INTRODUCTION
TO SEISMIC METHODS

Maria Amélia (1994)

Lecture notes

Ivan Pšenčík

PPPG/UFBa

Salvador-1994

ÍNDICE

PREFACE

These lectures notes were written for the course, which I had at PPPG/UFBa during my two year stay in Salvador, Bahia in 1992-94. Originally, the notes were handwritten and they served for the basic orientation of students in topics related to seismic methods and elastic body wave propagation. Only the decision to retype the text made me to fill many gaps in the text but the result is still incomplete in many respects. I wish that many problems are explained more clearly than they are. Many formulations could also be improved and more details given. Some very important methods like matrix methods, methods of finite differences and elements are not included at all, for example. The text should, therefore, still serve as an introduction to more specialized textbooks. Those textbooks which served as a source for these notes and several additionals related to the topic are listed in the end of the text.

The notes start with basics of elastodynamic theory and the elastodynamic equation. Search for and study of its body wave solutions fills the rest of the notes. Relatively large part of the text is devoted to the plane waves since many wave phenomena can be easily demonstrated on them. Another reason for an extensive study of plane waves is that the plane waves play an important role in many seismic methods. Further topics, which are more emphasized than others are body wave propagation in anisotropic media and the calculation of various forms of the Green function. Both topics play a very important role in contemporary seismic prospecting. In the notes, both topics merge in the end of the last chapter, where the ray Green function for inhomogeneous anisotropic media is presented.

The English of the lecture notes did not pass any language revision, which it surely needs.

This text would not exist without strong assistance of Prof. Sláva Červený. The structure of the notes is based on the handwritten notes, which Sláva prepared for his course during his stay at PPPG/UFBa. In the begining, my handwritten text followed Sláva's notes quite closely. I only tried to make the derivations and proofs differently then Sláva did so that students could compare different approaches and see the problem from various points of view. Later, I started to add new paragraphs or remove some of Sláva's. Sláva's lecture notes covered roughly chapters 2-6. In preparation of Chap.7 concerning the ray method, I used extensively another Sláva's text - the manuscript of his new book "Seismic ray theory". Thus Sláva's influence is everywhere in these notes. I am very grateful to Sláva for having access to both materials as well as for years of close and friendly cooperation from which I gained tremendously.

Another person, without whom these notes would not exist, is Joaquim. Without knowing well English, he managed to retype horribly written manuscript into the present form and still remained meu amigo - parabens. Zaira was equally patient and helpfull in typing my numerous modifications.

My thanks also go to my students who helped me enormously in removing many errors and misprints from the text, suggested better formulations or computed and plotted the presented numerical examples. I am most grateful to Aggio, Leo, Rogério and Telesson. Daniel drew all the remaining figures.

Last but not least, I am grateful to all my colleagues at PPPG who prepared for me excellent conditions and made my stay very pleasant, and to Petrobrás and CNPq for support.

Salvador, June 1994 Ivan Pšenčík

CHAPTER 1

SELECTED CALCULUS

In this chapter some basic concepts used throughout the notes are given. First, basics of the tensorial calculus are discussed in Cartesian coordinates. Then, some useful identities are also introduced in curvilinear orthogonal coordinates. Finally, some properties of the analytic signal, broadly used in the notes, are discussed.

1.1 Cartesian tensors

Physical quantities can be mathematically represented by tensors. The equations describing physical laws are thus tensor equations. In the following, we are going to concentrate on tensors in 3-D Euclidean space. Mostly, we shall work in Cartesian coordinates. Therefore, we shall mostly deal with *Cartesian tensors.*

Typical examples of tensors are the following quantities:

temperature, density - scalars (tensor of the zero rank);
force, displacement, velocity - vectors (tensors of the first rank);
stress tensor, strain tensor - tensors of the second rank;
tensor of elastic parameters - tensor of the fourth rank.

1.1.1 Transformation of coordinates

As mentioned above, we shall work in 3-D Euclidean space. Each point of this space can be characterized by three numbers, *coordinates* of the point. We shall work exclusively in *orthogonal* coordinate systems, most often in *Cartesian coordinates.*

For arbitrary two points A and B specified by Cartesian coordinates $[x_1^A, x_2^A, x_3^A]$ and $[x_1^B, x_2^B, x_3^B]$ respectively, we define the distance $d(A,B)$ as follows

$$d(A,B) = \sum_{i=1}^{3} [(x_i^B - x_i^A)(x_i^B - x_i^A)]^{1/2} \quad .$$

In order to simplify the notation, we are going to use the so-called *Einstein summation convention*. In every expression, where two equally denoted indices repeat (it is not permitted that more than two indices repeat), summation over these indices is understood. According to this rule,

$$u_i v_i$$

is a shorthand notation for

$$\sum_{i=1}^{3} u_i v_i$$

and

$$u_i v_i w_i$$

is not permitted. Non-repeated indices, called *free indices*, can attain values 1,2 and 3. We can thus use shorthand notation u_i for the vector (u_1, u_2, u_3). Similarly, e_{ij} denotes the 3×3 matrix.

Using Einstein summation convention, we can rewrite the expression for the distance as

$$d(A,B) = [(x_i^B - x_i^A)(x_i^B - x_i^A)]^{1/2} \quad .$$

The distance function has the following properties:

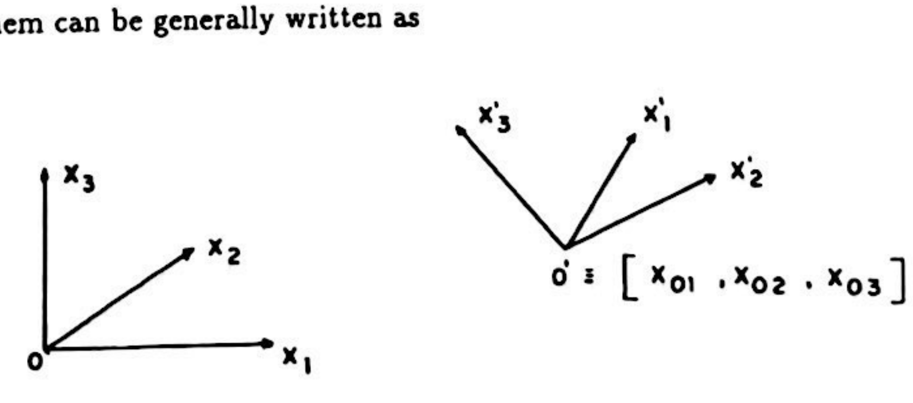
$d(A,B)$ is real-valued and non-negative;

$d(A,A) = 0$;

$d(A,B) = d(B,A)$;

$d(A,B) \leq d(A,C) + d(B,C)$.

With the above definitions, we can start to consider coordinate transformations. Let us consider the two Cartesian coordinate systems x_i and x_i' shown in the picture. The relation between them can be generally written as



$$x_i' = \alpha_{ij}(x_j - x_{oj}) \quad ,$$

where x_{oj} are the coordinates of the origin of the system x'_i in the coordinates x_i and α_{ij} are elements of a transformation matrix.

Let us introduce so-called *Kronecker's symbol* δ_{ij}: It is defined as

$$\delta_{ij} = \begin{cases} 0 & \text{for} \quad i \neq j \,, \\ 1 & \text{for} \quad i = j \,. \end{cases}$$

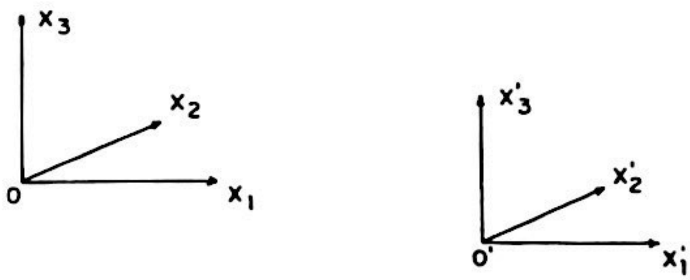
Let us note that due to the Einstein convention,

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \quad .$$

If we put $\alpha_{ij} = \delta_{ij}$, the coordinate transformation will be

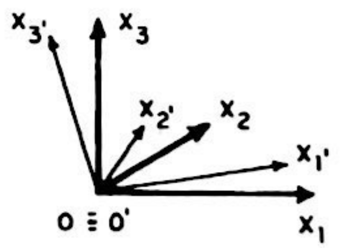
$$x'_i = x_i - x_{oi} \,.$$

This transformation decribes *parallel translation* of one coordinate system with respect to



the other, see the picture. If we put $x_{oi} = 0$, the original transformation relation reduces to

$$x'_i = \alpha_{ij} x_j \quad .$$



This relation describes the *rotation* of the primed coordinate system with respect to the unprimed one around their common origin, see the picture.

where x_{oj} are the coordinates of the origin of the system x'_i in the coordinates x_i and α_{ij} are elements of a transformation matrix.

Let us introduce so-called *Kronecker's symbol* δ_{ij}: It is defined as

$$\delta_{ij} = \begin{cases} 0 & \text{for} \quad i \neq j\,, \\ 1 & \text{for} \quad i = j\,. \end{cases}$$

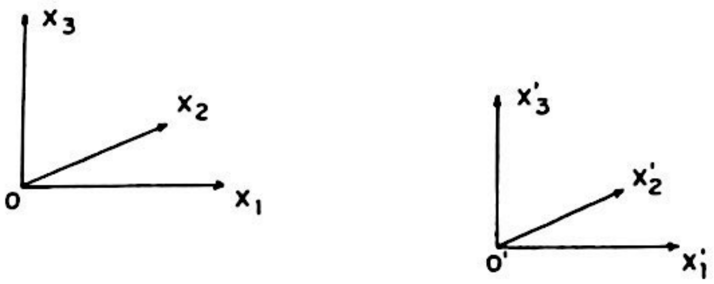
Let us note that due to the Einstein convention,

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \quad .$$

If we put $\alpha_{ij} = \delta_{ij}$, the coordinate transformation will be

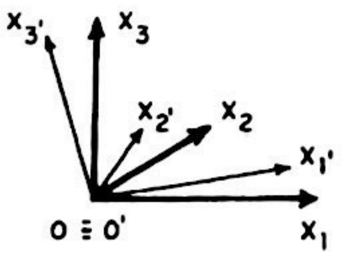
$$x'_i = x_i - x_{oi}\,.$$

This transformation decribes *parallel translation* of one coordinate system with respect to



the other, see the picture. If we put $x_{oi} = 0$, the original transformation relation reduces to

$$x'_i = \alpha_{ij} x_j \quad .$$



This relation describes the *rotation* of the primed coordinate system with respect to the unprimed one around their common origin, see the picture.

Let us investigate the meaning of the elements of the transformation matrix α_{ij}. Let us introduce the *radius vector* r_i pointing from the origin of a coordinate system to a considered point x_i, i.e. $r_i = x_i$. We further introduce so-called *base vectors*, $\vec{i}_1$, $\vec{i}_2$, $\vec{i}_3$, which are three mutually perpendicular unit vectors oriented along axes x_1, x_2, x_3, respectively. We can express the radius vector r_i in both, primed and unprimed coordinate systems

$$\vec{r} = x_1\vec{i}_1 + x_2\vec{i}_2 + x_3\vec{i}_3 = x'_1\vec{i}'_1 + x'_2\vec{i}'_2 + x'_3\vec{i}'_3 \quad .$$

If we use, on the right-hand side, the transformation relation

$$x'_i = \alpha_{ij}x_j \quad ,$$

we get

$$\vec{r} = x_1\vec{i}_1 + x_2\vec{i}_2 + x_3\vec{i}_3 = \alpha_{1j}x_j\vec{i}'_1 + \alpha_{2j}x_j\vec{i}'_2 + \alpha_{3j}x_j\vec{i}'_3 \quad .$$

If we multiply the equation by $\vec{i}'_k$, we get

$$\alpha_{kj} = \vec{i}_j\vec{i}'_k \quad .$$

Thus the elements of the matrix α_{ij} are cosines and we call them *directional cosines.* Let us note that the elements α_{ij} can also be understood as components of the base vectors of one coordinate system in the other system,

$$\begin{aligned} \vec{i}'_k &= (\vec{i}_j\vec{i}'_k)\vec{i}_j = \alpha_{kj}\vec{i}_j \quad , \\ \vec{i}_k &= (\vec{i}'_j\vec{i}_k)\vec{i}'_j = \alpha_{jk}\vec{i}'_j \quad . \end{aligned}$$

Let us now consider an inverse transformation to

$$x'_i = \alpha_{ij}(x_j - x_{oj}) \quad .$$

We shall show that the matrix α_{ij} is always regular and thus there exists an inverse matrix $(\alpha^{-1})_{ij}$ such that

$$\alpha_{ij}(\alpha^{-1})_{jk} = \delta_{ik} \, , \quad (\alpha^{-1})_{jk}\alpha_{kl} = \delta_{jl} \, .$$

With the inverse matrix, we can write inverse transformation in the following form

$$x_i = x_{oi} + (\alpha^{-1})_{ij}x'_j \quad .$$

Since the distance between the points A and B must be independent of the coordinate system, in which we are measuring, we have

$$(x_i^B - x_i^A)(x_i^B - x_i^A) = (x_i'^B - x_i'^A)(x_i'^B - x_i'^A) \quad ,$$

which yields

$$(x_i^B - x_i^A)(x_i^B - x_i^A) = \alpha_{kj}\alpha_{kl}(x_j^B - x_j^A)(x_l^B - x_l^A) \quad .$$

Thus

$$\alpha_{kj}\alpha_{kl} = \delta_{jl} \quad .$$

If we take into account relation, which we derived above,

$$(\alpha^{-1})_{jk}\alpha_{kl} = \delta_{jl} \quad ,$$

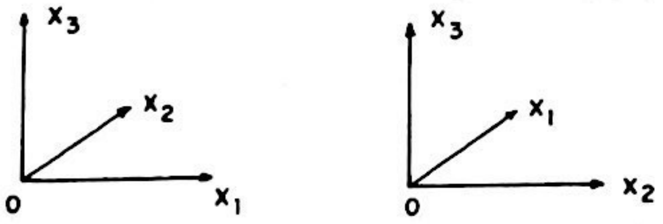
both above equations imply

$$(\alpha^{-1})_{jk} = \alpha_{kj} \quad .$$

This means that the inverse matrix to α_{ij} can be determined simply by transposing the matrix α_{ji}. The matrix with such a property is called *orthogonal* matrix. Orthogonality of the matrix α_{ij} has several interesting consequences. We can write

$$\det[(\alpha^{-1})_{ik}\alpha_{kj}] = \det(\alpha_{ki})\det(\alpha_{kj}) = [\det(\alpha_{kj})]^2 = 1 \quad .$$

From this, we can see that the matrix α_{ij} is *regular*, its determinant being equal to ± 1. We shall always work with the right-handed coordinate systems and thus we shall always have $\det(\alpha_{ij}) = 1$. We would obtain $\det(\alpha_{ij}) = -1$ only for the transformation from the right-hand system to the left-hand one and vice versa, see the picture below.



