Computational Learning Theory

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2021 Spring

Thanks to the slides of Prof. Yu, Tian-Li from NTU.

Outline

- Sample Complexity
- Errors of a Hypothesis
- PAC Learnability
- Exhausting the Version Space
- Mistake Bounds

Computational Learning Theory

- What general laws constrain inductive learning?
- We seek theory to relate:
  - Complexity of hypothesis space considered by the learner
  - Accuracy to which target concept is approximated
  - Probability that the learner outputs a successful hypothesis
  - Manner in which training examples presented to the learner
- Goals:
  - Sample complexity: How many training examples are needed for successful learning?
  - Computational complexity: How much computational effort is needed for a learner to converge to a successful hypothesis?
  - Mistake bound: How many examples will the learner misclassify before the convergence?

Q1:

- Which of the following statements below is not the goal that computational learning theory want to achieve?
- (A) Learning successfully in polynomial time.
- (B) Finding out the upper and lower bound of error.
- (C) Deriving sample complexity.
- (D) All of the above.

Sample Complexity

- How many training examples are sufficient to learn the target concept?
- 3 settings:
  1. Learner proposes instances, as queries to teacher:
     Learner proposes instance x, teacher provides $c(x)$.
  2. Teacher provides training examples:
     Teacher provides sequence of examples of form $\langle x, c(x) \rangle$.
  3. Some random process (*e.g.*, nature) proposes instances:
     Instance x generated randomly, teacher provides $c(x)$.

Cross-validation

Sample Complexity: Setting 1

- Learner proposes instance x, teacher provides $c(x)$ (assume c is in learner's hypothesis space H)
- Optimal query strategy: play 20 questions
  - Pick instance x such that half of hypotheses in VS classify x positive, half classify x negative.
  - When this is possible, need $\lceil \log_2 |H| \rceil$ queries to learn c. => Best case
  - When not possible, need even more.

Sample Complexity: Setting 2

- Teacher (who knows c) provides training examples (assume c is in learner's hypothesis space H)
- Optimal teaching strategy: depends on H used by learner.
- Consider the case where H is conjunctions of up to n boolean literals (positive or negative).
  - e.g., $(AirTemp = Warm) \wedge (Wind = Strong)$, where $AirTemp, Wind, \ldots$ each has 2 possible values.
  - if n possible boolean attributes in H, $(n+1)$ examples suffice.
  - Why?

The size of hypothesis space (|H|) : 3^n (Attribute is +, -, or ?)
The number of examples: log(|H|) => Worst case

如果concept有don 't care? (1/2)

	A_1	A_2	A_3	.....	A_n
Concept:	+	-	?	?...	?

要學會這樣的concept，需要提供幾個example??

Step1: 學don' t care

A_1	A_2	A_3	.....	A_n	Class
+	-	+	+…	+	=> +
+	-	-	-…	-	=> +

需要兩個example來學所有的don't care

同時包含+ & - ，在conjunction做不到
=> 所以就會是don 't care

Step2: 學A_1只能是+ & A_2只能是-

+	+	+	+…	+	=> -
-	-	+	+…	+	=> -

如果concept有don　't care? (2/2)

A_1	A_2	A_3	.....	A_n
+	-	?	?...	?

要學會這樣的concept，需要提供幾個example??

Step1: 學don’　t care

A_1	A_2	A_3	.....	A_n	Class
+	-	+	+...	+	=> +
+	-	-	-...	-	=> +

n-k

花兩個example來學k個don　't care

Step2: 學A_1只能是+　& A_2只能是-

+	+	+	+...	+	=> -
-	-	+	+...	+	=> -

n-k個 example

Total example: n-k+2. If there is don’t care, k>=1　=> n-k+2 <=n+1

如果concept都沒有don 't care?

	A_1	A_2	A_3	.....	A_n
Concept:	+	+	+	+...	+

要學會這樣的concept，需要提供幾個example??

A_1	A_2	A_3	.....	A_n	Class
+	+	+	+...	+	=> +
-	+	+	+...	+	=> -
+	-	+	+...	+	=> -
+	+	-	+...	+	=> -
⋮					

1 example

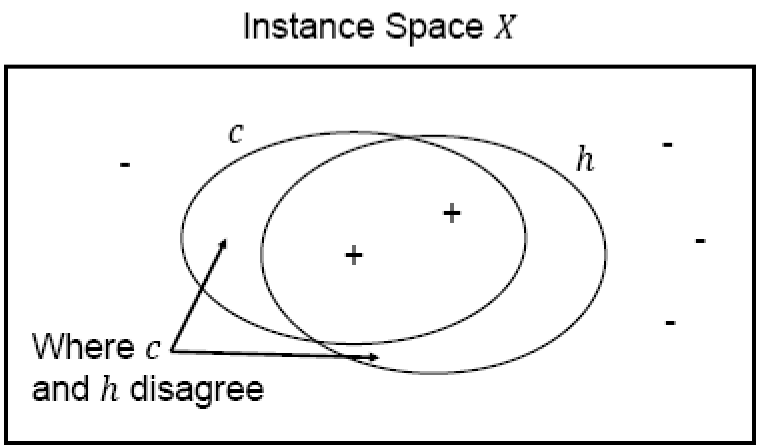
n example

Total example: n+1

Sample Complexity: Setting 3

- **Given:**
  - Set of instances X.
  - Set of hypotheses H.
  - Set of possible target concepts C.
  - Training instances generated by a fixed, unknown probability distribution $\mathbb{D}$ over X.
- Learner observes a sequence D of training examples of form $\langle x, c(x) \rangle$, for some target concept $c \in C$.
  - Instances x are drawn from distribution $\mathbb{D}$.
  - Teacher provides target value $c(x)$ for each x.
- Learner must output a hypothesis h estimating c
  - h is evaluated by its performance on subsequent instances drawn according to $\mathbb{D}$
- **Note:** randomly drawn instances, noise-free classifications.

