x} \mid \mathrm{Tm}((\Pi_t^{N,n})) \in \mathrm{d}\tau)$$

$$= \underbrace{\mathbb{P}(Y_1^N(-\tau_{u_0}) \in \mathrm{d}\xi_{u_0})}_{\substack{\mathrm{location of most recent common ancestor (MRCA)}} \times \underbrace{\prod_{u \in \mathrm{nd}(F) \backslash \mathrm{rt}(F)} p(\tau_{\mathrm{pr}_F(u)} - \tau_u, (\boldsymbol{x}\xi)_u - \xi_{\mathrm{pr}_F(u)}) \, \mathrm{d}(\boldsymbol{x}\xi)_u}_{\substack{\mathrm{locations of all other particles at branch points}}$$

$$= d\xi_{u_0} \times f_{\mathrm{sp}}(\xi \mid \tau, \boldsymbol{x}) \left( \prod_{u \in \mathrm{nd}^{\circ}(F) \backslash \mathrm{rt}(F)} d\xi_u \right) d\boldsymbol{x}$$
$$= f_{\mathrm{sp}}(\xi \mid \tau, \boldsymbol{x}) d\xi d\boldsymbol{x},$$

where we used that the first (in fact, any) level of  $\mathbf{Y}^N$  at stationarity is uniformly distributed in E. We conclude

$$\mathbb{P}(Z \in d(F, \tau, \xi), \boldsymbol{X}_0^{N,n} \in d\boldsymbol{x}) = f_{\text{tm}}^{\boldsymbol{\lambda}}(F, \tau) f_{\text{sp}}(\xi \mid \tau, \boldsymbol{x}) d\xi d\tau d\boldsymbol{x}.$$

Integrating out  $(F, \tau, \xi)$  gives

$$\mathbb{P}(\boldsymbol{X}_0^{N,n} \in d\boldsymbol{x}) = N^{\boldsymbol{\nu}}(\boldsymbol{x}) d\boldsymbol{x}, \tag{3.46}$$

so

$$\mathbb{P}(Z \in d(F, \tau, \xi) \mid \boldsymbol{X}_0^{N,n} \in d\boldsymbol{x}) = \frac{1}{N^{\nu}(\boldsymbol{x})} f_{\mathrm{tm}}^{\lambda}(F, \tau) f_{\mathrm{sp}}(\xi \mid \tau, \boldsymbol{x}) d\xi d\tau.$$

Recalling (3.34), this shows that the law of  $\operatorname{Dec}((\Pi_t^{N,n}, \boldsymbol{X}_t^{N,n}))$  conditional on  $\boldsymbol{X}_0^{N,n} = \boldsymbol{x}$  is equal to  $\operatorname{Dec} \# \mathbb{P}^{\boldsymbol{x}}$  where  $\mathbb{P}^{\boldsymbol{x}}$  is the law of a Brownian spatial coalescent with transition measures  $\boldsymbol{\nu}$  started from  $\boldsymbol{x}$ .

Finally, we already know that  $(Y_1^N,\ldots,Y_n^N)$  is at stationarity, and  $(Y_1^N(0),\ldots,Y_n^N(0))=(\boldsymbol{X}_0^{N,n}(\{1\}),\ldots,\boldsymbol{X}_0^{N,n}(\{n\}))$ , so the final claim follows from (3.46).

**Lemma 3.32.** Let  $n, N \in \mathbb{N}$ ,  $n \leq N$ , and denote by  $\mathbb{Z}^n$  the n'th level Brownian spatial coalescent with resampling associated with transition measures

$$\nu_{n,\vec{k}}(\mathrm{d}z) = T_N p_{n,\vec{k}}^N \,\mathrm{d}z.$$

Then  $(Y_1^N(-t), \ldots, Y_n^N(-t))_{t\geq 0}$  and  $\mathbf{Z}^n$  have the same law.

Proof. Let  $k \in \{2, ..., n\}$  and  $J \subset \{1, ..., n\}$  with |J| = k. We want to compute the probability  $q_{n,k}$  that at any given reproduction event in the spatial Cannings model, all levels in J look down to level min J, and all levels in  $\{1, ..., n\} \setminus J$  do not look down. For a given realisation  $(O_1, ..., O_N) = (o_1, ..., o_N)$ , and indices  $i_J \in [N]$  and  $i_j \in [N]$ ,  $j \in [n] \setminus J$ , all pairwise distinct, the probability that level j for  $j \in [n] \setminus J$  falls into "bucket"  $i_j$ , and all levels in J fall into bucket  $i_J$ , is

$$\frac{1}{(N)_n}(o_{i_J})_k \prod_{j \notin J} o_j.$$

There are  $(N)_{n'}$  choices for the indices  $i_J$ ,  $(i_j)_{j\notin J}$  where n'=n-k+1, so by exchangeability of  $(O_1,\ldots,O_N)$ ,

$$q_{n,k} = \frac{(N)_{n'}}{(N)_n} \mathbb{E}\left[ (O_1)_k O_2 \dots O_{n'} \right] = p_{n,k}^N.$$

Thus the rate at which such a lookdown event happens when observing  $(Y_1^N, \ldots, Y_n^N)$  forwards in time is equal to  $T_N p_{n,\vec{k}}^N$ . A similar argument holds for general lookdown events. With this observation, and the fact that we know the stationary distribution of  $(Y_1^N, \ldots, Y_n^N)$  from Lemma 3.31, the rest of the proof is identical to that of Theorem 3.30.

The final ingredients in the proof of Theorem 1.26 are the following two lemmas.

**Lemma 3.33.** If  $\lambda_{n,\vec{k}}^N \to \lambda_{n,\vec{k}}$ , then the Brownian spatial coalescent with transition measures  $\nu_{n,\vec{k}}^N \, dz$  converges weakly to those with transition measures  $\nu_{n,\vec{k}} \, dz = \lambda_{n,\vec{k}} \, dz$ .

*Proof.* For  $x \in \mathcal{X}$  and  $N \in \mathbb{N}$  put  $P^{x,N} := \operatorname{Dec} \# \mathbb{P}^{x,N}$ , and  $P^x := \operatorname{Dec} \# \mathbb{P}^x$ . Suppose we can show that  $P^{x,N} \Longrightarrow P^x$  for every  $x \in \mathcal{X}$ . Then, for any  $f \in C_b(\Omega)$ ,

$$\int_{\Omega} f(\omega) \mathbb{P}^{x,N} (d\omega) = \int_{\operatorname{dec}(\mathbb{F})} K_{x}(F^{\star}, f) P^{x,N} (dF^{\star}), \tag{3.47}$$

where  $K_{\boldsymbol{x}}(F^{\star}, f) = \int_{\Omega} f(\omega) K_{\boldsymbol{x}}(F^{\star}, d\omega)$  is continuous in  $\boldsymbol{x}$  and  $F^{\star}$  (recall Lemma 2.9), and bounded by  $||f||_{\infty}$ . Hence the right-hand side of (3.47) converges to the same expression with  $P^{\boldsymbol{x},N}$  replaced by  $P^{\boldsymbol{x}}$ , so to  $\int_{\Omega} f(\omega) \mathbb{P}^{\boldsymbol{x}}(d\omega)$ . This proves  $\mathbb{P}^{\boldsymbol{x},N} \Longrightarrow \mathbb{P}^{\boldsymbol{x}}$  as  $N \to \infty$ . Now recall from (3.34) that

$$P^{\boldsymbol{x}}(F, d\tau, d\xi) = \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} f_{\mathrm{tm}}^{\boldsymbol{\lambda}}(F, \tau) f_{\mathrm{sp}}^{F}(\xi \mid \tau, \boldsymbol{x}) d\tau d\xi,$$

and analogously for  $P^{x,N}$ . By Lemma 3.35,  $N^{\nu^N}(x) \to N^{\nu}(x)$  as  $N \to \infty$  for fixed x, so

$$\frac{1}{N^{\boldsymbol{\nu}^N}(\boldsymbol{x})} f_{\mathrm{tm}}^{\boldsymbol{\lambda}^N}(F,\tau) f_{\mathrm{sp}}(\xi \mid \tau, \boldsymbol{x}) \to \frac{1}{N^{\boldsymbol{\nu}}(\boldsymbol{x})} f_{\mathrm{tm}}^{\boldsymbol{\lambda}}(F,\tau) f_{\mathrm{sp}}(\xi \mid \tau, \boldsymbol{x}).$$

Hence both  $P^{x,N}$  and  $P^x$  have densities w.r.t. the same (uniform) measure, and the densities converge pointwise, so Scheffé's lemma implies  $P^{x,N} \implies P^x$  for every  $x \in \mathcal{X}$ .

**Lemma 3.34.** If  $\lambda_{n,\vec{k}}^N \to \lambda_{n,\vec{k}}$ , then the n'th level Brownian spatial coalescent with resampling associated with transition measures  $\nu_{n,\vec{k}}^N(\mathrm{d}z) = \lambda_{n,\vec{k}}^N\,\mathrm{d}z$  converges weakly to that associated with transition measures  $\nu_{n,\vec{k}}(\mathrm{d}z) = \lambda_{n,\vec{k}}\,\mathrm{d}z$ .

*Proof.* Exactly by the same argument that proved Lemma 3.33. The resampling causes no issues because  $N^{\nu^N} \to N^{\nu}$  pointwise (see Lemma 3.35).

Proof of Theorem 1.26. The first statement has been proved in Lemma 3.31. Statement (ii) implies (iii), and (iii) and (iv) are equivalent by [29]. Statement (iv) implies (ii) by Lemmas 3.31 and 3.33. The coalescent process  $(\Pi^{N,n}, \mathbf{X}^{N,n})$  is a continuous image of  $(Y_1^N, \ldots, Y_n^N)$ , so (i) implies (ii) by the continuous mapping theorem. On the other hand, by Lemma 3.32 the time reversal  $\mathbf{Z}^{N,n}$  of  $(Y_1^N, \ldots, Y_n^N)$  is the Brownian spatial coalescent with resampling associated with the transition measures  $\boldsymbol{\nu}^N$  defined in Theorem 1.26. Thus if (iv) holds, then by Lemma 3.34 the time reversal of, and thus  $(Y_1^N, \ldots, Y_n^N)$  itself converge weakly. This proves equivalence of (i) to (iv).

