Dynamic Stability and Levitation Control in Maglev Systems

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This paper explores the dynamic stability and levitation control of Magnetic Levitation (Maglev) trains, focusing on Electromagnetic Suspension (EMS) technology. Through theoretical models, a tailored experimental setup, and numerical simulations, we demonstrate effective levitation control strategies that enhance system stability. A scaled-down EMS train model was constructed to deepen our understanding about the theory behind magnetic levitation. Our study also emphasizes the practical implications of these findings and suggests directions for future research, paving the way for advancements in Maglev technology.

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I. INTRODUCTION

Magnetic levitation (or Maglev) trains are represented as floating forms of transportation that is supported by electromagnetic attraction or repulsion. In this paper, we will dive into questions concerning its properties, the forces involved to make it levitate and their current advantages and drawbacks. It will help the reader get a clearer understanding of the functioning of Electromagnetic Suspension (EMS) trains and the following concepts of classical electrodynamics: magnetic field produced by magnets, an energy approach thanks to Earnshaw's theorem and the levitation force. The critical questions addressed involve understanding the dynamics of EMS: How does it levitate and stabilize the train at both high speeds and low maintenance costs, and what are the limitations imposed by physical laws such as Earnshaw's theorem?
We first created a model of the Maglev trains to have a first insight of the project and to describe the link relationship between the weight imposed on the train and its levitation height. Then we used Biot-Savart's law to study the magnetic field produced by the magnets to induce a levitation force. Then, by using our formulas, we were able to plot various functions such as the current as a function of the mass, magnetic field lines of the track, torque vectors in the magnetic field and the potential energy.

II. TWO LEVITATION SYSTEMS

The Maglev trains rely on levitation systems that aim to eliminate all friction with the rails and offer a rapid transportation solution. There are two main systems for the Maglev train, which we will both study in this course: The Electrodynamic Suspension Train (EDS) and the Electromagnetic Suspension Train (EMS). We will study the EMS train in our report and will briefly present the EDS train.

The EDS relies on a levitation system with conductors who are exposed to time varying magnetic fields. To obtain time varying magnetic fields we have one permanent magnetic field (a magnet) and the other magnetic field is induced from the evolution of the field that happen because the magnet moves relative to a conductor. This interaction induces what is called an eddy current. An eddy current (also Foucault current) is a loop of electric current induced in a conductor because of a changing magnetic field in the conductor or due to the relative motion between the conductor and a magnetic field. The current generates a repulsive magnetic field which will repel the two objects (The rails and the train). To be powered, there is an interaction between superconducting magnets on the train with conductors on the rails so there is no need for another power source to levitate the train.

The magnets used in EDS trains are superconducting, which means that when they are cooled to less than 450 degrees Fahrenheit below zero, they can generate magnetic fields up to 10 times stronger than normal electromagnets, enough to make a train levitate.

These magnetic fields interact with coils set into the concrete walls of the Maglev guideway. The coils are made of conductive materials, like aluminum, and when a magnetic field moves past, it creates an electric current that generates another magnetic field. Three types of coils are set into the guideway at specific intervals: one creates a field that makes the train levitate about 13 cm above the guideway; a second keeps the train stable horizontally. Both coils use magnetic repulsion to keep the train car in the optimal spot; the further it gets from the center of the guideway or the closer to the bottom, the more magnetic resistance pushes it back on track.

The third set of loops is a propulsion system run by alternating current power. Here, both magnetic attraction and repulsion are used to move the train car along the guideway. [1] One Maglev train that uses the EDS system is the Japanese train " SCMaglev".

We will proceed to study the EMS train in more depth in our report.

III. EXPERIMENT OF THE ELECTROMAGNETIC SUSPENSION TRAINS

For the sake of our project and the comprehension of physical principles, we created our real life electromagnetic train. In this section, we will explain our process in realizing the train, collecting data to later be able to compare with our theoretical predictions and what we learned from the experiment.

III.1. Realization of the train

[2] For the experiment we will need the following materials:

1. Monopolar Magnetic Tape
2. 90° angle pieces
3. Wooden block
4. Cartboard
5. Neodymium Magnets

It is essential in this experiment to use monopolar magnetic tape however in hardware stores only bipolar magnetic tape is sold. Monopolar magnetic tape consists of a single magnetic pole and a single magnetic field is generated. However, in dipolar magnetic tape there are two magnetic dipoles with usually opposite polarity. But, this is not a problem though as we can easily repolarize tape the right way using a neodymium magnet.

Repolarization of a magnetic strip with a neodymium magnet involves altering the orientation of magnetic domains within the strip to align with the magnetic field of the neodymium magnet. We first had to understand the magnetic domain, meaning the initial orientation of the net magnetic field. For this, we could use a compass or directly try to make the train levitate on top of the "tracks". If it was attracted or sideways, we knew we had to repolarize them. When a neodymium magnet is brought near the magnetic strip, it generates a strong external magnetic field due to its own magnetic properties. This external magnetic field interacts with the magnetic domains in the strip. It therefore exorts a torque which allows the magnetic domains to rotate and align themselves with in the direction of the external magnetic field. We do this by rubbing the neodymium magnet along the strip and the more we do it the stronger becomes the magnetic field. Once the majority of domains are aligned, the magnetic strip becomes magnetized. It retains this magnetization even after the neodymium magnet is removed, due to the alignment of the magnetic domains.

Once the tape is repolarized and the magnets repell each other, we can proceed with setting up our train and its rails. We glue down two stripes of magnetic tape onto the extremities of one side of our wooden block. This will become our train. We then use cardboard as the base of our train rails onto which we will glue down two long strips of magnetic tape (that have also been repolarized). Then we place the two angle 90° angle pieces on the exterior sides to the train. Now comes the part where we can start testing our train and make some adjustments to our setup. At the start, our train directly shifted to the side as we attempted to make it levitate on top of the rails, so we reduced the distance between the plastic pieced. We see that the importance of these pieces is essential for the stability and guidance of the train which we will try to understand with theory. After repositioning our plastic pieces, we observe that the train levitates perfectly over the rails.

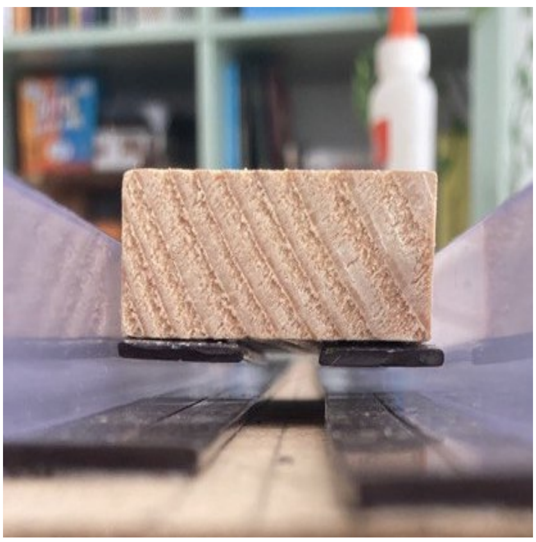

Figure 1: Experimental Maglev Train

III.2. Measurements and Analysis

In this part of our study, we explored how additional mass influences the levitation height of our Maglev train model. We will proceed to place a cardboard cup on top of our train and progressively add weight using rice to see how the weight of the train influences the height of levitation.

The scale used to measure the added mass had a precision of 1g so we consider an uncertainty $u(m) = 1g$ for our measurements. The levitation height was measured using a ruler with a resolution of 1 mm. Based on the resolution of our measuring instrument, the uncertainty in each levitation height measurement is considered to be $u(h) = 0.5$ mm. This estimate assumes a standard practice of measuring to the nearest half division of the least count of the ruler, which is a common approach in precision measurements to account for observer error.

Our experimental dataset for mass and levitation height can be found in Appendix A.

We then plot the levitation height as a function of the mass:

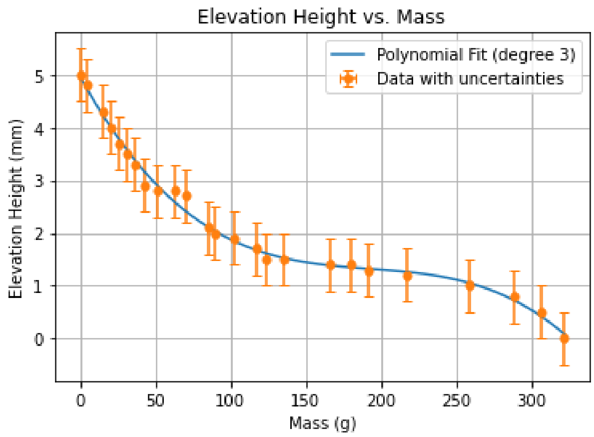

Figure 2: Experimental data with polynomial fit

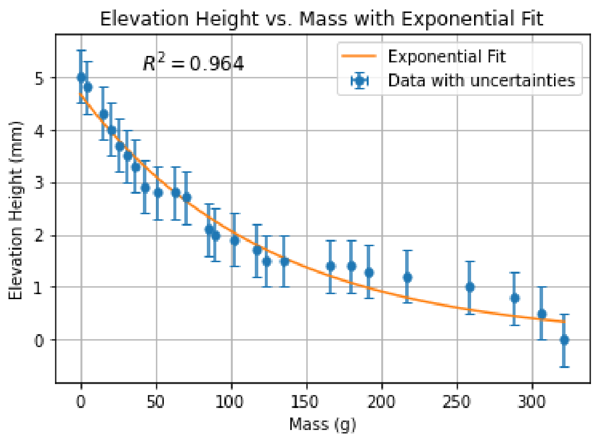

Figure 3: Experimental data with exponential fit

We analyzed the data using two models: a polynomial fit and an exponential decay model. The second fit model seems more appropriate as the coefficients obtained for the polynomial fit do not showcase any relevance. The exponential model provided a better fit, which is reflective of physical phenomena where the response variable changes proportionally with the influence of an exponentially decaying factor—in this case, the magnetic field strength with respect to mass. The appropriateness of the exponential decay model can be attributed to the magnetic field's nonlinear response to changes in distance caused by the added mass. This response is not linear due to the nature of magnetic field interactions and the physical design of the levitation system, which may include non-linear materials properties and geometrical configurations that affect field distribution.

