Center for Statistics and the Social Sciences Math Camp 2021

Lecture 4: Integral Calculus

Peter Gao & Jessica Kunke

Department of Statistics
University of Washington

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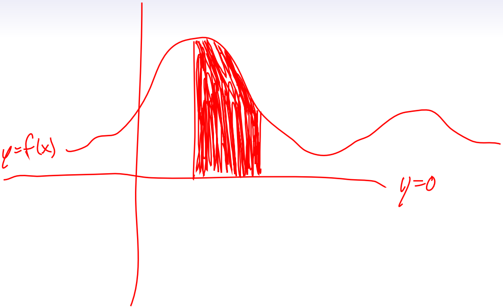
Outline



- Motivation for Integrals
- Rules of Integration
- Lots of Examples

Motivation for Integrals in Statistics

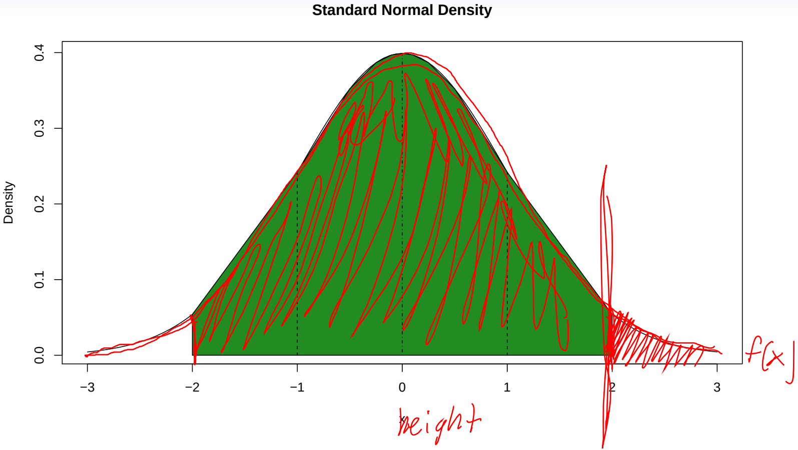


Figure: Standard Normal Density (N(0,1)). Approximately 68% of the probability lies within 1 standard deviation and 95% within 2 standard deviations. The area under the whole curve (from $-\infty$ to ∞) is 1.

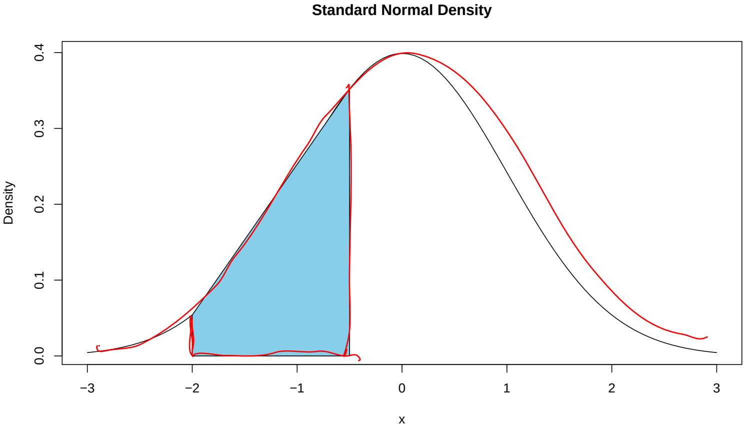
Motivation for Integrals in Statistics

Integral calculus...

- is a tool for computing areas under curves.
- can be used to compute percentile rankings.
- is also used to compute probabilities of events.
- will be needed to compute the expected values and variance of probability distributions.
- is heavily used in statistical theory!