Now suppose that (iii) holds. Then the limit of  $T_N p_{n,\vec{k}}^N$  must be the rates of a  $\Xi$ -coalescent by [29], so by Lemmas 3.31 and 3.33 the limit of  $(\Pi^{N,n}, \mathbf{X}^{N,n})$  is a Brownian spatial  $\Xi$ -coalescent (with initial condition drawn from  $N^\Xi(\mathbf{x}) \, \mathrm{d}\mathbf{x}$ ). Similarly, by Lemmas 3.32 and 3.34 the time reversal of  $(Y_1^N, \ldots, Y_n^N)$  converges to the Brownian spatial  $\Xi$ -coalescent with resampling, which is itself the time reversal of the first n levels  $(Y_1, \ldots, Y_n)$  of the  $\Xi$ -Fleming-Viot process by Theorem 3.30, so  $(Y_1^N, \ldots, Y_n^N)$  converges weakly to  $(Y_1, \ldots, Y_n)$ .

**Lemma 3.35.** Suppose that  $\lambda^N \to \lambda$  in  $R^{\circ}$ , in the sense that  $\lambda_{n,\vec{k}}^N \to \lambda_{n,\vec{k}}$  for every  $(n,\vec{k}) \in M$ . Define  $\nu^N \in R$  by  $\nu_{n,\vec{k}}(\mathrm{d}\xi) = \lambda_{n,\vec{k}} \, \mathrm{d}\xi$  for  $(n,\vec{k}) \in M$  and  $N \in \mathbb{N}$ , and  $\nu$  similarly. Then,

$$N^{oldsymbol{
u}^N}(oldsymbol{x}) o N^{oldsymbol{
u}}(oldsymbol{x})$$

for every  $x \in \mathcal{X}$ .

*Proof.* Recall that for  $(\Pi, x) \in \mathcal{X}$ ,

$$N^{\boldsymbol{\nu}}(\boldsymbol{x}) = \sum_{F \in \mathbb{F}(\Pi)_{\operatorname{tm}(F)}} \int_{\operatorname{sp}(F)} f_{\operatorname{tm}}^{\boldsymbol{\lambda}}(\tau) f_{\operatorname{sp}}(\xi \mid \tau, \boldsymbol{x}) \, \mathrm{d}\xi \, \mathrm{d}\tau.$$

The map  $\lambda \mapsto f_{\operatorname{tm}}^{\lambda}(\tau)$  is continuous for every  $\tau \in \operatorname{tm}(F)$ , and it is easy to see that there exists  $\lambda' \in R^{\circ}$  and  $N_0 \in \mathbb{N}$  such that  $\sup_{N \geq N_0} f_{\operatorname{tm}}^{\lambda'} \leq f_{\operatorname{tm}}^{\lambda'}$ . Then the claim follows from the dominated convergence theorem.

# APPENDIX A. TECHNICAL LEMMATA

Recall that for finite measures  $m_1, m_2$ , we write  $m_1 \sim m_2$  if  $m_1 = cm_2$  for some c > 0.

**Lemma A.1.** Suppose  $x_0 \in E$ ,  $s_0 > 0$ ,  $f: E \times (s_0, \infty) \to (0, \infty)$  is continuous, and  $\mu$  is a finite measure on E. Then

$$B^{(x_0,s_0)+}(\cdot) \sim \iint B^{(x_0,s_0)\to(x,s)+}(\cdot)f(x,s)\mu(dx) ds$$

if and only if  $f(x,s)\mu(dx) \sim p_{s-s_0}(x-x_0) dx$  for all  $s > s_0$ . If further  $x_1 \in E$ ,  $s_1 > s_0$ , and  $f: E \times (s_0, s_1) \to (0, \infty)$  is continuous, then

$$B^{(x_0,s_0)\to(x_1,s_1)}(\cdot) \sim \iint B^{(x_0,s_0)\to(x,s)\to(x_1,s_1)}(\cdot)f(x,s)\mu(\mathrm{d}x)\,\mathrm{d}s$$

if and only if  $f(x, s)\mu(dx) \sim p_{s-s_0}(x-x_0)p_{s_1-s}(x_1-x) dx$  for all  $s \in (s_0, s_1)$ .

Proof. The "if" direction is clear in both cases: for fixed  $s > s_0$ , if we sample  $z \sim \mathcal{N}(0, s)$  and, independent of z, a Brownian bridge from  $(s_0, x_0)$  to (s, z) followed by a Brownian motion started from (s, z), then the law is the same as that of a Brownian motion started from  $(s_0, x_0)$ . Similarly for the second claim. From now assume that d = 1, so E is the one-dimensional torus. The proof is analogous in higher dimensions. We start with the first statement. Without loss of generality  $x_0 = s_0 = 0$  and  $\iint f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient reformulation is obtained by putting  $g(s) \coloneqq \int f(x, s)\mu(\mathrm{d}x)\,\mathrm{d}s = 1$ . A first convenient  $g(s) \vdash f(s) \vdash f(s) \vdash f(s) \vdash f(s) \vdash f(s) \vdash f(s) \vdash f$ 

So let  $g:(0,\infty)\to(0,\infty)$  with  $\int g(s)\,\mathrm{d}s=1$  be continuous,  $\nu_s$  for s>0 be probability measures on  $\mathbb{R}$ , continuous in s w.r.t. the 1-Wasserstein distance, such that

$$B^{(0,0)+}(\cdot) = \iint B^{(0,0)\to(x,s)+}(\cdot)\nu_s(\mathrm{d}x)g(s)\,\mathrm{d}s.$$

Denote by  $\varphi_s(u) := \int \mathrm{e}^{\mathrm{i}ux} \nu_s(\mathrm{d}x)$  the characteristic function of  $\nu_s$ , which is bounded and differentiable in u, and both  $\varphi_s$  and  $\partial_u \varphi_s$  are jointly continuous in u and s by assumptions on  $\nu_s$ . This will justify all interchanges of differentiation and integration that follow. Evaluating the equation at time t>0 and taking characteristic functions on both sides gives, after a short calculation,

$$e^{-tu^2/2} = \int_0^t \varphi_s(u)e^{-(t-s)u^2/2}g(s) ds + \int_t^\infty \varphi_s(ut/s)e^{-t(s-t)u^2/(2s)}g(s) ds.$$

Cancelling  $e^{-tu^2/2}$  on both sides and taking a derivative in t gives

$$0 = \int_{t}^{\infty} \partial_{t} \left( \varphi_{s} \left( \frac{ut}{s} \right) e^{(ut)^{2}/(2s)} \right) g(s) ds$$

$$= \frac{u}{t} \partial_{u} \left( \int_{t}^{\infty} \varphi_{s} \left( \frac{ut}{s} \right) e^{(ut)^{2}/(2s)} g(s) ds \right).$$
(A.1)

This implies that  $\int_t^\infty \varphi_s(ut/s) \mathrm{e}^{(ut)^2/(2s)} g(s) \,\mathrm{d}s$  is constant in u, so letting  $u \to 0$  for fixed t > 0 implies by dominated convergence that

$$\int_{1}^{\infty} g(s) \, \mathrm{d}s = \int_{1}^{\infty} \varphi_s(ut/s) \mathrm{e}^{(ut)^2/(2s)} g(s) \, \mathrm{d}s,$$

for every u, t > 0. Taking a derivative in t for fixed u gives, reusing (A.1),

$$-g(t) = -g(t)\varphi_t(u)e^{u^2t/2} + \underbrace{\int_t^\infty \partial_t \left(\varphi_s(ut/s)e^{(ut/s^2/(2s)}\right)g(s)\,\mathrm{d}s}_{=0},$$

that is  $\varphi_t(u) = e^{-u^2t/2}$  and therefore  $\nu_t = \mathcal{N}(0, t)$ .

Consider the second statement with  $x_0 = x_1 = s_0 = 0$  and  $s_1 = 1$ , then by similar arguments we need to prove that

$$B^{(0,0)\to(0,1)}(\cdot) = \int_0^1 \int_{\mathbb{R}} B^{(0,0)\to(x,s)\to(1,0)}(\cdot)\nu_s(\mathrm{d}x)g(s)\,\mathrm{d}s \tag{A.2}$$

implies  $\nu_s = \mathcal{N}(0, s(1-s))$ , with assumptions on  $\nu_s$  and g analogous to before. Consider the measurable map  $C([0,1],\mathbb{R}) \to C([0,\infty),\mathbb{R})$  that maps a path y to  $t \mapsto (1+t)y(\frac{t}{1+t})$ ; it sends  $B^{(0,0)\to(0,1)}$  to  $B^{(0,0)\to(0,1)}$  to  $B^{(0,0)\to(x,s)\to(0,1)}$  to  $B^{(0,0)\to(\frac{x}{1-s},\frac{s}{1-s})+}$  for  $x\in\mathbb{R}, s\in(0,1)$ . The easiest way to see this is that the pushforward in both cases is a Gaussian process with the correct mean and covariance. Applying this map to (A.2) and substituting  $u=\frac{s}{1-s}$  gives

$$B^{(0,0)+}(\cdot) = \int_0^\infty \int_{\mathbb{R}} B^{(0,0)\to((1+u)x,u)+}(\cdot)\nu_{u/(1+u)}(\mathrm{d}x) \frac{g(\frac{u}{1+u})}{(1+u)^2} \,\mathrm{d}u.$$

By what we've already proved this implies  $\nu_{u/(1+u)}(\mathrm{d}x) \sim p_u(\frac{x}{1+u})\,\mathrm{d}x$ , that is

$$\nu_s(\mathrm{d} x) \sim p_{s/(1-s)}(\frac{x}{1-s})\,\mathrm{d} x \sim p_{s(1-s)}(x)\,\mathrm{d} x.$$

**Lemma A.2.** Let A be a measurable space and B a Borel space, m a probability measure on B with no atoms, and  $f: A \times B \to (0, \infty)$  measurable with  $\int_B f(a,b)m(\mathrm{d}b) = 1$  for every  $a \in A$ , so that  $m_a(\mathrm{d}b) := f(a,b)m(\mathrm{d}b)$  defines a probability measure on B for every  $a \in A$ . Then there exists a measurable function  $h: A \times B \to B$  such that

$$\forall a \in A \colon h(a, \cdot) \# m = m_a.$$

*Proof.* Let  $\varphi \colon B \to [0,1]$  be a Borel isomorphism,

$$f': A \times [0,1] \to (0,\infty); (a,x) \mapsto f(a,\varphi^{-1}(x)),$$

and  $m' := \varphi \# m$ . Then  $\int_{[0,1]} f'(a,x) m'(\mathrm{d}x) = \int_B f(a,b) m(\mathrm{d}b) = 1$  for every  $a \in A$ , so  $m'_a(\mathrm{d}x) := f'(a,x) m'(\mathrm{d}x)$  is a probability measure on [0,1], and

$$\int_{[0,1]} g(x)m'_a(\mathrm{d}x) = \int_{[0,1]} g(x)f(a,\varphi^{-1}(x))m'(\mathrm{d}x) = \int_B g(\varphi(b))f(a,b)m(\mathrm{d}x),$$

so  $m'_a = \varphi \# m_a$ . Now if there exists a measurable  $h': A \times [0,1] \to [0,1]$  with  $h'(a,\cdot) \# m' = m'_a$  for every  $a \in A$ , then  $h: A \times B \to B$  defined by  $(a,b) \mapsto \varphi^{-1}(h'(a,\varphi(b)))$  satisfies

$$\int_{B} g(b)(h(a,\cdot)\#m)(\mathrm{d}b) = \int_{B} g(\varphi^{-1}(h'(a,\varphi(b))))m(\mathrm{d}b) = \int_{[0,1]} g(\varphi^{-1}(h'(a,x)))m'(\mathrm{d}x) 
= \int_{[0,1]} g(\varphi^{-1}(x))m'_{a}(\mathrm{d}x) 
= \int_{B} g(b)m_{a}(\mathrm{d}b),$$

so  $h(a,\cdot)\#m=m_a$ . This shows that it suffices to show the statement if B=[0,1].