True Error of a Hypothesis



Definition

The **true error** (denoted $error_{\mathbb{D}}(h)$) of hypothesis h with respect to target concept c and distribution $\mathbb{D}$ is the probability that h misclassifies an instance drawn at random according to $\mathbb{D}$.

$$error_{\mathbb{D}}(h) \equiv \Pr_{x \in \mathbb{D}} (c(x) \neq h(x))$$

Two Notations of Error

多常錯? =>100個training example 錯2個 =>2%

- Training error, denoted $error_D(h)$, of hypothesis h with respect to c: How often $h(x) \neq c(x)$ over training instances.

機率

- True error, denoted $error_{\mathbb{D}}(h)$, of hypothesis h with respect to c: How often $h(x) \neq c(x)$ over future random instances.

- Our concerns:

  Training error: 2% => True error不高於3%的機率是多少?

  - Can we bound the true error of h given its training error?
  - First consider when training error of h is zero (*i.e.*, $h \in VS_{H,D}$)

PAC Learning

- Consider a class C of possible target concepts defined over a set of instances X of length n, and a learner L using hypothesis space H.
- We desire that the learner probably learns a hypothesis that is approximately correct.

Definition

C is **PAC-learnable** by L using H if for all $c \in C$, distributions $\mathbb{D}$ over X, ϵ such that $0 < \epsilon < 1/2$, and δ such that $0 < \delta < 1/2$, learner L will with probability at least $(1 - \delta)$ output a hypothesis $h \in H$ such that $error_{\mathbb{D}}(h) \leq \epsilon$, in time that is polynomial in $1/\epsilon$, $1/\delta$, n and $size(c)$.

- To prove any concept is PAC-learnable or not, we need to derive the sample complexity needed for setting 3.

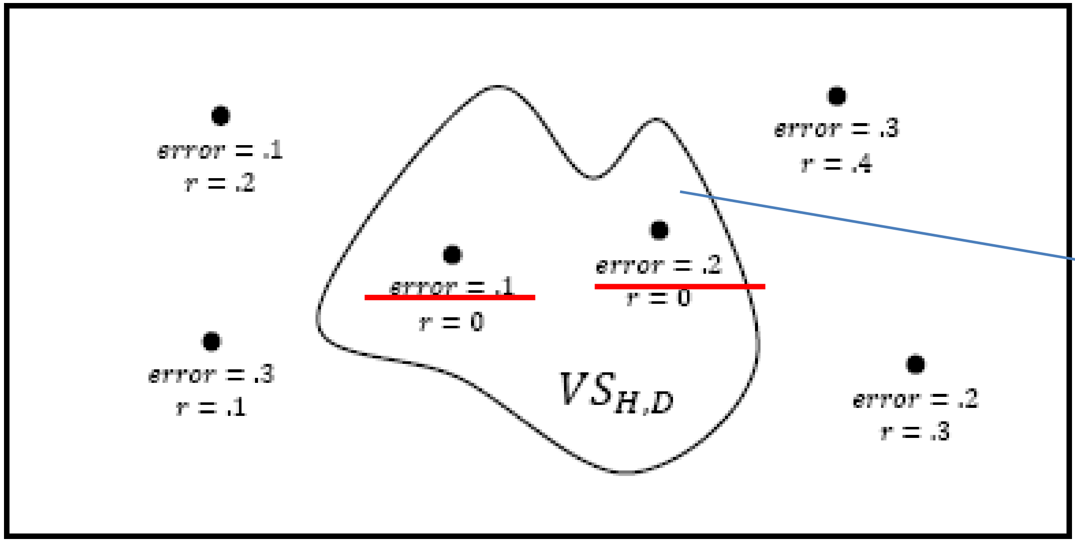
如果一個concept是PAC-learnable，代表此concept沒有很難，可以在夠短的時間內， 夠高的機率輸出一個夠準確的hypothesis

Q2:

- Which of the following statements is true about PAC learning?
- (A) The parameters ε should be less than ½.
- (B) The algorithm is expected to output a hypothesis that is approximately correct.
- (C) If the concept is PAC learnable, we can get an accurate hypothesis with a high enough probability in a short time.
- (D) All of the above.

Exhausting the Version Space

Hypothesis Space H



**
r: training error
error: true error

This version space is **0.3-exhausted**.

(r is training error, *error* is true error)

Definition

The version space $VS_{H,D}$ is ϵ-**exhausted** with respect to c and $\mathbb{D}$, if every hypothesis h in $VS_{H,D}$ has error less than ϵ with respect to c and $\mathbb{D}$.

$$(\forall h \in VS_{H,D})\ error_{\mathbb{D}}(h) < \epsilon$$

所有

Question

- Given training error is 0 (i.e. hypothesis is in version space), what is the true error?

- => How many examples can make version space

ε-exhausted?

