Complex Numbers

Advanced

$7 + 3i$

Real *Imaginary*

A Complex Number

A Complex Number is a combination of a
Real Number and an **Imaginary Number**

Real Numbers are numbers like:

1 12.38 −0.8625 3/4 √2 1998

Nearly any number you can think of is a Real Number!

Imaginary Numbers when **squared** give a **negative** result.

Normally this doesn't happen, because:

- when we square a positive number we get a positive result, and
- when we square a negative number we also get a positive result (because a negative times a negative gives a positive), for example **−2 × −2 = +4**

But just imagine such numbers exist, because we will need them.

The "unit" imaginary number (like 1 for Real Numbers) is i, which is the square root of −1

$$i = \sqrt{-1}$$

Because when we square i we get -1

$$i^2 = -1$$

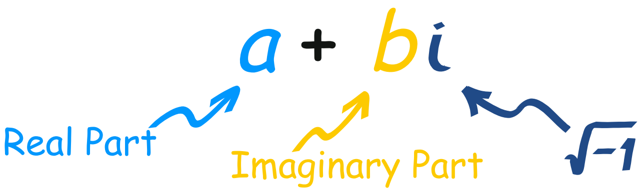
Examples of Imaginary Numbers:

$3i \quad 1.04i \quad -2.8i \quad 3i/4 \quad (\sqrt{2})i \quad 1998i$

And we keep that little "i" there to remind us we need to multiply by $\sqrt{-1}$

Complex Numbers

A Complex Number is a combination of a Real Number and an Imaginary Number:



Examples:

$1 + i \quad 39 + 3i \quad 0.8 - 2.2i \quad -2 + \pi i \quad \sqrt{2} + i/2$

Can a Number be a Combination of Two Numbers?

Can we make up a number from two other numbers? Sure we can!

We do it with fractions all the time. The fraction $^3/_8$ is a number made up of a 3 and an 8. We know it means "3 of 8 equal parts".

Well, a Complex Number is just **two numbers added together** (a Real and an Imaginary Number).

Either Part Can Be Zero

So, a Complex Number has a real part and an imaginary part.

But either part can be **0**, so all Real Numbers and Imaginary Numbers are also Complex Numbers.

Complex Number	Real Part	Imaginary Part
3 + 2i	3	2
5	5	**0**
−6i	**0**	−6

Complicated?

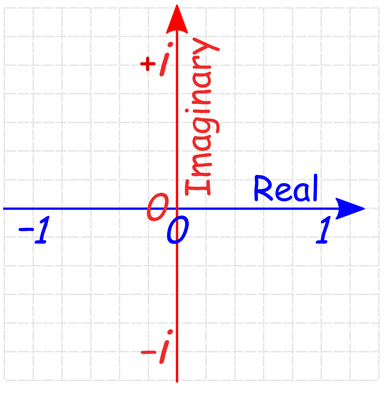
Complex does **not** mean complicated.

It means the two types of numbers, real and imaginary, together form a **complex**, just like a building complex (buildings joined together).

A Visual Explanation

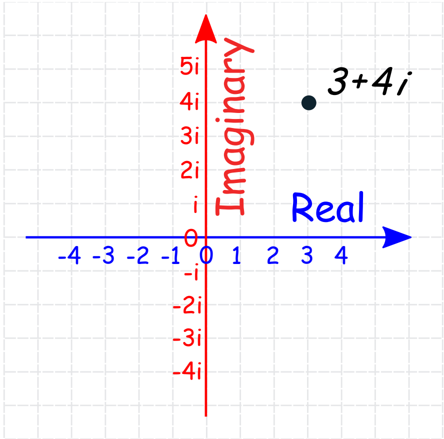
You know how the number line goes **left-right**?