Define  $F: A \times [0,1] \to [0,1]$  by  $F(a,y) = \int_0^y f(a,x) m(\mathrm{d}x)$ , which is measurable by Tonelli's theorem. Then  $F^{-1}(a,u) \coloneqq \inf\{y \in [0,1]: F(a,y) \ge u\}$  defines a function  $A \times [0,1] \to [0,1]$  which is measurable because F is càdlàg. Furthermore, if  $\lambda$  denotes Lebesgue measure on [0,1], then  $F^{-1}(a,\cdot)\#\lambda = m_a$ . Let  $G: [0,1] \to [0,1]$  be defined by  $G(y) = \int_0^y m(\mathrm{d}x)$ , so that  $G\#m = \lambda$  because m has no atoms. Then  $h(a,x) \coloneqq F^{-1}(a,G(x))$  is a measurable map  $A \times [0,1] \to [0,1]$ , and

$$h(a,\cdot)\#m = (F^{-1}(a,\cdot)\circ G)\#m = F^{-1}(a,\cdot)\#(G\#m) = F^{-1}(a,\cdot)\#\lambda = m_a.$$

**Lemma A.3.**  $\Omega_0$  and  $\Omega$  are closed subsets of  $D([0,\infty),\mathcal{P})$  and  $D([0,\infty),\mathcal{X})$ , respectively. In particular,  $\Omega_0$  and  $\Omega$  are Polish.

Proof. Suppose  $(\Pi_s^n) \in \Omega_0$ ,  $n \in \mathbb{N}$ ,, and  $(\Pi_s^n) \to (\Pi_s) \in D([0,\infty), \mathcal{P})$ , and fix t > 0. Since  $(\Pi_s)$  is càdlàg and has only countably many discontinuities, we can find u < t < r such that  $\Pi_u = \Pi_{t-}$  and  $\Pi_t = \Pi_r$  and  $(\Pi_s)$  is continuous at u and r. Then  $\Pi_t^n \to \Pi_t$  and  $\Pi_r^n \to \Pi_r$ , so for sufficiently large n,

$$\Pi_{t-} = \Pi_u = \Pi_u^n \ge \Pi_r^n = \Pi_r = \Pi_t.$$

This proves that  $(\Pi_s) \in \Omega_0$ .

If  $(\Pi_s^n, \boldsymbol{x}_s^n) \in \Omega$ ,  $n \in \mathbb{N}$ , and  $(\Pi_s^n, \boldsymbol{x}_s^n) \to (\Pi_s, \boldsymbol{x}_s) \in D([0, \infty), \mathcal{X})$ , then  $(\Pi_s^n) \to (\Pi_s)$  in  $D([0, \infty), \mathcal{P})$ , so  $(\Pi_s) \in \Omega_0$ , so  $(\Pi_s, \boldsymbol{x}_s) \in \Omega$ .

**Lemma A.4.** For  $F \in \mathbb{F}$  and T > 0, the set

$$\{\operatorname{Fr} = F, \text{ no jumps after } T\} = \{\omega = (\Pi_t) \in \Omega_0 \colon \operatorname{Fr}(\omega) = F, \Pi_t = \Pi_T \, \forall t > T\}$$

is closed. In particular, Fr:  $\Omega_0 \to \mathbb{F}$  is measurable.

Proof. Let  $\omega_n = (\Pi_t^n) \in \{\text{Fr} = F, \text{ no jumps after } T\}$  for  $n \in \mathbb{N}$ , and  $\omega_n \to \omega = (\Pi_t)$  in  $\Omega_0$ . Say  $F = (\Pi_0^F, \dots, \Pi_m^F)$  and assume that F is non-trivial so  $m \geq 1$ . Then each  $\omega_n$  has exactly m jump times  $0 < t_1^n < \dots < t_m^n \leq T$ , so there is a subsequence along which the tuple of jump times converges to some  $0 \leq t_1 \leq \dots \leq t_m \leq T$ . Recall that  $\Omega_0$  is endowed with the Skorokhod topology, so for any  $i \in [m]$  and sufficiently large n,

$$\Pi^{n}(t_{i}^{n}), \Pi^{n}(t_{i}^{n}-) \in \{\Pi(t_{i}-), \Pi(t_{i})\}$$

if  $t_i > 0$ , and  $\Pi^n(t_i^n) = \Pi(0)$  if  $t_i = 0$ . Now assume that  $t_1 = 0$ , then  $\Pi_1^F = \Pi^n(t_1^n) \to \Pi(0)$ , but also  $\Pi_0^F = \Pi^n(0) \to \Pi(0)$ , a contradiction. Now assume that  $t_1 = t_2$ , then for sufficiently large n, all of  $\Pi^n(t_1^n)$ ,  $\Pi^n(t_1^n)$ , and  $\Pi^n(t_2^n)$  are in  $\{\Pi(t_1), \Pi(t_1)\}$ , but they are

respectively equal to  $\Pi_0^F$ ,  $\Pi_1^F$ ,  $\Pi_2^F$  hence all different, a contradiction. Inductively we obtain  $0 < t_1 < \ldots < t_m \le T$ . Then for any  $i \in [m]$  and large n,

$$\Pi_{i}^{F} = \Pi^{n}(t_{i}^{n}) \in \{\Pi(t_{i}-), \Pi(t_{i})\}, \Pi_{i-1}^{F} = \Pi^{n}(t_{i}^{n}-) \in \{\Pi(t_{i}-), \Pi(t_{i})\},$$

and since  $\omega \in \Omega_0$  we must have  $\Pi(t_i) \leq \Pi(t_i)$ , so  $\Pi(t_i) = \Pi_{i-1}^F$  and  $\Pi(t_i) = \Pi_i^F$ . Since  $\omega$  can't have any additional jumps, we proved  $\omega \in \{Fr = F, \text{ no jumps after } T\}$ .

If F is trivial, then

$$\{\operatorname{Fr} = F, \text{ no jumps after } T\} = \{\operatorname{Fr} = F\} = \{\omega = (\Pi_t) \in \Omega_0 \colon \Pi_t = \Pi_0 \,\forall t > 0\},$$

which is obviously closed.

Finally, for any  $F \in \mathbb{F}$  we obtain that

$$\{\operatorname{Fr}=F\}=\bigcup_{T>0}\left\{\operatorname{Fr}=F, \text{ no jumps after } T\right\}$$

is measurable.  $\Box$ 

**Lemma A.5.** For  $F \in \mathbb{F}$ , the map

$$\operatorname{Tm}_F : \Omega_0 \supset \{\operatorname{Fr} = F\} \to \operatorname{tm}(F); \quad (\Pi_t) \mapsto [\Pi \mapsto \inf\{t > 0 : \Pi_t = \Pi\}]$$

is measurable.

*Proof.* Say  $F = (\Pi_0^F, \dots, \Pi_m^F)$  with  $m \ge 1$ . By definition of the topology on  $\operatorname{tm}(F)$ , measurability of  $\operatorname{Tm}_F$  is equivalent to measurability of  $\operatorname{Tm}_F(\cdot)(\Pi_i^F)$  for all  $i \in [m]$ . Then for  $i \in [m]$  and t > 0,

$$\left\{\omega\colon\operatorname{Tm}_F(\omega)(\Pi_i^F)\leq t\right\}=\left\{\omega(t)\leq\Pi_i^F\right\}=\bigcup_{\Pi\leq\Pi_i^F}\left\{\omega(t)=\Pi\right\},$$

is measurable as a finite union of measurable sets.

Note with this definition,  $\operatorname{Tm}: \Omega_0 \to \operatorname{tm}(\mathbb{F})$  is given by  $\omega \mapsto (\operatorname{Fr}(\omega), \operatorname{Tm}_{\operatorname{Fr}(\omega)}(\omega))$ .

**Lemma A.6.** The map  $Tm: \Omega_0 \to tm(\mathbb{F})$  is bijective and bimeasurable.

*Proof.* Bijectivity is obvious. By definition of the topology on  $\operatorname{tm}(\mathbb{F})$ , measurability of Tm is equivalent to measurability of  $\operatorname{Tm}_F$  for all  $F \in \mathbb{F}$ , which was proved above. For measurability of the inverse, recall that the topology on  $\Omega_0$  is generated by sets of the form  $\{\omega \colon \omega(t) = \Pi\}$  for  $t \geq 0$  and  $\Pi \in \mathcal{P}$ . The preimage of such a set under the inverse of Tm is (recall (3.1))

$$\bigcup_{\substack{F_1 \in \mathbb{F} \\ \operatorname{rt}(F_1) = \Pi \ \operatorname{lf}(F_2) = \Pi}} \bigcup_{\substack{F_2 \in \mathbb{F} \\ \operatorname{rt}(F_2) = \Pi}} \left\{ (F_1 F_2, (\tau_1/t/\tau_2)) \colon \tau_1 \in \operatorname{tm}(F_1), \tau_2 \in \operatorname{tm}(F_2), \tau_1(\operatorname{rt}(F_1)) \leq t \right\},$$

which is a countable union over sets that are measurable by the definition of the topologies on  $\operatorname{tm}(\mathbb{F})$  and  $\operatorname{tm}(F)$ ,  $F \in \mathbb{F}$ .