For the exponential fit, we fit the data according the function $f\colon x \mapsto ae^{-bx}$ which can be rewritten

$$h(m) = h_0 e^{-m/2m_0} \tag{1}$$

where $h_0 = a$ and $m_0 = 1/2b$ are respectively the levitation height and the mass of the train when there is no added mass on it ($m = 0$). In our model, the factor of 2 in the exponential term $e^{-m/2m_0}$ comes from mathematical convenience and can be interpreted as a scaling factor. This scalability could reflect the symmetrical design of the system, where interactions with components such as dual rails might contribute equally to the levitation mechanism. Each rail could potentially affect the magnetic field distribution or the force exerted on the train, necessitating a cumulative effect in the model.

IV. PROPULSION SYSTEM

The Electrodynamic Suspension train (EDS) and the Electromagnetic suspension train (EMS) can use the same propulsion systems but require some differences in for example the sizes of the motor, and other parameters. These systems have been studied to operate rapid and safe transportation. The systems are flexible in function of the different weight loads and have to operate the same way whether the train if filled with passengers or empty.

There exists two main Propulsion systems: The Linear Synchronous System Motor (LSM) and the Linear Induction System (LIM).

We will explain the processes of the Linear Synchronous Motor and the briefly the Linear Induction Motor system. Our train, The German Transrapid model TR1 to TR4 used the Linear Induction Motor system and the models TR5 to TR7 used the Linear Synchronous Motor system.

IV.1. Linear Synchronous Motor

By varying the magnetic fields along the track, the train can be propelled forward or slowed down as needed. It uses a linear motor system for propulsion, where the track itself acts as the stator (stationary part) of the motor, and the train serves as the moving rotor (the part that rotates or moves). Eliminating the need for traditional wheels and mechanical propulsion systems.

Indeed, Maglev trains use an intricate system used both for propulsion and braking called the synchronous longstator linear motor (LSM) directly integrated in the rails. There are magnets on the train that create a permanent magnetic field and windings along the track that generate a travelling magnetic field. It is the interaction of both these magnetic fields that will generate a propulsive force. Th orientation of these fields is chosen so that the train will be propulses along the horizontal axis in the direction of choice. The EDS train, uses magnetic field generated by superconducting magnets with no iron in the magnetic path so the winding used by the propulsion can be any size and adapted to each train. For the EMS train the magnetic flux is carried in an isolated area with the iron core so the winding of the train must be fixed.

To better understand what is it, we can break down the name of the motor to have the following information:

- Synchronous motor: generates motion by synchronizing the rotation of a magnetic field with the rotation of the motor's rotor (the support magnets)

- Long stator design: the stator (stationary part of the motor which contains coils of wire) extends along the entire length of the track. It ensures that the magnetic field generated by the stator interacts well with the rotor on the train providing continuous propulsion

- Linear motion: generates motion in a straight line (unlike traditional rotational motors). This is done by arranging the magnetic fields in such way that they push or pull the train along the tracks without the need of physical contact

We can therefore link the stator of the motor to packs with three-phase motor winding situated in the tracks and the rotor being the support magnets on the train. The motor works by supplying alternating currents to the three-phase motor winding which generates an electromagnetic travelling field which moves the the vehicle, pulled along by its support magnets which act as the excitation component. Therefore, the speed can be easily modified by varying the frequency of the alternating current. This also explains how the braking works. If the travelling field is reversed, the motor becomes a generator which brakes the vehicle without any contact.

Furthermore, the motor in the tracks is divided in sections which are only supplied when the train passes. Then, the power of the motor can be adjusted along certain areas only if there are expected acceleration, braking or gradients. Therefore, the power of the substations is higher than on level sections which are traveled at constant speed. [3]

Then, according to the following article [4], we can write the propulsion energy the following way:

$$E_p = \frac{3\pi}{2\tau_s}(\Phi_{sm} i_q + (L_d - L_q) i_d i_q) \quad (2)$$

with

- Φ_{sm}: mutual flux linkage between the excitation winding and the d-axis (in $kg.m^2.s^{-2}.A^{-1}$)
- i_q and i_d: currents on direct-quatrature axis (in Ampères)
- L_q and L_d: stator total inductance on direct-quatrature axis (in $kg.m^2.s^{-2}.A^{-2}$)
- τ_s: pole pitch (in meters)

The d-q axis is defined the following way:

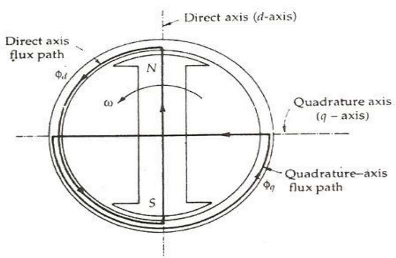

Figure 4: d-q axis

Furthermore, we can rewrite Φ_{sm} as $K\frac{i_m(t)}{h(t)}$ where K is a total proportionality constant, i_m the field windings current and $h(t)$ the levitation height. Hence,

$$E_p = \frac{3\pi}{2\tau_s}(K\frac{i_m(t)}{h(t)} i_q + (L_d - L_q) i_d i_q) \quad (3)$$

Therefore, we can obtain the propulsion force by taking minus the derivative with respect to h of the propulsion energy:

$$\begin{aligned} F_p &= -\frac{d}{dh}(\frac{3\pi}{2\tau_s}(K\frac{i_m(t)}{h(t)} i_q + (L_d - L_q) i_d i_q)) \quad (4) \\ &= \frac{3K\pi}{2\tau_s}\frac{i_m(t) i_q}{h^2(t)} \end{aligned}$$

Indeed, this is only the force created by one side of the motor on the train. So, since the train is propelled by double side motors, the motion formula, according to Newton's 2nd law, can be give by:

$$ma = 2F_p - F_z - Mg\sin(\alpha)$$

with:

- F_z: overall resistance force (air resistance force, magnet resistance force and generator resistance force)
- α: gradient degree

Hence, we can notice that the maglev train is subjected to 3 main forces: its weight depending on the inclines, the overall resistance force and the propulsion force. We develop this part with some modeling in Section VI applied to our model of the train.

IV.2. Linear Induction Motor

The linear induction Motor (LIM) consists of two main parts: the rotor and the stator as pictured on the following image:

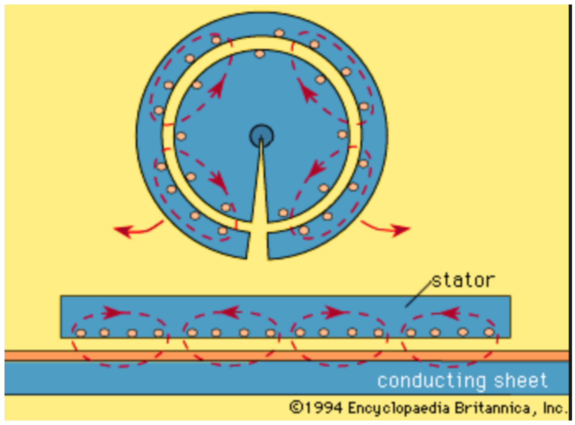

Figure 5: Schematic of the LIM system from Britannica

This method of propulsion allows strong acceleration and deceleration without relying on the friction between the railings and the bottom of the train so the system can be subjected to all types of weather conditions. The stator is made of iron laminations with three section, multipole winding with conductors oriented perpendicularly to the direction of motion of the train. The rotor is made of iron laminations with a winding, with copper and iron sheets. In our train, the rotor is placed on the railings and the stator located underneath the train.

V. THE STABILITY OF THE SYSTEM

As we see on our experiment of the Electromagnetic train, it cannot levitate without the plastic bars that keep the train along the rails. Without these angles pieces, the train would shift off to the side. We will see the physical principles that explain this phenomenon.

The theorem that explains this is called Earnshaw's Theorem presented by Samuel Earnshaw in 1839 in a paper *Transactions of the Cambridge Philosophical Society.* He showed that it is not possible to place a collection of bodies subject to only electrostatic forces so that they are in a stable equilibrium configuration. We will show show and demonstrate this theorem which is valid for magnetic forces. [5]

V.1. Energy approach

We consider W the energy of our magnetic dipole, $\mathbf{M}$ the magnetic dipole moment which is therefore constant and $\mathbf{B}$ the external magnetic field generated by our magnets. We know that :

$$W = -\mathbf{M} \cdot \mathbf{B} \tag{5}$$

Our dipole can be levitated and stable when its energy reaches a minimum. We know that our energy reaches a minimum when our Laplacian of W is greater than zero. So we look for:

$$\begin{aligned}
&\Delta W > 0 \\
\iff &\nabla^2 W > 0 \\
\iff &\nabla^2 U = \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} > 0 \\
\iff &\nabla^2 W = -\left(\frac{\partial^2}{\partial x^2}(M_x B_x + M_y B_y + M_z B_z)\right) \\
&\qquad - \left(\frac{\partial^2}{\partial y^2}(M_x B_x + M_y B_y + M_z B_z)\right) \\
&\qquad - \left(\frac{\partial^2}{\partial z^2}(M_x B_x + M_y B_y + M_z B_z)\right) > 0 \\
\iff &\nabla^2 W = -M_x \nabla^2 B_x - M_y \nabla^2 B_y - M_z \nabla^2 B_z > 0
\end{aligned}$$

We can simplify to the last equivalence because the dipole moment $\mathbf{M}$ is constant.