1.1.2 Definition and properties of Cartesian tensors

We now define what we shall understand under the Cartesian tensor and list operations, which we can perform with them.

At each point of the space, the Cartesian tensor is a finite sequence of numbers which corresponds, in a unique way, to the Cartesian coordinates of the point. The sequence represents a tensor if it satisfies the following two conditions:

1. The sequence contains 3^k numbers where the number 3 represents the dimension of the space and k is a non-negative integer called *rank*:

$$T_{m_1,m_2,\dots m_k}\ , \qquad m_i = 1,2,3\ .$$

2. An element of the tensor in a coordinate system x_i' is related to the elements of the tensor in the coordinate system x_i by the relationship:

$$T'_{p_1,p_2,\dots p_k} = \alpha_{p_1 m_1}\alpha_{p_2 m_2}\dots\alpha_{p_k m_k}T_{m_1,m_2,\dots m_k}\ .$$

Tensor of the zeroth rank, *scalar*, transforms according to this definition as follows

$$T' = T\ .$$

As we could expect, a scalar quantity is independent of the coordinate transformation. Tensor of the first rank, *vector*, and tensor of the second rank transform as follows

$$T_i' = \alpha_{ij}T_j \qquad T_{ij}' = \alpha_{im}\alpha_{jn}T_{mn}\ .$$

Within a rank, the tensors can be *added* or *subtracted* by adding or subtracting corresponding elements

$$A_{ij} + B_{ij} = C_{ij}\ .$$

It is easy to show that the resulting quantity C_{ij} is also tensor. Remember that operation like

$$A_{ij} + B_{ijk}$$

is not permitted.

In contrast to summation, the *multiplication* does not require all the involved tensors to be of the same rank. If all the indices are different, the multiplication increases the rank of the resulting tensor. The resulting rank is a sum of ranks of tensors involved in the multiplication,

$$A_{ij}B_{klm} = C_{ijklm}\ .$$

Again, it is easy to show that the resulting quantity is tensor. Multiplication of two tensors with two equal indices reduces the rank of the resulting tensor by 2,

$$A_{ij}B_j = C_i\ .$$

The reduction of the rank in this way is called *contraction*. A typical example is *scalar product* of two vectors

$$u_i v_i\ ,$$

which results in a scalar quantity.

In the following, we shall often work with *symmetric* and *antisymmetric* tensors. Under the symmetric tensor, we understand a tensor with the following property

$$T_{ijk} = T_{jik} \quad .$$

This tensor of the third rank is symmetric in its indices i and j. Under the antisymmetric tensor, we understand tensor with the property

$$T_{ijk} = -T_{ikj} \quad .$$

This tensor of the third rank is antisymmetric in indices j and k.

Every tensor B_{ij} of the second rank can be expressed in the following way

$$B_{ij} = \frac{1}{2}(B_{ij} + B_{ji}) + \frac{1}{2}(B_{ij} - B_{ji}) = S_{ij} + A_{ij} \quad ,$$

i.e., as a sum of the symmetric tensor S_{ij} and the antisymmetric tensor A_{ij}. Note that the symmetry or antisymmetry of a tensor decreases the number of its independent elements. For example, a symmetric tensor of the second rank has 6 independent elements, an antisymmetric tensor of the second rank has only 3 independent elements. In the latter case all the diagonal terms are zero.

Symmetry and antisymmetry of a tensor is preserved even after coordinate transformation. For $T_{mnl} = T_{nml}$ we get after transformation

$$T'_{ijk} = \alpha_{im}\alpha_{jn}\alpha_{kl}T_{mnl} = \alpha_{im}\alpha_{jn}\alpha_{kl}T_{nml} = \alpha_{jn}\alpha_{im}\alpha_{kl}T_{nml} = T'_{jik} \quad .$$

Similarly we could show this for an antisymmetric tensor.

1.1.3 Isotropic (special) tensors

The components of these tensors reproduce themselves during the transformation of coordinates.

We show that the Kronecker's symbol δ_{ij} introduced above is the isotropic tensor of the second rank. We can prove it in the following way

$$\delta'_{ij} = \alpha_{im}\alpha_{jn}\delta_{mn} = \alpha_{im}\alpha_{jm} = \delta_{ij} \quad .$$

An important isotropic tensor of the third rank is the *Levi-Civita's symbol* ε_{ijk}. It is defined as follows:

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{when } i,j,k \text{ form an even permutation of 1, 2, 3;} \\ -1 & \text{when } i,j,k \text{ form an odd permutation of 1, 2, 3;} \\ 0 & \text{when two indices have the same value.} \end{cases}$$

Thus, $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$ and other elements are zero.

It is easy to prove the following identity

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \quad .$$

This yields

$$\varepsilon_{ijk}\varepsilon_{imn} = \begin{vmatrix} 3 & \delta_{im} & \delta_{in} \\ \delta_{ji} & \delta_{jm} & \delta_{jn} \\ \delta_{ki} & \delta_{km} & \delta_{kn} \end{vmatrix} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \quad ,$$

which is a very important identity. We can proceed from it to

$$\varepsilon_{ijk}\varepsilon_{ijn} = 3\delta_{kn} - \delta_{kn} = 2\delta_{kn}$$

and

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6 \quad .$$

For an arbitrary matrix a_{ij}, using the above identity for $\varepsilon_{ijk}\varepsilon_{lmn}$, we can write

$$\varepsilon_{ijk}a_{1i}a_{2j}a_{3k} = \varepsilon_{ijk}\varepsilon_{123}a_{1i}a_{2j}a_{3k} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} a_{1i}a_{2j}a_{3k} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \det(a_{ij}).$$

The above relation can be simply generalized to get

$$\varepsilon_{ijk}a_{mi}a_{nj}a_{lk} = \det(a_{pq})\varepsilon_{mnl} \quad .$$

This identity can be used to prove that the Levi-Civita's symbol is the isotropic tensor of the third rank. We can write

$$\varepsilon'_{ijk} = \alpha_{im}\alpha_{jn}\alpha_{kl}\varepsilon_{mnl} = \det(\alpha_{pq})\varepsilon_{ijk} = \varepsilon_{ijk} \quad .$$

The previous identity can also be used for the derivation of a useful expression for the value of a determinant. If we remember that $\varepsilon_{mnl}\varepsilon_{mnl} = 6$, the identity yields

$$\det(a_{pq}) = \frac{1}{6}\varepsilon_{mnl}\varepsilon_{ijk}a_{mi}a_{nj}a_{lk} \quad .$$

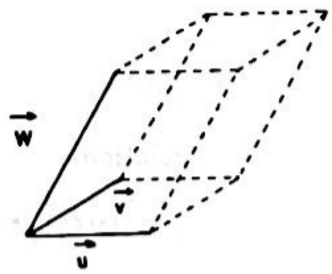
Levi-Civita's symbol can be used in several other important relations. For example, *vectorial product* of two vectors can be expressed in terms of it:

$$(\vec{u} \times \vec{v})_i = \varepsilon_{ijk} u_j v_k \quad .$$

Another application is in the so-called *tripple scalar product*

$$\vec{u}(\vec{v} \times \vec{w}) = u_i \varepsilon_{ijk} v_j w_k \quad ,$$

which represents the volume of the presented parallelepiped.



1.1.4 Eigenvalues and eigenvectors of a symmetric tensor of second rank

Similarly as we can use a vector u_i to define a plane normal to it,

$$u_i x_i = \pm 1 \quad ,$$

we can use symmetric tensor of the second rank T_{ij} to define a quadrics - surface of second order with its centre at the origin of the coordinate system

$$T_{ij} x_i x_j = \pm 1 \quad .$$

Let us first show that the above equation is invariant with respect to the coordinate transformation

$$x'_i = \alpha_{ij} x_j \quad .$$

We can write

$$T'_{ij} x'_i x'_j = \alpha_{im} \alpha_{jn} \alpha_{ik} \alpha_{jl} T_{mn} x_k x_l = \delta_{mk} \delta_{nl} T_{mn} x_k x_l = T_{mn} x_m x_n = T_{ij} x_i x_j \quad .$$

For a special choice of the coordinates x'_i, the equation of quadrics can be written as follows

$$T'_{11} x'^2_1 + T'_{22} x'^2_2 + T'_{33} x'^2_3 = \pm 1 \quad .$$

In this case, the coordinate axes x_i' coincide with the *principal axes* of the quadrics. This implies that the normal to the quadrics at the point where the axis x_i' intersects quadrics must be parallel to the axis x_i'. Let us specify the principal axes by the unit vectors $g_i^{(j)}$, i.e., $\vec{g}^{(1)} = \vec{i}_1'$, $\vec{g}^{(2)} = \vec{i}_2'$, $\vec{g}^{(3)} = \vec{i}_3'$ and let us try to determine them. From the parallelness of the normal to the quadrics and $g_i^{(k)}$, we get

$$g_i^{(k)} \sim 2T_{ij}x_j(g_m^{(k)}) \quad ,$$

where $x_j(g_m^{(k)})$ denotes coordinates of the point of intersection of the quadrics with the axis x_k'. If we denote by a the lenght of the considered principal axis, we can write

$$x_j(g_m^{(k)}) = ag_j^{(k)} \quad .$$

With this, we can rewrite the above relation of proportionality as

$$(T_{ij} - \lambda^{(k)}\delta_{ij})g_j^{(k)} = 0 \quad .$$

This equation can be used to determine $g_j^{(k)}$ and thus to find the principal axis x_k'. The problem of the determination of the principal axis of the symmetric tensor of the second rank represents an *eigenvalue problem*. Solutions of such an eigenvalue problem are *eigenvalues* $\lambda^{(k)}$ and corresponding *eigenvectors* $g_i^{(k)}$. From the mathematical point of view, the above tensorial equation represents a system of three homogeneous linear algebraic equations for $g_j^{(k)}$. The condition of its solvability leads to the equation

$$\det(T_{ij} - \lambda^{(k)}\delta_{ij}) = 0$$

known as the *characteristic equation*. We can see that in our case the equation is cubic with three roots $\lambda^{(k)}$. For each eigenvalue $\lambda^{(k)}$, there is one eigenvector $g_i^{(k)}$.