Probability of Exhausting the Version Space

- How many examples ϵ-exhaust the VS?

Theorem (Haussler, 1988)

If H is finite, and D is a sequence of $m \geq 1$ independent random examples (from distribution $\mathbb{D}$) of some target concept c, then for any $0 \leq \epsilon \leq 1$, the probability that $VS_{H,D}$ is not ϵ-exhausted is less than or equal to

$$|H|e^{-\epsilon m}.$$

- The above theorem bounds the probability that any consistent learner will output a hypothesis h with $error_{\mathbb{D}}(h) \geq \epsilon$.
- If we want to this probability to be below δ

$$|H|e^{-\epsilon m} \boxed{\leq \delta} \overset{\log}{\Rightarrow} m \geq \frac{1}{\epsilon}(\ln|H| + \ln(1/\delta))$$

1-δ的機率輸出夠準確的 hypothesis 所需要的example

充分但不必要條件!!

Q3:

- Which of the following statements is true about the probability of the version space is not ε-exhausted?
- (A) By this theorem , we can know the most number of example drawn from distribution, that we can get a hypothesis such that the true error is large than or equal to ε.
- (B) According to this, we can infer that if, Pr will be large than or equal to $|H|e^{-\varepsilon m}$.
- (C) m is the symbol of the number of the examples.
- (D) The theorem is still true, if H is infinite.

Proof of ε-exhausting (1/2)

- What is the probability that version space is not ε-exhausted if m examples are given?

Proof: ϵ-exhausting the version space.

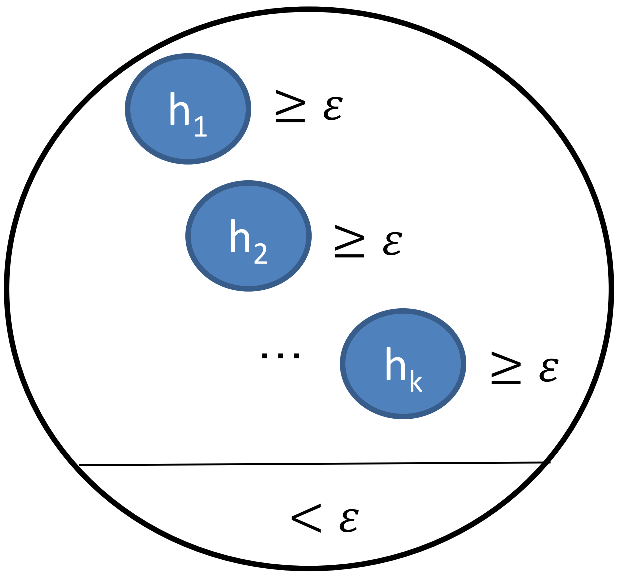
- Let $h_1, \cdots, h_k$ be all hypotheses in H with true errors greater than ϵ with respect to c.
- Fail to ϵ-exhausting the VS iff at least one of these hypotheses consistent with all m examples.
- Such prob. for a single hypothesis and a single random example is $(1-\epsilon)$; or $(1-\epsilon)^m$ for all m examples.
- The prob. that fail to ϵ-exhausting is at most $k(1-\epsilon)^m$.

For k 個hypothesis

$$k(1-\epsilon)^m \leq |H|(1-\epsilon)^m \leq |H|e^{-\epsilon m}$$

□

Proof of ε-exhausting (2/2)



k個h $error_{ID}(h_i) \geq \varepsilon$

$h(x_1)$: +
$c(x_1)$: +

h has to consistent with c
Otherwise, h is not in the version space.
The probability of h consistent with c based on x_1 is $1-\varepsilon$

$h(x_2)$: -
$c(x_2)$: -

The probability of h consistent with c based on x_2 is $1-\varepsilon$

⋮ m examples

After asking m times, the probability of h consistent with c is $(1-\varepsilon)^m$

Learning Conjunctions of Boolean Literals

- Recall that $m \geq \frac{1}{\epsilon}(\ln|H| + \ln(1/\delta))$ examples are sufficient to assure with probability at least $(1-\delta)$ that every h in $VS_{H,D}$ satisfies $error_{\mathbb{D}}(h) \leq \epsilon$.

- Suppose H contains conjunctions of constraints on up to n boolean attributes.
  - $|H| = 3^n$. Every attribute can be (+, -, don't care)
  - $m \geq \frac{1}{\epsilon}(n \ln 3 + \ln(1/\delta))$
  - Boolean conjunctions is PAC-learnable!

Polynomial in $\frac{1}{\varepsilon}$.

Polynomial in $\frac{1}{\delta}$.

Polynomial in n

EnjoySport Revisit

- Inn *EnjoySport*, if we consider only conjunctions, $|H| = 973$.

$$m \geq \frac{1}{\epsilon}(\ln 973 + \ln(1/\delta))$$

- If want to assure that with probability $\boxed{95\%}$, *VS* contains only hypotheses with $error_{\mathbb{D}}(h) \leq 0.1$, then it is sufficient to have m examples, where

$$m \geq \frac{1}{0.1}\left(\ln 973 + \ln \frac{1}{0.05}\right)$$

$$m \geq 98.8$$

⇒ m=99 就充分

⇒ 給99個example，就有95%以上的機率可以輸出一個true error<10%的hypothesis

Agnostic Learning (Learning Inconsistent Hypotheses)

- The equation $m \geq \frac{1}{\epsilon}(\ln|H| + \ln(1/\delta))$ tells us how many training examples suffice to ensure that every hypotheses in H having zero training error will have true error of at most ϵ.

