

# Center for Statistics and the Social Sciences

## Math Camp 2021

### Lecture 1: Algebra, Functions, & Limits

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# A typical day

## Schedule

- **Before class:** Review the posted lectures and challenge problems
- **10:00am-10:45am** Review lectures and practice problems (First day, introduction)
- **10:45am-11:30am** Breakout Rooms: Practice problems
- **1:30pm-3:00pm** R labs
- **3:00pm-4:00pm** Additional problem session/office hours (if needed)

# Lecture material

- Will be reviewed in brief each morning.
- Set realistic goals.
- Be patient with yourself.
- Communicate with us early and often.

# Intros

- Name/how you'd like to be addressed
- Program
- One goal for math camp
- If you'd like: one thing you're nervous about!

# Plans

- Breakout rooms
- Pre-lab quiz
- Feedback form
- Speed up/slow down

# Day 1

## Outline

- Math notation
- Order of operations
- Equation of a line
- Functions, domain, range, examples
- Function transformations
- Rules of exponents, logarithms
- Continuous and piecewise functions
- Limits

# Notation

## Real Numbers

- Any number that falls on the continuous line. Often represented by  $a, b, c, d$
- Examples:  $2, 3.234, 1/7, \sqrt{5}, \pi$
- The set of real numbers is denoted by  $\mathbb{R}$ . Then  $a \in \mathbb{R}$  means  $a$  is in the set of real numbers.

## Integers

- Any whole number. Often represented by  $i, j, k, l$
- Examples:  $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$

## Variables

- Can take on different values
- Often represented by  $x, y, z$

# Notation

## Functions

- Often represented by  $f, g, h$
- Examples:  $f(x) = x^2 + 3$ ,  $g(y) = 6y^2 - 2y$ ,  $h(z) = z^3$

## Summations

- Often represented by  $\sum$  and summed over some integer
- Example:

$$\sum_{i=1}^3 (i+1)^2 = (1+1)^2 + (2+1)^2 + (3+1)^2 = 2^2 + 3^2 + 4^2 = 29$$

## Products

- Often represented by  $\prod$  and multiplied over some integer
- Example:  $\prod_{k=1}^3 (y_k + 1)^2 = (y_1 + 1)^2 \times (y_2 + 1)^2 \times (y_3 + 1)^2$



# Order of Operations

Please **E**xcuse **M**y **D**ear **A**unt **S**ally

- **P**arentheses
- **E**xponents
- **M**ultiplication
- **D**ivision
- **A**ddition
- **S**ubtraction

# Order of Operations

## Examples

When looking at an expression, work from the left to right following **PEMDAS**. Note: multiplication and division are interchangeable; addition and subtraction are interchangeable.

- $\left((1 + 2)^3\right)^2 = (3^3)^2 = 27^2 = 729$
- $4^3 \cdot 3^2 - 10 + 27/3 = 64 \cdot 9 - 10 + 9 = 576 - 10 + 9 = 575$
- $(x + x)^2 - 2x + 3 = (2x)^2 - 2x + 3 = 4x^2 - 2x + 3$

# Fractions

## Multiplying & Dividing

Fractions are used to describe parts of numbers. They are comprised of two parts:

$$\frac{\text{numerator}}{\text{denominator}}$$

Examples:  $\frac{2}{3}, \frac{16}{4}(=4), \frac{2}{4} = \frac{1}{2}, \frac{8}{1}(=8)$ .

**Multiplication:** Multiply the numerators; multiply the denominators. Examples:  $\frac{1}{2} \times \frac{3}{4} = \frac{1 \cdot 3}{2 \cdot 4} = \frac{3}{8}$

**Division:** Best to change it into a multiplication problem by multiplying the top fraction by the inverse of the bottom fraction.

Examples:  $\frac{1/2}{7/8} = \frac{1}{2} \times \frac{8}{7} = \frac{1 \cdot 8}{2 \cdot 7} = \frac{8}{14}$ .