We shall now show several properties of eigenvalues and eigenvectors of a real and symmetric tensor T_{ij}. Real and symmetric tensor satisfies the following relations

$$T_{ij} = T_{ji} = T_{ji}^* \quad ,$$

where the symbol $*$ denotes complex conjugacy ($a = c + id, a^* = c - id \Rightarrow a + a^* = 2c$ $-$real, $(a^*)^* = a$). We shall show that the tensor T_{ij} has three real eigenvalues. Let us multiply the eigenvalue equation

$$T_{ij}g_j^{(k)} = \lambda^{(k)}g_i^{(k)}$$

by $g_i^{(k)*}$ and the complex conjugate equation

$$T_{ij}^*g_j^{(k)*} = \lambda^{(k)*}g_i^{(k)*}$$

by $g_j^{(k)}$. We get

$$T_{ij} g_j^{(k)} g_i^{(k)*} = \lambda^{(k)} g_i^{(k)} g_i^{(k)*} \quad , \quad T_{ij}^* g_j^{(k)*} g_i^{(k)} = \lambda^{(k)} g_i^{(k)*} g_i^{(k)} \ .$$

Since T_{ij} is real, i.e. $T_{ij} = T_{ij}^*$, and symmetric, the left-hand sides of the above equations are equal and thus

$$(\lambda^{(k)} - \lambda^{(k)*}) g_i^{(k)} g_i^{(k)*} = 0 \quad .$$

Since $g_i^{(k)} g_i^{(k)*} \neq 0$, this equation implies that $\lambda^{(k)} = \lambda^{(k)*}$, which means that the eigenvalue $\lambda^{(k)}$ is *real-valued.*

Now we show that

$$T_{ij} g_j^{(k)} = \lambda^{(k)} g_i^{(k)}$$

must be satisfied by a real eigenvector. Let us again consider the complex conjugate equation and let us take into account that T_{ij} and λ are real. Then we can rewrite it as

$$T_{ij} g_j^{(k)*} = \lambda^{(k)} g_i^{(k)*} \quad .$$

After adding this equation to the original one, we get

$$T_{ij}(g_j^{(k)} + g_j^{(k)*}) = \lambda^{(k)}(g_i^{(k)} + g_i^{(k)*}) \quad .$$

Since $g_i^{(k)} + g_i^{(k)*}$ is real vector, we proved that eigenvalue equation is also satisfied by a *real-valued* eigenvector. Of course, any vector $A(g_i^{(k)} + g_i^{(k)*})$ with A real or complex also satisfies the eigenvalue equation. In the following, we are going to consider eigenvectors of the real tensor T_{ij} to be unit vectors, i.e., $g_i^{(k)} g_i^{(k)} = 1$.

An important property of positively definite tensor T_{ij} is that its eigenvalues are positive. The tensor T_{ij} is positively definite if $T_{ij} x_i x_j > 0$ for any x_i. Let us again multiply the eigenvalue equation by $g_i^{(k)}$. We get

$$T_{ij} g_i^{(k)} g_j^{(k)} - \lambda^{(k)} g_i^{(k)} g_i^{(k)} = 0 \quad .$$

Since the first term is positive due to the positive definiteness of T_{ij} and $g_i^{(k)} g_i^{(k)} = 1 > 0$ by definition, we can see that $\lambda^{(k)}$ must be *positive.*

Let us note that this property holds even for a *complex-valued* tensor T_{ij} satisfying the inequality $T_{ij} x_i x_j^* > 0$ with $T_{ij} x_i x_j^*$ being real-valued quantity. The proof is as above. We must introduce a new normalization condition for the eigenvectors, $g_i^{(k)} g_i^{(k)*} = 1$, and multiply the eigenvalue equation by $g_i^{(k)*}$.

Finally, let us show that eigenvectors belonging to different eigenvalues of a real and symmetric tensor T_{ij} are mutually orthogonal. We can write

$$T_{ij} g_j^{(1)} - \lambda^{(1)} g_i^{(1)} = 0 \quad , \quad T_{ij} g_j^{(2)} - \lambda^{(2)} g_i^{(2)} = 0 \quad .$$

We multiply the first equation by $g_i^{(2)}$ and the second one by $g_i^{(1)}$. If we subtract the resulting equations, we arrive at

$$(\lambda^{(1)} - \lambda^{(2)})g_i^{(1)}g_i^{(2)} = 0 \quad .$$

For $\lambda^{(1)} \neq \lambda^{(2)}$, this yields $g_i^{(1)}g_i^{(2)} = 0$, i.e. the eigenvectors $g_i^{(1)}$ and $g_i^{(2)}$ are *perpendicular.*

If all the three eigenvalues are different, i.e., $\lambda^{(1)} \neq \lambda^{(2)} \neq \lambda^{(3)}$, three mutually perpendicular unit vectors $g_i^{(k)}$ can be uniquely determined. The corresponding tensor surface for positively definite T_{ij} is ellipsoidal with all three principal axes of different length.

If two eigenvalues, e.g., $\lambda^{(1)}$ and $\lambda^{(2)}$ coincide, i.e., $\lambda^{(1)} = \lambda^{(2)} \neq \lambda^{(3)}$, we speak about *degenerate* case and we *cannot uniquely* determine the eigenvectors $g_i^{(1)}$ and $g_i^{(2)}$. Only the eigenvector $g_i^{(3)}$ can be determined uniquely. The eigenvectors $g_i^{(1)}$ and $g_i^{(2)}$ can be chosen arbitrarily as two mutually perpendicular unit vectors in the plane perpendicular to $g_i^{(3)}$. The corresponding tensor surface for positively definite T_{ij} is rotational ellipsoid.

For $\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)}$, any three mutually perpendicular unit vectors can serve as eigenvectors. The corresponding tensor surface is a sphere and the corresponding tensor is proportional to δ_{ij}.

The above considerations can be extended to *complex-valued* tenstors T_{ij}. The resulting relation $g_i^{(1)}g_i^{(2)} = 0$ must then be interpreted as follows,

$$Re(g_i^{(1)}g_i^{(2)}) = 0, Im(g_i^{(1)}g_i^{(2)}) = 0 \, .$$

Let us now find the elements T'_{ij} of the tensor T_{ij} in the coordinate system with its axes parallel to principal axes of the tensor. In the primed coordinate system x'_i, the eigenvectors $g_i^{(k)}$ can be written as

$$g_i^{(1)} \equiv (1,0,0) \, , \quad g_i^{(2)} \equiv (0,1,0) \, , \quad g_i^{(3)} \equiv (0,0,1) \, .$$

If we use them in the eigenvalue equation, we get from

$$T'_{ij}g_j^{(1)} = \lambda^{(1)}g_i^{(1)}$$

the following identities

$$T'_{11} = \lambda^{(1)} \, , \quad T'_{12} = T'_{13} = 0 \quad .$$

Similarly from equations for $\lambda^{(2)}$ and $\lambda^{(3)}$ we get

$$T'_{22} = \lambda^{(2)} \, , \quad T'_{23} = 0 \, , \quad T'_{33} = \lambda^{(3)} \quad .$$

We thus get the expression for the tensor in the *diagonal form*

$$\begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix} \quad .$$

The equation of the tensor surface attains the form

$$\lambda^{(1)} x_1'^2 + \lambda^{(2)} x_2'^2 + \lambda^{(3)} x_3'^2 = \pm 1 \quad .$$

We can see that $(\lambda^{(i)})^{-1/2}$ represents halflength of the i-th principal axis of the tensor T_{ij}.

We have proved above that the equation $T_{ij}x_i x_j = \pm 1$ is invariant with respect to the coordinate transformation and thus the tensor surface is invariant. This means that the orientation of the principal axes and the eigenvalues are also invariants. This in turn implies that the characteristic equation is also invariant and thus

$$\begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{12} & T_{22}-\lambda & T_{23} \\ T_{13} & T_{23} & T_{33}-\lambda \end{vmatrix} = \begin{vmatrix} \lambda^{(1)}-\lambda & 0 & 0 \\ 0 & \lambda^{(2)}-\lambda & 0 \\ 0 & 0 & \lambda^{(3)}-\lambda \end{vmatrix} = 0 \quad .$$

Left-hand side of the above equation yields:

$$\begin{aligned} &-\lambda^3 + \lambda^2(T_{11}+T_{22}+T_{33}) - \lambda(T_{11}T_{22}+T_{22}T_{33}+T_{11}T_{33}-T_{23}^2-T_{13}^2-T_{12}^2) \\ &+(T_{11}T_{22}T_{33}+2T_{12}T_{23}T_{13}-T_{13}^2T_{22}-T_{23}^2T_{11}-T_{12}^2T_{33}) = 0 \quad . \end{aligned}$$

Right-hand side of the above equation yields:

$$-\lambda^3 + \lambda^2(\lambda^{(1)}+\lambda^{(2)}+\lambda^{(3)}) - \lambda(\lambda^{(2)}\lambda^{(3)}+\lambda^{(1)}\lambda^{(3)}+\lambda^{(1)}\lambda^{(2)}) + \lambda^{(1)}\lambda^{(2)}\lambda^{(3)} = 0 \quad .$$

By comparing individual coefficients of the above cubic equations, we get *invariants of the tensor* T_{ij},

$$\begin{aligned} &T_{ii} = \lambda^{(1)}+\lambda^{(2)}+\lambda^{(3)} \quad , \\ &\frac{1}{2}\varepsilon_{ijk}\varepsilon_{imn}T_{jm}T_{kn} = \lambda^{(1)}\lambda^{(2)}+\lambda^{(2)}\lambda^{(3)}+\lambda^{(1)}\lambda^{(3)} \quad , \\ &\frac{1}{6}\varepsilon_{ijk}\varepsilon_{lmn}T_{il}T_{jm}T_{kn} = \lambda^{(1)}\lambda^{(2)}\lambda^{(3)} \quad . \end{aligned}$$

1.1.5 Tensor differentiation

Tensors are generally functions of location and time. We shall, therefore, investigate their differentiation with respect to time (parameter) and spatial coordinates.

We consider a tensor of rank k and we shall differentiate it with respect to time. We shall keep in mind that directional cosines α_{ij} are independent of time. We can write

$$T'_{m_1,m_1,\ldots,m_k} = \alpha_{m_1 j_1}\alpha_{m_2 j_2}\ldots\alpha_{m_k j_k} T_{j_1,j_2,\ldots,j_k} \,.$$

Differentiating this with respect to time, we get

$$\frac{\partial}{\partial t}(T'_{m_1,m_1,\ldots,m_k}) = \alpha_{m_1 j_1}\alpha_{m_2 j_2}\ldots\alpha_{m_k j_k}\frac{\partial}{\partial t}(T_{j_1,j_2,\ldots,j_k}) \,.$$

We can see that the time derivative of the tensor is again a tensor of the same rank.

Let us now differentiate the same tensor as above but with respect to spatial coordinates. We keep in mind that the directional cosines α_{ij} are independent of position and that $\partial x'_i/\partial x_j = \alpha_{ij}$ and $\partial x_j/\partial x'_i = \alpha_{ij}$, which simply follows from $x'_i = \alpha_{ij}(x_j - x_{oj})$. Differentiating the relation between the tensor in primed and unprimed coordinates with respect to x'_l, we get

$$\begin{aligned}\frac{\partial}{\partial x'_l}(T_{m_1,m_2,\ldots,m_k}) &= \alpha_{li}\frac{\partial}{\partial x_i}(\alpha_{m_1 j_1}\ldots\alpha_{m_k j_k}T_{j_1,j_2,\ldots,j_k}) \\ &= \alpha_{li}\alpha_{m_1 j_1}\ldots\alpha_{m_k j_k}\frac{\partial}{\partial x_i}(T_{j_1,j_2\ldots,j_k}) \,.\end{aligned}$$

We can see that the derivative of a tensor with respect to a spatial coordinate increases rank of the resulting tensor by one.

1.2 Differential operators and integral theorems

In this section, we introduce differential operators with which we are going to work frequently in these notes. We first introduce them in Cartesian coordinates, then we derive them also in curvilinear orthogonal coordinates.

1.2.1 Cartesian coordinates

Operator *gradient*: Applied to a scalar quantity A, it can be written as

$$\partial A/\partial x_i \equiv A_{,i} \quad .$$

Notice the shorthand notation $A_{,i}$, in which the derivative with respect to the spatial coordinate x_i is denoted by comma and index i. Vector notation of the operator gradient is ∇ or grad (∇A, grad A).

Operator *divergence*: Applied on a vector u_i, it can be written as

$$\partial u_i/\partial x_i \equiv u_{i,i} \,.$$

Vector notation is ∇ or div ($\nabla\vec{u}$, div $\vec{u}$).

Operator rotation: Applied on a vector u_i, it can be written as

$$\varepsilon_{ijk}\partial u_k/\partial x_j \equiv \varepsilon_{ijk}u_{k,j} \quad .$$

Vector notation is $\nabla\times$, rot or curl ($\nabla\times\vec{u}$, rot $\vec{u}$, curl $\vec{u}$).

Laplace operator: It is a combination of the operators of divergence and gradient. Applied to a scalar A, it can be written as

$$\partial^2 A/\partial x_i^2 \equiv A_{,ii} \quad .$$

Vector notation is Δ or ∇^2 (ΔA, $\nabla^2 A$).

Let us note that

$$\text{div rot } \vec{u} = 0 \,, \quad \text{rot grad } A = 0 \quad ,$$

which can be easily proved.