C ≠H

- However, if $\boxed{c \notin H}$, zero training error may not be achievable.
- We desire to know how many examples suffice to ensure $error_{\mathbb{D}}(h) > error_D(h) + \epsilon$.
- **Hoeffding bounds:** $|\bar{X} - \mu|$

$$\Pr\left(error_{\mathbb{D}}(h) > error_D(h) + \epsilon\right) \leq e^{-2m\epsilon^2}$$

- Sample complexity in this case:

$$\Pr\left((\exists h \in H)\; error_{\mathbb{D}}(h) > error_D(h) + \epsilon\right) \leq \underline{|H|e}^{-2m\epsilon^2} \leq \delta$$

$$m \geq \frac{1}{2\epsilon^2}(\ln|H| + \ln(1/\delta))$$

H個

Infinite Hypothesis Space

- The above sample complexity has two drawbacks:
  1. Weak bounds.
  2. H has to be finite.
- We need another measure of the complexity of H.

Definition

A **dichotomy** of a set S is a partition of S into two disjoint subsets.

Definition

A set of instances S is **shattered** by hypothesis space H iff for every dichotomy of S there exists some hypothesis in H consistent with this dichotomy.

S ={a,b,c} => {a}
{b,c}

$h \in H$ {a}:+ {b,c}:-

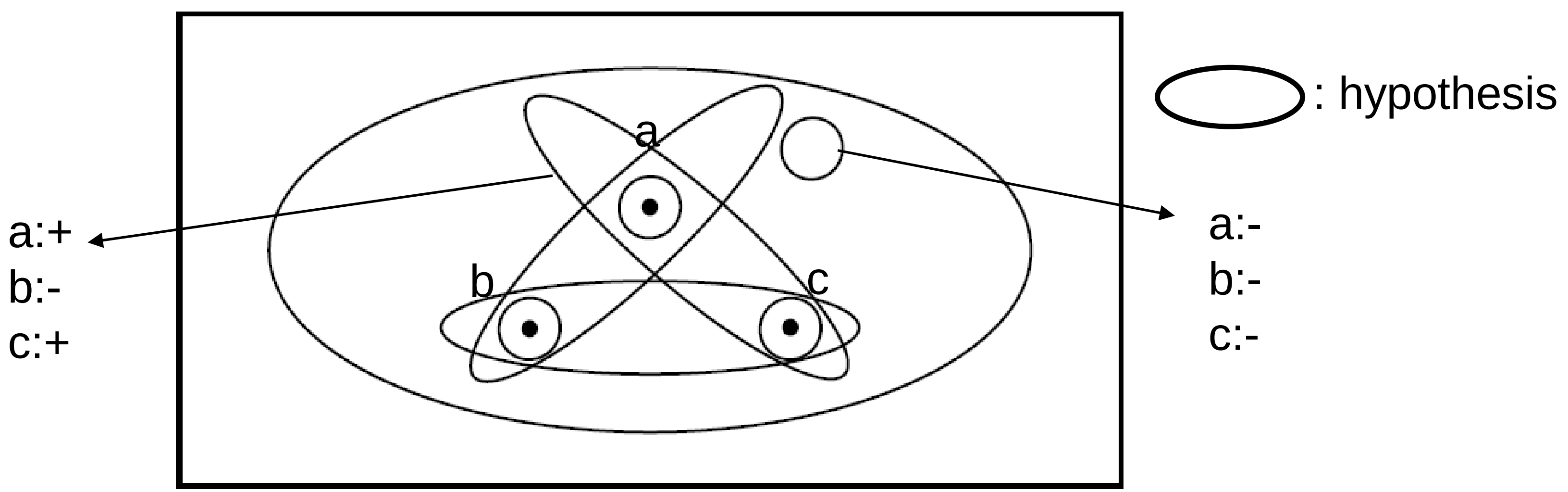
Shattering a Set of Instances (1/2)

- S is a subset of instances, $S \subseteq X$; $2^{|S|}$ distinct dichotomies in total.
- Each $h \in H$ imposes a dichotomy on S:

$$\{x \in S | h(x) = 0\} \text{ and } \{x \in S | h(x) = 1\}$$

- H shatters S iff every dichotomy of S is represented by some $h \in H$.

Instance Space X



a, b, c instances have 8 dichotomies.

=>如果8個dichotomies對應的h都在H裡
=>S is shattered by H

Shattering a Set of Instances (2/2)

- H shatter S => $|H| \geq 2^{|S|}$

a	b	C	
+	+	+	h_1
+	+	-	h_2
…			…
-	-	-	h_8

8個h
均屬於H

The Vapnik-Chervonenkis (VC) Dimension

- The ability to shatter a set of instances is closely related to the inductive bias of the hypothesis space.
- An unbiased hypothesis space can represent every possible concept (dichotomy) over X: An unbiased hypothesis space shatters X.
- What if H cannot shatter X, but can shatter a subset S?
- Intuitively, the larger S is, the more expressive H is.

Definition

The **Vapnik-Chervonenkis dimension**, $VC(H)$, of hypothesis space H is the size of the largest finite subset of instance space X shattered by H. If arbitrarily large finite sets of X can be shattered by H, then $VC(H) \equiv \infty$.