Simplify:  $\frac{8}{14} = \frac{2 \cdot 4}{2 \cdot 7} = \frac{2}{2} \times \frac{4}{7} = 1 \times \frac{4}{7} = \frac{4}{7}$

# Fractions

## Adding & Subtracting

Adding and subtracting requires that fractions must have the same denominator. If not, we need to find a common denominator (a larger number that has both denominators as factors) and convert the fractions. Then add (or subtract) the two numerators.

Examples:  $\frac{1}{7} + \frac{4}{7} = \frac{5}{7}$

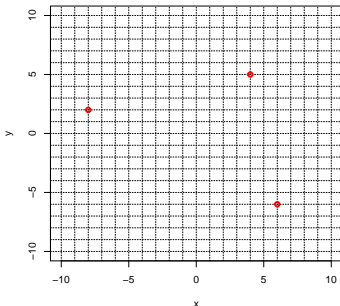
$$\frac{1}{3} + \frac{1}{4} = \frac{1}{3} \times \frac{4}{4} + \frac{1}{4} \times \frac{3}{3} = \frac{1 \cdot 4}{3 \cdot 4} + \frac{1 \cdot 3}{4 \cdot 3} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$$

$$\frac{17}{20} - \frac{3}{4} = \frac{17}{20} \times \frac{1}{1} - \frac{3}{4} \times \frac{5}{5} = \frac{17 \cdot 1}{20 \cdot 1} - \frac{3 \cdot 5}{4 \cdot 5} = \frac{17}{20} - \frac{15}{20} = \frac{2}{20} = \frac{1}{10}$$

# Coordinate plane

- The collection of all points  $(x, y)$ , such that  $x \in (-\infty, \infty)$  and  $y \in (-\infty, \infty)$ .
- **Coordinates**  $(x, y)$  provide an “address” for a point in  $\mathbb{R}^2$ .
- The point  $(0,0)$  is where the  $x$  and  $y$  axes intersect and is called the **origin**.
- Other names: Cartesian plane, two-dimensional (2-D) space,  $\mathbb{R}^2$

**Examples:**  $(-8,2), (4,5), (6,-6)$



# Equation of a Line

## Linear Equations

If we have two pairs of points  $(x_1, y_1)$ ,  $(x_2, y_2)$ , we can find a line between the two points.

A common equation for a line is:

$$y = mx + b$$

where  $m$  is the **slope** and  $b$  is the **y-intercept**. A line is also a way to define a variable  $y$  in terms of another variable  $x$ .

Another common form (often used in the regression setting) is

$$y = \beta_0 + \beta_1 x$$

, where  $\beta_0$  is the **y-intercept** and  $\beta_1$  is the **slope**.

# Slopes

The **slope** is the ratio of the difference in the  $y$ -values to the difference in the two  $x$ -values for any two points on a line. Commonly referred to as **rise** over **run**.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- $m$  measures of the steepness of a line, e.g. how high does the line “rise” in “ $y$ -land” when we move one unit to the “right” (toward  $\infty$ ) in “ $x$ ”-land.
- The sign of  $m$  indicates whether we’re going “uphill” (+) or “downhill” (-) when we move to the “right” in “ $x$ ”-land.

# Intercepts

The **intercept**, often denoted  $b$ , is the value of  $y$  when  $x = 0$ .

- i.e. every line (that isn't a vertical line) has a point  $(0, b)$ .
- the vertical height where the line crosses the  $y$ -axis.

Find the intercept by plugging in one point on the line and the slope into the equation and then solving for the intercept.