The above introduced operators divergence and rotation play an important role in the Gauss and Stokes theorem, which we are going to use broadly. If we consider a volume V surrounded by a closed surface Σ with an outer unit normal $\vec{n}$ at any point of Σ, the *Gauss theorem* indicates how to transform a voluminal integral into a surface one and vice versa. It can be expressed as follows

$$\iiint\limits_V \text{div } \vec{u}dV = \iint\limits_\Sigma \vec{u}\vec{n}dS \quad .$$

If we consider a surface Σ surrounded by a closed line L with unit tangent $\vec{t}$ at any point of L, the *Stokes theorem* indicates how to transform a surface integral into a line one and vice versa. It can be expressed as follows

$$\iint\limits_\Sigma \vec{n}\text{rot } \vec{u}dS = \int\limits_L \vec{u}\vec{t}dl \,.$$

1.2.2 Curvilinear orthogonal coordinates

We denote the curvilinear coordinates by γ_k, $k = 1,2,3$. The transformation relation of curvilinear coordinates into the Cartesian ones can be expressed as follows

$$x_i = x_i(\gamma_k) \quad .$$

For a length element along the x_i-axis, we can write

$$dx_i = \frac{\partial x_i}{\partial \gamma_k} d\gamma_k \quad .$$

For a general length element, we have

$$(ds)^2 = dx_i dx_i = \frac{\partial x_i}{\partial \gamma_k}\frac{\partial x_i}{\partial \gamma_l} d\gamma_k d\gamma_l = g_{kl} d\gamma_k d\gamma_l \ , \quad g_{kl} = \frac{\partial x_i}{\partial \gamma_k}\frac{\partial x_i}{\partial \gamma_l}$$

If $g_{kl} \neq 0$ only for $k = l$, we speak about *orthogonal curvilinear coordinates*. In this case $g_{kl} = 0$ for $k \neq l$, which implies that the vector $\partial x_i/\partial \gamma_k$, tangent to the coordinate line γ_k is perpendicular to the vector $\partial x_i/\partial \gamma_l$, tangent to the coordinate line γ_l. Thus the coordinate lines γ_k, γ_l are orthogonal at any point. In such coordinates,

$$(ds)^2 = h_1^2(d\gamma_1)^2 + h_2^2(d\gamma_2)^2 + h_3^2(d\gamma_3)^2 \quad ,$$

where, for example h_1^2 is given by

$$h_1^2 = \frac{\partial x_1}{\partial \gamma_1}\frac{\partial x_1}{\partial \gamma_1} + \frac{\partial x_2}{\partial \gamma_1}\frac{\partial x_2}{\partial \gamma_1} + \frac{\partial x_3}{\partial \gamma_1}\frac{\partial x_3}{\partial \gamma_1} \quad .$$

A length element along the coordinate line γ_1 is thus

$$ds_1 = h_1 d\gamma_1$$

and similarly along remaining coordinates. We can thus write

$$\frac{\partial}{\partial s_1} \Leftrightarrow \frac{1}{h_1}\frac{\partial}{\partial \gamma_1}$$

and the operator gradient "∇" can be written as

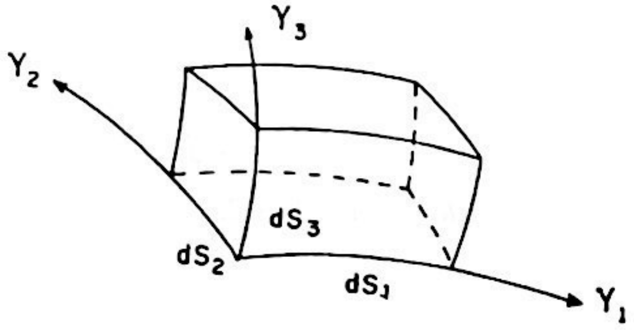
$$\nabla \equiv \left(\frac{1}{h_1}\frac{\partial}{\partial \gamma_1}, \quad \frac{1}{h_2}\frac{\partial}{\partial \gamma_2}, \quad \frac{1}{h_3}\frac{\partial}{\partial \gamma_3}\right) \quad .$$

We now also derive the operator divergence in curvilinear orthogonal coordinates since we shall need it later. We shall use the Gauss theorem for this purpose:

$$\iiint\limits_V u_{i,i} dV = \iint\limits_S u_i n_i dS \quad ,$$

where n_i is a unit outer normal to the surface S surrounding the volume V. Let us consider an elementary volume $dV = ds_1 ds_2 ds_3$. The contributions of the closer surfaces on the following picture to the surface integral are:

$$-u_1 ds_2 ds_3, \ -u_2 ds_1 ds_3, \ -u_3 ds_1 ds_2 \quad .$$



The contributions of the more distant surfaces are:

$$u_1 ds_2 ds_3 + \frac{\partial}{\partial s_1}(u_1 ds_2 ds_3) ds_1 \quad ,$$
$$u_2 ds_1 ds_3 + \frac{\partial}{\partial s_2}(u_2 ds_1 ds_3) ds_2 \quad ,$$
$$u_3 ds_1 ds_2 + \frac{\partial}{\partial s_3}(u_3 ds_1 ds_2) ds_3 \quad .$$

Summing up the contributions of all the six surfaces and equating them to the contribution of the volume integral in the Gauss theorem, we get

$$\begin{aligned} u_{i,i} ds_1 ds_2 ds_3 &= \frac{1}{h_1}\frac{\partial}{\partial \gamma_1}(h_2 h_3 u_1) d\gamma_2 d\gamma_3 ds_1 + \frac{1}{h_2}\frac{\partial}{\partial \gamma_2}(h_1 h_3 u_2) d\gamma_1 d\gamma_3 ds_2 + \\ &+ \frac{1}{h_3}\frac{\partial}{\partial \gamma_3}(h_1 h_2 u_3) d\gamma_1 d\gamma_2 ds_3 \quad . \end{aligned}$$

Using $ds_1 = h_1 d\gamma_1$, etc., this can be further rewritten as

$$u_{i,i} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \gamma_1}(h_2 h_3 u_1) + \frac{\partial}{\partial \gamma_2}(h_1 h_3 u_2) + \frac{\partial}{\partial \gamma_3}(h_1 h_2 u_3)\right] .$$

Combining the expressions for the gradient and divergence in orthogonal curvilinear coordinates, we get the expression for Laplacian $\Delta \equiv \partial^2/\partial x_i^2$

$$\Delta \equiv \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \gamma_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial \gamma_1}\right) + \frac{\partial}{\partial \gamma_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial \gamma_2}\right) + \frac{\partial}{\partial \gamma_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial \gamma_3}\right)\right] .$$

1.3 Analytic signal

In the study of transient waves in the following chapters, we shall work with *analytic signal* $F(\xi)$. The function $F(\xi)$ is defined as follows,

$$F(\xi) = g(\xi) + ih(\xi) \quad .$$

Here $g(\xi)$ is the actual transient signal for which the analytic signal is constructed, $h(\xi)$ is the Hilbert transform of $g(\xi)$,

$$h(\xi) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(\sigma)}{\sigma-\xi}d\sigma \quad .$$

The functions $g(\xi)$ and $h(\xi)$ defined in the above way form so-called *Hilbert transform pair.*

Before we discuss the role of analytic signal in the theory of propagation of transient plane waves, let us recall several important facts from the Fourier analysis.

With exception of some special functions or distributions (as e.g. Dirac δ function), we shall mostly consider *Fourier functions of the standard type*. Under such functions we understand those which are *absolutely integrable* in the interval $(-\infty,+\infty)$ and which satisfy *the Dirichlet's conditions* in any finite interval. The function is absolutely integrable if it satisfies the following inequality

$$\int_{-\infty}^{+\infty}|g(t)|dt \le A \quad ,$$

where A is a real positive constant. The Dirichlet conditions require the continuity of the function g in a finite interval (situated anywhere in $(-\infty,+\infty)$) with a possibility of finite number of discontinuities of the first kind (at which finite limits from the left and right exist) and finite number of maxima and minima. Under these conditions, we can define the Fourier transform of function $g(t)$ as

$$g(f) = \int_{-\infty}^{+\infty} g(t)e^{i2\pi ft}dt \quad , \quad g(t) = \int_{-\infty}^{+\infty} g(f)e^{-i2\pi ft}df \, .$$

For further explanations, we need to know the *convolution theorem* and the inverse Fourier transform of the function $sgn(f)$.

Under the convolution $c(t)$ of two functions $a(t)$, $b(t)$, we understand the function

$$c(t) = \int_{-\infty}^{+\infty} a(\tau)b(t-\tau)d\tau \quad ,$$

which is usually symbolically denoted as $c(t) = a(t) * b(t)$. The convolution theorem says that the Fourier transform of the convolution is equal to the product of Fourier transformed functions $a(t)$, $b(t)$,

$$c(f) = a(f)b(f) \quad .$$

This can be easily proved:

$$\begin{aligned} c(f) &= \int_{-\infty}^{+\infty} c(t)e^{i2\pi ft}dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} a(\tau)b(t-\tau)e^{i2\pi ft}d\tau dt \\ &= \int_{-\infty}^{+\infty} a(\tau)e^{i2\pi f\tau}d\tau\int_{-\infty}^{+\infty} b(\xi)e^{i2\pi f\xi}d\xi = a(f)b(f) \quad . \end{aligned}$$

The signum function $sgn(f)$ is defined as follows:

$$sgn(f) = \begin{cases} 1 & f > 0 \\ 0 & f = 0 \\ -1 & f < 0 \end{cases} .$$

As we can see, the function sgn is not absolutely integrable and thus requires a special treatment. We shall find its inverse Fourier transform as a limiting case of an auxiliary function which in the limit approaches the function $sgn(f)$. Let us introduce the auxiliary function in the following way,

$$g(f) = e^{-\alpha|f|} i \, sgn(f)$$

with $\alpha > 0$. This function is integrable and for $\alpha \to 0$ it yields $i\,sgn(f)$. Thus, when we find its inverse Fourier transform, we shall let $\alpha \to 0$ and we get the inverse Fourier transform of the function $isgn(f)$. Let us look for $g(t)$ corresponding to $g(f) = e^{-\alpha|f|} i \, sgn(f)$:

$$\begin{aligned} g(t) &= \int_{-\infty}^{+\infty} e^{-\alpha|f|} isgn(f) e^{-i2\pi ft} df = -\int_{-\infty}^{0} e^{\alpha f} e^{-i2\pi ft} i df \\ &+ \int_0^{\infty} e^{-\alpha f} e^{-i2\pi ft} i df = -\int_0^{\infty} e^{-\alpha f'} e^{i2\pi f' t} i df' + \int_0^{\infty} e^{-(\alpha + i2\pi t) f} i df \\ &= -\int_0^{+\infty} e^{(-\alpha + 2i\pi t) f} i df + \int_0^{\infty} e^{(-\alpha 2i\pi t) f} i df = \frac{i}{i2\pi t - \alpha} + \frac{i}{i2\pi t + \alpha} . \end{aligned}$$

In the limit, for $\alpha \to 0$ this yields

$$g(t) = \int_{-\infty}^{+\infty} isgn(f) e^{-i2\pi ft} dt = \frac{1}{\pi t} .$$

Thus, we found a Fourier pair

$$g(t) = (\pi t)^{-1} \quad , \quad g(f) = isgn(f) .$$

From the Fourier transform equations given above, we can immediately derive a property of the Fourier transform of the real signals $g(t)$ which will be useful in the following considerations. We can see that

$$g^*(f) = \int_{-\infty}^{+\infty} g(t) e^{-i2\pi ft} dt = g(-f) .$$

Using this property, we can rewrite the expression for the real signal in the following way:

$$\begin{aligned} g(t) &= \int_{-\infty}^{+\infty} g(f) e^{-i2\pi ft} df = \int_{-\infty}^{0} g(f) e^{-i2\pi ft} df + \int_0^{\infty} g(f) e^{-i2\pi ft} df \\ &= \int_0^{\infty} g(-f') e^{i2\pi f' t} df + \int_0^{\infty} g(f) e^{-i2\pi ft} df \\ &= \int_0^{\infty} g^*(f) e^{i2\pi ft} df + \int_0^{\infty} g(f) e^{-i2\pi ft} df = 2Re\int_0^{\infty} g(f) e^{-i2\pi ft} df . \end{aligned}$$

Thus,

$$g(t) = 2Re \int_0^\infty g(f)e^{-i2\pi ft}df \quad .$$

It is natural to introduce the complex signal

$$F(t) = 2\int_0^\infty g(f)e^{-i2\pi ft}df \quad .$$

It can be written as

$$F(t) = g(t) + ih(t) \quad ,$$

where

$$h(t) = 2Im \int_0^\infty g(f)e^{-i2\pi ft}df \quad .$$

This expression can be rewritten as follows

$$\begin{aligned} h(t) &= -i\left[\int_0^\infty g(f)e^{-i2\pi ft}df - \int_0^\infty g^*(f)e^{i2\pi ft}df\right] \\ &= -i\left[\int_0^\infty g(f)e^{-i2\pi ft}df - \int_{-\infty}^0 g(f')e^{-i2\pi f't}df'\right] \\ &= -i\left[\int_{-\infty}^{+\infty} g(f)sgn(f)e^{-i2\pi ft}df\right] = -g(t) * \frac{1}{\pi t} \\ &= -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{g(\sigma)}{t-\sigma}d\sigma = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{g(\sigma)}{\sigma - t}d\sigma \quad . \end{aligned}$$

Thus, the function $h(t)$ is the Hilbert transform of the real signal $g(t)$, and $g(t)$ and $h(t)$ form a Hilbert pair. The complex signal $F(t)$,

$$F(t) = g(t) + ih(t)$$

is the *analytic signal.*

We can see that the functions $g(t)$, $h(t)$ and $F(t)$ behave like $\cos t$, $-\sin t$ and $\exp(-it)$, respectively. By assuming

$$g(f) = \frac{1}{2}\delta(f - f_o) \quad , \quad f_o \epsilon (0, +\infty)$$

we get

$$F(t) = \exp(-i2\pi f_o t)\,,\; g(t) = \cos(2\pi f_o t)\,,\; h(t) = -\sin(2\pi f_o t)\,.$$

Thus, $\cos t$ and $-\sin t$ form a Hilbert pair.