**Lemma A.7.** The map  $\operatorname{Sp}: \Omega \to \bigcup_{F \in \mathbb{F}} \operatorname{sp}(F)$  is measurable.

*Proof.* It is enough to show that for any  $F \in \mathbb{F}$  and  $u \in \operatorname{nd}^{\circ}(F)$ , the map  $\{\operatorname{Fr} = F\} \to E$  given by  $\omega = (\boldsymbol{x}_t) \mapsto \boldsymbol{x}_{\operatorname{Tm}_F(\omega)_u}(u)$  is measurable. It can be written as the composition of the three maps

$$\begin{cases}
\{\operatorname{Fr} = F\} & \to & \{\operatorname{Fr} = F\} \times (0, \infty) & \to & \mathcal{X} & \to & E \\
\\
\omega & \mapsto & (\omega, \operatorname{Tm}_F(\omega)(u)) \\
(\omega, t) & \mapsto & \omega(t) \\
& (\Pi, \boldsymbol{x}) & \mapsto & \begin{cases} \boldsymbol{x}(u), & u \in \Pi, \\ x_0, & \text{else,} \end{cases}
\end{cases}$$

for some fixed, arbitrary  $x_0 \in E$ . The first map is measurable by Lemma A.5, the second by standard facts about the Skorokhod topology, and the third because it is continuous.

**Lemma A.8.** The map  $\operatorname{Dec}: \Omega \to \operatorname{dec}(\mathbb{F})$  is measurable.

*Proof.* Follows from Lemmas A.5 and A.7.

Proof of Lemma 2.9. Uniqueness follows because the collection of maps  $(\operatorname{Path}_u^{F^*}: u \in \operatorname{nd}(F))$  uniquely determines an element of  $\{\operatorname{Dec} = F^*\}$ . For existence and the remaining properties, let

$$(B_t^u)_{t\in[0,1]}: u\subset\mathbb{N}, \qquad ((W_t^u)_{t\geq0}: u\subset\mathbb{N}),$$

be independent families of, respectively, i.i.d. standard Brownian bridges and i.i.d. standard Brownian motions, defined on some common probability space  $(S, \mathcal{A}, \mathbb{P})$ . For any  $\boldsymbol{x}$  and  $F^* = (F, \tau, \xi) \in \text{dec}(\mathbb{F})$ , we define a random element  $(\Pi_t, \boldsymbol{X}_t) \in \{\text{Dec} = F^*\}$  with  $\boldsymbol{X}_0 = \boldsymbol{x}$  in the following way:  $(\Pi_t) := \text{Tm}^{-1}(F, \tau)$ , and if  $u \in \text{nd}(F)$  and  $t \in [\tau_u, \tau_{\text{pr}_F(u)})$ ,

$$\boldsymbol{X}_{t}(u) \coloneqq \begin{cases} (\boldsymbol{x}\xi)_{u} + (\xi_{\operatorname{pr}_{F}(u)} - (\boldsymbol{x}\xi)_{u})B_{(t-\tau_{u})/(\tau_{\operatorname{pr}_{F}(u)}-\tau_{u})}^{u}, & \operatorname{pr}_{F}(u) \neq \emptyset, \\ (\boldsymbol{x}\xi)_{u} + W_{t-\tau_{u}}^{u}, & \operatorname{else.} \end{cases}$$

Then  $X_t \in E^{\Pi_t}$  for all  $t \geq 0$ , and by standard properties of Brownian motion, there is a  $\mathbb{P}$ -null set  $N_{\boldsymbol{x},F^*} \in \mathcal{A}$  outside of which  $X_t \in E_{\circ}^{\Pi_t}$  for all  $t \geq 0$ . On  $N_{\boldsymbol{x},F^*}$ , we let  $(\Pi_t, X_t)$  be defined by  $(\Pi_t) = \operatorname{Tm}^{-1}(F,\tau)$ , and  $X_t$  linearly interpolates between nodes of the forest (any other particular element of  $\{\operatorname{Dec} = F^*\}$  starting at  $\boldsymbol{x}$  would do). Then the law  $K_{\boldsymbol{x}}(F^*,\cdot)$  of  $(\Pi_t, X_t)$  has the stated properties.

Let  $F = (\Pi_0, \dots, \Pi_m) \in \mathbb{F}$  be fixed,  $\tau^n \to \tau$  in  $\operatorname{tm}(F)$ ,  $\xi^n \to \xi$  in  $\operatorname{sp}(F)$ , and  $\boldsymbol{x}_n \to \boldsymbol{x}$  in  $E_\circ^{\operatorname{lf}(F)}$ ,  $F_n^\star \coloneqq (F, \tau^n, \xi^n)$ ,  $F^\star \coloneqq (F, \tau, \xi)$ , and denote the associated paths by  $(\Pi_t^n, \boldsymbol{X}_t^n)$ ,  $n \in \mathbb{N}$ , and  $(\Pi_t, \boldsymbol{X}_t)$ . This gives a coupling of the laws  $K_{\boldsymbol{x}_n}(F_n^\star, \cdot)$ ,  $n \in \mathbb{N}$ , and  $K_{\boldsymbol{x}}(F^\star, \cdot)$  on the probability space  $(S, \mathcal{A}, \mathbb{P})$ , so almost-sure convergence on S implies weak convergence of the laws. Put  $\tau_i \coloneqq \tau(\Pi_i)$  for  $i \in \{0, \dots, m\}$ , and  $\tau_{m+1} \coloneqq \infty$ , and similarly for  $\tau_i^n$ ,  $n \in \mathbb{N}$ . We show that  $(\Pi^n, \boldsymbol{X}^n) \to (\Pi, \boldsymbol{X})$  in  $\Omega$  holds outside  $N_{\boldsymbol{x}, F^\star} \cup \bigcup_{n \in \mathbb{N}} N_{\boldsymbol{x}_n, F_n^\star}$ , using Lemma A.9. Let  $t \geq 0$  and  $r_n \leq t_n \leq s_n$  be non-negative sequences all converging to t. If t = 0 then  $t_n < \tau_1^n$  for large n so  $\Pi^n(t_n) = \Pi_0 = \Pi(0)$ , and  $\boldsymbol{X}^n(t_n) \to \boldsymbol{x} = \boldsymbol{X}(0)$ . Now suppose t > 0. By construction, the only discontinuities of  $(\Pi, \boldsymbol{X})$  are the jump times of  $(\Pi_t)$ . If t is no such jump time, say  $\tau_i < t < \tau_{i+1}$  for some  $i \in \{0, \dots, m\}$ , then for large n also

 $\tau_i^n < t_n < \tau_{i+1}^n$ , so  $\Pi^n(t_n) = \Pi_i = \Pi(t)$ . If t is a jump time, say  $t = \tau_i$  for some  $i \in [m]$ , then  $\tau_{i-1}^n < t_n < \tau_{i+1}^n$  for sufficiently large n so

$$\Pi^n(t_n) \in \{\Pi_{i-1}, \Pi_i\} = \{\Pi(t-), \Pi(t)\}.$$

If  $\Pi^n(t_n) \to \Pi_i$ , that is  $\Pi^n(t_n) = \Pi_i$  for all sufficiently large n, then  $\Pi^n(s_n) \leq \Pi^n(t_n) = \Pi_i$ , but also  $\Pi^n(s_n) \in \{\Pi_{i-1}, \Pi_i\}$  for large n, so  $\Pi^n(s_n) = \Pi_i$ . Similarly if  $\Pi^n(t_n) \to \Pi_{i-1}$  then  $\Pi^n(r_n) \to \Pi_{i-1}$ . To finish confirming the conditions of Lemma A.9 in the t > 0 case, it remains to show that whenever  $t_n \to t > 0$  and  $\Pi^n(t_n) = \Pi(t) =: \Pi$  for all  $n \in \mathbb{N}$ , then for any  $u \in \Pi$  we have  $|X^n(t_n)_u - X(t)_u| \to 0$ . If  $\operatorname{pr}_F(u) = \emptyset$ , then

$$|\boldsymbol{X}^{n}(t_{n})_{u} - \boldsymbol{X}(t)_{u}| \leq |(\boldsymbol{x}_{n}\xi^{n})_{u} - (\boldsymbol{x}\xi)_{u}| + \left|W_{t_{n}-\tau_{u}^{n}}^{u} - W_{t-\tau_{u}}^{u}\right| \stackrel{n \to \infty}{\longrightarrow} 0,$$

and similarly if  $\operatorname{pr}_F(u) \neq \emptyset$ .

We have proved that whenever  $F_n^* \to F^*$  in  $\operatorname{dec}(\mathbb{F})$ , and  $\boldsymbol{x}_n \to \boldsymbol{x}$  in  $E_{\circ}^{\operatorname{lf}(F)}$ , then  $K_{\boldsymbol{x}_n}(F_n^*,\cdot) \to K_{\boldsymbol{x}}(F^*,\cdot)$  weakly. Now fix arbitrary choices  $(\boldsymbol{x}_F \in E_{\circ}^{\operatorname{lf}(F)})_{F \in \mathbb{F}}$ , and put for  $F \in \mathbb{F}$  and  $\boldsymbol{x} \in \mathcal{X}$  with  $\boldsymbol{x} \notin E_{\circ}^{\operatorname{lf}(F)}$ ,

$$K_{\boldsymbol{x}}(F^{\star},\cdot) := K_{\boldsymbol{x}_{F}}(F^{\star},\cdot).$$

This defines  $K_{\boldsymbol{x}}(F,\cdot)$  for any  $\boldsymbol{x}\in\mathcal{X}$  and  $F\in\mathbb{F}$ . Now let  $F_n^\star\to F^\star$  in  $\operatorname{dec}(\mathbb{F})$ , and  $\boldsymbol{x}_n\to\boldsymbol{x}$  in  $\mathcal{X}$ . If  $\boldsymbol{x}\in E_\circ^{\operatorname{lf}(F)}$ , then for large n we must have  $\operatorname{dom}(\boldsymbol{x}_n)=\operatorname{dom}(\boldsymbol{x})=\operatorname{lf}(F)$  and  $F_n=F$ , so also  $\boldsymbol{x}_n\in E_\circ^{\operatorname{lf}(F_n)}$ , in which case  $K_{\boldsymbol{x}_n}(F_n^\star,\cdot)\to K_{\boldsymbol{x}}(F^\star,\cdot)$  is already proved. If  $\boldsymbol{x}\not\in E_\circ^{\operatorname{lf}(F)}$ , then by the same argument also  $\boldsymbol{x}_n\not\in E_\circ^{\operatorname{lf}(F_n)}$  and  $F_n=F$  for large n, so

$$K_{\boldsymbol{x}_n}(F_n^\star,\cdot)=K_{\boldsymbol{x}_{F_n}}(F_n^\star,\cdot)=K_{\boldsymbol{x}_F}(F_n^\star,\cdot)\overset{n\to\infty}{\longrightarrow}K_{\boldsymbol{x}_F}(F^\star,\cdot)=K_{\boldsymbol{x}}(F^\star,\cdot),$$

by what we have already proved, since  $\boldsymbol{x}_F \in E_{\circ}^{\mathrm{lf}(F)} = E_{\circ}^{\mathrm{lf}(F_n)}$  for large n.