Well let's have the imaginary numbers go **up-down**:



And we get the Complex Plane

And a complex number can now be shown as a point:



The complex number 3 + 4**i**

Adding

To add two complex numbers we add each part separately:

$$(a+b\mathbf{i}) + (c+d\mathbf{i}) = (a+c) + (b+d)\mathbf{i}$$

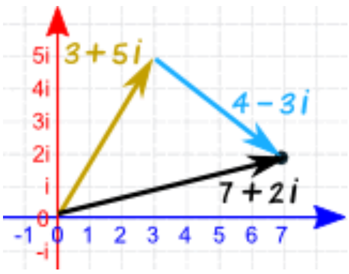
Example: add the complex numbers **3 + 2*i*** and **1 + 7*i***

- add the real numbers, and
- add the imaginary numbers:

$$(3 + 2i) + (1 + 7i)$$
$$= 3 + 1 + (2 + 7)i$$
$$= (4 + 9i)$$

Let's try one visually:

Example: add the complex numbers **3 + 5*i*** and **4 − 3*i***



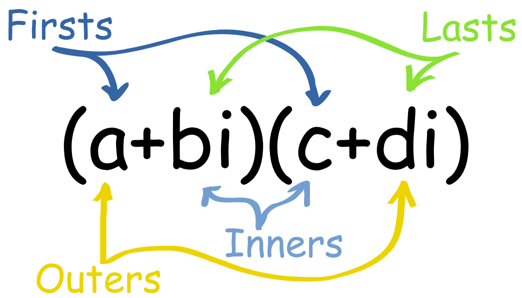
$$(3 + 5i) + (4 - 3i)$$
$$= 3 + 4 + (5 - 3)i$$
$$= 7 + 2i$$

Multiplying

To multiply complex numbers:

Each part of the first complex number gets multiplied by
each part of the second complex number

Just use "FOIL", which stands for "**F**irsts, **O**uters, **I**nners, **L**asts" (see [Binomial Multiplication]) for more details):



- Firsts: $\mathbf{a \times c}$
- Outers: $\mathbf{a \times d}\boldsymbol{i}$
- Inners: $\mathbf{b}\boldsymbol{i}\mathbf{ \times c}$
- Lasts: $\mathbf{b}\boldsymbol{i}\mathbf{ \times d}\boldsymbol{i}$

$$(a+b\boldsymbol{i})(c+d\boldsymbol{i}) = ac + ad\boldsymbol{i} + bc\boldsymbol{i} + bd\boldsymbol{i}^{\mathbf{2}}$$

Like this:

Example: (3 + 2i)(1 + 7i)

$$\begin{aligned}(3 + 2i)(1 + 7i) &= 3\times1 + 3\times7i + 2i\times1 + 2i\times7i \\ &= 3 + 21i + 2i + 14i^2 \\ &= 3 + 21i + 2i - 14 \qquad \text{(because } i^2 = -1\text{)} \\ &= -11 + 23i\end{aligned}$$

And this:

Example: $(1 + i)^2$

$$\begin{aligned}(1 + i)^2 = (1 + i)(1 + i) &= 1\times1 + 1\times i + 1\times i + i^2 \\ &= 1 + 2i - 1 \qquad \text{(because } i^2 = -1\text{)} \\ &= 0 + 2i\end{aligned}$$

But There is a Quicker Way!

Use this rule:

$$(a+b\mathbf{i})(c+d\mathbf{i}) = (ac-bd) + (ad+bc)\mathbf{i}$$

Example: $(3 + 2i)(1 + 7i) = (3\times1 - 2\times7) + (3\times7 + 2\times1)i = -11 + 23i$

Why Does That Rule Work?

It is just the "FOIL" method after a little work:

$$\begin{aligned} (a+b\mathbf{i})(c+d\mathbf{i}) &= ac + ad\mathbf{i} + bc\mathbf{i} + bd\mathbf{i}^2 && \text{FOIL method} \\ &= ac + ad\mathbf{i} + bc\mathbf{i} - bd && \text{(because } \mathbf{i}^2 = -1\text{)} \\ &= (ac - bd) + (ad + bc)\mathbf{i} && \text{(gathering like terms)} \end{aligned}$$

And there we have the $(ac - bd) + (ad + bc)\mathbf{i}$ pattern.