Another important Hilbert pair and corresponding analytic signal are related to the real signal in the form of the Dirac δ function. For $g(t) = \delta(t)$, the equation for $h(t)$ yields

$$h(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\delta(\sigma)}{\sigma - t}d\sigma = -\frac{1}{\pi t} \quad .$$

The corresponding analytic signal has thus the form

$$F(t) = \delta(t) - \frac{i}{\pi t} \quad .$$

Finally, let us mention another property of the Hilbert transform, which we are going to use. We show that the Hilbert transform of the derivative $dg(t)/dt$ of the signal $g(t)$ is the derivative $dh(t)/dt$ of the Hilbert transform of the signal $g(t)$. It can be easily proved. Let us write the derivatives of h and g

$$\frac{dh(t)}{dt} = 2Im\int_0^{+\infty} g(f)(-i2\pi f)e^{-i2\pi ft}df \quad ,$$

$$\frac{dg(t)}{dt} = 2Re\int_0^{+\infty} g(f)(-i2\pi f)e^{-i2\pi ft}df \quad .$$

From the first equation we can see that the derivative of the Hilbert transform $h(t)$ corresponds to a "new" signal whose Fourier transform is $g(f)(-i2\pi f)$. From the second equation we can see that the "new" signal is just the derivative $dg(t)/dt$.

In the following, we shall mostly work with the analytic signal $F(t)$ but the physically meaningful signal is $g(t)$.

CHAPTER 2

ELASTODYNAMIC THEORY

Under an external load, a body can be translated, rotated or deformed. In the following, we shall concentrate on the latter effect: deformation.

The deformation of the material is a process during which *distances* between individual points of the material *change*. We shall consider temporary loads which cause temporary deformations. In a realistic material, the application of a force at a particular place causes deformation, first in a vicinity of this place and successively also in more distant parts. This process is called *wave propagation*.

The wave propagation must overcome the resistence of the material caused by its consistency and the resistence caused by inertia. If the consistency of the material were such that the material is nondeformable (rigid), the effect of application of external forces at one point of the body would be felt immediately at each point of the material: the body would be shifted. If the material were deformable but without inertia, then all the particles of the material would be excited simultaneously. We shall consider realistic materials which are deformable and which, after the load is removed, return to the state, which is the same or similar to the state before loading. In the latter case we speak about *imperfectly elastic* materials, in the former case about *perfectly elastic* materials. Now, we shall fully concentrate on perfectly elastic materials.

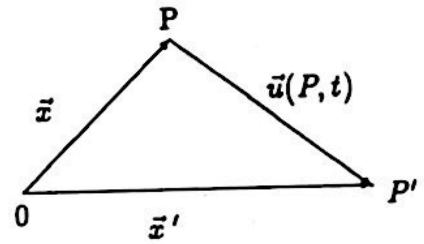
The wave propagation is connected with transmission of energy. The material is assumed to be formed of a continuous system of small elements, particles, which have, however, nothing common with the microscopic structure of the material. The energy is transported from one particle of the material to another, it is not transported by the flow of the particles. The particles of the material oscillate around their mean positions.

In classical mechanics, the wave motion is assumed to be a small disturbance of an initial state of the material, called *natural state*. In natural state there are no stresses and no deformations. The internal parts of the Earth's body, where the seismic waves propagate are, however, under a large stress (the hydrostatic pressure reaches values of $10^{10} - 10^{11}\,Pa$) and the material there is, therefore, in a deformed state. Thus, we shall consider the initial state to be *the state of static equilibrium* in which the deformation does not change with time.

We shall study how the deformation changes relations between two close particles of the medium. The change will be described by *the tensor of deformation or strain tensor*. Before it, we introduce *the displacement vector* describing a displacement of a single particle under a deformation. We shall study both in Cartesian coordinate system x_1, x_2, x_3 with the origin at a point 0.

2.1 Displacement vector

Let us consider a particle, which is in the initial state situated at the point $P[x_1, x_2, x_3]$. In the deformed material at time t, this particle will be shifted to the point $P'[x_1', x_2', x_3']$. The



shift from the point P to P' is specified by *the displacement vector* $u_i(x_k, t)$ such that

$$x_i' = x_i + u_i(x_k, t) = x_i + u_i(P, t) \quad .$$

In this way, the displacement vector is specified as a vectorial function of the coordinates x_i of the point P (the position of the particle in the initial state) and time t. This approach is widely used in seismology and seismic prospecting and it is known as *Lagrangian approach*. The displacement vector defined in this way is also sometimes called the *particle motion vector*.

It would be possible to consider the displacement vector as a function of coordinates x_i' of the point P' (the position of the particle in the deformed state) and time t, $u_i(x_k', t)$. In such a case, u_i depends on time also through the coordinates x_k', which are time dependent. This approach is known as *Euler's approach* and it is used more in hydrodynamics. Here we shall use the Lagrangian approach.

In the Lagrangian approach, the velocity of the motion of a particle can be simply evaluated by partial differentiation of $u_i(x_k, t)$ with respect to time t

$$v_i(x_k, t) = \dot{u}_i(x_k, t) = \frac{\partial u_i(x_k, t)}{\partial t} \quad .$$

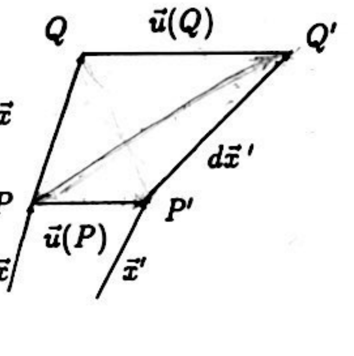
Similarly, for acceleration of the particle we get

$$a_i(x_k,t) = \ddot{u}_i(x_k,t) = \frac{\partial^2 u_i(x_k,t)}{\partial t^2} \quad .$$

The quantity $v_i(x_k,t)$ is known as the *particle velocity* and $a_i(x_k,t)$ as the *particle acceleration.*

2.2 Strain tensor

In addition to the point P, we now consider a point $Q[x_1+dx_1, x_2+dx_2, x_3+dx_3]$ situated in a close vicinity of the point P in the non-deformed medium. The point Q will be shifted during the deformation to the point $Q'[x_1'+dx_1', x_2'+dx_2', x_3'+dx_3']$. The displacement



vector of the point Q can be expressed in terms of the displacement vector $u_i(P)$ as follows

$$u_i(Q) = u_i(x_k+dx_k) \sim u_i(x_k) + \frac{\partial u_i(x_k)}{\partial x_j}dx_j = u_i(P) + \frac{\partial u_i(P)}{\partial x_j}dx_j \quad .$$

Let us investigate the change of the distance between the points P and Q due to the deformation. We shall do it by comparing squares of the distances $\overline{PQ}$ and $\overline{P'Q'}$.

For $\overline{PQ}^2$, we immediately have

$$\overline{PQ}^2 = dx_i dx_i \quad .$$

Let us determine dx_i'. We can write

$$dx_i' = dx_i + u_i(Q) - u_i(P) \sim dx_i + \frac{\partial u_i(P)}{\partial x_j}dx_j \quad .$$

For $\overline{P'Q'}^2$, we can thus write

$$\overline{P'Q'}^2 = dx_i'dx_i' \sim \left(dx_i + \frac{\partial u_i}{\partial x_j}dx_j\right)\left(dx_i + \frac{\partial u_i}{\partial x_k}dx_k\right)$$

$$= dx_i dx_i + \frac{\partial u_i}{\partial x_j} dx_j dx_i + \frac{\partial u_i}{\partial x_k} dx_k dx_i + \frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_j} dx_j dx_k$$
$$= dx_i dx_i + \left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\right) dx_j dx_k \quad .$$

For $\overline{P'Q'}^2 - \overline{PQ}^2$, we get

$$\overline{P'Q'}^2 - \overline{PQ}^2 \sim 2E_{jk} dx_k dx_j \quad ,$$

where

$$E_{jk} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_k}\right) \quad .$$

The quantity E_{jk} is a tensor of the second rank. It is called the *tensor of finite strain*. Since all the derivatives in the expression for E_{jk} are taken at P, the tensor of finite strain characterizes the deformation in the vicinity of the point P. The tensor E_{jk} is symmetric ($E_{jk} = E_{kj}$) and thus it is specified by 6 independent components only. Due to the third term in the expression for E_{jk}, the tensor of finite strain is *nonlinear*.

In the following, we shall consider only such wave processes, in which the deformation is small, i.e.,

$$\left|\frac{\partial u_i}{\partial x_j}\right| \ll 1 \quad .$$

The elements of the strain tensor usually do not exceed the values of the order 10^{-4} (the elements are non-dimensional). Exceptions may be large earthquakes. The zone of strains larger than 10^{-4} around the earthquake source represents, however, only several wavelength. This value is comparable with strains caused by the Earth's tides. Strains connected with tectonic proceses may reach values of 10^{-5}.

It is thus possible to neglect the nonlinear term in the expression for E_{jk} since it is of second order with respect to $|\partial u_i/\partial x_j|$. The linearization substantially simplifies all the mathematical operations. We must, however, keep the above approximation always in mind in applications.

We shall denote the linearized tensor of finite strain e_{jk}

$$e_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) \quad ,$$

and we shall call it the *tensor of small strain*. In the following, we shall work exclusively with the tensor of small strain and we shall call it briefly *the strain tensor*. It is symmetric and linear. Elements of the strain tensor are dimensionless quantities.

Let us note that the tensor of small strain e_{ik} has the same form in Lagrangian and Euler formulation (the form of the tensor E_{ik} in both formulations differs).

2.2.1 Physical meaning of the elements of the strain tensor pula

The strain tensor describes "pure" deformation, it does not contain information about displacement or rotation of the deformed body as a whole. We can expect that the physical interpretation of diagonal and off-diagonal elements of e_{ik} will be different.

a) The meaning of the elements e_{11}, e_{22}, e_{33}

Let us consider points P and Q situated along the x_1 - axis so that $dx_i = (dx_1, 0, 0)$ and $\overline{PQ} = dx_1$. From the definition of the (small) deformation, we have in our case

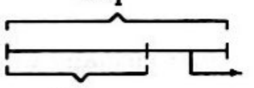
$$\overline{P'Q'}^2 - \overline{PQ}^2 \sim 2e_{ij}dx_i dx_j = 2e_{11}(dx_1)^2 \quad ,$$

from which we get for $\overline{P'Q'}$

$$\overline{P'Q'} \sim \sqrt{1+2e_{11}}\, dx_1$$

If we define relative extension e_r as $(\overline{P'Q'} - \overline{PQ})/\overline{PQ}$, we get in our approximation

$$e_r \sim \frac{\sqrt{1+2e_{11}}\, dx_1 - dx_1}{dx_1} \sim e_{11}$$



Thus the element e_{11} of the strain tensor represents approximately a *relative extension (contraction)* of the material along the x_1 - axes. The components e_{22} and e_{33} have their corresponding similar meaning.

b) The meaning of the elements e_{12}, e_{13}, e_{23}

Let us consider two points Q and R in a vicinity of the point P. Let us specify the points Q and R in the initial state in the following way:

$$dx_i^Q = (dx_1, 0, 0) \quad , \quad dx_i^R = (0, dx_2, 0) \quad .$$

Thus, the vectors dx_i^Q and dx_i^R are mutually perpendicular. After deformation, the points Q and R will be shifted to the new positions specified approximately as follows

$$dx_i'^Q \sim dx_1\delta_{1i} + \frac{\partial u_i}{\partial x_1}dx_1\, , \quad dx_i'^R \sim dx_2\delta_{i2} + \frac{\partial u_i}{\partial x_2}dx_2 \quad .$$

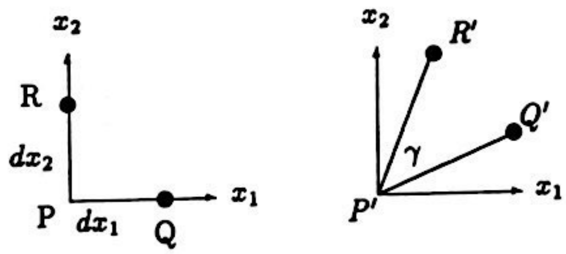
Let us now investigate how the deformation changes the mutual orientation of the lines $\overline{PQ}$ and $\overline{PR}$, which were originally perpendicular. For this purpose, let us consider the scalar product of the vectors $dx_i'^Q$, $dx_i'^R$:

$$dx_i'^Q dx_i'^R \sim \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) dx_1 dx_2 + \frac{\partial u_i}{\partial x_1}\frac{\partial u_i}{\partial x_2} dx_1 dx_2 \quad .$$

The last term can be neglected since we consider small deformations and it is of higher order in $|\partial u_i/\partial x_j|$. This gives

$$dx_i'^Q dx_i'^R = |dx_i'^Q||dx_i'^R| \cos\gamma \sim 2e_{12} dx_1 dx_2 \quad .$$

Here γ is the angle between the vectors $dx_i'^Q$, $dx_i'^R$.