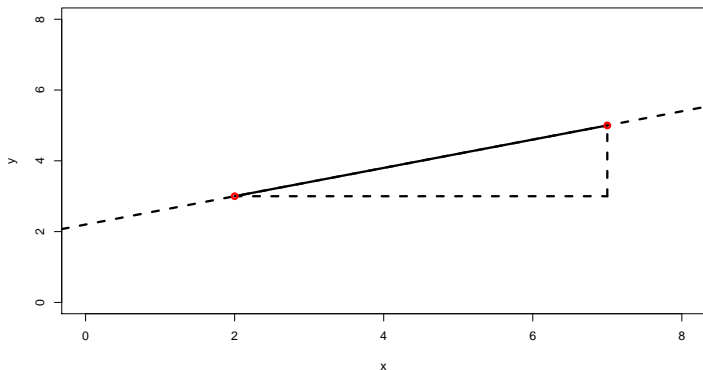
$$y_1 = m \cdot x_1 + b \Rightarrow b = y_1 - m \cdot x_1$$

In a simple linear regression setting  $\beta_0$  can be interpreted as the average value of a dependent variable,  $y$ , when the dependent variable  $x$  is equal to 0, *if* 0 is a observed or sensible value of your independent variable.



## Find the equation of a line using two points

- **Points:**  $(2, 3), (7, 5)$ :
- **Slope:**  $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{7 - 2} = \frac{2}{5}$
- **Intercept:**  $b = y_1 - mx_1 = 3 - \frac{2}{5} \cdot 2 = 3 - \frac{4}{5} = \frac{11}{5}$
- **Equation of the line:**  $y = \frac{2}{5}x + \frac{11}{5}$



# Functions and their Limits

A **function** is a formula or rule of correspondence that maps each element in a set  $X$  to an element in set  $Y$ .

The **domain** of a function is the set of all possible values that you can plug into the function. The **range** is the set of all possible values that the function  $f(x)$  can return.

Examples:

$$f(x) = x^2$$

- **Domain:** all real numbers  $\mathbb{R}$
- **Range:** zero and all positive real numbers,  $f(x) \geq 0$

# Functions and their Limits

## Examples continued

$$f(x) = \sqrt{x}$$

- **Domain:** zero and all positive real numbers,  $x \geq 0$
- **Range:** zero and all positive real numbers,  $x \geq 0$

$$f(x) = 1/x$$

- **Domain:** all real numbers except zero
- **Range:** all real numbers except zero

# Solving Linear Equations

Often we would like to find the **root** of a linear equation. This is the value of  $x$  that maps  $f(x)$  to 0 (where the line crosses the  $x$ -axis, or the value of  $x$  when  $y = 0$ ).

$$f(x) = mx + b$$

Setting  $f(x) = 0$ , to find the root we need to solve for  $x$ .

$$\begin{array}{ll} 0 = mx + b & \text{[subtract } b \text{ from both sides]} \\ -b = mx & \text{[divide both sides by } m\text{]} \\ \frac{-b}{m} = x & \end{array}$$

The value  $-b/m$  is the **root** of  $f(x) = mx + b$ , i.e. most lines (except horizontal lines) have a point  $(\frac{-b}{m}, 0)$  on them.

# Solving Linear Equations

Why do we do operations on both sides?

On the previous slide, we subtracted  $b$  from both sides or added  $-b$  to both sides. Why is that okay?

$$\begin{aligned}0 &= mx + b \\ \Rightarrow 0 &= mx + b + (\textcolor{red}{b} - \textcolor{blue}{b}) \\ \Rightarrow -\textcolor{red}{b} + 0 &= mx + (b - \textcolor{blue}{b}) \\ \Rightarrow -\textcolor{red}{b} &= mx + 0 \\ \Rightarrow -\textcolor{red}{b} &= mx\end{aligned}$$

The number zero is called the **additive identity**. For any number  $a \in \mathbb{R}$ ,

$$a + 0 = a.$$

# Solving Linear Equations

Why do we do operations on both sides?

Then, we divided both sides by  $m$  or multiplied both sides by  $\frac{1}{m}$ .  
Why is that okay?

$$\begin{aligned} -b &= mx \\ \Rightarrow -b &= mx \cdot \frac{1/m}{1/m} \\ \Rightarrow -b \cdot \frac{1}{m} &= mx \cdot \frac{1}{m} \\ \Rightarrow \frac{-b}{m} &= x \end{aligned}$$

The number one is called the **multiplicative identity**. For any number  $a \in \mathbb{R}$ ,

$$a \times 1 = a.$$

# Solving Linear Equations

## Examples

We may be interested in solving linear equations for values other than zero.