We know the divergence of the magnetic field $\nabla \cdot \mathbf{B} = 0$ from Maxwell's equation because there is an absence of monopole in magnetism.

So the Laplacian of the Magnetic field is: $\nabla^2 \mathbf{B} = \nabla \cdot \nabla \mathbf{B} = \nabla \cdot \mathbf{0} = 0$. Since we are in the magnetic dipole configuration, our magnetic moment is constant so $\Delta U = 0$

We showed that there exists no minimum so there exists no point in space where ou dipole would levitate in a configuration that is stable in all directions. This means that that all systems that are in levitation cannot be subjected to static magnetic fields alone as any perturbation would move it away from its equilibrium position. For this reason, the angle pieces are necessary for our train to remain perfectly aligned on top of the magnets and stay in the air.

There are no exceptions to this theorem but there are some factors that can be changed in our system to have stable levitation: The Levitron.
The levitron is an example of how rotation in addition to the repulsive magnetic forces can make the object levitate in a stable configuration. A Levitron is made of two parts: a main body and a top. The main body is a magnet with North oriented North and the top is a magnet Norht oriented South. As we saw in the previous sections, these two magnets with the same polarization created a magnetic force which is responsible for the levitation. However according to Earnshaw's theorem this levitation would be unstable but we will explain the effects of the rotation of the levitron. It is only the top magnet that spins and is responsible for a gyroscopic effect which allow the levitron to levitate in a stable configuration.

As with our train, if the system is tilting towards the side then since there are two forces vertically there will be the creation of a torque that will pull the top back to the ground. However, our top is spinning so the angular momentum reacts against this torque and the levitating system remains straight. The rotation however is required to be in a certain interval to create these effects that will allow stable levitation. If it is too low, the torque created is not important significant anymore and we are back to our original situation with unstable levitation. On the contrary, if the angular momentum is too high then the top cannot correctly sit straight anymore and the slightest disturbance will destroy the stable levitation.

V.2. Stability analysis of our Train Model

The guideway of a Maglev system contains electromagnetic coils or superconducting magnets arranged in a specific pattern. These coils generate magnetic fields that interact with the magnetic fields produced by the magnets on the train.
The interaction between the magnetic fields produced by the train's magnets and the coils in the track creates a stable levitation effect. The system is designed to maintain a specific distance between the train and the track, allowing for smooth and stable levitation even at high speeds.

It is also extremely interesting to look at the tracks for our model and study their generated magnetic field in order to understand where our stability issue is coming from. Modeling our track as an infinite sum of magnets laid along the y axis, in the x-z plane we can then analytically extrapolate the behaviour of the magnetic field in order to show there is no stable equilibrium with our current disposition of the model. We therefore take an imaginary model with two infinite tracks made up of upwards magnetized (z) ferromagnets and analytically compute the total magnetic field or vector potential field. Considering that our model is invariant with y, we then have by symmetry argument that the generated magnetic field has no y component. Moreover, we could use the course formula for continuous dipole distribution for the Vector Potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3 r'$$

a better trick we can use is to consider hypothetical electromagnets instead of ferromagnets. This enables us to model our experiment and our two tracks as two individual pairs of parallel wires carrying a current I in opposite directions, therefore acting as infinitely long magnets.

The magnetic field around parallel wires can then be analyzed using the Biot-Savart Law.

The magnetic field $\mathbf{B}$ at a point due to a small current element is given by:

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I \, d\mathbf{l} \times \mathbf{r}}{|\mathbf{r}|^3}$$

For parallel wires along the x-axis carrying currents I_1 and I_2 in opposite directions:

$$B = \frac{\mu_0 I}{2\pi r}$$

The total magnetic field at any point due to these wires is the vector sum of the fields due to each wire. The resultant magnetic field vector at each point is computed from the superposition of fields due to individual wires:

$$\mathbf{B}(\mathbf{r}) = \sum_i \mathbf{B}_i(\mathbf{r})$$

In the case of our model considering two rails positioned at y positions $y \pm \frac{d}{2}$. The equation for the magnetic field $\mathbf{B}_i$ is then given by:

$$\mathbf{B}_i(x,z) = \frac{\mu_0(\pm I_i)}{2\pi}\left(\frac{z}{\sqrt{(x \pm \frac{d}{2} \pm d_s)^2 + z^2}}\mathbf{e}_x + \frac{-x \pm \frac{d}{2} \pm d_s}{\sqrt{(x \pm \frac{d}{2} \pm d_s)^2 + z^2}}\mathbf{e}_z\right)$$

With the separation changing from $+\frac{d}{2}$ to $-\frac{d}{2}$ depending on the wire we are considering and with d_s a small offset to differentiate the wires forming our individual rails.

This then enables us to plot those field lines in python which gives us a much clearer visualisation of the generated magnetic field.

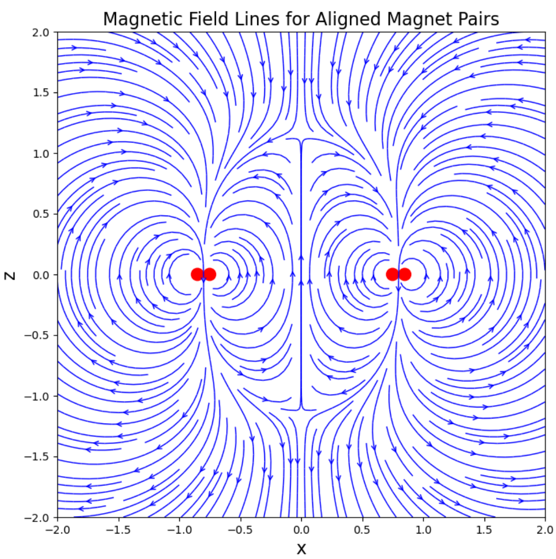

Figure 6: Visualization of the generated magnetic field lines by a pair of aligned rails

In order to make the stability issue even more apparent we can then plot the Potential energy levels of such a system plotting the energy $W = -\mathbf{M}\cdot\mathbf{B}$. Doing so we can observe the absence of potential "well" in our Model train track for an electromagnet with magnetic moment $\mathbf{M} = -M\mathbf{e}_z$.

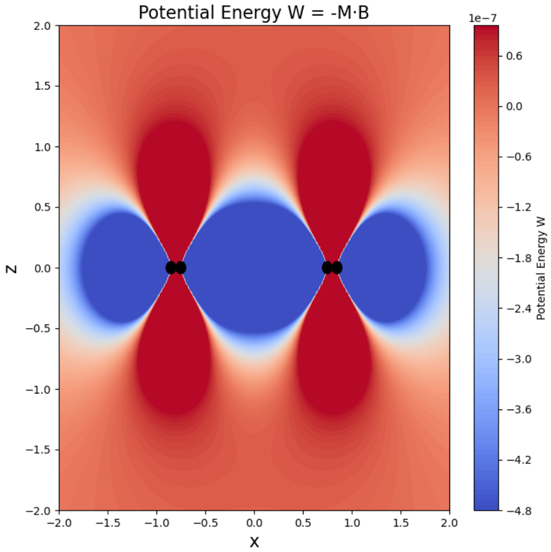

Figure 7: Potential energy for a pair of aligned rails

In order to analyse this further we can also consider the force that would be applied on our electromagnet $\mathbf{M}$, being $-\nabla(W)$. Looking at our plot we can therefore see that there is no stable position for the Magnet in our current model.

Moreover plotting the torque for the same electromagnet $\mathbf{M} = -M\mathbf{e}_z$ using $\tau = \mathbf{M} \times \mathbf{B}$ we can study the instability with respect to rotation about the axis of the train.

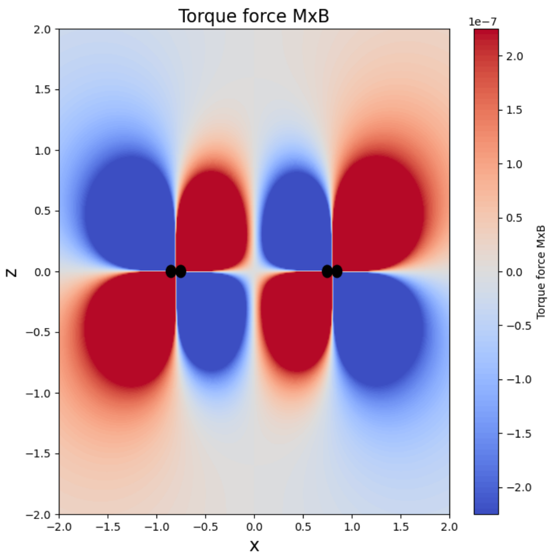

Figure 8: Torque for a pair of aligned rails

Looking at the torque for the track we can understand that a small displacement laterally of the train would induce a positive or negative torque that would flip the train over. It is therefore easy to see the instability of orientation with respect to the axis of the train

As such we can clearly see there is no potential "well" in the way our model track is arranged nor is there a stable orientation about the axis of the train, which underlines the stability of our model.

V.3. Application to the real life train

Nowadays, the EMS tracks are constructed in a very similar way as our model train using guidance magnets on the side of the track to allow stability of the system. A key difference between our constructed track and the working version of EMS is the fact that where our track uses the repulsion between the track and the train electromagnets, the working German version of the Maglev train uses attraction to pull the train upwards rather than pushing it upwards.