For $\cos\gamma$ we can write

$$\cos\gamma \sim \frac{2e_{12} dx_1 dx_2}{|dx_i'^Q||dx_i'^R|} .$$

For $|dx_i'^Q|$ we have

$$|dx_i'^Q| \sim \sqrt{(dx_1)^2 + 2\frac{\partial u_1}{\partial x_1}(dx_1)^2 + \left(\frac{\partial u_1}{\partial x_1}\right)^2 (dx_1)^2} = dx_1\sqrt{1 + 2\frac{\partial u_1}{\partial x_1} + \left(\frac{\partial u_1}{\partial x_1}\right)^2} \, .$$

Since $|\partial u_i/\partial x_j| \ll 1$,

$$|dx_i'^Q| \sim dx_1 \overset{\left(1+\frac{\partial u_1}{\partial x_1}\right)}{} \text{ and similarly } |dx_i'^R| \sim dx_2 \quad .$$

If, in the expression for $\cos\gamma$, we use an angle $\alpha_{12} = \frac{\pi}{2} - \gamma$, then the above expression for $\cos\gamma$ yields

$$\sin\alpha_{12} \sim \alpha_{12} \sim 2e_{12} \, ,$$

and from it approximately

$$e_{12} \sim \frac{1}{2}\alpha_{12} \quad .$$

The approximation $\sin\alpha_{12} \sim \alpha_{12}$ is possible because $\gamma \sim \frac{\pi}{2} \Rightarrow \alpha_{12} \ll 1$.

Thus, the element e_{12} of the strain tensor represents a *half of the change* (decrease, increase) *of the right angle* between the directions which were originally perpendicular. The elements e_{13}, e_{23} have a similar meaning.

Let us note that the off-diagonal terms are sometimes called *shearing strains.*

2.2.2 Strain quadrics

Strain tensor surface can be generally writen in the form

$$e_{ij}x_ix_j = \pm 1 \quad .$$

After an appropriate rotation of coordinates, this can be diagonalized so that

$$e'_{11}x_1'^2 + e'_{22}x_2'^2 + e'_{33}x_3'^2 = \pm 1 \quad .$$

The axes of this quadrics are *principal axes of the strain tensor*, the values e'_{11}, e'_{22}, e'_{33} are *the principal strains.* The shearing strains are zero in the primed coordinate system.

2.2.3 Volume dilatation

Let us consider the Cartesian coordinate system x_i chosen so that its axes coincide with principal axes of the strain tensor and let us consider a change of an elementary volume dV during the deformation. In the initial state we have

$$dV = dx_1dx_2dx_3 \quad .$$

After the deformation, dV will change into dV'

$$dV' = dx'_1dx'_2dx'_3 \quad .$$

For dx'_i we derived above

$$dx'_i \sim dx_i + \frac{\partial u_i}{\partial x_k}dx_k \quad .$$

In our coordinate system this will give for dx'_1 (shearing strains are zero),

$$dx'_1 \sim dx_1 + \frac{\partial u_1}{\partial x_1}dx_1 = dx_1(1 + e_{11})$$

and similarly for dx'_2 and dx'_3. For dV' we can thus write

$$dV' \sim (1 + e_{11})(1 + e_{22})(1 + e_{33})dx_1dx_2dx_3 \sim dV + (e_{11} + e_{22} + e_{33})dV \ ,$$

where we have neglected the terms of higher order (like $e_{11}e_{22}, \ldots$). From this we have

$$\theta = \text{div}\ \vec{u} = (e_{11} + e_{22} + e_{33}) \sim \frac{dV' - dV}{dV} \,.$$

The quantity θ is called *volume dilatation* or briefly *dilatation* and it represents approximately a relative change of the volume during the deformation. Since θ is a trace of the strain tensor and that is invariant, also θ is invariant.

2.3 Stress tensor

The resistence of the material to deformations is caused by forces, which are called *internal stress*. In the state without deformation, they are zero. They appear after the force causing the deformation is applied and they tend to prevent it. They exist only between neighbouring particles of the material, which are in direct contact. Because of this, they are sometimes called *contact forces*. Their radius of action is very small. They can also be viewed as acting across an "imaginary" surface, dividing an elementary volume of the deformed body, on the other part of the considered volume. Because of this, they are also called *surface forces*. An example of such a force is hydrostatic pressure.

The considered elementary volume can also be affected by forces with larger radius of action. Examples are gravitational and electromagnetic forces. Since they are generally proportional to the volume of the considered part of the material, they are called *voluminal* or *body forces*. Another voluminal force, which appears in dynamic problems, is inertial force. A seismic source is just another example of voluminal force.

Let us consider an elementary volume ΔV of the deformed material and let us intersect it by an elementary surface ΔS so small that the surface forces and moments of forces acting on ΔS can be substituted by an equivalent force $\Delta\vec{T}$ and moment $\Delta\vec{M}$ acting on ΔS. We then define

$$\vec{T} = \lim_{\Delta S \to 0} \Delta\vec{T}/\Delta S \quad , \quad \vec{M} = \lim_{\Delta S \to 0} \Delta\vec{M}/\Delta S = 0 \quad ,$$

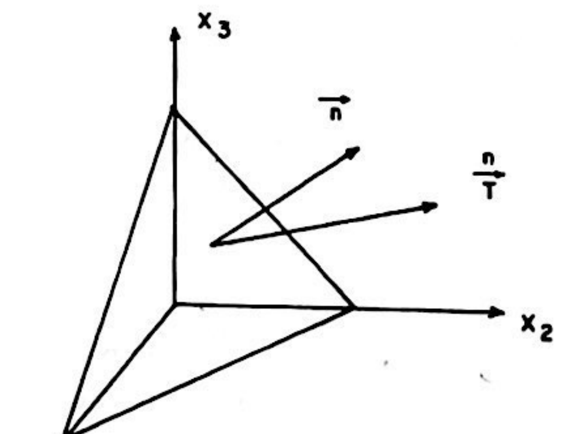
where $\vec{T}$ is *the stress vector* or *traction*. In a similar way, we can define voluminal force and the corresponding moment. They will be scalled by ΔV, i.e. the resulting quantities will be related to unit volume. We assume again the resulting moment to be zero. In the following, when we speak about stress and voluminal forces we *must remember* that in fact, we are dealing with densities of these forces. In case of stress, we deal with surface density and in case of voluminal forces with voluminal density of the forces.

The traction $\vec{T}$ depends on the orientation of the surface ΔS at a given point. The traction on a surface different from ΔS but at the same point will generally differ both in its size and orientation. We shall, therefore, denote the traction acting on ΔS with unit normal $\vec{n}$ by $\overset{n}{\vec{T}}$. The unit normal $\vec{n}$ is introduced so that it points to that part of the volume ΔV which is considered to act on the other part of ΔV. Due to the principle of action and reaction, the traction acting on the other side of the surface ΔS, $\vec{T}(-\vec{n})$ must be of the same size as $\vec{T}(\vec{n})$ but of opposite orientation

$$\vec{T}(-\vec{n}) = -\vec{T}(\vec{n}) \quad .$$

The traction generally does not coincide with the direction of the normal $\vec{n}$. It can be, therefore, projected on the normal and on the surface. The component of $\overset{n}{\vec{T}}$ into the normal $\vec{n}$ is called *normal stress* or *tension* (tensile stress) if $\overset{n}{\vec{T}}$ tends to separate the two parts of the considered elementary volume (and thus it is positive) or *pressure* if it has oposite orientation (and it is negative). An example of pressure is hydrostatic pressure, for which $\vec{T}\vec{n} = -|\vec{T}|$ since $\vec{T}||\vec{n}$. The components tangent to the surface ΔS are called *tangential* or *shearing stresses.*

We shall introduce stress components τ_{ij} as components of traction acting on an elementary surface with normal along the x_i axis, $\tau_{ij} = \overset{i}{T}_j$. We shall now try to determine the relationship between the traction $\overset{n}{T}_i$ and the stress components τ_{ij}. Let us consider our elementary volume ΔV as a tetrahedron bounded by the surface ΔS, characterized by the



normal n_i, and elementary surfaces ΔS_i in the planes perpendicular to the coordinate axes. For a sufficiently small tetrahedron, the body forces, which are proportional to the size of the volume can be neglected in comparison with surface forces, which are proportional to the size of the surface. The condition of equilibrium for tetrahedron then contains only surface forces acting on it:

$$\overset{n}{T}_i\, \Delta S + \overset{-1}{T}_i\, \Delta S_1 + \overset{-2}{T}_i\, \Delta S_2 + \overset{-3}{T}_i\, \Delta S_3 = 0 \quad ,$$

where $\overset{-k}{T}_i$ denotes traction acting on the elementary surface ΔS_k with normal $-i_k$ (i_k being basis vector of the Cartesian coordinate system). Since $\overset{-n}{T}_i = -\overset{n}{T}_i$ and $\overset{j}{T}_i = \tau_{ji}$, we can rewrite the equation as

$$\overset{n}{T}_i\, \Delta S - \tau_{1i}\Delta S_1 - \tau_{2i}\Delta S_2 - \tau_{3i}\Delta S_3 = 0 \quad .$$

There is a straightforward relation between ΔS and ΔS_j,

$$\Delta S_j = \Delta \vec{S}\vec{i}_j = \Delta S \vec{n}\vec{i}_j = \Delta S n_j \quad .$$

After insertion of this into the condition of equilibrium, we get

$$\overset{n}{T}_i = \tau_{ji} n_j$$

This important relation says that a traction acting on *an arbitrary oriented* surface specified by the unit normal n_j can be expressed in terms of 9 stress components τ_{ij}.

Let us now consider a rotated Cartesian system x_i' and let us choose its x_k' axis along the normal $\vec{n}$. Then we have,

$$\overset{k}{T'}_m = \alpha_{mi}\, \overset{n}{T}_i = \alpha_{mi}\tau_{ji}n_j = \alpha_{mi}\tau_{ji}\alpha_{lj}n_l' = \alpha_{mi}\alpha_{lj}\tau_{ji}\delta_{lk} = \alpha_{mi}\alpha_{kj}\tau_{ji} \,.$$

It remains to realize that $\overset{k}{T'}_m$ is the m-th component of the traction acting on the elementary surface perpendicular to the axis x_k'. $\overset{k}{T'}_m$ is thus a stress component τ_{km}' and we have

$$\tau_{km}' = \alpha_{kj}\alpha_{mi}\tau_{ji} \quad .$$

This is, however, the transformation equation of the tensors of the second rank. Thus, the stress components are elements of a tensor. We call it the *stress tensor* τ_{ij}. It describes the stress field in a vicinity of the point in which it is specified. Once it is known, a traction acting on an arbitrary oriented surface element can be determined from it.

2.3.1 Condition of equilibrium

Let us consider a small volume V of a deformed material in a static equilibrium. The volume V is surrounded by a closed surface S characterized by the unit external normal $\vec{n}$.

The volume can be considered as rigid and, therefore, the conditions of equilibrium required for the rigid body can be applied to V. The volume V will be in equilibrium if the sum of all forces acting on the volume V and sum of their moments vanish. If we denote the resulting volume force by f_i then the above conditions applied to our volume V read

$$\iiint_V f_j dV + \iint_S \overset{n}{T}_j \, dS = 0 \quad ,$$

$$\iiint_V \varepsilon_{ijk} X_j f_k dV + \iint_S \varepsilon_{ijk} X_j \, \overset{n}{T}_k \, dS = 0 \quad .$$

These are conditions of equilibrium in an integral form. In the above equations, the symbol X_i denotes the radius vector of a displaced point of the volume V in the Euler coordinates, $X_i = x_i + u_i$. The arguments of the forces f_i and $\overset{n}{T}_i$ are also considered to be specified in the Euler coordinates. We shall perform the transfer from the Euler to Lagrangian coordinates, which we are using, in the final formulae.

Let us now consider the condition that all forces acting on the volume V must vanish. It can be rewriten in terms of the stress tensor τ_{ij}

$$\iiint_V f_j dV + \iint_S \tau_{ij} n_i dS = 0 \quad .$$

We can apply the Gauss theorem on the surface integral to get

$$\iiint_V (f_j + \partial \tau_{ij} / \partial X_i) dV = 0 \quad .$$

Since the integrand of the above integral is a continuous function of spatial coordinates and the volume can be chosen arbitrarily, we can write for any point of the deformed volume

$$f_j + \partial \tau_{ij} / \partial X_i = 0 \quad .$$

Similarly as we considered the term $\partial u_i / \partial x_j$ small and neglected products of $\partial u_i / \partial x_j$, we also assume that $\partial \tau_{ij} / \partial X_k$ are small and their products with $\partial u_i / \partial x_j$ can be neglected. Then we have

$$\frac{\partial \tau_{ij}}{\partial x_i} = \frac{\partial \tau_{ij}}{\partial X_k} \frac{\partial X_k}{\partial x_i} = \frac{\partial \tau_{ij}}{\partial X_k} \left(\delta_{ik} + \frac{\partial u_k}{\partial x_i} \right) \sim \frac{\partial \tau_{ij}}{\partial X_i} \quad .$$

We can thus write

$$f_j + \tau_{ij,i} = 0 \quad .$$

This is the *equation of static equilibrium* of deformed material in differential form.