Say you are at the Garage on Capitol Hill (pre-Covid) and you have \$40.00 with you. If shoes are \$7.00 and a lane is \$11.00/hr how long can you bowl?

Let's take  $x$  is hours and  $f(x)$  total price.

$$f(x) = 7 + 11x$$

How long can you bowl?

$$40 = 11x + 7$$

$$40 - 7 = 11x$$

$$33 = 11x$$

$$33/11 = 3 = x$$

# Solving Systems of Linear Equations

We often are interested in finding the **intersection** of two lines or the point  $(x, y)$  where two lines cross. This is called solving the system of linear equations.

Suppose we have two equations

$$y = 3 + 0.6x \quad y = 8 - 0.8x$$

Since these lie on the same plane (i.e.  $x$  and  $y$  represent the same dimension in both equations), we now have three different ways to “call”  $y$ :

- Given name:  $y$
- Nicknames:  $3 + 0.6x$ ,  $8 - 0.8x$ .

This means

$$3 + 0.6x = 8 - 0.8x.$$



# Solving Systems of Linear Equations

We use the fact that we have two different definitions of  $y$  to our advantage. Instead of two equations and two unknowns we now have one equation with one unknown!

$$3 + 0.6x = 8 - 0.8x$$

$$3 - 3 + 0.6x + 0.8x = 8 - 3 - 0.8x + 0.8x$$

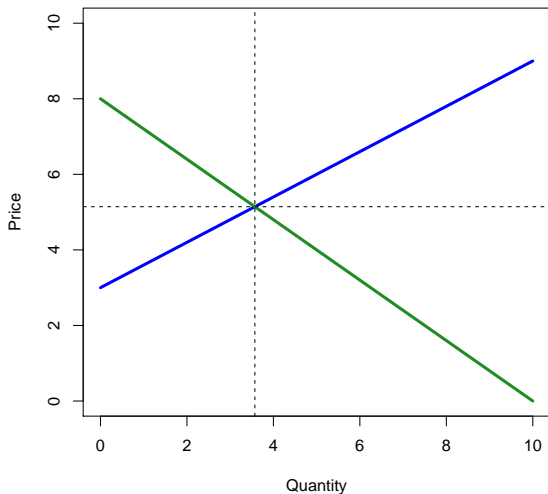
$$1.4x = 5$$

$$x = 5/1.4 = 3.571429$$

The  $y$ -value is found by plugging the found value of  $x$  into either original equation:  $y = 3 + 0.6(3.571429) = 5.142857$

# Solving Systems of Linear Equations

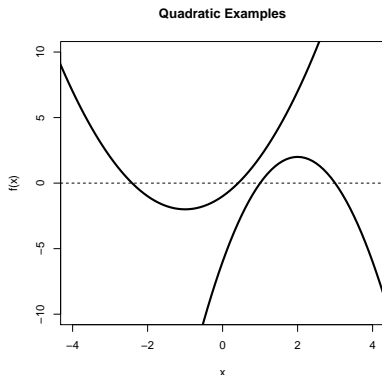
**Supply and Demand**



# Quadratic Equations

Linear functions of  $x$  or lines, always take the form  $f(x) = mx + b$ , where the maximum power of  $x$  is 1.

A **quadratic** function has the form  $f(x) = ax^2 + bx + c$ , where the maximum power  $x$  is raised to is 2. Quadratic functions often take the shape of parabolas.



# Quadratic Equations

## Examples

For any quadratic equation  $f(x) = ax^2 + bx + c$ , we find the **root(s)** (values of  $x$  such that  $f(x) = 0$ , or where the function crosses the  $x$ -axis) by using the **quadratic equation**:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \& \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac$  is called the **discriminant**. If the discriminant is

- positive, there will be two roots.
- zero, there will be one root.
- negative, there will be no real roots.

# Quadratic Equations

## Factoring and FOIL

Many quadratic equations can be factored into a more simple form. For example:

$$2x^2 - 6x - 8 = (x - 4)(2x + 2)$$

To see that they are equivalent we can FOIL to multiply the two terms on the right hand side of the equation.