- Note that for any finite H, $VC(H) \leq \log_2 |H|$. $\Rightarrow |H| \geq 2^{|S|} \Rightarrow |H| \geq 2^{|VC(H)|}$
  =>雙邊取log

Why VC Dimension?

- Make VC dimension to define sample complexity.
- Since $m \geq log|\text{H}|$ is too weak, we will use VC Dimension to bound.

Q4:

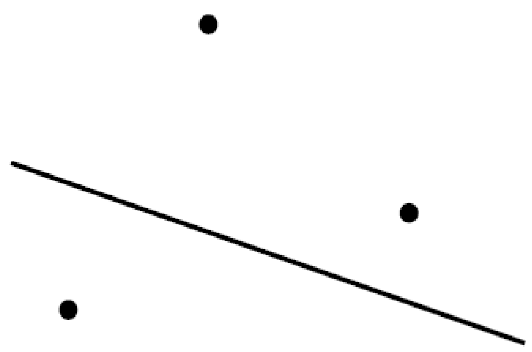
- Which of the following statements is the application of VC dimension?
- (A) The complexity of the model.
- (B) The accuracy of the prediction.
- (C) The speed of the computation.
- (D) The upper bound of the training examples.

VC Dimension (1/3)

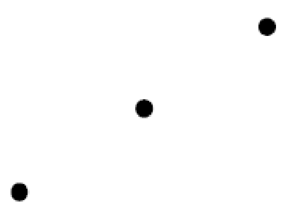
- Instances are real numbers: $X = \mathbb{R}$
- Hypotheses are real intervals: $h_{ab} = a < x < b$; $H = \{\forall a, b\ h_{ab}\}$
- Consider $S = \{3.1, 5.7\}$. H shatters S, why?
- For any set of 3 instances: $S = \{x, y, z\}$, where $x < y < z$. There is no way for H to represent this dichotomy: $\{x, z\}$ and $\{y\}$.

$$VC(H) = 2$$

- For 2D points (X) and line separations (H), $VC(H) = 3$.



(a)

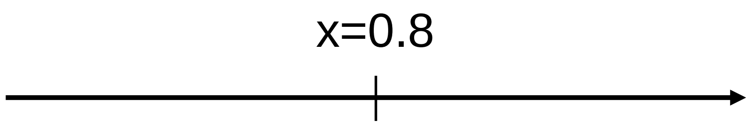


(b)

Example: 1 Instance on a Line

$X = \mathbb{R}$

$|H| = \infty$



{x} => Dichotomy: $\emptyset, \{x\}$

$\{x\}, \emptyset$

Is there h can make $\emptyset: +, \{x\}: -$? =>don' t include x: $h_{10,20}$

Is there h can make $\{x\}: +, \emptyset: -$? =>include x: $h_{0,1}$

$h_{10,20}$ and $h_{0,1}$ are belong to H => H shatter {x}

VC(H)=? $VC(H) \geq 1$

Example: 2 Instances on a Line

$X = \mathbb{R}$
$|H| = \infty$

Dichotomy: 4 =>	a	b
	+	+
	+	-
	-	+
	-	-

Is there h can get $+ \ +$? => Include a and b: $h_{5,5}$

Is there h can get $+ \ -$? =>Include a and not include b: $h_{-5,0.5}$

Is there h can get $- \ +$? =>not include a and include b: $h_{0.5,5}$

Is there h can get $- \ -$? =>not include a and b: $h_{20,40}$

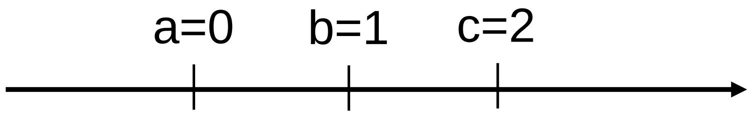
All h are belong to H => H shatter {a,b}

VC(H)=? $VC(H) \geq 2$

Example: 3 Instances on a Line

$X = \mathbb{R}$

$|H| = \infty$



Dichotomy: 8

Is there h can get $+ \quad - \quad +$? => Include a, c and not include b:??

=> We cannot get a "h" to shatter **any** 3 instances in the line.