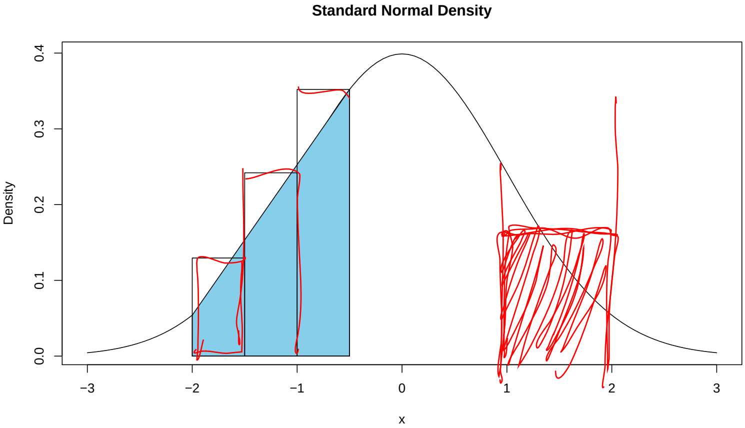
Motivation for Integrals in Statistics

What if we wanted to find the area under the curve from -2 to -0.5?



Motivation for Integrals in Statistics

We could approximate with rectangles or trapezoids. Narrower rectangles would give better approximations.



Differentiation Example

distance, velocity, acceleration

Let's take d=distance, v=velocity, a=acceleration. You may remember from physics, the distance travel after time t

$$\boxed{d(t) = \frac{a}{2}t^2}$$

The velocity at any time t is the instantaneous rate of change of the distance, $v(t) = d'(t)$:

$$\boxed{v(t) = 2 \cdot \frac{a}{2}t = at}$$

The acceleration at any time t is the instantaneous rate of change of the velocity, $a(t) = \boxed{v'(t)} = \boxed{d''(t)}$:

$$\boxed{a(t) = a}$$

Distance

$d(t) = \frac{a}{2}t^2$

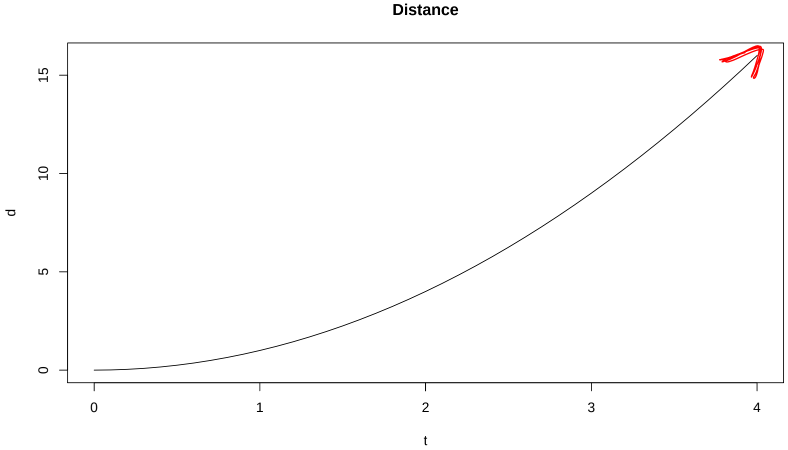


Figure: Distance over time, when $\boxed{a(t) = 2,}$ $v(t) = 2t$, and $\boxed{d(t) = t^2}$.

Velocity

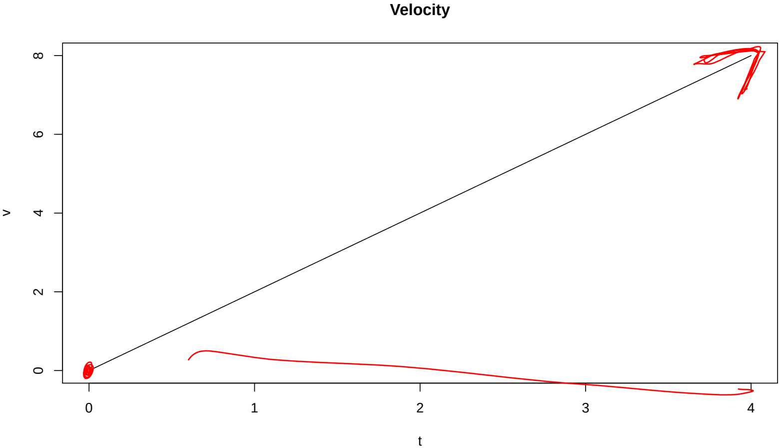


Figure: Velocity over time, when $a(t) = 2$, $v(t) = 2t$, and $d(t) = t^2$.