Let us now investigate the implications of the vanishing sum of moments. Using stress tensor instead of the traction in the condition for moments, we can write

$$\iiint\limits_V \varepsilon_{ijk} X_j f_k dV + \iint\limits_S \varepsilon_{ijk} X_j \tau_{lk} n_l dS = 0 \quad .$$

Applying the Gauss theorem on the surface integral, we get

$$\begin{aligned}
\iint\limits_S \varepsilon_{ijk} X_j \tau_{lk} n_l dS &= \iiint\limits_V \varepsilon_{ijk} \partial (X_j \tau_{lk}) / \partial X_l dV \\
&= \iiint\limits_V \varepsilon_{ijk} (\delta_{lj} \tau_{lk} + X_j \tau_{lk,l}) dV \\
&= \iiint\limits_V \varepsilon_{ijk} \tau_{jk} dV - \iiint\limits_V \varepsilon_{ijk} X_j f_k dV \quad .
\end{aligned}$$

In the last term, we used the equation of static equilibrium. If we insert this result into the original equation, we get

$$\iiint\limits_V \varepsilon_{ijk} \tau_{jk} dV = 0 \quad .$$

Since V can be chosen arbitrarily, we can write for any point of the material in an equilibrium,

$$\varepsilon_{ijk} \tau_{jk} = 0 \quad .$$

The only non-zero terms on the LHS of this equation are those for which $i \neq j \neq k$. This yields

$$\tau_{jk} - \tau_{kj} = 0 \quad .$$

The stress tensor is symmetric and can thus be specified by 6 independent components only.

2.3.2 Equation of motion

The equation

$$f_j + \tau_{ij,i} = 0$$

describes *static equilibrium*. To get the dynamic equations - equation of motion, we can use the d'Alembert principle. According to it, the static equations become dynamic ones when we add the inertial forces to the static equations.

The density of the inertial force can be written as $\rho \frac{\partial^2 u_i}{\partial t^2}$ so that the dynamic equation takes the form

$$f_j + \tau_{ij,i} = \rho \frac{\partial^2 u_j}{\partial t^2} \quad .$$

This is the *equation of motion*. It is the basic equation for solving wave propagation problems. Before it can be solved, it is necessary to specify it in terms of displacements u_i. For this purpose, it is necessary to find the relations between the stress and strain. This will be done in Sec. 2.4.

2.3.3 Stress quadrics

Stress surface can be generally written in the form

$$\tau_{ij}x_ix_j = \pm 1 \quad ,$$

which can be diagonalized into the form

$$\tau'_{11}x_1'^2 + \tau'_{22}x_2'^2 + \tau'_{33}x_3'^2 = \pm 1 \quad .$$

The axes of this quadrics are called the *principal directions of the stress* and the values τ'_{11}, τ'_{22}, τ'_{33} are the *principal stresses*. The shearing stresses are zero in the primed coordinate system.

Let us note that the elements of the stress tensor as well as tractions are measured in pascals (Pa): $1Pa = N/m^2 = kgm^{-1}s^{-2}$, voluminal forces are measured in N/m^3.

2.4 Stress-strain relations

It is evident that the stress and the strain are mutually dependent. The mutual relation depends on the character of the material and is, therefore, an important characteristics of the material. The determination of this relation which is also called *constitutive relation* is subject of *rheology*. According to their constitutive relations, the materials can be called elastic, viscous, viscoelastic, plastic.

The study of the deformation and flow matter

We shall concentrate on elastic materials, i.e. on materials which, after removing load, return to the state before application of the load. We can do that since the additional stresses and strains (additional to the stresses and strains existing in the initial state of the material) connected with the wave propagation are small and permanent deformations due to the wave process are negligible.

We shall assume that at each point of the medium, the stress tensor is uniquely specified by strain, i.e.

$$\tau_{ij} = \tau_{ij}(e_{kl}) \quad .$$

This is a generalization of the law formulated by Hooke 300 years ago on the basis of experiments with springs. It said: "Extension is proportional to the force". The above equation is a generalization: Each component of the stress tensor depends generally on each component of the strain tensor and the dependence is generally nonlinear. We shall, however expand the above equation and keep only its linear terms since the higher order terms can be neglected due to the assumed smallness of e_{jk}. Since we consider τ_{ij} as a perturbation of the initial prestressed state, we can put $\tau_{ij}(0) = 0$. In such a way we get *the generalized Hooke law*:

$$\tau_{ij}(x_m) = c_{ijkl}(x_m)e_{kl}(x_m) \quad .$$

Here c_{ijkl} are constants of proportionality, which we call *elastic parameters*. It is easy to prove (taking into account the tensor character of τ_{ij} and e_{kl}) that c_{ijkl} represents a tensor of the 4-th rank with 3^4, i.e. 81 elements. This tensor is sometimes called the *stiffness tensor*. Before we proceed with the discussion of the generalized Hooke law, we introduce the strain energy.

2.4.1 Strain energy

External forces acting on a body deform it. Due to this, the internal energy and the temperature of the body may change. In the following, we shall consider *adiabatic processes*, i.e. processes in which there is no heat exchange with the surrounding medium. Wave processes can be considered adiabatic since the oscillations are so fast that there is no time for exchange of heat. In such a situation, the first law of thermodynamics can be written as

$$dU + dK = dA \quad .$$

This equation says that the change of internal energy U and kinetic energy K in a volume V of the considered deformed body must be balanced by the work of the surface and volume forces acting on the volume. From the form of the above equation we can deduce that the internal energy U is related to the potential energy. We introduce a quantity W such that

$$dU = \iiint_V W dV \quad .$$

We shall call W the *density of strain energy*.

If we relate the quantities in the first thermodynamic law to a time unit, we get

$$\frac{d}{dt}\iiint_V W dV + \frac{dK}{dt} = \frac{dA}{dt} \quad ,$$

matter - materia

and from it (because V is related to the original position - Lagrangian coordinates):

$$\iiint\limits_V \frac{\partial W}{\partial t} dV + \frac{dK}{dt} = \frac{dA}{dt} \quad .$$

First, we calculate the work done by the surface and volume forces on the volume V. Displacement of a point of volume V in the time interval $(t, t+dt)$ is $\frac{\partial u_i}{\partial t} dt$. The work of volume forces f_i in the given time interval and in a volume unit is thus $f_i \frac{\partial u_i}{\partial t} dt$. Similar considerations for the work done by surface forces lead to $\overset{n}{T}_i \frac{\partial u_i}{\partial t} dt$. For $\frac{dA}{dt}$ in the volume V we finally get

$$\frac{dA}{dt} = \iiint\limits_V f_i \frac{\partial u_i}{\partial t} dV + \iint\limits_S \tau_{ji} n_j \frac{\partial u_i}{\partial t} dS \quad .$$

Here the symbol S denotes the closure of the volume V. Vector n_j denotes the unit outer normal to it. Applying the Gauss theorem to the surface integral, we get

$$\begin{aligned} \iint\limits_S \tau_{ji} n_j \frac{\partial u_i}{\partial t} dS &= \iiint\limits_V \left(\tau_{ji} \frac{\partial u_i}{\partial t} \right)_{,j} dV = \iiint\limits_V \left(\tau_{ji,j} \frac{\partial u_i}{\partial t} + \frac{\partial u_{i,j}}{\partial t} \tau_{ji} \right) dV \\ &= -\iiint\limits_V f_j \frac{\partial u_j}{\partial t} dV + \iiint\limits_V \rho \frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_j}{\partial t} dV + \iiint\limits_V \tau_{ij} \frac{\partial}{\partial t} e_{ij} dV \quad . \end{aligned}$$

Thus,

$$\frac{dA}{dt} = \iiint\limits_V \rho \frac{\partial^2 u_j}{\partial t^2} \frac{\partial u_j}{\partial t} dV + \iiint\limits_V \tau_{ij} \frac{\partial e_{ij}}{\partial t} dV \quad .$$

Kinetic energy of the volume V is

$$K = \frac{1}{2} \iiint\limits_V \rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} dV$$

and its time change (assuming that ρ does not vary with time)

$$\frac{dK}{dt} = \iiint\limits_V \rho \frac{\partial^2 u_i}{\partial t^2} \frac{\partial u_i}{\partial t} dV \quad .$$

The first law of thermodynamics now gets the form

$$\iiint\limits_V \frac{\partial W}{\partial t} dV - \iiint\limits_V \tau_{ji} \frac{\partial e_{ij}}{\partial t} dV = 0 \quad .$$

Since the volume V can be chosen arbitrarily, to satisfy the above equation requires

$$\frac{\partial W}{\partial t} = \tau_{ji} \frac{\partial e_{ij}}{\partial t} \quad .$$

For a specific point of the volume V, this gives a change of W in the time interval dt,

$$dW = \tau_{ji} de_{ij} \qquad \left(de_{ij} = \frac{\partial e_{ij}}{\partial t} dt\right) \quad .$$

Since we consider W to be a function of e_{ij}, dW is the total differential and for τ_{ij} we have

$$\tau_{ij} = \partial W / \partial e_{ij} \quad .$$

Since τ_{ij} are linear functions of e_{kl} (generalized Hooke law), the above derivatives of strain energy must be linear functions of e_{kl} and W must be a quadratic function of e_{kl}. We shall define that in the state without deformation, i.e. when $e_{kl} = 0$, the strain energy is zero, $W = 0$. Then, W is a homogeneous quadratic function of components of the strain tensor. According to the Euler's theorem on homogeneous functions

$$\frac{\partial W}{\partial e_{kl}} e_{kl} = 2W \quad .$$

From this we have

$$W = \frac{1}{2} \frac{\partial W}{\partial e_{ij}} e_{ij} = \frac{1}{2} \tau_{ij} e_{ij} = \frac{1}{2} c_{ijkl} e_{kl} e_{ij} \quad .$$

This is the final equation for *the strain energy density*. It is interesting to note that instataneous values of strain and kinetic energy are in phase, i.e., when kinetic energy is zero, strain energy is zero too, when kinetic energy reaches maximum, strain energy reaches it too.

2.4.2 Energy flux

We shall again consider the volume V of a deformed body. It is the volume which is during the deformation shifted together with all the particles which it contained in the initial state. Now, we are going to investigate *how is kept* the energy balance in the volume V. As shown above, the *elastic energy* ϵ (the sum of the strain and kinetic energy) in the volume V is

$$\epsilon = \iiint_V E dV = \frac{1}{2} \iiint_V \left(\rho \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial t} + \tau_{ij} e_{ij}\right) dV \quad .$$

The time rate of ϵ is

$$\frac{d\epsilon}{dt} = \iiint_V \frac{\partial E}{\partial t} dV = \iiint_V \left[\rho \frac{\partial u_i}{\partial t} \frac{\partial^2 u_i}{\partial t^2} + \tau_{ij} \frac{\partial}{\partial t} \left(\frac{\partial u_i}{\partial x_j}\right)\right] dV \, .$$

The second integrand on the RHS was obtained by direct differentiation of $\tau_{ij} e_{ij}$ and by taking into account the symmetry of the stress tensor. It can be rearranged:

$$\begin{aligned} \iiint_V \tau_{ij} \frac{\partial}{\partial t} \left(\frac{\partial u_i}{\partial x_j}\right) dV &= \iiint_V \frac{\partial}{\partial x_j} \left(\tau_{ij} \frac{\partial u_i}{\partial t}\right) dV - \iiint_V \frac{\partial u_i}{\partial t} \tau_{ij,j} dV \\ &= -\iiint_V \frac{\partial S_j}{\partial x_j} dV - \iiint_V \frac{\partial u_i}{\partial t} \left(\rho \frac{\partial^2 u_i}{\partial t^2} - F_i\right) dV \, . \end{aligned}$$

After inserting this into the expression for $d\epsilon/dt$, we get

$$\frac{d\epsilon}{dt} + \iiint\limits_V \text{div } \vec{S} dV = \iiint\limits_V \frac{\partial u_i}{\partial t} f_i dV \quad .$$

In order to give the physical meaning of the introduced vector $S_j = -\tau_{ij}\partial u_i/\partial t$, let us put $f_i = 0$, i.e. let us neglect voluminal forces. Then, the above equation reduces to

$$\frac{d\epsilon}{dt} + \iiint\limits_V \text{div } \vec{S} dV = 0 \Leftrightarrow \frac{d\epsilon}{dt} + \iint\limits_\Sigma S_j n_j d\Sigma = 0 \quad ,$$

where Σ denotes the surface surrounding the volume V (we use the symbol Σ instead of S to avoid confusion with the vector $\vec{S}$). The above equations say that the time change of energy ϵ in the volume V is balanced by the flux of the vector $\vec{S}$ across the surface Σ. Therefore, we call the vector $\vec{S}$ - *elastic energy flux* (or more precisely the density of the elastic energy flux). It is an equivalent of the Poyinting vector from the theory of the electromagnetic field. The direction of the vector $\vec{S}$ specifies the direction of the flux of energy at the given point and its length corresponds to the amount of energy passing within unit time interval across a unit surface perpendicular to $\vec{S}$. The vector $\vec{S}$ divided by the density of elastic energy E gives a vector specifying the velocity of the energy flux.