- **F**irst:  $x \cdot 2x = 2x^2$
- **O**uter:  $x \cdot 2 = 2x$
- **I**nnner:  $-4 \cdot 2x = -8x$
- **L**ast:  $-4 \cdot 2 = -8$

$$\text{Thus, } (x - 4)(2x + 2) = 2x^2 + 2x - 8x - 8 = 2x^2 - 6x - 8$$

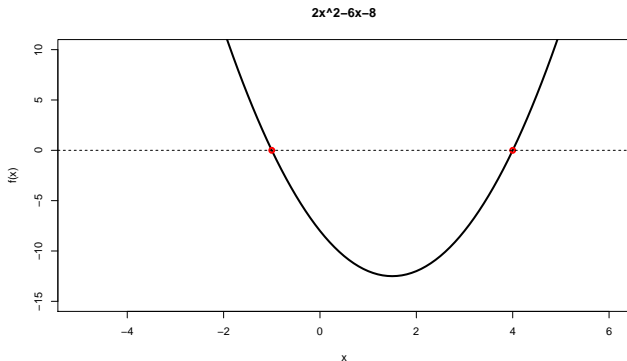
# Quadratic Equations

## Factoring and FOIL

When your quadratic has been factored you can find the roots by solving each term for zero. For example:

$$2x^2 - 6x - 8 = (x - 4)(2x + 2)$$

has roots when  $x - 4 = 0$  and  $2x + 2 = 0$ . Thus, the roots are found at  $x = -1, 4$ .



# Quadratic Equations

## Factoring and FOIL

Hunting for the FOIL factors can be tricky! Remember the quadratic equation always works!!

- If  $b^2 - 4ac$  is a whole number, a fraction, a squared number, then it can be factored into something simple, if not use the quadratic formula.

Examples:

- $2x^2 + 4x - 16 \Rightarrow b^2 - 4ac = 4^2 - 4 \cdot 2 \cdot (-16) = 144$ ; 2 roots; factors
- $3x^2 - 2x + 9 \Rightarrow b^2 - 4ac = (-2)^2 - 4 \cdot 3 \cdot 9 = -104$ ; no real roots

# Exponents

$a^n$  is ' $a$  to the power of  $n$ '.  $a$  is multiplied by itself  $n$  times. Often  $a$  is called the base,  $n$  the exponent. Examples:

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

$$6^4 = 6 \cdot 6 \cdot 6 \cdot 6 = 1296$$

Exponents do not have to be whole numbers. They can be fractions or negative.

Examples:

$$4^{1/2} = \sqrt{4} = 2$$

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$



# Common Rules

- $a^1 = a$
- $a^k \cdot a^l = a^{k+l}$
- $(a^k)^l = a^{kl}$
- $(ab)^k = a^k \cdot b^k$
- $\left(\frac{a}{b}\right)^k = \left(\frac{a^k}{b^k}\right)$
- $a^{-k} = \frac{1}{a^k}$
- $\frac{a^k}{a^l} = a^{k-l}$
- $a^{1/2} = \sqrt{a}$
- $a^{1/k} = \sqrt[k]{a}$
- $a^0 = 1$

# Logarithms

A logarithm is the power ( $x$ ) required to raise a base ( $c$ ) to a given number ( $a$ ).

$$\log_c(a) = x \Rightarrow c^x = a$$

Examples:

- $2^3 = 8 \Rightarrow \log_2(8) = 3$
- $4^6 = 4096 \Rightarrow \log_4(4096) = 6$
- $9^{1/2} = 3 \Rightarrow \log_9(3) = \frac{1}{2}$

# Logarithms

The three most common bases are 2, 10, and  $e \approx 2.718$ , the natural logarithm. It is often called Euler's number after Leonhard Euler.