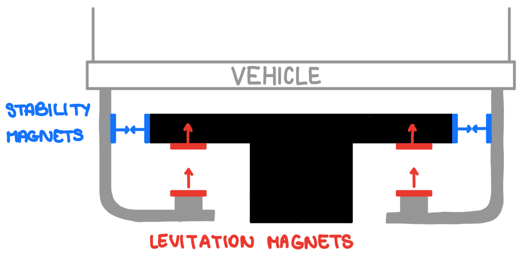

Figure 9: Representation of the German Maglev train magnet Disposition

Taking our model further we can slightly arrange our code so that we can plot the magnetic field lines generated by the actual train tracks used in the EMS Maglev system today. The difference with our model is the use of guidance rails on the side. Using this we can then show that the "enhanced" track does indeed show a potential well, indicating the possibility for stability. Plotting the field lines we obtain:

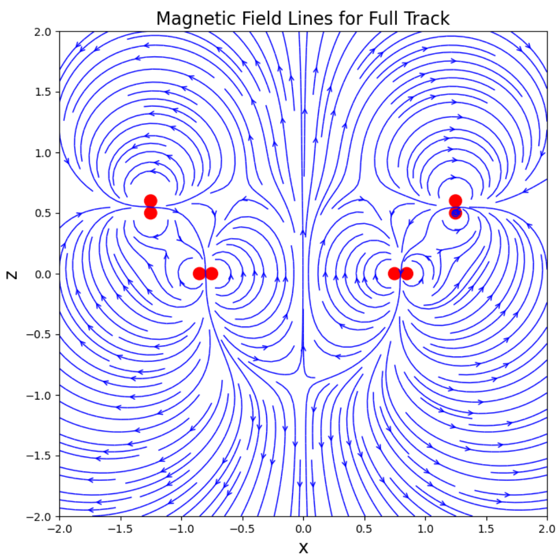

Figure 10: Visualization of the generated magnetic field lines by the railway and the guidance (stability) magnets

Similarly as our constructed model we can analyse the Potential Energy and Torque $W = -\mathbf{M} \cdot \mathbf{B}$ and $\tau = \mathbf{M} \times \mathbf{B}$ with a considered electromagnet $\mathbf{M} = M\mathbf{e}_z$. Doing so we can then look at the stability of the actual track in use today. A notable difference to mark between this model and the previous analysed model is the fact that attractive forces are being used for the German EMS Maglev train so that our considered electro magnet has an upwards facing magnetization and is supposed to be suspended under the track. Looking at the Potential Energy and Torque we then obtain:

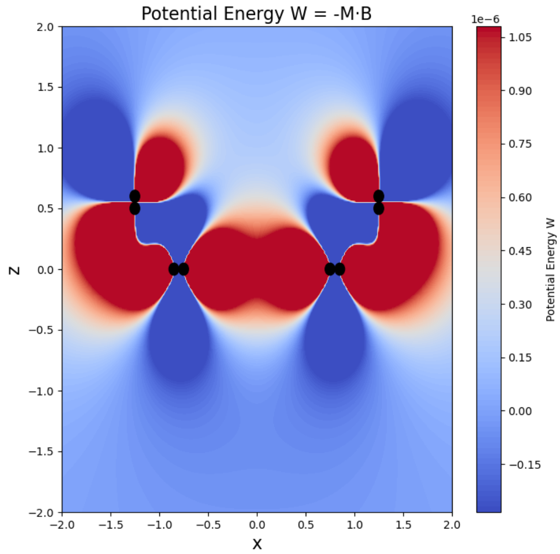

Figure 11: Potential Energy for German EMS Model

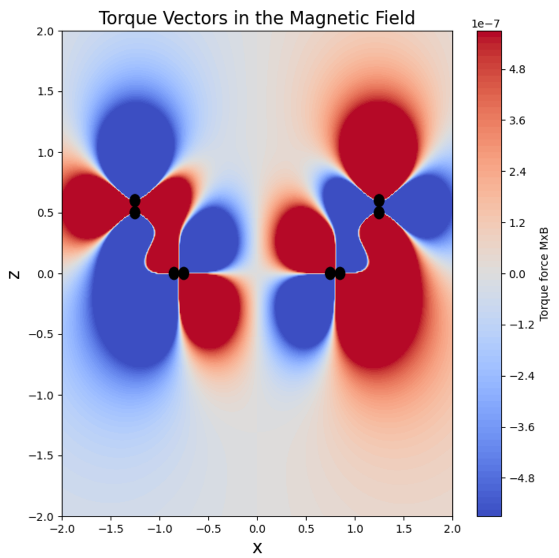

Figure 12: Torque for German EMS Model

The plot of potential energy W reveals regions of low and high energy around the magnet pairs. The presence of potential wells (regions of low potential energy) indicates that the system can achieve stable equilibrium positions. Moreover, the alternating colors represent regions where the magnetic forces exert stabilizing torques on the train. The symmetric distribution of torque vectors around the equilibrium positions contributes to maintaining the train's orientation and preventing it from tipping over.

Moreover an important comment should be made on the fact that this track uses attractive forces. The disadvantage with attractive forces is that the equilibrium position for levitation is not made naturally by the sum of the gravitational force and the applied force on our magnet. This is then achieved through a controlled current in our electromagnet, inducing a controlled magnetic moment for our magnet which is then computed to keep a steady distance between the track and the train. This choice for the construction of the system is purely an engineering choice in order not to have to shield the wagon of the train from the strong magnetic fields generated.

V.4. Forces approach

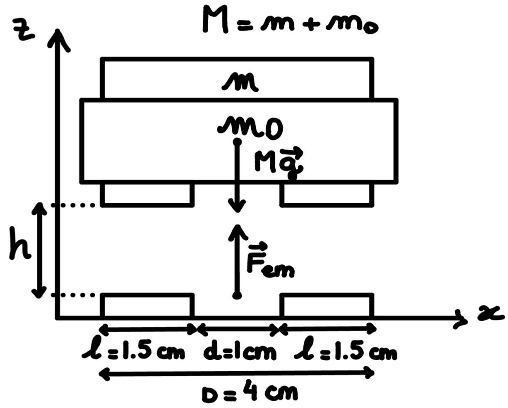

Figure 13: Sketch of the situation

We then consider $\mathbf{F_{em}}$ the electromagnetic force applied by the track on the magnetic dipole in our experimental setup. Since the system is not mobile, thus the only forces acting on our pseudo-train are its weight $(m_0 + m)\mathbf{g}$ and $\mathbf{F_{em}}$ where m_0 is the mass of the train itself and m the added mass on it (which is considered fixed here).
Therefore, by Newton's second law we have at its equilibrium position:

$$\mathbf{F_{em}} = (m_0 + m)g\mathbf{e_z} \tag{6}$$

We know that the magnetic moment $\mathbf{M}$ is constant from the previous subsection, which allows us to write the following expression for the electromagnetic force:

$$\mathbf{F_{em}} = -\nabla(-\mathbf{M} \cdot \mathbf{B})$$

One can also write $\mathbf{M} = M\mathbf{e_z}$ and since the electromagnetic force is only along $\mathbf{e_z}$, we have :

$$\mathbf{F_{em}} = (\mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial z})\mathbf{e_z}$$

Then since the magnetic field lies in the xz-plane as we showed previously, its derivative with respect to z also lies in that plane so the dot product simplifies as follows:

$$\mathbf{F_{em}} = M\frac{\partial B_z(x,z)}{\partial z}\mathbf{e_z}$$

Therefore, by combining these two expressions of the magnetic force we get:

$$dB_z = \frac{(m_0 + m)g}{M}dz \tag{7}$$

V.5. Qualitative analysis

We know now that the levitation height h decreases exponentially with added mass m which means that when m increases, the B-field along the vertical direction, which is decreasing with h, increases with m. Then, the electromagnet system produces a stronger magnetic field for a heavier object. We now question ourselves on a limit added mass m after which the train falls. In that case, the magnetic force becomes of the opposite side and in the same direction as weight.

VI. MATHEMATICAL MODEL OF MAGNETIC LEVITATION SYSTEM

VI.1. Model Assumptions

In order to obtain these results, we did a few assumptions which may or not lower its accuracy:

- We neglect the action of the 90° plastic angle pieces
- The magnetic flux leakage and edge effect are ignored, and the magnetic flux is uniformly distributed in the air gap consequently
- The electromagnetic force generated by the electromagnet is considered to concentrate on the center of mass
- We do not consider propulsion since our train was stable and did not have any motion in the horizontal direction
- We neglect the height of the electromagnetic tape

VI.2. Theoretical calculation of the induced current

In order to a establish a mathematical model for our magnetic levitation system, we consider induction coils with N turns and let S be the polar surface of our electromagnet, as described in the scheme below.

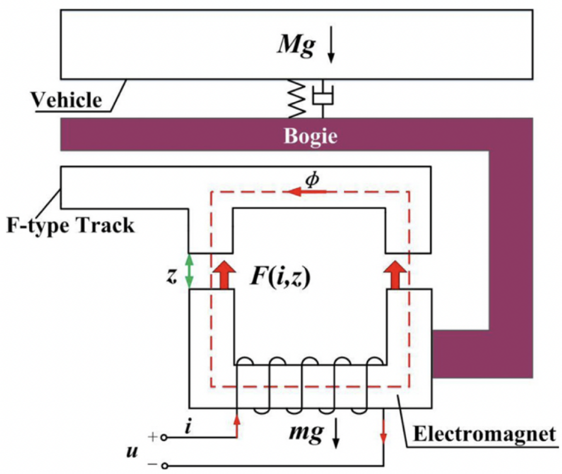

Figure 14: Mathematical model

Then, according to the International Journal of Applied Electromagnetics and Mechanics vol. 61, as demonstrated in the calculations by Zhai et al. [6], we can write the electromagnetic force as follows:

$$\mathbf{F_{em}} = \frac{\mu_0 N^2 S I^2}{4h(m)^2}\mathbf{e_z} \tag{8}$$

.