Acceleration

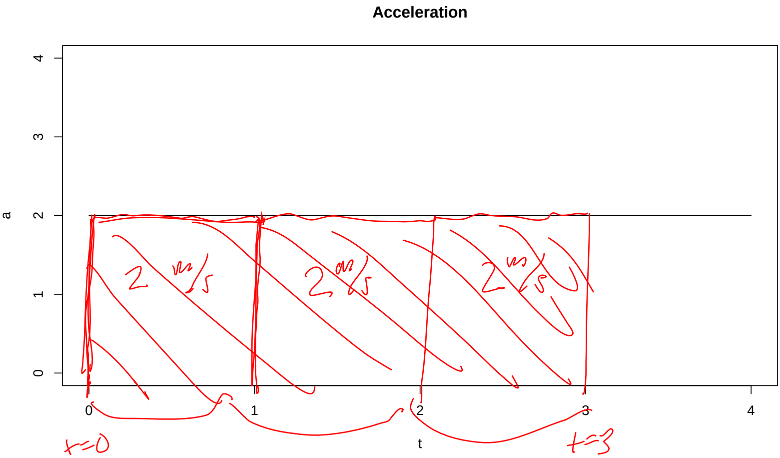


Figure: Acceleration over time, when $a(t) = 2$, $v(t) = 2t$, and $d(t) = t^2$.

What is the velocity at t=3 when a=2?

acceleration = 2 m/s/s

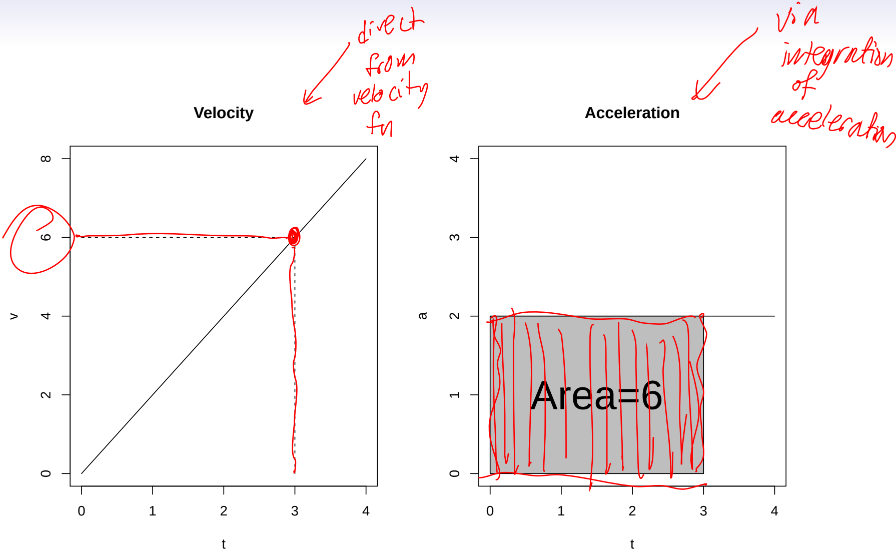
We know that $\boxed{v(t) = 2t}$, so clearly

$$\boxed{v(3) = 2 \cdot 3 = 6.}$$

However we can also find the velocity, by looking at the area under the acceleration curve from $t = 0$ to $t = 3$. This would just be the area of a rectangle (`base X height`),

$$\boxed{(3 - 0) \cdot 2 = 3 \cdot 2 = 6.}$$

What is the velocity at t=3 when a=2?



What is the distance at t=3 when a=2?