It remains to show that  $(\boldsymbol{x}, F^\star) \mapsto K_{\boldsymbol{x}}(F^\star, A)$  is measurable for measurable  $A \subset \Omega$ . The set of such A is a  $\lambda$ -system, so by the  $\pi$ - $\lambda$  lemma it suffices to show the claim for A taken from a  $\pi$ -system that generates the  $\sigma$ -algebra on  $\Omega$ , such as the family of closed sets. By what we have already showed and the Portmanteau theorem, if  $A \subset \Omega$  is closed, and  $\boldsymbol{x}_n \to \boldsymbol{x}$  and  $F_n^\star \to F^\star$ ,

$$\overline{\lim_{n\to\infty}} K_{\boldsymbol{x}_n}(F_n^{\star}, A) \le K_{\boldsymbol{x}}(F^{\star}, A),$$

which implies that  $(\boldsymbol{x}, F^{\star}) \mapsto K_{\boldsymbol{x}}(F^{\star}, A)$  is upper semi-continuous and hence measurable.

**Lemma A.9** (Proposition 3.6.5 in [20]). Let (M,d) be a metric space. Then  $y_n \longrightarrow y$  in  $D([0,\infty),M)$  with respect to the Skorokhod topology if and only if, whenever  $t_n \to 0$  then  $y_n(t_n) \to y(0)$ , and when  $t_n \to t > 0$ , then the following hold.

- (i)  $d(y_n(t_n), y(t)) \wedge d(y_n(t_n), y(t-)) \to 0$
- (ii) If  $d(y_n(t_n), y(t)) \to 0$  and  $s_n \ge t_n$ ,  $s_n \to t$ , then also  $d(y_n(s_n), y(t)) \to 0$ ,
- (iii) If  $d(y_n(t_n), y(t-)) \to 0$  and  $s_n \le t_n$ ,  $s_n \to t$ , then also  $d(y_n(s_n), y(t-)) \to 0$ .

If y is continuous at t, then (i)-(iii) reduce to  $y_n(t_n) \to y(t)$ .

**Lemma A.10.** Suppose  $h: (\mathbb{Q} \cap [0,\infty)) \times A \times A \to \mathbb{R}$  is a function for a set A, and that there is a family of sets  $(A_x \subset A)_{x \in A}$  such that  $A_x \cap A_{x'} \neq \emptyset$  for all  $x, x' \in A$ , and for every  $s, t, \in \mathbb{Q} \cap [0,\infty)$ ,  $x, y \in A$ ,  $z \in A_x$ ,

$$g(t+s,x,y) = g(s,x,z) + g(t,z,y).$$

Suppose further that g(0, x, y) = b(x) - b(y) for some function  $b: A \to \mathbb{R}$ . Then there is  $\beta \in \mathbb{R}$  such that for all  $t \in \mathbb{Q} \cap [0, \infty)$ ,  $x \in A$  and  $z \in A_x$ ,

$$g(t, x, z) = \beta t + b(x) - b(z).$$

*Proof.* Fix  $x \in A$ . Then for  $t, s \in \mathbb{Q} \cap [0, \infty)$  and  $z \in A_x$  we have g(t + s, x, z) = g(s, x, z) + g(t, z, z), and thus

$$g(t, z, z) + g(s, z, z) = \left[g(t, x, z) - g(0, x, z)\right] + \left[g(t + s, x, z) - g(t, x, z)\right] = g(t + s, z, z),$$

for any  $t, s, \in \mathbb{Q} \cap [0, \infty)$  and  $z \in A_x$ . This implies that there exists  $\beta(z) \in \mathbb{R}$  such that  $g(t, z, z) = \beta(z)t$  for  $t \in \mathbb{Q} \cap [0, \infty)$ . Then for all  $z \in A_x$  and  $t \in \mathbb{Q} \cap [0, \infty)$ ,

$$g(t, x, z) = g(0, x, z) + g(t, z, z) = b(x) - b(z) + \beta(z)t.$$
(A.3)

We show that  $\beta(z) = \beta(z')$  for all  $z, z' \in \bigcup_{x \in A} A_x$ . By assumption and (A.3), for every  $z \in A_x$  we have  $g(t, x, x) = g(t, x, z) + g(0, z, x) = \beta(z)t$ , in particular  $\beta(z) = \beta(z') =: \beta(x)$  for all  $z, z' \in A_x$ .

Finally, if  $x, x' \in A$  then there is  $z \in A_x \cap A_{x'}$ , so  $\beta(x) = \beta(z) = \beta(x')$ , so in fact there is one  $\beta \in \mathbb{R}$  such that  $\beta(z) = \beta$  for all  $z \in \bigcup_{x \in A} A_x$ .

### APPENDIX B. PROOF OF LEMMA 2.12

Throughout this section, fix  $\boldsymbol{\nu} \in R$  with  $\boldsymbol{\nu}(\mathrm{d}\xi) = \boldsymbol{\lambda} \, \mathrm{d}\xi$  for some  $\boldsymbol{\lambda} \in R^{\circ}$ . In that case,  $N_{\boldsymbol{\nu}}^F$  for  $F \in \mathbb{F}$  is a map  $E_{\circ}^{\mathrm{lf}(F)} \to (0, \infty)$  defined by

$$N_{\boldsymbol{\nu}}^F(\boldsymbol{x}) = \int_{\mathrm{tm}(F)} \int_{\mathrm{sp}(F)} f_{\mathrm{tm}}^{\boldsymbol{\lambda}}(\tau) f_{\mathrm{sp}}(\xi \mid \tau, \boldsymbol{x}) \,\mathrm{d}\xi \,\mathrm{d}\tau.$$

The goal of this section is to show that  $N_{\nu}^{F}$  is (finite and) continuous.

**Lemma B.1.** Let  $F \in \mathbb{F}$ . For fixed  $\tau \in \text{tm}(F)$ , the map

$$E_{\circ}^{\mathrm{lf}(F)} \to (0, \infty); \quad \boldsymbol{x} \mapsto \int_{\mathrm{sp}(F)} f_{\mathrm{sp}}(\xi \mid \tau, \boldsymbol{x}) \,\mathrm{d}\xi$$

is continuous.

*Proof.* For fixed  $\tau \in \text{tm}(F)$ ,  $f_{\text{sp}}(\xi \mid \tau, \boldsymbol{x})$  is continuous in  $\boldsymbol{x}$  for every  $\xi \in \text{sp}(F)$ , and bounded uniformly in  $\boldsymbol{x}$  and  $\xi$ , so the statement follows from the dominated (or bounded) convergence theorem.

Let  $\Pi \in \mathcal{P}$ . Then  $E_{\circ}^{\Pi}$  is an open subset of  $E^{\Pi}$ , so the distance between some  $\boldsymbol{x} \in E^{\Pi}$  to  $E^{\Pi} \setminus E_{\circ}^{\Pi}$  is well-defined and denoted  $\varepsilon_d(\boldsymbol{x})$ . More explicitly,

$$\varepsilon_d(\boldsymbol{x}) = \begin{cases} \min_{u \neq v \in \Pi} \rho(\boldsymbol{x}_u, \boldsymbol{x}_v), & d \ge 2, \\ \max_{u_0, v_0 \in \Pi} \min_{u \neq v \in \Pi, \{u, v\} \neq \{u_0, v_0\}} \rho(\boldsymbol{x}_u, \boldsymbol{x}_v), & d = 1, \end{cases}$$
(B.1)

where  $\rho$  denotes the Euclidean metric on E, see (2.1). If  $d \geq 2$  then  $\varepsilon_d(x)$  is the smallest, if d = 1 it is the second smallest distance among all pairs of two particles.

**Lemma B.2.** Let  $F \in \mathbb{F}$ . Then there exists  $C = C(F, \lambda, d) > 0$  such that, for any  $A \subset E_{\circ}^{\mathrm{lf}(F)}$ ,

$$\int_{\operatorname{tm}(F)} f_{\operatorname{tm}}^{\lambda}(\tau) \sup_{\boldsymbol{x} \in A} \left( \int_{\operatorname{sp}(F)} f_{\operatorname{sp}}(\xi \mid \tau, \boldsymbol{x}) \, \mathrm{d}\xi \right) \mathrm{d}\tau \le C \left[ 1 + \left( \inf_{\boldsymbol{x} \in A} \varepsilon_d(\boldsymbol{x}) \right)^{-C} \right]. \tag{B.2}$$

In particular,  $N_{\nu}^F \colon E_{\circ}^{\mathrm{lf}(F)} \to (0, \infty)$  is continuous and, specialising A in (B.2) to a singleton,

$$N_{\boldsymbol{\nu}}^F(\boldsymbol{x}) \leq C \left(1 + \varepsilon_d(\boldsymbol{x})^{-C}\right), \qquad \boldsymbol{x} \in E_{\circ}^{\mathrm{lf}(F)}$$

The continuity of  $N^F_{\nu}$  follows from (B.2) and Lemma B.1 by the dominated convergence theorem. We do not prove (B.2) directly. Instead, we prove an analogous statement when the torus is replaced by  $\mathbb{R}^d$ , where a certain heat kernel trick can be used that isn't available on the torus. Write  $\tilde{E} = \mathbb{R}^d$ , and denote by  $\tilde{p}_t \colon \mathbb{R}^d \to (0, \infty)$  for t > 0 the heat kernel, defined by

$$\widetilde{p}_t(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right), \quad x \in \mathbb{R}^d.$$