This rule is certainly faster, but if you forget it, just remember the FOIL method.

Let us try i^2

Just for fun, let's use the method to calculate i^2

Example: i^2

i can also be written with a real and imaginary part as $0 + i$

$$\begin{aligned} i^2 = (0 + i)^2 &= (0 + i)(0 + i) \\ &= (0\times0 - 1\times1) + (0\times1 + 1\times0)\mathbf{i} \\ &= -1 + 0\mathbf{i} \\ &= \mathbf{-1} \end{aligned}$$

And that agrees nicely with the definition that $i^2 = -1$

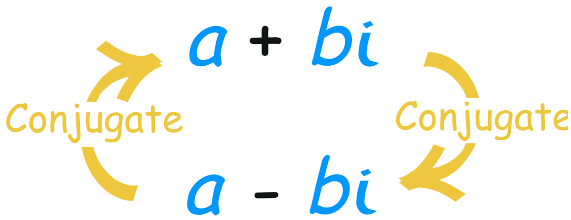
So it all works wonderfully!

Learn more at Complex Number Multiplication.

Conjugates

We will need to lnow about conjugates in a minute!

A conjugate is where we **change the sign in the middle** like this:



A conjugate is often written with a bar over it:

Example:

$$\overline{5 - 3\mathbf{i}} = 5 + 3\mathbf{i}$$

Dividing

The conjugate is used to help complex division.

The trick is to **multiply both top and bottom** by the **conjugate of the bottom**.

Example: Do this Division:

$$\frac{2 + 3\mathbf{i}}{4 - 5\mathbf{i}}$$

Multiply top and bottom by the conjugate of $4 - 5\mathbf{i}$:

$$\frac{2 + 3\mathbf{i}}{4 - 5\mathbf{i}} \times \frac{4 + 5\mathbf{i}}{4 + 5\mathbf{i}} = \frac{8 + 10\mathbf{i} + 12\mathbf{i} + 15\mathbf{i}^2}{16 + 20\mathbf{i} - 20\mathbf{i} - 25\mathbf{i}^2}$$

Now remember that $\mathrm{i}^2 = -1$, so:

$$= \frac{8 + 10\mathbf{i} + 12\mathbf{i} - 15}{16 + 20\mathbf{i} - 20\mathbf{i} + 25}$$

Add Like Terms (and notice how on the bottom $20\mathbf{i} - 20\mathbf{i}$ cancels out!):

$$= \frac{-7 + 22\mathbf{i}}{41}$$

We should then put the answer back into $a + b\mathbf{i}$ form:

$$= \frac{-7}{41} + \frac{22}{41}\mathbf{i}$$

DONE!

Yes, there is a bit of calculation to do. But it **can** be done.

Multiplying By the Conjugate

There is a faster way though.

In the previous example, what happened on the bottom was interesting:

$$(4 - 5\mathbf{i})(4 + 5\mathbf{i}) = 16 + 20\mathbf{i} - 20\mathbf{i} - 25\mathbf{i}^2$$

The middle terms cancel out!

And since $\mathbf{i}^2 = -1$ we ended up with this:

$$(4 - 5\mathbf{i})(4 + 5\mathbf{i}) = 4^2 + 5^2$$

Which is really quite a simple result

In fact we can write a general rule like this:

$$(a + b\mathbf{i})(a - b\mathbf{i}) = a^2 + b^2$$

So that can save us time when do division, like this:

Example: Let's try this again

$$\frac{2 + 3\mathbf{i}}{4 - 5\mathbf{i}}$$

Multiply top and bottom by the conjugate of $4 - 5\mathbf{i}$:

$$\frac{2+3\mathbf{i}}{4-5\mathbf{i}} \times \frac{4+5\mathbf{i}}{4+5\mathbf{i}} = \frac{8+10\mathbf{i}+12\mathbf{i}+15\mathbf{i}^2}{16+25}$$