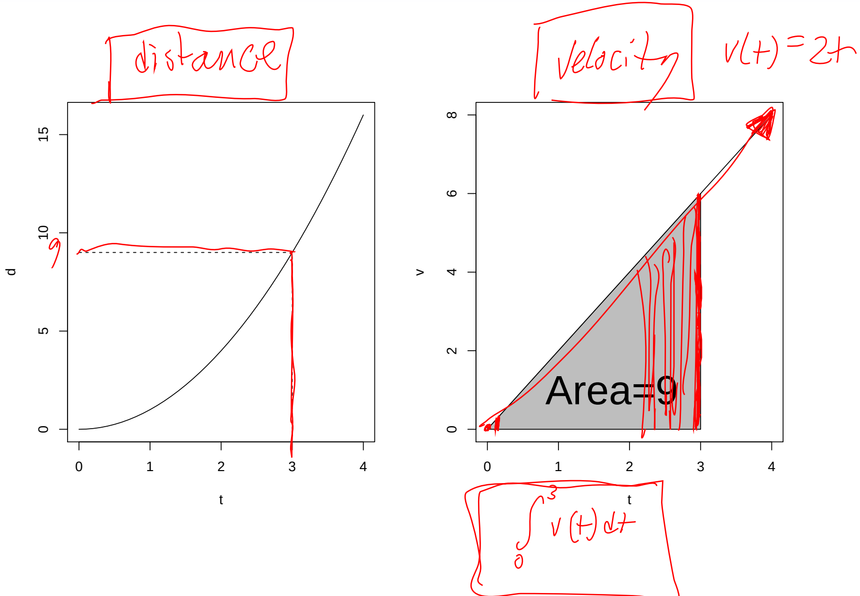
We know that $\boxed{d(t) = 2/2t^2 = t^2}$, so clearly

$$\boxed{d(3) = 3^2 = 9.}$$

However we can also find the distance, by looking at the area under the velocity curve from $t = 0$ to $t = 3$. This would just be the area of a triangle (`1/2 X base X height`),

$$1/2 \cdot (3 - 0) \cdot 6 = 3/2 \cdot 6 = 18/2 = 9.$$

What is the distance at t=3 when a=2?



Integration

The area under a curve is written:

$$\boxed{\int_a^b f(x)dx}$$

$x=$ $x=b$

This formula is called the **definite integral** of $f(x)$ from a to b.

Here a and b are our endpoints of interest. You can think of the integral as the 'opposite' of the derivative.

Integration

F antiderivative
or indefinite integral

More specifically,

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)} \text{ where } F'(x) = f(x)$$

$F(x)$ is called the **indefinite integral** of $f(x)$. The important relationships between derivatives and integrals are:

$$F'(x) = f(x) \quad \& \quad \int f(x)dx = \boxed{F(x)}$$

What is an integral?

You can think of integrating as looking at a derivative and trying to find the original function.

- $\int 3dx$. What function has a derivative equal to 3? $\boxed{3x}$.
- $\int 2xdx$. What function has a derivative equal to $2x$? $\boxed{x^2}$.
- $\int e^x dx$. What function has a derivative equal to e^x? $\boxed{e^x}$.

In practice, you don't have to search for the right function. We have handy shortcuts (rules).

Integration Rules

Integrating a Constant

$f(x) = c$ $\quad \int c\,dx = \boxed{cx}$

$$\boxed{\int c dx = cx}$$

$\int_a^b c\,dx = cb - ca$

Examples:

- $\int 1dx = x$
- $\int 6dx = 6x$
- $\int ydx = yx$

Integration Rules

Integrating a Power of x

$\frac{d}{dx} ax^n = n \cdot ax^{n-1}$

differentiate

$\downarrow$

integrate

$\uparrow$

$f(x) = x^n$

$$\int x^n dx = \boxed{\frac{1}{n+1} x^{n+1}}$$

$F(x) = \frac{1}{n+1} x^{n+1}$

$$\boxed{\frac{d}{dx} F(x) = x^n}$$

Examples:

- $\int x dx = \frac{1}{2} x^2$ $f(x) = x$ $F(x) = \frac{1}{2} x^2$
- $\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{1}{-1} x^{-1} = -\frac{1}{x}$

Exponential:

$$\boxed{\int e^x dx = e^x}$$

$\frac{d}{dx} e^x = e^x$

(Natural) Logarithm:

$$\boxed{\int \frac{1}{x} dx = log(x)}$$

$\frac{d}{dx} log(x) = \frac{1}{x}$

Remember, $\boxed{\frac{d}{dx}cos(x) = -sin(x)}$, thus

$$\boxed{\int sin(x)dx = -cos(x)}$$

and $\boxed{\frac{d}{dx}sin(x) = cos(x)}$, thus

$$\boxed{\int cos(x)dx = sin(x).}$$

Integration Rules

Multiple of a Function

$\int g(x)dx \qquad g(x) = af(x)$

$$\boxed{\int af(x)dx} = \boxed{a \cdot \int f(x)dx} = \boxed{aF(x)}$$

Examples:

- $\int 4x^2 dx = 4\int x^2 dx = 4\left(\frac{1}{3}x^3\right) = \frac{4}{3}x^3$
- $\int \frac{3}{x^2}dx = 3\int \frac{1}{x^2}dx = 3\int x^{-2}dx = \frac{3}{-1}x^{-1} = -\frac{3}{x}$
- $\int \mu y dy = \mu \int y dy = \mu\left(\frac{1}{2}y^2\right) = \frac{\mu}{2}y^2$

$\int 4x^2 dx = 4\int x^2 dx = 4 \cdot \left(\frac{1}{3}x^3\right) = \frac{4}{3}x^3$

$\frac{d}{dx}\frac{4}{3}x^3$

$= \boxed{4x^2}$

Integration Rules

Sums of Functions

$$\boxed{\int (f(x)+g(x))\,dx} = \boxed{\int f(x)dx} + \boxed{\int g(x)dx} = \boxed{F(x)} + \boxed{G(x)}$$

Examples:

- $\int 4x + 3x^2 dx = \int 4xdx + \int 3x^2 dx = 4\int xdx + 3\int x^2 dx = 4\cdot\frac{1}{2}x^2 + 3\cdot\frac{1}{3}x^3 = \boxed{2x^2 + x^3}$
- $\int e^x - \frac{2}{x} dx = \int e^x dx - 2\int \frac{1}{x} dx = e^x - 2log(x)$

$$\int (4x+3x^2)dx = \int 4x dx + \int 3x^2 dx$$
$$= 4\int x dx + 3\int x^2 dx = 4\left(\frac{1}{2}x^2\right) + 3\left(\frac{1}{3}x^3\right)$$

Sometimes the function we are integrating is similar to a simpler function with an easy derivative.

For example, $\int \frac{1}{1-x}dx$ is similar to $\int \frac{1}{x}dx$ which we know is $log(x)$. Similar to the chain rule, we can think about functions within functions.

Let's set $u = 1 - x$. If we differentiate the left with respect to u and the right with respect to x we have $du = -1dx$. Solving for dx we have $dx = -1du$. Now we can substitute these values into our original integral.

$u = u(x) = 1-x$ $\frac{du}{dx} = -1 \Rightarrow du = -1dx$

$\Rightarrow dx = -1du$

$$\int \frac{1}{1-x}dx = \int \frac{1}{u} \cdot (-1)du = -1 \int \frac{1}{u}du$$

$= -1 \cdot \log u = -\log(1-x)$

$u = 1 - x \qquad dx = -du$

Now let's take the integral with respect to u:

$$\boxed{\int \frac{1}{1-x} dx} = \boxed{-1 \int \frac{1}{u} du} = \boxed{-log(u)}$$

Then we can plug in the value for $u = 1 - x$:

$$\int \frac{1}{1-x} dx = -1 \int \frac{1}{u} du = -log(u) = \boxed{-log(1-x)}$$

Integration Rules

u-substitution continued

$f(x) = (2x+4)^3$

Example:

$$\int (2x+4)^3 dx$$

$\int u^3 du$

$\boxed{u = 2x+4}$

$u(x) = 2x+4$

$\frac{du}{dx} = 2$

$du = 2dx$

$\boxed{dx = \frac{1}{2} du}$

We can take $u = 2x + 4$. Then $du = 2dx$ or $\frac{1}{2}du = dx$.