By definition of VC, we have to shatter "every" dichotomy

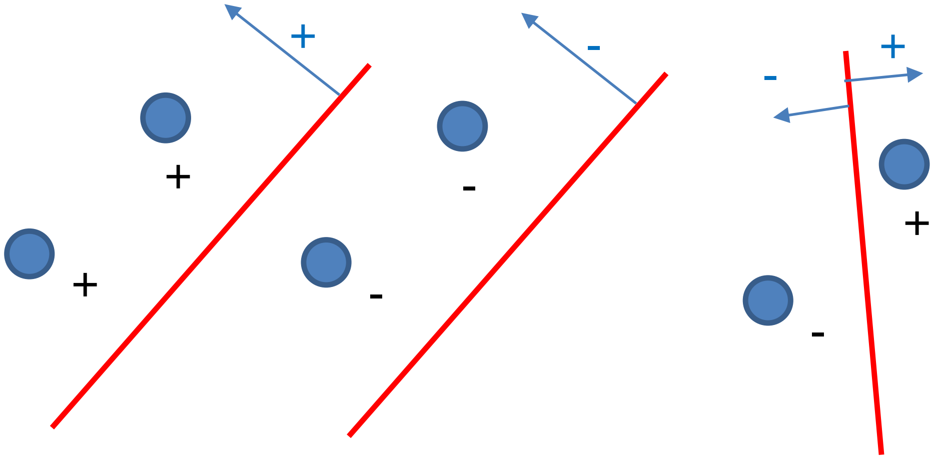
=> $\mathrm{VC(H)} \neq 3$

=> $VC(H) = 2$

Example: Linear Classifier with 2 Instances

$X = \mathbb{R}^2 = \{(x,y) | x,y \in R\}$

$m(H) = \{(x,y) | ax+by+c \geq 0,\ a,b,c \in R\}$



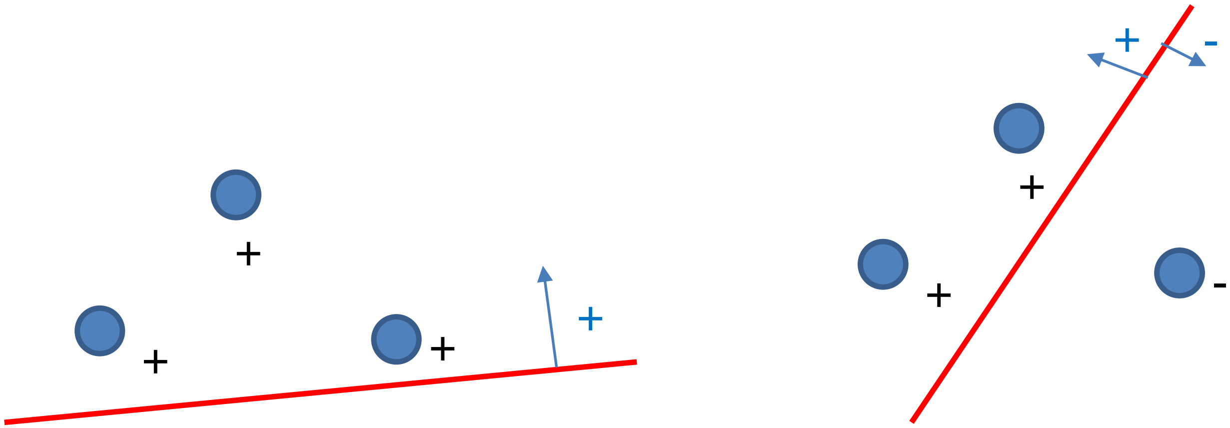
VC(H)=?

$\Rightarrow VC(H) \geq 2$

Example: Linear Classifier with 3 Instances

$X = \mathbb{R}^2 = \{(x,y) | x,y \in R\}$

$m(H) = \{(x,y) | ax+by+c \geq 0, a,b,c \in R\}$



VC(H)=?

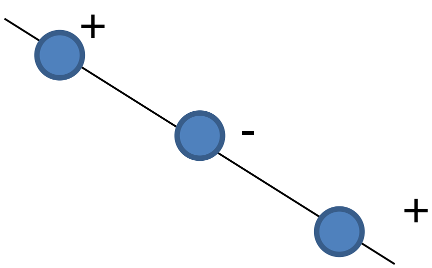
$\Rightarrow VC(H) \geq 3$

Example: Linear Classifier with 3 Instances

$$X = \mathbb{R}^2 = \{(x,y) | x,y \in R\}$$

$$m(H) = \{(x,y) | ax+by+c \geq 0,\ a,b,c \in R\}$$

If 3 instances are on a line??



We cannot find a linear classifier to shatter 3 instances on a line.
So $VC(H) \geq 2$??

Definition

The **Vapnik-Chervonenkis dimension**, $VC(H)$, of hypothesis space H is the size of the largest finite subset of instance space X shattered by H. If arbitrarily large finite sets of X can be shattered by H, then $VC(H) \equiv \infty$.

So $VC(H) = 3$

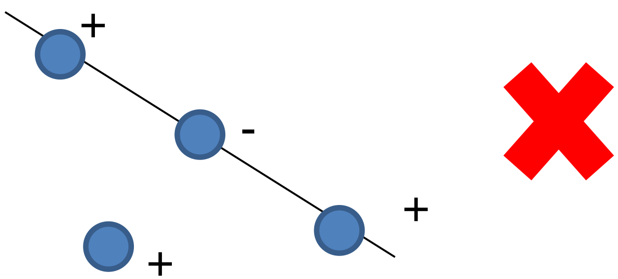
Q5:

- Consider the case on the 2D plane. VC(H)=?
- (A) 2
- (B) 3
- (C) 4
- (D) 8

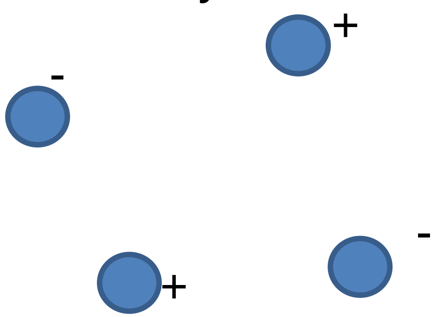
Example: Linear Classifier with 4 Instances

$$X = \mathbb{R}^2 = \{(x,y) | x,y \in R\}$$
$$m(H) = \{(x,y) | ax+by+c \geq 0,\ a,b,c \in R\}$$

Case 1: Any 3 instances are on a line.



Case 2: Any 3 instances are not on a line.