In this equation:

- μ_0 is the permeability of free space,
- N represents the number of turns in the coil,
- S is the polar surface area of the electromagnet,
- I is the current flowing through the electromagnet,
- $h(m)$ is the distance between the electromagnet and the magnetic track given by Equation (1), and
- $\mathbf{e_z}$ is the unit vector in the direction of the magnetic force (vertically upward).

Thus, since $h(m) = h_0 e^{-\frac{m}{2m_0}}$, the electromagnetic force is given by:

$$\mathbf{F_{em}} = \frac{\mu_0 N^2 S I^2}{4h_0^2} e^{\frac{m}{m_0}}\mathbf{e_z} \tag{9}$$

.

The derivation of the electromagnetic force formula is rooted in the fundamental principles of electrodynamics, beginning with Maxwell's equations, which emphasize the magnetic flux linkage caused by the current in the coils. This is followed by the application of a magnetic circuit model, where the magnetic flux generated by an electromagnet with N turns and carrying current I travels through the air gap and interacts with the magnetic track. The force itself is calculated based on the Lorentz force law, which states that the electromagnetic force on a charge moving within a magnetic field is proportional to the current, the magnetic field, and the length of the conductor in the field, here represented by the surface area S of the electromagnet. The model further simplifies the interaction of fields in three-dimensional space to an inverse square law, reflecting the decrease in magnetic field strength with increasing distance h from the electromagnet, thereby allowing for the linearization of the field's effects over distance. Lastly, the exponential term $e^{\frac{2m}{m_0}}$ adjusts for variations in the levitation height and current (cf. Equation (1) and Equation (10), based on experimental data, which ensures that the force remains linearly related to the total mass of the system. These steps combine to provide a theoretically sound yet practically applicable method for estimating the magnetic levitation force in maglev systems, making it a valuable tool for engineering design and predictive modeling.

Moreover, by Newton's second law, we have

$$\mathbf{F_{em}} = (m_0 + m)g\mathbf{e_z}$$

, which yields

$$(m + m_0)g = \frac{\mu_0 N^2 S I^2}{4h_0^2} e^{\frac{m}{m_0}}$$

And furthermore we have:

$$I = ((m + m_0)g\frac{4h_0^2}{\mu_0 N^2 S})^{1/2} e^{-\frac{m}{2m_0}} \tag{10}$$

.

Then we plot I with respect to m in two different contexts, first in our experiment then we apply what we know so far to the actual Maglev situation.

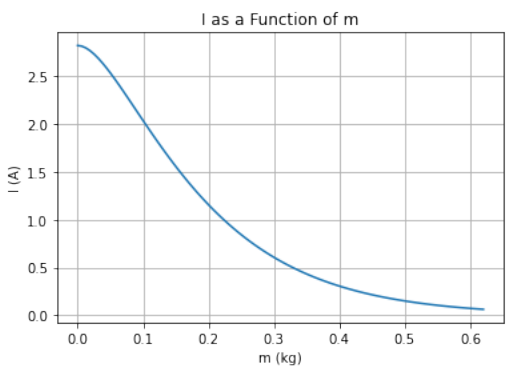

Figure 15: Plot of the current against the added mass m with (a) $m_0 = 0.062kg, h_0 = 5.10^{-3}m, N = 40, S = 0.00381m^2$

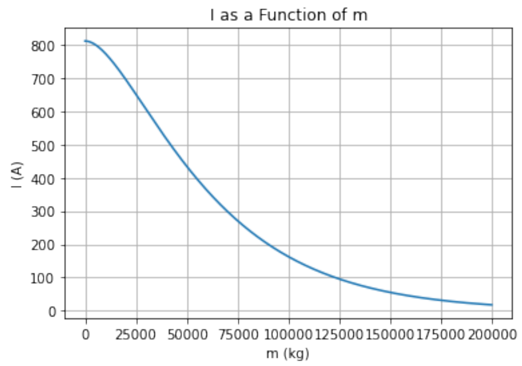

Figure 16: Plot of the current against the added mass m with (b) $m_0 = 20.10^3 kg, h_0 = 8.10^{-2} m, N = 400, S = 0.038 m^2$

We are then interested to know how many passengers can the train carry while levitating.

VI.3. Numerical Analysis of the Maglev Train's Carrying Capacity

To determine the maximum carrying capacity of our Maglev train model under specific conditions, we performed a numerical simulation to find the critical mass m_c where the current $I(m)$ required to maintain levitation drops below a critical threshold of $\epsilon = 5$ A. This critical current threshold represents the operational limit beyond which the electromagnetic system cannot generate sufficient force to counteract the gravitational pull on the increased mass.

Using the fsolve function from Python's scipy.optimize module, we determined the critical mass at which the train can no longer sustain levitation. The calculation involves solving the equation:

$$I(m) = \left(\frac{(m + m_0)g \cdot 4h_0^2}{\mu_0 N^2 S} \right)^{1/2} e^{-\frac{m}{2m_0}} - \epsilon = 0$$

The result, $m_c \approx 2.6 \times 10^5$ kg, signifies the total mass at which the system's performance degrades below the required levitation threshold. Given that the average mass of a Maglev passenger is estimated at 80 kg, this translates to a carrying capacity of approximately 3200 passengers. This result seems surprisingly high given that an EMS train can carry up to 1000 passengers. In particular, the Shanghai Maglev train which relies on EMS technology has 'the maximum capacity is 959 passengers each train' [7].This discrepancy can largely be attributed to our model's assumptions. Specifically, the model calculations only consider the mass of the passengers and do not account for additional significant weight contributions from the train's infrastructure itself, such as the weight of seats, onboard services, fuel, or other operational equipment. Furthermore, the actual construction and design limitations, such as train size, safety standards, and practical engineering constraints, were not included in our simulation.

A more realistic approach would be to come back to our experimental findings described in Equation (1) and to consider the levitation height h function with the parameters of an actual EMS train which we will perform in Section VIII.

VII. FUTURE OF MAGLEV TRAINS

Some research is currently being done to develop faster working trains. A future possibility for maglev trains is known as evacuated tube transport. This involves the trains traveling in enclosed vacuum tubes with very little air resistance. Implementing this involves permanently removing air along the travel route. Having no air present in the tunnels means that there is no air friction limiting the speed of the trains, leaving them free to reach projected top speeds as high as 6400 to 8000 km/h. this idea was already thought of in the late 19th century by the son of Jules Verne but always remained fiction as it was not at all close to being feasible. Now, in 2003, engineers realised that it might actually be feasible thanks to the development of technology. Their idea was to connect London to New York as this region of Atlantic is the shallowest (below 3km deep) and the narrowest. There are two possible routes with both their benefits and drawbacks. The first route would minimize underwater construction but it would be the longest route and it would have to deal with terrible winter weather conditions. This is why this route is not ideal. The second route would have a larger underwater portion (3km), but its directness and simplicity makes it the prefered option. Engineers are looking into 3 underwater tunnel structures.

VII.1. A subterranean tunnel

A subterranean tunnel would theoretically be possible as we already have the technologies to do so. This tunnel would be built under the ocean but this presents a lot of difficulties as this would be done at more than 4km deep. Indeed, the pressure at this depth is around 400 atm, it would have to cross the mid-atlantic ridge (a highly seismic region in the middle of the ocean), to have ventilation shafts would be nearly impossible and lastly, tunnel digging is incredibly slow. So, if we look at a normal pace drilling, this tunnel would take 300 years to finish. Therefore, this solution has been dismissed.

VII.2. A tunnel on the ocean's floor

Another option was taking pre made tunnel parts that would then be layed on the ocean's floor. However, we would encounter the same problems for pressure as seen above which deams the project unfeasible.

VII.3. A floating tunnel

Finally, engineers had to come up with ideas that did not have the pressure as an issue. They came up with a tunnel being at 200m beneath the surface and tied to the seafloor with cables. These would not be too tight to be able to adjust to currents and the passing of large sea animals. This could be done relatively fast as the sections of the tunnel could be done on land and then submerged and placed thanks to submersibles. The only problem is that with current maglev trains going at a speed around 600km/h, it would still take 10h to go from london to NY. But, some companies are already trying to develop faster Maglev trains reaching speeds of around 1200km/h. At this pace, the journey would take around 5h thanks to hyperlook technologies.

VII.4. A very expensive project

However, this project would need a lot of resources (1 billion tons of steel, cables would add up to 400000km and to remove all the air, it would take 100 jet engines to work 24h over 2 weeks), international collaboration for approval, funding and construction. This project would take around 100 years counting the tests and construction and cost around 12 trillion USD. Furthermore, this project stays conceptual at the moment as it still has a lot of risks associated to it: it a fire would break out or in case of an accident, it would take hours to get the passengers, it could only be built during summer and good weather conditions, the submersibles would have to dive 4km deep, 12trillion USD is a very important amount, to pay back the initial tunnel cost it would take around 674 years if a ticket is worth 200 dollars and if there would be 89 million passengers. Therefore, for all these reasons, this project is still just an idea as it is still deemed unfeasible with two many risks. But it is not completely off the table as it might be possible in some near future when it is deemed feasible. [8] [9]

VII.5. Advantages and disadvantages

As any other model, the EMS trains presents their advantages and drawbacks.