Define  $\widetilde{E}_{\circ}^{\Pi}$  for  $\Pi \in \mathcal{P}$  analogously to  $E_{\circ}^{\Pi}$ , and for a forest  $F \in \mathbb{F}$ , denote by  $\widetilde{\operatorname{sp}}(F)$  the set of maps  $\xi \colon \operatorname{nd}^{\circ}(F) \to \widetilde{E}$  such that  $\xi \big|_{\Pi} \in \widetilde{E}_{\circ}^{\Pi}$  for all  $\Pi \in F$ . For  $\xi \in \widetilde{\operatorname{sp}}(F)$ ,  $\tau \in \operatorname{tm}(F)$  and  $x \in \widetilde{E}_{\circ}^{\operatorname{lf}(F)}$ , write

$$\widetilde{f_{\rm sp}}(\xi \mid \tau, \boldsymbol{x}) \coloneqq \prod_{u \in {\rm nd}(F) \backslash {\rm rt}(F)} \widetilde{p}(\tau_{{\rm pr}_F(u)} - \tau_u, (\boldsymbol{x}\xi)_u - \xi_{{\rm pr}_F(u)}).$$

For every  $x,y \in E$  there exists a  $\vec{k}(x,y) \in \{-1,0\}$  (not necessarily unique) such that  $\rho(x,y) = |x-y+\vec{k}(x,y)|$  (on the left is the Euclidean metric on the torus E, on the right is the Euclidean norm on  $\mathbb{R}^d$ ). Here and in the following, if we treat an element  $x \in E$  as an element of  $\mathbb{R}^d$ , then we identify it with its unique representative in  $[0,1)^d$ . Then, for some C = C(d) > 0,

$$p_{t}(x-y) \leq C(1+\sqrt{t})^{d} \widetilde{p}_{t}(x-y+\vec{k}(x,y))$$

$$\leq C(1+\sqrt{t})^{d} \sum_{\vec{k}\in\{-1,0,1\}^{d}} \widetilde{p}_{t}(x-y+\vec{k}), \tag{B.3}$$

where the first inequality follows e.g. from [3, Thm. 8].

**Lemma B.3.** For every  $F = (\Pi_0^F, \dots, \Pi_m^F) \in \mathbb{F}$  there is C = C(d, F) > 0 such that for every  $\tau \in \operatorname{tm}(F)$  and  $\mathbf{x} \in E_{\circ}^{\operatorname{lf}(F)}$ ,

$$f_{\rm sp}(\xi \mid \tau, \boldsymbol{x}) \le C \left(1 + \sqrt{\max \tau}\right)^C \sum_{\vec{\boldsymbol{k}} : \ \operatorname{hd}^{\circ}(F) \to \{-m, \dots, m\}^d} \widetilde{\boldsymbol{k}} \cdot \operatorname{hf}(F) \to \{-m, \dots, m\}^d$$
(B.4)

for all  $\xi \in \operatorname{sp}(F)$ , and

$$\int_{\operatorname{sp}(F)} f_{\operatorname{sp}}(\xi \mid \tau, \boldsymbol{x}) \, \mathrm{d}\xi \le C (1 + \sqrt{\max \tau})^C \sum_{\vec{\boldsymbol{k}} : \operatorname{lf}(F) \to \{-m, \dots, m\}^d} \int_{\widetilde{\operatorname{sp}}(F)} \widetilde{f_{\operatorname{sp}}}(\xi \mid \tau, \boldsymbol{x} + \vec{\boldsymbol{k}}) \, \mathrm{d}\xi. \tag{B.5}$$

*Proof.* Denote by C > 0 a constant depending only on d and F whose value may increase from line to line. Let  $v \in \text{rt}(F)$  with  $\text{ch}_F(v) \neq \emptyset$ , and suppose for simplicity that  $\text{ch}_F(u) \neq \emptyset$  for all  $u \in \text{ch}_F(v)$ . Then, using (B.3),

$$\prod_{u \in \operatorname{ch}_{F}(v)} p_{\tau_{v} - \tau_{u}}(\xi_{u} - \xi_{v}) \leq \prod_{u \in \operatorname{ch}_{F}(v)} \left( C(1 + \sqrt{\tau_{v} - \tau_{u}})^{d} \sum_{\vec{k}_{u} \in \{-1, 0, 1\}^{d}} \widetilde{p}_{\tau_{v} - \tau_{u}}(\xi_{u} - \xi_{v} + \vec{k}_{u}) \right) \\
\leq C(1 + \sqrt{\max \tau})^{C} \prod_{u \in \operatorname{ch}_{F}(v)} \sum_{\vec{k}_{u} \in \{-1, 0, 1\}^{d}} \widetilde{p}_{\tau_{v} - \tau_{u}}[(\xi_{u} + \vec{k}_{u}) - \xi_{v}].$$

We put  $\vec{\ell}_u := \vec{k}_u$  for  $u \in \operatorname{ch}_F(v)$ . Now let  $u \in \operatorname{ch}_F(v)$  and suppose again for simplicity that  $\operatorname{ch}_F(w) \neq \emptyset$  for all  $w \in \operatorname{ch}_F(u)$ . Then

$$\prod_{w \in \text{ch}_{F}(u)} p_{\tau_{u} - \tau_{w}}(\xi_{w} - \xi_{u})$$

$$= \prod_{w \in \text{ch}_{F}(u)} p_{\tau_{u} - \tau_{w}}[\xi_{w} + \vec{\ell}_{u} - (\xi_{u} + \vec{\ell}_{u})]$$

$$\leq C(1 + \sqrt{\max \tau})^{C} \prod_{w \in \text{ch}_{F}(u)} \sum_{\vec{k}_{w} \in \{-1, 0, 1\}^{d}} \widetilde{p}_{\tau_{u} - \tau_{w}}[(\xi_{w} + \vec{k}_{w} + \vec{\ell}_{u}) - (\xi_{u} + \vec{\ell}_{u})].$$

We put  $\vec{\ell}_w := \vec{k}_w + \vec{\ell}_u$  for  $w \in \operatorname{ch}_F(u)$ . Proceeding inductively in this way down towards the leaves yields (B.4), from which the second claim easily follows (note that the integral on the left-hand side is w.r.t. Lebesgue measure on the torus, and on the right-hand side w.r.t. Lebesgue measure on all of  $\mathbb{R}^d$ ).

This reduction lets us make use of the following statement. Define  $\widetilde{\epsilon_d}(\boldsymbol{x})$  for  $\boldsymbol{x} \in \widetilde{E}^{\Pi}$  for some  $\Pi \in \mathcal{P}$  analogously to  $\varepsilon_d$ , that is by replacing  $\rho(\cdot, \cdot)$  in (B.1) by the Euclidean distance  $|\cdot - \cdot|$  on  $\mathbb{R}^d$ .

**Lemma B.4.** Let  $F \in \mathbb{F}$  and K > 0. Then there exists  $C = C(F, \lambda, d, K) > 0$  such that, for any  $A \subset \widetilde{E}^{\mathrm{lf}(F)}_{\circ}$ ,

$$\int_{\operatorname{tm}(F)} f_{\operatorname{tm}}^{\lambda}(\tau) (1 + \sqrt{\max \tau})^K \sup_{\boldsymbol{x} \in A} \left( \int_{\widetilde{\operatorname{sp}}(F)} \widetilde{f}_{\operatorname{sp}}(\xi \mid \tau, \boldsymbol{x}) \, \mathrm{d}\xi \right) \mathrm{d}\tau \le C \left[ 1 + \left( \inf_{\boldsymbol{x} \in A} \widetilde{\varepsilon}_d(\boldsymbol{x}) \right)^{-C} \right].$$

The proof of Lemma B.4 hinges on an explicit representation of  $\int \widetilde{f_{\rm sp}}(\xi \mid \tau, \boldsymbol{x}) d\xi$  based on the following heat kernel trick.

**Lemma B.5.** Let  $x_1, \ldots, x_n \in \mathbb{R}^d$  and  $s_1, \ldots, s_n > 0$ . Then,

$$\prod_{i=1}^{n} \widetilde{p}_{s_i}(x_i - z) = \frac{\widetilde{p}_s(z - \overline{x})}{\widetilde{p}_s(0)} \prod_{i=1}^{n} \widetilde{p}_{s_i}(x_i - \overline{x}),$$

where  $\frac{1}{s} = \sum_{i=1}^{n} \frac{1}{s_i}$  and  $\overline{x} = \sum_{i=1}^{n} \frac{s}{s_i} x_i$ .

*Proof.* By definition of the heat kernel,

$$\prod_{i=1}^{n} \widetilde{p}_{s_i}(x_i - z) = \left(\prod_{i=1}^{n} (2\pi s_i)^{-d/2}\right) \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \frac{(x_i - z)^2}{s_i}\right),\tag{B.6}$$

and

$$\sum_{i=1}^{n} \frac{(x_i - z)^2}{s_i} = \frac{z^2}{s} - 2\frac{\overline{x}z}{s} + \sum_{i=1}^{n} \frac{x_i^2}{s_i} = \frac{1}{s}(z - \overline{x})^2 - \frac{1}{s}\overline{x}^2 + \sum_{i=1}^{n} \frac{x_i^2}{s_i}$$

$$= \frac{1}{s}(z - \overline{x})^2 + \sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{s_i},$$
(B.7)

where in the last step we used that

$$\sum_{i=1}^{n} \frac{(x_i - \overline{x})^2}{s_i} = \sum_{i=1}^{n} \frac{x_i^2}{s_i} - 2\overline{x} \sum_{i=1}^{n} \frac{x_i}{s_i} + \frac{1}{s} \overline{x}^2 = \sum_{i=1}^{n} \frac{x_i^2}{s_i} - \frac{1}{s} \overline{x}^2.$$

Plugging (B.7) back into (B.6) finishes the proof.

**Lemma B.6.** For a non-trivial forest  $F \in \mathbb{F}$  and  $\mathbf{x} \in \widetilde{E}_{\circ}^{\mathrm{lf}(F)}$ ,  $\tau \in \mathrm{tm}(F)$ ,

$$\int_{\widetilde{\operatorname{sp}}(F)} \widetilde{f_{\operatorname{sp}}}(\xi \mid \tau, \boldsymbol{x}) \, \mathrm{d}\xi = \prod_{v \in \operatorname{nd}^{\circ}(F)} \frac{1}{\widetilde{p}_{r_{v}}(0)} \prod_{u \in \operatorname{ch}_{F}(v)} \widetilde{p}_{r_{u} + \tau_{v} - \tau_{u}}(\overline{x}_{v} - \overline{x}_{u}), \tag{B.8}$$

where  $\overline{x}_u = x_u$  and  $r_u = 0$  for  $u \in lf(F)$  and, inductively for  $v \in nd^{\circ}(F)$ 

$$\frac{1}{r_v} := \sum_{u \in \operatorname{ch}_F(v)} \frac{1}{r_u + \tau_v - \tau_u}, \qquad \overline{x}_v := \sum_{u \in \operatorname{ch}_F(v)} \frac{r_v}{r_u + \tau_v - \tau_u} \overline{x}_u.$$

The following hold.