Examples:

- $10^2 = 100 \Rightarrow \log_{10}(100) = 2$
- $2^3 = 8 \Rightarrow \log_2(8) = 3$
- $e^2 = 7.3891... \Rightarrow \log(7.3891) = 2$

The natural logarithm ( $\log_e$ ) is the most common; used to model exponential growth (populations, etc). If no base is specified, i.e.  $\log(a)$ , most often the base is  $e$ . Sometimes written as  $\ln(a)$ .

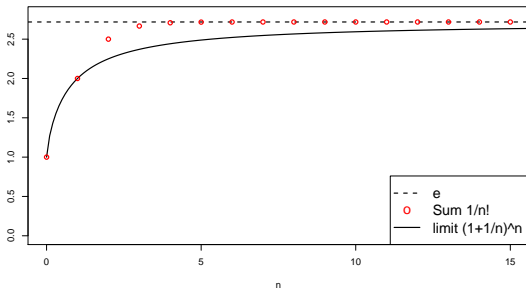
# Logarithms

## What is e?

The number  $e$  is a famous irrational number. The first few digits are  $e = 2.718282\dots$

Two ways to express  $e$ :

- $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$
- $\sum_{n=0}^{\infty} \frac{1}{n!}$



# Logarithms

## Rules

$$\log_c(a \cdot b) = \log_c(a) + \log_c(b)$$

$$x = \log_c(a \cdot b) \iff c^x = a \cdot b$$

$$\Rightarrow c^{x_1+x_2} = a \cdot b \text{ where } x_1 + x_2 = x$$

$$\Rightarrow c^{x_1} \cdot c^{x_2} = a \cdot b \Rightarrow c^{x_1} = a; c^{x_2} = b$$

$$\Rightarrow x_1 = \log_c(a); x_2 = \log_c(b)$$

$$\Rightarrow x = x_1 + x_2 \Rightarrow \log_c(a \cdot b) = \log_c(a) + \log_c(b)$$

# Logarithms

## Rules

$$\log_c(a^n) = n \cdot \log_c(a)$$

For  $n = 2$ :

$$x = \log_c(a^2) \iff c^x = a^2$$

$$\Rightarrow c^{x_1+x_2} = a \cdot a \text{ where } x_1 + x_2 = x$$

$$\Rightarrow c^{x_1} \cdot c^{x_2} = a \cdot a \Rightarrow c^{x_1} = a; c^{x_2} = a$$

$$\Rightarrow x_1 = \log_c(a); x_2 = \log_c(a)$$

$$\Rightarrow x = x_1 + x_2 \Rightarrow \log_c(a^2) = \log_c(a) + \log_c(a) = 2 \cdot \log_c(a)$$

# Logarithms

## Rules

$$\log_c \left( \frac{a}{b} \right) = \log_c(a) - \log_c(b)$$

$$x = \log_c \left( \frac{a}{b} \right) \iff c^x = \frac{a}{b}$$

$$\Rightarrow c^{x_1+x_2} = \frac{a}{b} \text{ where } x_1 + x_2 = x$$

$$\Rightarrow c^{x_1} \cdot c^{x_2} = \frac{a}{b} \Rightarrow c^{x_1} = a; c^{x_2} = \frac{1}{b} = b^{-1}$$

$$\Rightarrow x_1 = \log_c(a); x_2 = (-1) \cdot \log_c(b)$$

$$\Rightarrow x = x_1 + x_2 \Rightarrow \log_c \left( \frac{a}{b} \right) = \log_c(a) - \log_c(b)$$

# Logarithms

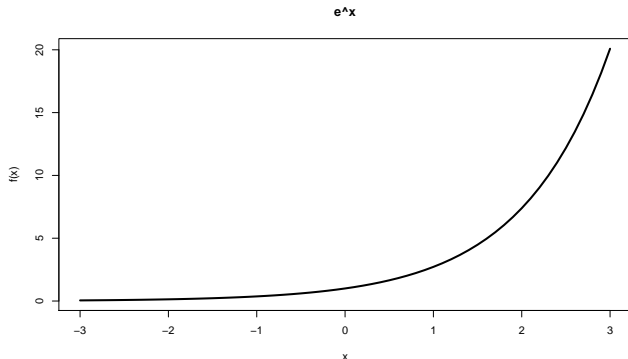
## Examples

- $\log_2(8 \cdot 4) = \log_2(8) + \log_2(4) = 3 + 2 = 5$
- $\log_{10}\left(\frac{1000}{10}\right) = \log_{10}(1000) - \log_{10}(10) = 3 - 1 = 2$
- $\log_4(6^4) = 4 \cdot \log_4(6)$
- $\log(x^3) = 3 \cdot \log(x)$