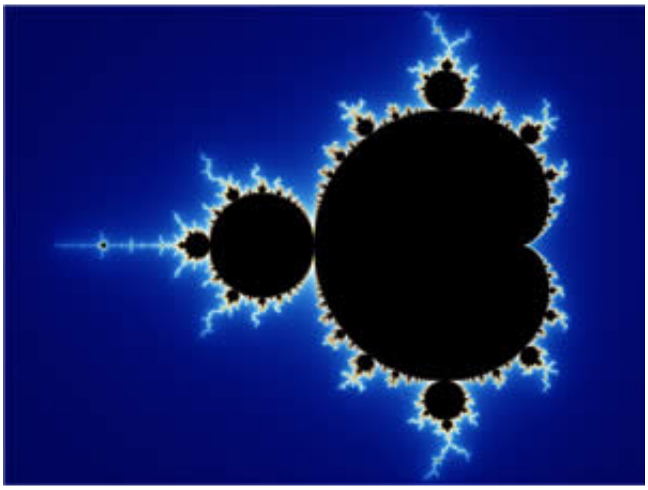
$$= \frac{-7+22\mathbf{i}}{41}$$

And then back into $a + b\mathbf{i}$ form:

$$= \frac{-7}{41} + \frac{22}{41}\mathbf{i}$$

DONE!

The Mandelbrot Set

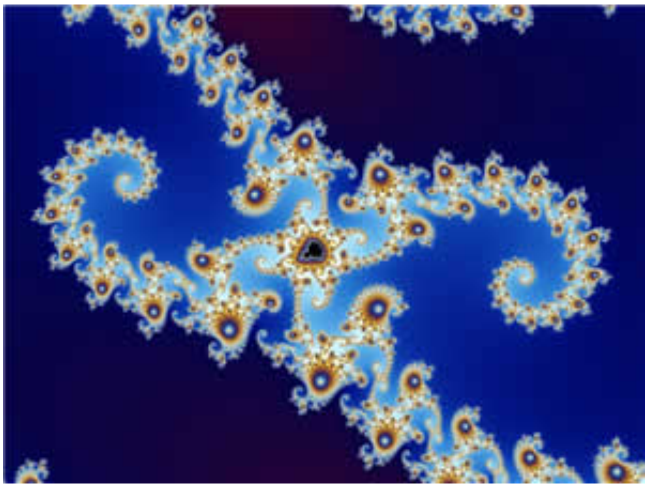


The beautiful Mandelbrot Set (pictured here) is based on Complex Numbers.

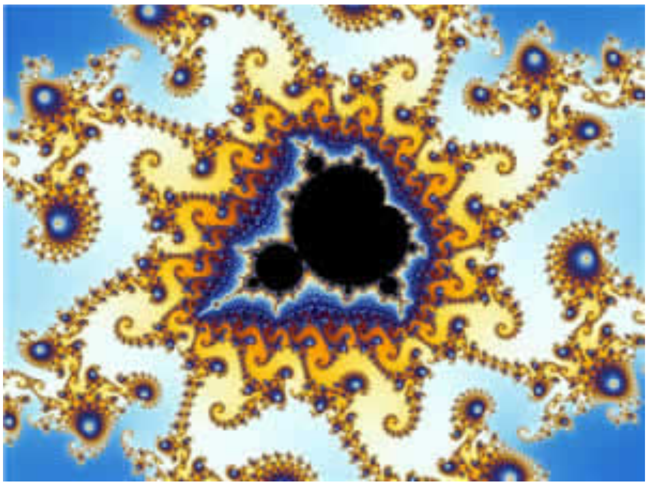
It is a plot of what happens when we take the simple equation $\mathbf{z}^2+\mathbf{c}$ (both complex numbers) and feed the result back into **z** time and time again.

The color shows how fast $\mathbf{z}^2+\mathbf{c}$ grows, and black means it stays within a certain range.

Here is an image made by zooming into the Mandelbrot set



And here is the center of the previous one zoomed in even further:



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Question 10
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