First of all, they are less expensive to maintain and to operate as there is no rolling friction to wear out the materials. Furthermore, derailment is highly unlikely which leads to fewer accidents and troubles on the lines. The traffic won't be perturbed as much. They are also more spacious than conventional trains with their wider cars. One important benefit of these trains are that they lead to almost no air pollution as there is no fuel consumed and the frictionless system reduces significantly the noise made by the train. Lastly, they can operate on higher ascending grades (up to 10% which is 6% more than on conventional trains, which helps to accommodate a little more to nature.

Despite all these notable benefits, the trains also present a few drawbacks. As they rely on new technologies, they require an entirely new infrastructure that cannot be integrated with existing railroads. Those infrastructures require large amounts of rare earth elements (such as scandium, yttrium etc) which are expensive to recover and refine but necessary for the magnets used for the trains (as iron is not at all strong enough to make the train levitate). This leads to an estimated cost of $10 million per mile excluding the cost of expensive computer systems for monitoring magnetic separation.

VIII. RESULTS AND DISCUSSIONS

Considering that our model is invariant with y, we then have by symmetry argument that the generated magnetic field has no y component. Moreover, we could use the course formula for continuous dipole distribution for the Vector Potential

VIII.1. Experimental Outcomes and Theoretical Correlation

Our experiment and theoretical analysis have culminated in significant findings regarding the electromagnetic suspension (EMS) capabilities of the Maglev system. Indeed, the coefficients we obtain for the model function $h(m) = h_0 e^{-\frac{m}{2m_0}}$ are $h_0 = 4.7 \pm 0.1mm$ and $m_0 = 61.6 \pm 3.4g$. Remarkably, this numerical estimation closely aligns with the initial levitation height and the actual physical mass of our train, which are respectively measured to be 5.0 ± 0.5 mm and 62 ± 1 grams. The errors reported for h_0 and m_0 are the statistical standard errors, obatined from the curve fitting process and error propagation for m_0, as explained in Appendix B.

The close agreement between the numerically derived m_0 and the actual mass of the train validates the precision and accuracy of our experimental setup and methodology. This concordance also underscores the effectiveness of the exponential model in capturing the fundamental dynamics governing the electromagnetic levitation system used in our Maglev model. It suggests that the assumptions and simplifications made in developing the mathematical model are reasonable and do not significantly detract from its predictive power.

To quantitatively assess the accuracy of our exponential model in predicting levitation heights based on varying masses, we compute the coefficient of determination, denoted as R^2. This statistical measure is critical as it indicates the proportion of variance in the dependent variable (levitation height) that can be explained by the independent variable (mass) using our model. The derivation of the R-squared coefficient is explained in Appendix C, we compute this value numerically and obtain an accuracy score of 96.4% which once again translates the coherence of our exponential model. Furthermore, the close match between theoretical predictions and actual measurements suggests that our findings could be scalable, thanks to the m_0 parameter. This implies that similar experimental setups and models might be applicable in larger-scale versions, potentially guiding the development of commercial Maglev trains.

Application to a commercial Maglev train

We now use the theoretical model we derived in order to estimate the value of the critical mass m_c before the train stops levitating. We know from the director of Iran Maglev Technology (IMT) Dr. Hamid Yaghoubi's article in the Journal of Engineering that "In EMS system, the vehicle is levitated about 1 to 2 cm above the guideway using attractive forces" [10]. Therefore, we plot the following graph of h against m and numerically solve $h(m) < 1cm$ as we did for the current:

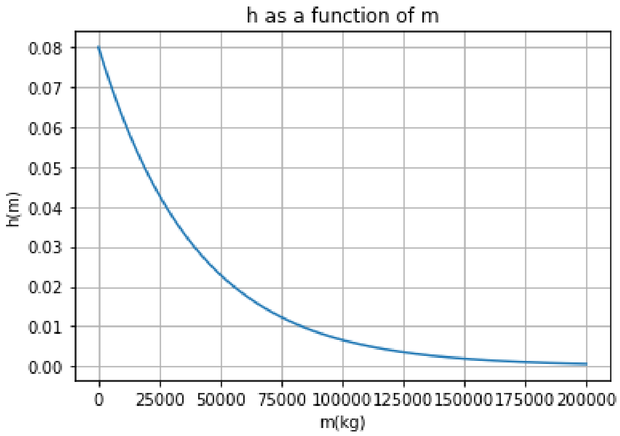

Figure 17: Plot of levitation height against added mass with $m_0 = 20.10^3 Kg$, $h_0 = 8cm = 0.08m$

In this configuration, we obtain a critical mass of $8.3 \times 10^4 kg$ which corresponds to 1040 people with an average mass of 80 kg. This result is close to the maximum capacity of the Shanghai Maglev train which may serve as a validation of the model when adjusted to reflect practical conditions, confirming that the theoretical predictions can indeed mirror practical outcomes when all relevant factors are considered.

Alternatively, the current-based control method outlined in Section VI estimated a significantly higher critical mass of approximately 2.6×10^5 kg. This figure exceeds the known capacity of similar Maglev systems by a substantial margin, indicating potential overestimations inherent in this approach. The current-based method, while theoretically sound, appears to overlook some practical constraints such as the additional weight from the train's infrastructure and the real-world environmental factors affecting magnetic field interactions. The divergence in results between the height-based and current-based methods can be attributed to uncertainties in several key parameters, which are critical in calculating the electromagnetic force necessary for levitation. The current-based method relies heavily on precise knowledge of the number of induction coils, the threshold current necessary for levitation, and the polar surface area of the electromagnet. These parameters directly influence the computed values of electromagnetic force, and any inaccuracies in these inputs can lead to significant deviations in the predicted levitation capabilities.

VIII.2. Stability

We have compared our experimental model and the real life train. In our model, we assumed that our model was invariant along y and that the magnetic field had only an x component. We considered our two rails to be aligned and computed the magnetic field. Observing the visualization of the magnetic field lines, the potential energy and the torques in the space. We observe no potential well and clear instability of our model under these assumptions. These observations confirm the constraints that are imposed by demonstrated Earnshaw's theorem. We then consider the real train as our new model which contains stability magnets and compute under these new conditions our magnetic field lines, potential energy and torques. We execute the same plots and observe clear differences. The graph that plots the potential energy in space shows a potential well which indicates a region of low potential energy). This well implies our system can achieve a stable equilibrium configuration. The real life train shows that some adaptations can be made to overcome the constraints imposed by Earnshaw's theorem and levitate in a stable configuration. These adaptations are the stability magnets which are on the EMS train.

IX. CONCLUSION

Overall, our model is an accurate representation of the real life German Transrapid Maglev train. This model of the EMS train was first used in Germany and has since been used in Shanghai with the Shanghai Transrapid train. Our experiment allowed us to understand the theoretical concepts which we proved theoretically in the report. Indeed, we studied the functioning of the Levitation Magnets and understood the interactions that generate a levitation force on our system. We showed that our train requires certain adaptations to levitate in a stable configuration by overcoming the limits imposed by Earnshaw's theorem. We computed the capacity of our train and studied how the weight load impacts the levitation of our train with numerical simulations. To understand the main concepts of levitation we simplified some parameters of our model. Indeed, we assumed the moment of our system was constant for the sake of our mathematical proofs, as well as considered a train with only one magnet at the bottom instead of multiple. With this assumption, we neglect the possible interactions between these magnets and their influence on our system. While, taking into account these parameters would alter our final results we are convinced to have acquired a clear understanding of the concepts of levitation and how our train is functioning. For curious students who wish to discover more about levitation, our report can be a good explanation of the main concepts. For students who are working on a similar subject and wish to deepen their knowledge on Maglev technologies, our experiment is a head-start [2] alongside the detailed explanation in our report. After having understood all the concepts we demonstrated they can then pursue to consider the parameters we neglected to simplify our model such as the moment,and the multiple magnets. We see the Maglev train project is dedicated to offering a rapid and eco-friendly alternative to our current transportation methods. This project has not yet been developed due to multiples limits we previously discussed (ex. costs) but we remain hopeful that one day the ambitious goals of crossing the Atlantic by train will become a reality.

ACKNOWLEDGEMENTS

We would like to express our gratitude to Professor Arnaud Couairon for his supervision and guidance throughout the project. His input and feedbacks during our weekly meetings and/or written progress reports have been greatly appreciated and beneficial to our research.

APPENDIX

A. EXPERIMENTAL DATA

Our experimental dataset for mass and levitation height is outlined in the table below.

Mass (±1g)	Levitation Height (±0.5 mm)
0	5.0
5	4.8
15	4.3
20	4.0
26	3.7
31	3.5
36	3.3
43	2.9
51	2.8
63	2.8
71	2.7
85	2.1
90	2.0
103	1.9
117	1.7
124	1.5
135	1.5
166	1.4
180	1.4
192	1.3
217	1.2
259	1.0
289	0.8
307	0.5
322	0

Table 1: Experimental Data for Mass and Levitation Height of Maglev Train Model

B. ERROR ANALYSIS FOR CURVE FITTING PARAMETERS

This appendix provides a detailed derivation of the error calculations for the parameters h_0 and m_0 obtained from the exponential decay model used in our study. These parameters were estimated using non-linear regression, specifically by fitting an exponential function to empirical data. The uncertainties in these parameters are crucial for assessing the reliability and robustness of our findings.

B.1. General Background on Error Propagation

When parameters are derived from data through a curve fitting process, their uncertainties can be quantitatively assessed using the covariance matrix provided by the fitting procedure. The standard error of a fitted parameter is the square root of the variance associated with that parameter, which is the diagonal element of the covariance matrix corresponding to that parameter.