Dichotomy: 16
$\Rightarrow$ There is one dichotomy cannot be shattered.
$\Rightarrow$ XOR problem.

VC(H)=?

=> $VC(H) \neq 4$

=> $VC(H) = 3$

Linear Classifier in n Dimension

- Linear classifier in n dimension => In general, the VC is n+1

VC Dimension and Sample Complexity

- How many randomly drawn examples suffice to ϵ-exhaust $VS_{H,D}$ with probability at least $(1-\delta)$? [Blumer *et al.*, 1989]

充分但不必要條件!!

Upper bound on sample complexity

$$m \geq \frac{1}{\epsilon}\left(4\log_2\frac{2}{\delta} + 8\,\underline{VC(H)}\log_2\frac{13}{\epsilon}\right)$$

$m \geq \frac{1}{\epsilon}(\ln|H| + \ln(1/\delta))$

- Similarly, m grows with $\log(1/\delta)$.
- Now, m grows with $(1/\epsilon)\log(1/\epsilon)$ rather than linear.
- Most importantly, $\ln|H|$ is replaced by $VC(H)$. Recall that $VC(H) \leq \log_2|H|$.

VC Dimension and Sample Complexity

- How about lower bound? [Ehrenfeucht *et al.*, 1989]

Lower bound on sample complexity

Consider any concept C where $VC(C) \geq 2$, any learner L, any $0 < \epsilon < \frac{1}{8}$, and $0 < \delta < \frac{1}{100}$. There exists a distribution $\mathbb{D}$ and target concept in C such that if L observes fewer examples than

Upper bound正比於VC(C)
Lower bound也正比於VC(C)

$$\max\left\{\frac{1}{\epsilon}\log_2(1/\delta), \frac{VC(C)-1}{32\epsilon}\right\}$$

then with prob. at least δ, L outputs a hypothesis h having $error_{\mathbb{D}}(h) > \epsilon$.

- Given the lower bound, we see that the upper bound in the previous slide is fairly tight.

Mistake Bounds

- So far, we discuss “How many examples you need to learn an accurate concept?”
- Now, we want to change the scenario.
- I give you an example without answer.
- Learner predict the result is positive or negative.
- And I tell you the answer.
- So under this scenario, how many errors will you encounter?

Mistake Bound for Find-S

- Consider FIND-S when H are conjunctions of n boolean literals $\ell_1, \cdots, \ell_n$.

FIND-S

- Initialize h to the most specific hypothesis

$\emptyset = \ \ell_1 \wedge \neg\ell_1 \wedge \ell_2 \wedge \neg\ell_2 \ldots \ell_n \wedge \neg\ell_n$

- For each positive training instance x
  - Remove from h any literal that is not satisfied by x
- Output hypothesis h.

- How many mistakes before converging to correct h?
  - Provided $c \in H$, FIND-S never misclassifies negative examples.
  - The first positive example reduce the $2n$ literals to n.
  - Then every misclassified positive examples removes at least one literal.
  - At most $(n+1)$ mistakes.

FIND-S Example

$\emptyset = \quad \ell_1 \wedge \neg\ell_1 \wedge \ell_2 \wedge \neg\ell_2 \ldots \ell_n \wedge \neg\ell_n$

Example x_1:

L_1	L_2	L_3	…..	Class
+	-	+	….	+

$h_1 = \quad \ell_1 \wedge \cancel{\neg\ell_1} \wedge \cancel{\ell_2} \wedge \neg\ell_2 \cancel{\neg l_3} \ell_n \wedge \neg\ell_n$

h_1 becomes x_1

Example x_2:

L_1	L_2	L_3	…..	Class
-	-	+	….	+

$h_2 = \quad \cancel{\ell_1} \wedge \cancel{\neg\ell_1} \wedge \cancel{\ell_2} \wedge \neg\ell_2 \cancel{\neg l_3} \ell_n \wedge \neg\ell_n$

Original hypothesis 2n $\longrightarrow$ 1st mistake: n $\longrightarrow$ 2nd mistake: -1 $\longrightarrow$ $\underbrace{\ldots}_{\text{Most: n times}}$

Q6

- Which of the following statement is true about the FIND-S algorithm for mistake bound?
- (A) Initially, we set h to the most general hypothesis.
- (B) If the concept c is in hypothesis space, the FIND-S probably misclassifies negative examples.
- (C) After first iteration, hypothesis space will become half of the original one.
- (D) There will be at most n mistake before finding the correct h.

Mistake Bound for Halving Algorithm

- Consider the HALVING Algorithm:
  - Learn concept with version space such as the CANDIDATE-ELIMINATION algorithm
  - Classify new instances by majority vote of version space members

70:+
30:-　=>+　=> Remove 30

- How many mistakes before converging to correct h?
  - Worst case: $\lfloor \log_2 |H| \rfloor$, why?
  - Best case: 0, why?

Original hypothesis space |H| $\longrightarrow$ 1st mistake: |H|/2 $\longrightarrow$ 2nd mistake: |H|/4 $\longrightarrow$ … $\longrightarrow$ Most: $\lfloor \log_2 |H| \rfloor$

Optimal Mistake Bound

- We define the mistake bound based on a specific algorithm.
- What is about the general case?

Optimal Mistake Bound

- Interested in the optimal mistake bound for an arbitrary concept class C, assuming $H = C$.
- Define $M_A(c)$ as the maximum over all possible sequence of training examples of the number of mistakes made by algorithm A and the target concept c.
- For any nonempty concept class C, define $M_A(C) = \max_{c \in C} M_A(c)$.