(i) If  $v \in \operatorname{nd}^{\circ}(F)$  with  $\operatorname{ch}(v) \subset \operatorname{lf}(F)$ , then

$$r_v = \frac{\tau_v}{|\operatorname{ch}_F(v)|}, \qquad \overline{x}_v = \frac{1}{|\operatorname{ch}_F(v)|} \sum_{u \in \operatorname{ch}_F(v)} x_u.$$

- (ii) There is c = c(F) > 0 such that  $c\tau_v \le r_v < \tau_v$  for all  $v \in \operatorname{nd}^{\circ}(F)$ .
- (iii) If  $v \in \operatorname{nd}^{\circ}(F)$  then

$$\max_{u \in \operatorname{ch}(v)} |\overline{x}_u - \overline{x}_v| \ge \frac{1}{2} \max_{u, u' \in \operatorname{ch}(v)} |\overline{x}_u - \overline{x}_{u'}|.$$

*Proof.* Eq. (B.8) follows by applying Lemma B.5 inductively from leaves to root, which is tedious but straightforward.

- (i) is obvious from the definitions.
- (ii) If v is not a leaf and  $r_u \leq \tau_u$  holds for all children, then

$$\frac{1}{r_v} \ge \sum_{u \in \operatorname{ch}_F(v)} \frac{1}{\tau_u + \tau_v - \tau_u} = \frac{|\operatorname{ch}_F(v)|}{\tau_v} \ge \frac{2}{\tau_v} > \frac{1}{\tau_v}.$$

If v is not a leaf and  $r_u \geq c\tau_u$  holds for all children, then

$$\frac{1}{r_v} \le \sum_{u \in \text{ch}_F(v)} \frac{1}{c\tau_u + \tau_v - \tau_u} \le \sum_{u \in \text{ch}_F(v)} \frac{1}{c(\tau_u + \tau_v - \tau_u)} = \frac{|\text{ch}_F(v)|}{c} \frac{1}{\tau_v}.$$

(iii) If  $v \in \operatorname{nd}^{\circ}(F)$  and  $u_1, u_2 \in \operatorname{ch}(v)$ , then

$$|\overline{x}_{u_1} - \overline{x}_{u_2}| \le |\overline{x}_{u_1} - \overline{x}_v| + |\overline{x}_v - \overline{x}_{u_2}| \le 2 \max_{u \in \operatorname{ch}(v)} |\overline{x}_u - \overline{x}_v|.$$

Proof of Lemma B.4. Fix a non-trivial forest  $F=(\Pi_0^F,\ldots,\Pi_m^F)\in\mathbb{F}$ , and abbreviate  $\mathrm{ch}=$  $\operatorname{ch}_F$ ,  $\operatorname{pr} = \operatorname{pr}_F$  and, for  $\tau \in \operatorname{tm}(F)$  and  $i \in [m]$ ,  $\tau_i = \tau_{\Pi_i^F}$ . Throughout the proof, c, C > 0 are constants that only depend on F, d,  $\lambda$ , and K, and whose value may respectively decrease and increase from line to line. First let  $d \geq 2$ , then for  $\boldsymbol{x} \in E_{\circ}^{\mathrm{lf}(F)}$  and  $\tau \in \mathrm{tm}(F)$ , using (B.8),

$$\int \widetilde{f_{\rm sp}}(\xi \mid \tau, \boldsymbol{x}) \, \mathrm{d}\xi \le C \prod_{i=1}^{m} \prod_{v \in \Pi_{i}^{F} \setminus \Pi_{i-1}^{F}} r_{v}^{d/2} \prod_{u \in \operatorname{ch}(v)} \widetilde{p}_{r_{u} + \tau_{v} - \tau_{u}} (\overline{x}_{v} - \overline{x}_{u}). \tag{B.9}$$

Denote  $G^{(i)}(\tau, \boldsymbol{x}) \coloneqq \prod_{j=i+1}^m G_j(\tau, \boldsymbol{x})$ , and  $f_{\operatorname{tm}}^{\boldsymbol{\lambda}}{}^{(i)}(\tau) \coloneqq \prod_{j=i+1}^m \lambda_{\Pi_{j-1}^F, \Pi_i^F} e^{-\lambda_{\Pi_j^F}(\tau_j - \tau_{j-1})}$  for  $i \in \{0, ..., m\}$ , so  $f_{\text{tm}}^{\lambda}(0) = f_{\text{tm}}^{\lambda}$  and  $f_{\text{tm}}^{\lambda}(0) \equiv 1$ . We show inductively that for  $i \in \{1, ..., m\}$  there exists  $b_i \geq 0$  such that

$$\int f_{\mathrm{tm}}^{\boldsymbol{\lambda}}(\tau)(1+\sqrt{\tau_m})^K \sup_{\boldsymbol{x}\in A} G^{(i)}(\tau,\boldsymbol{x}) \,\mathrm{d}\tau_{i+1} \dots \,\mathrm{d}\tau_m \le C\tau_1^{-b_i}(1+\tau_i)^C. \tag{B.10}$$

For i=m the LHS is equal to  $(1+\sqrt{\tau_m})^K \leq 2^K(1+\tau_m)^K$  and we can choose  $b_m=0$ . Suppose we have proved the claim for some fixed  $i \in \{2, ..., m\}$ , then by induction hypothesis

$$\int f_{\text{tm}}^{\boldsymbol{\lambda}} {}^{(i-1)}(\tau) (1 + \sqrt{\tau_m})^K \sup_{\boldsymbol{x} \in A} G^{(i-1)}(\tau, \boldsymbol{x}) \, d\tau_i \dots d\tau_m 
\leq C \tau_1^{-b_i} \int_{\tau_{i-1}}^{\infty} d\tau_i \, (1 + \tau_i)^C e^{-\lambda_i (\tau_i - \tau_{i-1})} \sup_{\boldsymbol{x} \in A} \prod_{v \in \Pi_i^F \setminus \Pi_{i-1}^F} r_v^{d/2} \prod_{u \in \text{ch}(v)} \widetilde{p}_{r_u + \tau_v - \tau_u} (\overline{x}_v - \overline{x}_u),$$

where we abbreviated  $\lambda_i := \lambda_{\Pi_i^F}$ . We bound  $r_v \leq \tau_v = \tau_i$ , and  $\widetilde{p}_{r_u + \tau_v - \tau_u}(\overline{x}_v - \overline{x}_u) \leq$  $C(r_u + \tau_v - \tau_u)^{-d/2}$ , and by Lemma B.6,

$$r_u + \tau_v - \tau_u \ge c\tau_u + \tau_v - \tau_u \ge c\tau_v \ge c\tau_1. \tag{B.11}$$

Let l be the total number of children of nodes  $v \in \Pi_i^F \setminus \Pi_{i-1}^F$ . Then our bound is no larger than  $\tau_1^{-b_i}$  times

$$C \int_{\tau_{i-1}}^{\infty} d\tau_i (1+\tau_i)^C e^{-\lambda(\tau_i-\tau_{i-1})} \tau_i^{|\Pi_i^F \setminus \Pi_{i-1}^F|d/2} (c\tau_1)^{-ld/2}$$

$$\leq C\tau_1^{-ld/2} \int_0^{\infty} e^{-\lambda_i s} (1+\tau_{i-1}+s)^C ds$$

$$\leq C\tau_1^{-ld/2} (1+\tau_{i-1})^C,$$

which completes the inductive step. Using (B.10) with i=1, recalling that  $r_u=\tau_u=\tau_1$  for  $u\in lf(F)$ , and putting  $s:=\tau_1,b:=b_1,\lambda:=\lambda_1$ , (B.9) becomes

$$\int f_{\text{tm}}^{\lambda}(\tau) (1 + \sqrt{\tau_m})^K \sup_{\boldsymbol{x} \in A} \int \widetilde{f_{\text{sp}}}(\xi \mid \tau, \boldsymbol{x}) \, d\xi \, d\tau 
\leq C \int_0^{\infty} s^{-b} (1 + s)^C e^{-\lambda s} \sup_{\boldsymbol{x} \in A} \prod_{v \in \Pi_1^F \setminus \Pi_0^F} s^{d/2} \prod_{u \in \text{ch}(v)} \widetilde{p}_{r_u + s_v - s_u} (\overline{x}_v - \overline{x}_u) \, ds 
\leq C \int_0^{\infty} s^{-b} (1 + s)^C e^{-\lambda s} \sup_{\boldsymbol{x} \in A} \prod_{v \in \Pi_1^F \setminus \Pi_0^F} \prod_{u \in \text{ch}(v)} \widetilde{p}_s (\overline{x}_v - \overline{x}_u) \, ds.$$

Fix any  $v_0 \in \Pi_1^F \setminus \Pi_0^F$  and pick, for fixed  $\boldsymbol{x} \in A$ ,  $u_0 \in \operatorname{ch}(v)$  for which  $|\overline{x}_{v_0} - \boldsymbol{x}_{u_0}|$  is largest, so that  $|\boldsymbol{x}_u - \boldsymbol{x}_{u'}| \leq 2|\boldsymbol{x}_{u_0} - \overline{x}_{v_0}|$  for all  $u, u' \in \operatorname{ch}(v)$ , in particular  $2|\boldsymbol{x}_{u_0} - \overline{x}_{v_0}| \geq \inf_{\boldsymbol{x} \in A} \min_{u \neq v} |\boldsymbol{x}_u - \boldsymbol{x}_v| =: D$ . Bound  $\widetilde{p}_s(\overline{x}_v - \overline{x}_u) \leq C s^{-d/2}$  if  $(u, v) \neq (u_0, v_0)$ , so we can further bound by

$$C \int_{0}^{\infty} s^{-b'} (1+s)^{C} e^{-\lambda s} e^{-cD^{2}/s} ds \le C \left(1 + D^{-2b'}\right), \tag{B.12}$$

for some b' > b, using Lemma B.8.