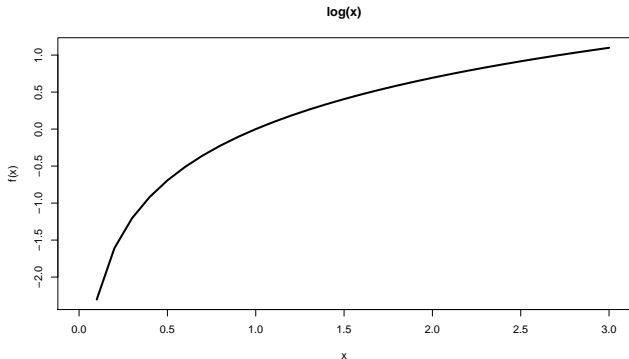
# Exponential Functions

Exponential Functions are of the form  $f(x) = ae^{bx}$ . Often used as a model for population increase where  $f(x)$  is the population at time  $x$ .



# Logarithmic Functions

Logarithmic Functions,  $f(x) = c + d \cdot \log(x)$ , can be used to find the time  $f(x)$  necessary to reach a certain population  $x$ . It can be thought of as an 'inverse' of the exponential function.



Note:  $c = -1/b \cdot \log(a)$  and  $d = 1/b$  from the previous exponential model.

# Continuous & Piecewise Functions

A **continuous** function behaves without break or interruption. If you can follow the ENTIRE graph of a function with your pencil without picking it up, the function is continuous. Examples:

- $f(x) = x^2$
- $f(x) = x + 4$

A **piecewise** function can either have 'jumps' in it or can be made up of different functions for different parts of the domain (possible  $x$ -values). Example:

- Absolute Value  $f(x) = |x|$  can be written as  $f(x) = x, x \geq 0$  and  $f(x) = -x, x < 0$

# Limits

Often we are interested in what a function does as it approaches a certain value. This behavior is called the **limit**.

The limit of  $f(x)$  as  $x$  approaches  $a$  is  $L$ :

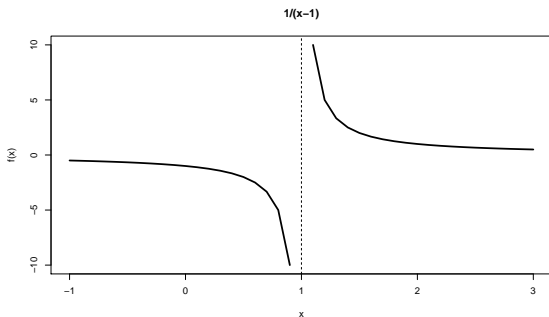
$$\lim_{x \rightarrow a} f(x) = L$$

It may be that  $a$  is not in the domain of  $f(x)$  but we can still find the limit by seeing what value  $f(x)$  is approaching as  $x$  gets very close to  $a$ . Examples:

- $\lim_{x \rightarrow 3} x^2 = 9$  (3 is in the domain)
- $\lim_{x \rightarrow \infty} (1 + 1/x)^x = e$

# Limits

Often limits are different depending on the direction from which you approach  $a$ . The limit 'from above' is approaching from the right ( $x \downarrow a$ ) and the limit 'from below' ( $x \uparrow a$ ) is approaching from the left.



If  $f(x) = \frac{1}{x-1}$  we have  $\lim_{x \downarrow 1} \frac{1}{x-1} = \infty$  and  $\lim_{x \uparrow 1} \frac{1}{x-1} = -\infty$

# The End

Questions?