小 c 屬於大 C 裡面最難最難的那一個

Definition

Let C be an arbitrary nonempty concept class. The **optimal mistake bound** for C, denoted $Opt(C)$, is the minimum over all possible learning algorithms A of $M_A(C)$.

$\min_A$:最聰明的那一個演算法

$$Opt(C) = \min_A M_A(C)$$

最聰明的那一個演算法在最困難的 concept，
concept class 裡面最難的那個 concept 裡面最糟的 sequence，所犯的錯誤

Bounds for Optimal Mistake Bound

- $VC(C) \leq Opt(C) \leq \log_2 |C|$ (Littlestone, 1987)

Proof.

Right: $Opt(C) \leq M_{\textsc{Halving}}(C) \leq \log_2 |C|$

Left (Adversarial):

1. Let $S = \{x_1, \ldots, x_{VC(C)}\} \subseteq X$ be a shattered set.
2. Suppose the environment reveals $x_i \in S$, and the algorithm outputs $\hat{y}_i$.
3. The environment selects a new target concept $c \in C$ such that $c(x_i) = y_i \neq \hat{y}_i$. 要唱反調，跟你預測的答案不同
4. Since S is shattered by C, there always exists such c, and no way the algorithm can tell the difference.
5. Therefore, the algorithm makes at least $VC(C)$ mistakes. $\square$

Example

Answer: 1234

Guess: 1567 => 1A

Guess: 1234 => I don’t want you to win so fast. I change the answer to 8097

Another guess

⋮

How many times can you change the answer?

Q7

- Which of the following statements is correct?
- (A) The algorithm makes at least VC(C) (assuming C=H).
- (B) MA(C) means the hardest concept to learn in C.
- (C) Worst case for the Halving algorithm is $\log_2|H|$, which is the upper bound of the mistakes.
- (D) All of the above.

Weighted-Majority Algorithm

WEIGHTED-MAJORITY

a_i: prediction algorithms; w_i: weights, initialized to all 1; $0 \le \beta < 1$

for each training example $\langle x, c(x) \rangle$
    $q_0 = 0$; $q_1 = 0$
    for each algorithm a_i
        If $a_i(x) == 0$ **then** $q_0 = q_0 + w_i$
        If $a_i(x) == 1$ **then** $q_1 = q_1 + w_i$
    If $q_0 > q_1$ **then** predict $\hat{c}(x) = 0$
    If $q_0 < q_1$ **then** predict $\hat{c}(x) = 1$
    If $q_0 == q_1$ **then** predict $\hat{c}(x) = 0$ or 1 at random
    for each algorithm a_i
        each $a_i(x) \neq c(x)$ **then** $w_i = \beta w_i$.

β is usually set to be 0.5

- Note that β is 0, WEIGHTED-MAJORITY reduces to HALVING.

Mistake Bound for Weighted-Majority

- For any sequence of training examples D, let A be any set of n prediction algorithms, and let k be the minimum number of mistakes made by any algorithm in A over D. The number of mistakes over D made by WEIGHTED-MAJORITY with $\beta = 1/2$ is at most

$$\boxed{2.4}(k + \log_2 n).$$

Proof.

- Let a_j be the best algorithm which yields k; its final weight $w_j = \frac{1}{2^k}$.
- Consider the sum $W = \sum_i w_i$. W initially n.
- Each mistake reduces W to at most $\frac{3}{4}W$.
- Let M be the total number of mistakes of WEIGHTED-MAJORITY.
- The final W is at most $n\left(\frac{3}{4}\right)^M$. So $\left(\frac{1}{2}\right)^k \leq n\left(\frac{3}{4}\right)^M$ □

1=>1/2=>1/4 =>k times=>$1/2^k$

聽一半的人，這一半的人犯錯，weight砍半，所以砍掉1/4

W -> 3/4 W -> 9/16 W

n -> 3/4 n -> 9/16 n->…..->$n(3/4)^M$

W/4	Correct	W/4	Correct	W/4
W/4		W/4	Wrong	W/8
W/4	Wrong	W/8	Correct	W/8
W/4		W/8	Wrong	W/16
Total: W		Total: 3/4W		Total: 9/16W

Q8

- Consider the Weighted-Majority algorithm with $\beta = 1/2.$

What is the total number's upper bound of the mistake? (where n is the number of total algorthis, and K is the minimum number of mistakes.)

(A) 2.4K

(B) 2.4K+2.4ln(n)

(C) 2.4[K+log(n)]

(D) $2.4K+2.4\log_2(n)$

Summary

- PAC considers algorithms that learns target concept using training examples randomly drawn from an unknown but fixed distribution.
- PAC: with high probability ($1-\delta$), the learner outputs a hypothesis that is approximately correct (within error ϵ) within computational time polynomial in $1/\delta$, $1/\epsilon$, the size of instances, and the size of target concept.
- For finite hypothesis spaces, sample complexity can be derived for a consistent and agnostic learners, respectively.
- VC dimension measures the expressiveness of a hypothesis space, and an alternative (usually tighter, and for infinite hypothesis space) upper bound is derived using VC-dimension.
- Optimal mistake is bounded by VC-dimension and HALVING.
- The number of mistakes of WEIGHTED-MAJORITY is bounded by its best predictor.