Now let d=1. The approach above fails when the first transition  $\Pi_0^F \to \Pi_1^F$  is a binary merger of two identical particles, say u and u' with parent  $v=u\cup u'$ , in which case  $\overline{x}_v=\overline{x}_u=\overline{x}_{u'}=x_u$ , so  $\widetilde{p}_{\tau_1}(\overline{x}_v-\overline{x}_u)=\widetilde{p}_{\tau_1}(\overline{x}_v-\overline{x}_{u'})=(2\pi\tau_1)^{-d/2}$ , and the final integral (of the form (B.12) but without the factor  $e^{-cD^2/s}$ ) diverges. We split A into a finite number of sets according to which pair of leaves are closest to each other (and potentially identical):

$$A_{uv} \coloneqq \Big\{ oldsymbol{x} \in A \colon |oldsymbol{x}_u - oldsymbol{x}_v| = \min_{\substack{u',v' \in \operatorname{lf}(F) \ \operatorname{distinct}}} |oldsymbol{x}_{u'} - oldsymbol{x}_{v'}| \Big\},$$

for distinct  $u, v \in lf(F)$ . There might be overlap due to ties, but it will always be true that in  $A_{uv}$  all particle locations are distinct except possibly  $\mathbf{x}_u = \mathbf{x}_v$ , and

$$\forall \boldsymbol{x} \in A_{uv} : \widetilde{\varepsilon_1}(\boldsymbol{x}) = \min_{\substack{u' \neq v' \in \mathrm{lf}(F) \\ \{u',v'\} \neq \{u,v\}}} |\boldsymbol{x}_{u'} - \boldsymbol{x}_{v'}|.$$

Since we can bound  $\sup_{\boldsymbol{x}\in A} \leq \sum_{u\neq v} \sup_{\boldsymbol{x}\in A_{uv}}$ , it suffices to restrict our attention to  $A_{u_0v_0}$  for fixed but arbitrary, distinct  $u_0, v_0 \in \operatorname{lf}(F)$ . With a slight abuse of notation we denote  $A = A_{u_0v_0}$  from now. If the first merge event is not a pure binary merge of  $u_0$  and  $v_0$ , that is  $\Pi_1^F \setminus \Pi_0^F \neq \{u_0 \cup v_0\}$ , then we can copy the proof of the  $d \geq 2$  case, and in the final step

we choose  $v \in \Pi_1^F \setminus \Pi_0^F$  with  $v \neq u_0 \cup v_0$ , so there are  $u, u' \in \operatorname{ch}(v)$  with  $\{u, u'\} \neq \{u_0, v_0\}$  and thus  $|\boldsymbol{x}_u - \boldsymbol{x}_{u'}| \geq \widetilde{\varepsilon_1}(\boldsymbol{x})$  for all  $\boldsymbol{x} \in A$ . Now suppose the first event is a binary merge of  $u_0$  and  $v_0$ , that is  $\Pi_1^F \setminus \Pi_0^F$  is a singleton set containing their parent  $w_0 := u_0 \cup v_0$ , so

$$G_1(\tau, \mathbf{x}) = (\tau_1/2)^{1/2} \widetilde{p}_{\tau_1}(\mathbf{x}_{u_0} - \overline{x}_{w_0}) \widetilde{p}_{\tau_1}(\mathbf{x}_{v_0} - \overline{x}_{w_0}) \le C\tau_1^{-1/2}$$
(B.13)

for all  $x \in A$ . Then as before, we can prove inductively that for  $i \in \{2, ..., m\}$  there is  $b_i \geq 0$  with

$$\int f_{\text{tm}}^{\lambda}(\tau)(1+\sqrt{\tau_m})^K \sup_{x\in A} G^{(i)}(\tau,x) \, d\tau_{i+1} \dots d\tau_m \le C\tau_2^{-b_i}(1+\tau_i)^C, \tag{B.14}$$

where we only need to replace the lower bound  $c\tau_1$  by  $c\tau_2$  in (B.11). Let  $s := \tau_1$  and  $r := \tau_2 - \tau_1$ , then plugging (B.13) and (B.14) into (B.9) gives

$$\int f_{\text{tm}}^{\lambda}(\tau) \sup_{\boldsymbol{x} \in A} \int \widetilde{f_{\text{sp}}}(\xi \mid \tau, \boldsymbol{x}) \, d\xi \, d\tau 
\leq C \int_{0}^{\infty} ds \, e^{-\lambda_{1} s} s^{-1/2} \int_{0}^{\infty} dr \, (s+r)^{-b_{2}} (1+s+r)^{C} e^{-\lambda_{2} r} \sup_{\boldsymbol{x} \in A} G_{2}(\tau, \boldsymbol{x}).$$
(B.15)

If the second merge involves at least two leaves, say  $w \in \Pi_2^F \setminus \Pi_1^F$  with  $u, v \in \operatorname{ch}(w) \cap \operatorname{lf}(F)$ , then necessarily  $\{u, v\} \cap \{u_0, v_0\} = \emptyset$ , so  $\max_{u' \in \operatorname{ch}(w)} |x_{u'} - \overline{x}_w| \ge \frac{1}{2} |x_u - x_v| \ge \widetilde{\varepsilon}_1(x)$  for all  $x \in A$  and  $\tau$ , so we can bound

$$\sup_{\boldsymbol{x} \in A} G_2(\tau, \boldsymbol{x}) \le C(1 + \tau_2)^C \tau_2^{-b_1} e^{-cD^2/\tau_2}$$

for some  $b_1 \geq 0$ , where  $D = \inf_{\boldsymbol{x} \in A} \widetilde{\varepsilon_1}(\boldsymbol{x})$ . With  $b := b_1 + b_2$  we can further bound (B.15) by

$$C \int_{0}^{\infty} ds \, e^{-\lambda_{1} s} s^{-1/2} \int_{0}^{\infty} dr \, \underbrace{(s+r)^{-b} e^{-cD^{2}/(s+r)}}_{\leq CD^{-2b} \, \forall s,r>0} (1+s+r)^{C} e^{-\lambda_{2} r}$$

$$\leq CD^{-2b} \int_{0}^{\infty} ds \, e^{-\lambda_{1} s} s^{-1/2} \int_{0}^{\infty} dr \, (1+s+r)^{C} e^{-\lambda_{2} r}$$

$$\leq CD^{-2b} \int_{0}^{\infty} e^{-\lambda_{1} s} s^{-1/2} (1+s)^{C} \, ds$$

$$\leq CD^{-2b}.$$
(B.16)

If the second merge involves at most one additional leaf, then it must be a binary merge of  $w_0$  and a leaf  $u \notin \{u_0, v_0\}$  with parent  $w_1 = u \cup w_0$ . Then  $|\boldsymbol{x}_u - \overline{x}_{w_1}| \ge \frac{1}{2}|\boldsymbol{x}_u - \overline{x}_{w_0}| \ge c|\boldsymbol{x}_u - \boldsymbol{x}_{u_0}| \ge c\varepsilon_1(\boldsymbol{x})$  for all  $\boldsymbol{x} \in A$  and  $\tau$ , where we used Lemma B.9 in the second step. Then we can bound

$$G_2(\tau, \boldsymbol{x}) = \underbrace{r_{w_1}^{1/2}}_{\leq \tau_2^{1/2}} \underbrace{\widetilde{p}_{\tau_2 - \tau_1/2}(\overline{x}_{w_0} - \overline{x}_{w_1})}_{\leq C\tau_2^{-1/2}} \underbrace{\widetilde{p}_{\tau_2}(\boldsymbol{x}_u - \overline{x}_{w_1})}_{\leq C\tau_2^{-1/2} e^{-cD^2/\tau_2}} \leq C\tau_2^{-1/2} e^{-cD^2/\tau_2},$$

and finish the proof as in (B.16).

The following lemma finishes the proof of Lemma B.2 together with (B.5) and Lemma B.4.

**Lemma B.7.** For every  $\Pi \in \mathcal{P}$ ,  $\boldsymbol{x} \in E_{\circ}^{\Pi}$ , and  $\vec{\boldsymbol{k}} \colon \Pi \to \mathbb{Z}^d$ ,

$$\widetilde{\varepsilon_d}(\boldsymbol{x} + \vec{\boldsymbol{k}}) \geq \varepsilon_d(\boldsymbol{x}).$$

*Proof.* If  $u, v \in \Pi$ , then

$$|x_u + \vec{k}_u - (x_v + \vec{k}_v)| \ge \inf\{|x_u - x_v + \vec{\ell}| : : \vec{\ell} \in \mathbb{Z}^d\} = \rho(x_u, x_v).$$

With that, the claim follows directly from the definitions of  $\varepsilon_d$  and  $\widetilde{\varepsilon_d}$ .

We finish by proving some technical lemmas used in the proof of Lemma B.4.

**Lemma B.8.** For  $a, b, \lambda, y > 0$ ,

$$\int_0^\infty e^{-\lambda s - y^2/s} (1+s)^a s^{-b} ds \le C (1+y^{-2b}),$$

where  $C = C(a, b, \lambda) > 0$ .

*Proof.* First note that

$$\sup_{s>0} \left( s^{-b} e^{-y^2/s} \right) = y^{-2b} \sup_{s>0} \left( s^{-b} e^{-1/s} \right) = Cy^{-2b},$$

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$$\int_0^\infty \mathrm{e}^{-\lambda s - y^2/s} (1+s)^a s^{-b} \, \mathrm{d}s \le 2^a \int_0^1 s^{-b} \mathrm{e}^{-y^2/s} \, \mathrm{d}s + \int_1^\infty \mathrm{e}^{-\lambda s} (1+s)^a \, \mathrm{d}s \le C y^{-2b} + C.$$

**Lemma B.9.** If  $x_1, \ldots, x_n \in \mathbb{R}^d$  and  $|x_1 - x_2| \leq |x_i - x_j|$  for all distinct  $i, j \in [n]$ , then

$$|x_i - \frac{x_1 + x_2}{2}| \ge \frac{3}{\sqrt{2}}|x_i - x_1|$$

for all  $i \in \{3, ..., n\}$ .

*Proof.* Put  $a := |x_1 - x_2|$  and  $\overline{x} := (x_1 + x_2)/2$ , and let  $i \in \{3, \dots, n\}$ . Then  $x_i \notin B(x_1, a) \cup B(x_2, a)$ , which means that  $|x_i - \overline{x}|$  is minimised if  $x_i \in B(x_1, a) \cap B(x_2, a)$ , in which case  $|x_i - x_1| = |x_i - x_2| = |x_1 - x_2| = a$ , so by Pythagoras  $|x_i - \overline{x}| = \frac{3}{\sqrt{2}}a$ .

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