B.2. Error Propagation in Non-Linear Regression

For a model function $f(x; a, b) = ae^{-bx}$, where parameters a and b correspond to h_0 and the reciprocal of twice m_0, respectively, the uncertainties σ_a and σ_b are directly derived from the covariance matrix. However, for parameters like $m_0 = \frac{1}{2b}$, the uncertainty must be calculated using error propagation techniques.

B.2.1. *Calculation of σ_{h_0}*

Since $h_0 = a$, the uncertainty in h_0, denoted as σ_{h_0}, is given by:

$$\sigma_{h_0} = \sigma_a$$

where σ_a is the standard error of parameter a from the covariance matrix of the fit.

B.2.2. *Calculation of σ_{m_0}*

The uncertainty in m_0, which is related to b by $m_0 = \frac{1}{2b}$, is calculated using the formula for error propagation:

$$\sigma_{m_0} = \left| \frac{d}{db} \left(\frac{1}{2b} \right) \right| \sigma_b = \left| -\frac{1}{2b^2} \right| \sigma_b = \frac{1}{2b^2} \sigma_b$$

Since $b = \frac{1}{2m_0}$, substituting in terms of m_0 gives:

$$\sigma_{m_0} = \frac{1}{2 \left(\frac{1}{2m_0} \right)^2} \sigma_b = 2m_0^2 \sigma_b$$

B.3. Summary

These calculations show that while σ_{h_0} is straightforwardly obtained from the fit's output, σ_{m_0} requires careful consideration of how changes in b influence m_0. This highlights the importance of accurate covariance estimation in non-linear regression to ensure the reliability of parameter estimates, especially in physical models where these parameters have direct implications on the system's behavior and interpretation.

C. COMPUTATION OF THE COEFFICIENT OF DETERMINATION (R^2)

In the evaluation of the exponential fit model applied to our Maglev train levitation height data, the coefficient of determination, R^2, plays a pivotal role in determining the accuracy and reliability of the model. The following steps detail the process used to compute R^2, providing a comprehensive look at the statistical underpinnings of our analysis.

C.1. Definition and Formula

The coefficient of determination, R^2, quantifies the amount of variance in the dependent variable (levitation height) that is predictable from the independent variable (mass). It is defined mathematically as:

$$R^2 = 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}} \tag{11}$$

where SS_{res} is the sum of squares of residuals, and SS_{tot} is the total sum of squares, representing the total variance in the observed data.

C.2. Detailed Computation Steps

1. **Calculate the mean of the observed data**: Compute the average levitation height from the experimental data:
$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$$
2. **Total Sum of Squares ($\mathbf{SS_{tot}}$)**: Represents the total variance in the dataset about the mean.
$$SS_{\text{tot}} = \sum_{i=1}^{n} (y_i - \bar{y})^2$$
3. **Sum of Squares of Residuals ($\mathbf{SS_{res}}$)**: Represents the variance not explained by the model.
$$SS_{\text{res}} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$
where $\hat{y}_i$ are the values predicted by the model:
$$\hat{y}_i = h_0 e^{-\frac{m_i}{2m_0}}$$

4. **Compute** R^2: Using the values obtained in the previous steps, calculate R^2:

$$R^2 = 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}}$$

C.3. Interpretation

The R^2 value provides a measure of how well our model explains the variability in levitation height based on the mass of the Maglev train. A value closer to 1 indicates that the model explains most of the variability, whereas a value closer to 0 indicates poor explanatory power.

D. PYTHON CODE USED FOR SIMULATIONS

D.1. Model fitting with polynomial and exponential regression

```

import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import curve_fit

data=pd.read_csv('./PHY204_EXPDATA.csv', sep=';')
data.iloc[:, 1] = data.iloc[:, 1].str.replace(',', '.').astype(float)
mass = data.iloc[:, 0]
#data.iloc[:, 1] = data.iloc[:, 1].str.replace(',', '.').astype(float)
height = data.iloc[:, 1]

# Define uncertainties
um = 1  # uncertainty in mass (g)
uz = 0.5  # uncertainty in height (mm)


# Fit a polynomial regression
degree = 3  # Degree of polynomial regression
coefficients = np.polyfit(mass, height, degree)

# Generate points for the fitted curve
mass_fit = np.linspace(min(mass), max(mass), 100)
height_fit = np.polyval(coefficients, mass_fit)

# Plot the fitted curve
plt.plot(mass_fit, height_fit, label='Polynomial Fit (degree {})'.format(degree))


plt.errorbar(mass, height, xerr=um, yerr=uz, fmt='o', markersize=5,
capsize=3, label='Data with uncertainties')
plt.xlabel('Mass (g)')
plt.ylabel('Elevation Height (mm)')
plt.title('Elevation Height vs. Mass')
plt.legend()
plt.grid(True)
plt.show()
print(coefficients)
```

Listing 1: Polynomial Regression

The resulting coefficients are: [-4.49484245e-07 2.60387032e-04 -5.23850803e-02 4.96259878e+00]

```

# Define the exponential function
def exponential_function(mass, a, b):
    return a * np.exp(-b * mass)

# Fit the exponential function to the filtered data
popt, pcov = curve_fit(exponential_function, mass, height)

# Generate points for the fitted curve
mass_fit = np.linspace(min(mass), max(mass), 100)
height_fit = exponential_function(mass_fit, *popt)
errors = np.sqrt(np.diag(pcov))
# Plot the fitted curve
plt.plot(mass_fit, height_fit, label='Exponential Fit')
plt.errorbar(mass, height, xerr=um, yerr=uz, fmt='o', markersize=5, capsize=3,
label='Data with uncertainties')
plt.xlabel('Mass (g)')
```

```
plt.ylabel('Elevation Height (mm)')
plt.title('Elevation Height vs. Mass with Exponential Fit')
plt.legend()
plt.grid(True)

h_0,m_0=popt[0], 0.5/popt[1]

#R-squared computation
residuals = height - exponential_function(mass, *popt)
ss_res = np.sum(residuals**2)
ss_tot = np.sum((height - np.mean(height))**2)
r_squared = 1 - (ss_res / ss_tot)

plt.figtext(0.25, 0.80, f'$R^2 = {r_squared:.3f}$', fontsize=12)  # Add R-squared to the plot

plt.show()
print("Function fit model: a * exp(-b * mass)")
print("Optimized coefficients (a, b):", popt)
print("Errors on coefficients (a_err, b_err):", errors)
print(f"R-squared value: {r_squared:.3f}")
```

Listing 2: Exponential Decay Regression

The printed results are:

```
Function fit model: a * exp(-b * mass)
Optimized coefficients (a, b): [4.66452001 0.00810954]
Errors on coefficients (a_err, b_err): [0.1256267  0.00045337]
R-squared value: 0.964
```

D.2. Solving the critical mass optimization problem

We first solve this question using the current I in both our experimental setup and in the real-world configuration. Then, we solve it using the exponential decay model for levitation height h with parameters h_0, m_0 that corresponds to that of the Maglev train.

D.2.1. *In our train*

```

# Our wooden train configuration
# Given parameters
m_0 = 0.062  # value of m_0
g = 9.81     # gravitational constant
h_0 = 0.005  # initial height
mu_0 = 4*np.pi*1e-7 # permeability constant
N = 40   # some constant
epsilon=0.5
L=0.127
l=0.015
S=2*L*l

# Define the function f(m) = I
def I(m):
    return ((m + m_0) * g * 4 * h_0**2 / (S*mu_0 * N**2))**0.5 * np.exp(-m /(2* m_0)) -epsilon

m_values = np.linspace(0, 10 * m_0, 100)  # adjust the range and number of points as needed

# Compute I for each m value
I_values = I(m_values)

# Plot I as a function of m
plt.plot(m_values, I_values+epsilon)
plt.xlabel('m (kg)')
plt.ylabel('I (A)')
plt.title('I as a Function of m')
plt.grid(True)
plt.show()

# Solve for the root of f(m) using fsolve
m_root = fsolve(I, 2*m_0)  # starting guess is 0

# Display the result
print("Root found at m =", m_root[0])
```

Listing 3: Solving critical mass in our train

Obtained result: Root found at m = 0.3285059286904645 which is close to 322g at which the height is null (cf. Table 1)

D.2.2. *In the actual Maglev train configuration*

```python

from scipy.optimize import fsolve
from scipy.misc import derivative
import numpy as np

# Given parameters
m_0 = 20e3  # value of m_0
g = 9.81    # gravitational constant
h_0 = 0.08  # initial height
mu_0 = 4*np.pi*1e-7 # permeability constant
N =400     # some constant
L= 153
l=1
S=0.038
epsilon=5

def I(m):
    return ((m + m_0) * g * 4 * h_0**2 / (S*mu_0 * N**2))**0.5 * np.exp(-m / (2*m_0)) -epsilon

# Solve for the root of f(m) using fsolve
m_root = fsolve(I, m_0)

# Display the result
print("Root found at m =", m_root[0])
print(m_root[0]/80)

m_values = np.linspace(0, 10 * m_0, 1000)

# Compute I for each m value
I_values = I(m_values)

# Plot I as a function of m
plt.plot(m_values, I_values+epsilon)

plt.xlabel('m (kg)')
plt.ylabel('I (A)')
plt.title('I as a Function of m')
plt.grid(True)
plt.show()
```

Listing 4: Solving critical mass in Maglev train

And the console prints:

```
Root found at m = 256039.482764046
3200.493534550575
```