When we make the substitutions in our integral we have:

$$\int (\overset{u}{\cancel{2x+4}})^3 \overset{\frac{1}{2}du}{\cancel{dx}} = \boxed{\int u^3 \cdot \frac{1}{2} du} = \boxed{\frac{1}{2}\int u^3 du}$$

Now we have an integral we can easily compute

$$\frac{1}{2}\int u^3 du = \frac{1}{2}\cdot\frac{1}{4}u^4 = \boxed{\frac{1}{8}u^4}$$

$u = 2x+4$

and then we just need to substitute back in for the functions of x.

$$\int (2x+4)^3 dx = \frac{1}{2}\int u^3 du = \boxed{\frac{1}{8}u^4} = \boxed{\frac{1}{8}(2x+4)^4}$$

Finding Definite Integrals

Often we will be interested in knowing the exact area under the curve $f(x)$, not just the function $F(x)$.

$$\int_a^b f(x)dx = F(x)|_a^b = \boxed{F(b) - F(a)}$$

Examples:

- $\boxed{\int_0^1 x^2 dx} = \boxed{\frac{1}{3}x^3}|_0^1 = \boxed{\frac{1}{3}1^3} - \boxed{\frac{1}{3}0^3} = \boxed{\frac{1}{3}}$
- $\boxed{\int_0^\infty e^{-x} dx} = \boxed{-e^{-x}}|_0^\infty = \boxed{-e^{-\infty}} - \boxed{-e^0} = -\frac{1}{e^\infty} + e^0 = \boxed{1}$
- $\int_2^8 \frac{1}{x} dx = log(x)|_2^8 = log(8) - log(2) = log\left(\frac{8}{2}\right) = log(4)$

Integration Example

distance, velocity, acceleration

Back to our original example, with $\boxed{a = 2}$. The velocity at any time $t = 3$ is the definite integral of of the acceleration, $\boxed{v(3) = \int\limits_0^3 a(t)dt}$:

$$v(3) = \boxed{\int_0^3 2dt} = \boxed{2t|_0^3} = \boxed{2 \cdot 3} - \boxed{2 \cdot 0} = (3 - 0) \cdot 2 = \boxed{6}$$

Similarly, the distance at any time $t = 3$ is the definite integral of of the velocity, $d(3) = \int\limits_0^3 v(t)dt$:

$$\boxed{d(3) = \int_0^3 v(t)dt = \int_0^3 2tdt = t^2|_0^3 = 3^2 - 0^2 = 9}$$

Example

$\int e^x dx = e^x$

$$\int_0^3 e^{x/3} dx = \int_{x=0}^{x=3} e^{x/3} dx$$

$u = x/3$

$u(x) = \frac{x}{3}$

$\frac{du}{dx} = \frac{1}{3}$

$du = \frac{1}{3} dx$

$dx = 3du$

We could take $\boxed{u = x/3}$. Then $du = 1/3dx$ and $\boxed{3du = dx}$.

When we substitute in for u and dx it is important to note that we must also substitute in for our limits of integration. The lower value $u = 0/3 = 0$ and the upper value would be $u = 3/3 = 1$.

$$\int_0^3 e^{x/3} dx = \int_0^1 e^u \cdot 3du = 3\int_0^1 e^u du = 3e^u|_0^1 = 3(e^1 - e^0) = \boxed{3(e-1)}$$

$\int e^u 3du$

$\int_{u=0}^{u=1} e^u \cdot 3du = 3\int_0^1 e^u du = 3e^u \Big|_{u=0}^{u=1} = \boxed{3e^1 - 3e^0}$

The End

Questions?