With the levitation height function:

```python

eps=0.01
m_0=20e3
h_0=0.08
m_values = np.linspace(0, 10 * m_0, 1000)

def h(m):
    return h_0*np.exp(-m/(2*m_0)) -eps

# Compute h for each m value
h_values = h(m_values)

# Plot h as a function of m
plt.plot(m_values, h_values+eps)

plt.xlabel('m(kg)')
plt.ylabel('h(m)')
plt.title('h as a function of m')
plt.grid(True)
plt.show()
```

```
m_root = fsolve(h, 2*m_0)  # starting guess is 0

# Display the result
print("Root found at m =", m_root[0],'Maximum number of passengers:',m_root[0]/80)
```

Listing 5: Levitation height function

And the console prints:

```
Root found at m = 83177.66166719345 Maximum number of passengers: 1039.720770839918
```

D.3. Code for plotting the field lines for the rails

Two important points have tow be made about the Code for the field lines and Potential and torque graphs can be made. Firstly The code was written with a model of tracks along the x axis in mind instead of the y axis but the computations remain the same and the subsequent plots and graphs as well. Secondly, taking in account how we modeled the track with infinite current carrying wires, we had extremely high values for the magnetic field close to the wires which made some extremely high values for the Potential energy and Torque when plotting them, close to the magnets, rendering the graphs very difficult to read with linear color gradients. This is why we added clippings to the values for the Energy and torque, ignoring the top highest and lowest 15 percentile, as the interest of the model was to observe the behaviour of the various forces at play far from the magnetic tracks and not infinitely close to them.

```

import numpy as np
import matplotlib.pyplot as plt

def plot_aligned_magnets(I, separation, pair_distance):
    y, z = np.meshgrid(np.linspace(-2, 2, 320), np.linspace(-2, 2, 320))
    positions = [(pair_distance, 0), (pair_distance + separation, 0), (-pair_distance, 0), (-pair_distance - separation, 0)]
    currents = [I, -I, -I, I]
    B_y = np.zeros_like(y)
    B_z = np.zeros_like(z)
    mu_0 = 4 * np.pi * 1e-7

    for (y_c, z_c), I_current in zip(positions, currents):
        r = np.sqrt((y - y_c) ** 2 + (z - z_c) ** 2)
        B_y += mu_0 * I_current / (2 * np.pi) * (z - z_c) / r ** 3
        B_z += mu_0 * -I_current / (2 * np.pi) * (y - y_c) / r ** 3

    M = -10
    W = M * B_z
    torque = M * B_y
    W_clipped = np.clip(W, np.percentile(W, 15), np.percentile(W, 85))
    torque_clipped = np.clip(torque, np.percentile(torque, 15), np.percentile(torque, 85))

    fig, ax = plt.subplots(figsize=(8, 8))
    ax.streamplot(y, z, B_y, B_z, color='blue', linewidth=1, density=2, arrowstyle='->')

    for y_c, z_c in positions:
        circle = plt.Circle((y_c, z_c), 0.05, color='red')
        ax.add_artist(circle)

    ax.set_xlim(-2, 2)
    ax.set_ylim(-2, 2)
    ax.set_xlabel('y', fontsize=16)
    ax.set_ylabel('z', fontsize=16)
    ax.set_title('Magnetic Field Lines for Aligned Magnet Pairs', fontsize=16)


    fig, ax2 = plt.subplots(figsize=(8, 8))
    c = ax2.contourf(y, z, W_clipped, levels=100, cmap='coolwarm')
    fig.colorbar(c, ax=ax2, label='Potential Energy W')

    for y_c, z_c in positions:
        circle2 = plt.Circle((y_c, z_c), 0.05, color='black')
        ax2.add_artist(circle2)

    ax2.set_xlim(-2, 2)
    ax2.set_ylim(-2, 2)
    ax2.set_xlabel('y', fontsize=16)
    ax2.set_ylabel('z', fontsize=16)
    ax2.set_title('Potential Energy W = -M B', fontsize=16)
```

```


    fig, ax3 = plt.subplots(figsize=(8, 8))
    d = ax3.contourf(y, z, torque_clipped, levels=100, cmap='coolwarm')
    fig.colorbar(d, ax=ax3, label='Torque force MxB')

    for y_c, z_c in positions:
        circle3 = plt.Circle((y_c, z_c), 0.05, color='black')
        ax3.add_artist(circle3)

    ax3.set_xlim(-2, 2)
    ax3.set_ylim(-2, 2)
    ax3.set_xlabel('y', fontsize=16)
    ax3.set_ylabel('z', fontsize=16)
    ax3.set_title('Torque force MxB', fontsize=16)

    plt.show()

plot_aligned_magnets(1.0, 0.1, 0.75)

def plot_full_track(I, I_2, separation, height, pair_distance, pair_distance_2):
    y, z = np.meshgrid(np.linspace(-2, 2, 320), np.linspace(-2, 2, 320))
    positions = [(pair_distance, 0), (pair_distance + separation, 0), (-pair_distance, 0), (-
    pair_distance - separation, 0),
                 (pair_distance_2, height), (pair_distance_2, height + separation),
                 (-pair_distance_2, height), (-pair_distance_2, height + separation)]
    currents = [I, -I, -I, I, -I_2, I_2, I_2, -I_2]
    B_y = np.zeros_like(y)
    B_z = np.zeros_like(z)
    mu_0 = 4 * np.pi * 1e-7

    for (y_c, z_c), I_current in zip(positions, currents):
        r = np.sqrt((y - y_c) ** 2 + (z - z_c) ** 2)
        B_y += mu_0 * I_current / (2 * np.pi) * (z - z_c) / r ** 3
        B_z += mu_0 * -I_current / (2 * np.pi) * (y - y_c) / r ** 3

    M = 10
    W = M * B_z
    torque = M * B_y
    W_clipped = np.clip(W, np.percentile(W, 15), np.percentile(W, 85))
    torque_clipped = np.clip(torque, np.percentile(torque, 15), np.percentile(torque, 85))

    fig, ax = plt.subplots(figsize=(8, 8))
    ax.streamplot(y, z, B_y, B_z, color='blue', linewidth=1, density=2, arrowstyle='->')

    for y_c, z_c in positions:
        circle = plt.Circle((y_c, z_c), 0.05, color='red')
        ax.add_artist(circle)

    ax.set_xlim(-2, 2)
    ax.set_ylim(-2, 2)
    ax.set_xlabel('y', fontsize=16)
    ax.set_ylabel('z', fontsize=16)
    ax.set_title('Magnetic Field Lines for Full Track', fontsize=16)


    fig, ax2 = plt.subplots(figsize=(8, 8))
    c = ax2.contourf(y, z, W_clipped, levels=100, cmap='coolwarm')
    fig.colorbar(c, ax=ax2, label='Potential Energy W')

    for y_c, z_c in positions:
        circle2 = plt.Circle((y_c, z_c), 0.05, color='black')
        ax2.add_artist(circle2)

    ax2.set_xlim(-2, 2)
    ax2.set_ylim(-2, 2)
    ax2.set_xlabel('y', fontsize=16)
    ax2.set_ylabel('z', fontsize=16)
    ax2.set_title('Potential Energy W = -M.B', fontsize=16)


    fig, ax3 = plt.subplots(figsize=(8, 8))
    d = ax3.contourf(y, z, torque_clipped, levels=100, cmap='coolwarm')
    fig.colorbar(d, ax=ax3, label='Torque force MxB')

    for y_c, z_c in positions:
        circle3 = plt.Circle((y_c, z_c), 0.05, color='black')
```

```python
        ax3.add_artist(circle3)

    ax3.set_xlim(-2, 2)
    ax3.set_ylim(-2, 2)
    ax3.set_xlabel('y', fontsize=16)
    ax3.set_ylabel('z', fontsize=16)
    ax3.set_title('Torque Vectors in the Magnetic Field', fontsize=16)


    plt.show()

plot_full_track(1.0, 1.0, 0.1, 0.5, 0.75, 1.25)
```

Listing 6: Plotting field lines for the rails

CONTRIBUTIONS

This project was the result of a collaborative effort, where each team member played essential roles. Peter organized the project by managing the coordination of tasks and maintaining an overview of the project progress.

Audrey and Jeanne worked together to build the Maglev train model, run the experiments, and collect data. Peter conducted the data analysis, applied model fitting, and discussed the accuracy of our results. Octave was responsible for the theoretical foundation, deriving magnetic field expressions and plotting field lines to analyze the stability of the Maglev system, while Audrey and Jeanne focused on explaining the propulsion mechanisms that allow the German Transrapid EMS train to move.

Then, Jeanne also developed the energy approach to system stability, applying Earnshaw's theorem to explain why stable levitation is not possible with static magnetic fields alone. This theoretical work underpins much of the project's analysis on system stability. She also wrote the section on Propulsion alongside Audrey, and differentiated the two different propulsion systems used by the different models of the German Transrapid. Peter and Octave collaborated to derive the expression for the electromagnetic force to complete Jeanne's explanation for the system's stability.

Furthermore, Peter developed the mathematical model involving induction coils, and conducted numerical simulations to plot the current as a function of the added mass in both our experimental and the actual Maglev configuration which allowed us to determine the critical mass for levitation. He then used the levitation height model function scaled to the Maglev train which gave a more feasible result.

Finally, Audrey listed the advantages and drawbacks of the current Maglev technology and discussed future prospects of Maglev trains.

All members actively participated in writing and revising the report to ensure consistency and accuracy. Audrey focused on the introduction, Peter wrote the abstract, the results discussion and this contributions section. Additionally, Jeanne compared the Electromagnetic Suspension (EMS) and Electrodynamic Suspension (EDS) systems, offering a detailed analysis of these technologies as well as wrote the conclusion.

Each member's contributions were integral to the project's success, ensuring a comprehensive approach to understanding and documenting electromagnetic levitation.

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