Let us note that the above equations can be written in the form of an equation of conservation. It is easy to show that for the case $f_i = 0$, we get

$$\frac{\partial E}{\partial t} + \frac{\partial S_i}{\partial x_i} = 0$$

and for the case $f_i \neq 0$, we get

$$\frac{\partial E}{\partial t} + \frac{\partial S_i}{\partial x_i} = \frac{\partial u_i}{\partial t} f_i \quad .$$

The units of energy quantities (energy density) are given in $J/m^3 = Pa = kgm^{-1}s^{-2}$. The units of density of the elastic energy flux S_i are given in $J/m^2s = W/m^2$.

2.4.3 Elastic parameters

We found that c_{ijkl} are elements of the tensor of the fourth rank with 81 elements. We shall show that not all of them are independent because of the symmetry of the matrix of elastic parameters c_{ijkl} in some indices.

Due to the symmetry of the stress tensor and the strain tensor, we get immediately from the Hooke law

$$c_{ijkl} = c_{jikl} = c_{ijlk} \quad .$$

Further symmetry conditions follow from the expression for the strain energy:

$$c_{ijkl} = \frac{\partial \tau_{ij}}{\partial e_{kl}} = \frac{\partial^2 W}{\partial e_{ij} \partial e_{kl}} = \frac{\partial^2 W}{\partial e_{kl} \partial e_{ij}} = c_{klij} \quad .$$

All the three symmetry conditions reduce the number of independent elastic parameters from 81 to 21.

Because of the symmetry in indices i, j and k, l, there are only 6 combinations of i, j and k, l (instead of 9 in case of no symmetry) specifying independent elastic parameters. These 6 combinations of i, j and k, l are sometimes substituted by integers 1-6 and the tensor of elastic parameters is thus substituted by a 6 x 6 matrix. We denote the elements of this matrix $C_{\alpha\beta}$ ($\alpha, \beta = 1, 2, \ldots 6$; the Greek indices thus differ from Latin indices which attain only values 1,2,3). The index α corresponds to indices i, j and the index β to k, l of the tensor c_{ijkl}. The indices are related as follows:

$$1 \leftrightarrow 1,1 \quad 2 \leftrightarrow 2,2 \quad 3 \leftrightarrow 3,3 \quad 4 \leftrightarrow 2,3 \quad 5 \leftrightarrow 3,1 \quad 6 \leftrightarrow 1,2 \quad .$$

Due to the symmetry $c_{ijkl} = c_{klij}$ of the matrix c_{ijkl} the matrix $C_{\alpha\beta}$ is also symmetric,

$$C_{\alpha\beta} = C_{\beta\alpha} \quad .$$

Substitution of indices i, j, k, l by α and β is sometimes called the *compressed notation*. Remember, $C_{\alpha\beta}$ is not a tensor!

We can introduce a similar compressed notation for the elements of the strain and stress tensor, e_{kl}, τ_{ij}. We introduce e_β and τ_α in such a way that the generalized Hooke's law and the expression for strain energy have a form

$$\tau_\alpha = C_{\alpha\beta} e_\beta \quad , \quad W = \frac{1}{2} \tau_\alpha e_\alpha \quad .$$

By comparing the expression for W in the standard notation and the compressed notation we find that

$$e_{11} \leftrightarrow e_1 \quad e_{22} \leftrightarrow e_2 \quad e_{33} \leftrightarrow e_3 \quad 2e_{12} \leftrightarrow e_6 \quad 2e_{13} \leftrightarrow e_5 \quad 2e_{23} \leftrightarrow e_4 \quad ,$$

$$\tau_{11} \leftrightarrow \tau_1 \quad \tau_{22} \leftrightarrow \tau_2 \quad \tau_{33} \leftrightarrow \tau_3 \quad \tau_{12} \leftrightarrow \tau_6 \quad \tau_{13} \leftrightarrow \tau_5 \quad \tau_{23} \leftrightarrow \tau_4 \quad .$$

The strain energy is minimum if there is no deformation. For this case we defined $W = 0$. This means that for any deformation the strain energy must be positive, $W > 0$. This inequality implies positive definiteness of the quadratic form

$$W = \frac{1}{2} C_{\alpha\beta} e_\alpha e_\beta > 0 \quad .$$

As a consequence of this, all the principal minors of the matrix $C_{\alpha\beta}$ are positive. (Principal minor of a matrix is the minor which remains after the removal of the same number of lines and columns with the same numbers.) It means that also all the diagonal elements of the matrix $C_{\alpha\beta}$ must be positive. Another consequence of the positive definiteness of the expression for the strain energy is

$$|C_{\alpha\beta}| = \det(C_{\alpha\beta}) > 0 \quad ,$$

which means that a matrix inverse to $C_{\alpha\beta}$ exists so that $C_{\alpha\beta}S_{\beta\gamma} = \delta_{\alpha\gamma}$. The 6 x 6 matrix $S_{\alpha\beta}$ has its counterpart in the tensor s_{ijkl} of 4-th rank which has the same symmetry properties as the tensor c_{ijkl}. With this matrix the generalized Hooke law can be rewritten

$$e_{ij} = s_{ijkl}\tau_{kl} \quad .$$

Let us note that s_{ijkl} is known as the *compliance tensor.*

2.4.4 Various types of anisotropic symmetries

The anisotropic materials can be of different degree of symmetry. With increasing degree of symmetry, the number of independent elastic parameters decreases. Under the symmetry of the material we understand that after a transformation of the coordinate system in which the tensor c_{ijkl} is specified, its properties remain the same. We can deal with mirror symmetry or symmetry with respect to a rotation around an axis.

There is a whole system of anisotropic materials with different degrees of symmetry. The most general material which is specified by 21 independent elastic parameters is called *triclinic.* The material which is specified by 13 independent parameters is called *monoclinic.* These systems are not considered in seismic applications.

The most complex anisotropy considered sometimes in seismology is *orthorhombic* symmetry with 9 independent elastic parameters. It is characterized by three mutually perpendicular axes of symmetry. Rotation by 180° around any of these axes does not change the tensor c_{ijkl}. This condition leads to the matrix $C_{\alpha\beta}$

$$C_{\alpha\beta} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{pmatrix} \quad .$$

The above matrix is fully specified when the coordinate system in which it is displayed is also given. In the above case, the coordinate system was chosen so that the coordinate axes coincide with the axes of symmetry. In a different coordinate system there would be less zero elements but the number of independent parameters would still remain 9.

Quite frequently used anisotropic symmetry is *hexagonal symmetry*. This system has higher symmetry then the orthorhombic one. This system has one of the axes of symmetry such that a rotation by an arbitrary angle around this axis does not change the tensor c_{ijkl}. It means that in the plane perpendicular to this axis the tensor behaves isotropically. Because of this, the symmetry is also sometimes called *transverse isotropy*, especially in case in which the axis of rotational symmetry coincides with the x_3 - axis of the coordinate system. Matrix $C_{\alpha\beta}$ of the hexagonally symmetric material with a vertical axis of symmetry has the form

$$C_{\alpha\beta} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & \frac{C_{11}-C_{12}}{2} \end{pmatrix} .$$

The hexagonal symmetry is described by 5 independent elastic parameters. Again the coordinate system in which the matrix $C_{\alpha\beta}$ is given must be specified. In the above case, the axis x_3 coincides with the axis of rotational symmetry, the axes x_1, x_2 being situated in the plane of isotropy.

Sometimes, so-called Love notation for the five independent elastic parameters of a transversely isotropic medium is used. In it

$$A = C_{11}\,, \quad C = C_{33}\,, \quad L = C_{44}\,, \quad N = C_{66}\,, \quad F = C_{13} \quad .$$

The highest symmetry among anisotropic materials posseses the *isotropic* medium. Isotropic material is invariant to any rotation. It is described by two independent elastic parameters. The matrix $C_{\alpha\beta}$ for isotropic material has a form

$$C_{\alpha\beta} = \begin{pmatrix} C_{11} & C_{11}-2C_{44} & C_{11}-2C_{44} & 0 & 0 & 0 \\ & C_{11} & C_{11}-2C_{44} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{44} \end{pmatrix} .$$

Instead of C_{11} and C_{44}, so-called Lamé's parameters λ and μ are used. The matrix $C_{\alpha\beta}$ has then the form

$$C_{\alpha\beta} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{pmatrix} .$$

The Hooke law can now be rewritten as follows:

$$\begin{aligned} \tau_{11} &= \tau_1 = \lambda(e_1+e_2+e_3) + 2\mu e_1 = \lambda\theta + 2\mu e_{11} , \\ \tau_{22} &= \tau_2 = \lambda(e_1+e_2+e_3) + 2\mu e_2 = \lambda\theta + 2\mu e_{22} , \\ \tau_{33} &= \tau_3 = \lambda(e_1+e_2+e_3) + 2\mu e_3 = \lambda\theta + 2\mu e_{33} , \\ \tau_{23} &= \tau_4 = \mu e_4 = 2\mu e_{23} , \\ \tau_{13} &= \tau_5 = \mu e_5 = 2\mu e_{13} , \\ \tau_{12} &= \tau_6 = \mu e_6 = 2\mu e_{12} , \end{aligned}$$

or by one equation

$$\tau_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij} .$$

For tensor of elastic parameters c_{ijkl} we get (if we take into account that $2e_{ij} = e_{ij} + e_{ji}$):

$$\begin{aligned} c_{ijkl} &= \frac{\partial \tau_{ij}}{\partial e_{kl}} = \lambda \frac{\partial e_{mm}}{\partial e_{kl}} \delta_{ij} + \mu \left(\frac{\partial e_{ij}}{\partial e_{kl}} + \frac{\partial e_{ji}}{\partial e_{kl}} \right) \\ &= \lambda \delta_{kl}\delta_{ij} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) \quad . \end{aligned}$$

Let us note that the Hooke law for anisotropic and isotropic media can also be rewritten in terms of displacement vector. We then have,

$$\begin{aligned} \tau_{ij} &= \frac{1}{2} c_{ijkl} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) = c_{ijkl} \frac{\partial u_k}{\partial x_l} , \\ \tau_{ij} &= \lambda\theta\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad , \end{aligned}$$

where $\theta = \frac{\partial u_k}{\partial x_k}$.

From the Hooke law for isotropic medium we get easily the specification of the relation

$$e_{ij} = s_{ijkl}\tau_{kl}$$

for the isotropic case. Let us first determine τ_{ii},

$$\tau_{ii} = (3\lambda + 2\mu)\theta \quad .$$

Assuming that $3\lambda + 2\mu \neq 0$, we can write

$$\theta = \frac{\tau_{ii}}{3\lambda + 2\mu} \quad .$$

If we insert this into the Hooke law, we find for e_{ij}

$$e_{ij} = \frac{1}{2\mu}\tau_{ij} - \lambda\delta_{ij}\frac{\tau_{kk}}{2\mu(3\lambda + 2\mu)} \quad .$$

Let us now choose the coordinate system so that it coincides with the principal axes of the strain tensor. Thus we have $e_{12} = e_{13} = e_{23} = 0$. From the Hooke law for isotropic media we get immediately $\tau_{12} = \tau_{13} = \tau_{23} = 0$. This means that our coordinate system also coincides with the principal axes of the stress tensor. In isotropic medium, principal axes of strain tensor coincide with principal axes of the stress tensor.

2.4.5 Elastic parameters for isotropic media

We saw above that the isotropic medium can be described by two parameters - Lamé's parameters λ and μ. The former parameter does not have a name, the latter one is called *rigidity*. Instead of λ and μ some other parameters can be introduced. We shall briefly describe them and we start with rigidity.

Rigidity μ. For $i \neq j$, we get form the Hooke law

$$\tau_{ij} = 2\mu e_{ij} \quad .$$

It means that the rigidity μ relates the shearing stress to the shearing strain. Because of this, μ is sometimes called *shear modulus*. It can be shown that the rigidity is a ratio of the shear stress to the change of the right angle caused by this stress.

Due to the positive definiteness of the strain energy W, rigidity of an *isotropic solid* is always positive, $\mu > 0$.

Bulk modulus k *and compressibility* κ. Let us consider a spherically symmetric strain, i.e. $e_{11} = e_{22} = e_{33} \neq 0$ and $e_{12} = e_{13} = e_{23} = 0$. This means that the matter is stretched in the same rate in all directions. The corresponding stress is called *hydrostatic*. From the Hooke law for isotropic media we have

$$\tau_{ii} = (3\lambda + 2\mu)\theta = 3k\theta \quad ,$$

where

$$k = \lambda + \frac{2}{3}\mu \quad .$$

The parameters k is called *bulk modulus* because k is the ratio of the mean stress $(\frac{1}{3}\tau_{ii})$ to the volume dilatation θ. Bulk modulus k is also known as *incompressibility*.

From the definition of k we see that for isotropic elastic solids, k is always positive - positive tension $(\tau_{ii} > 0)$ causes positive dilatation $(\theta > 0)$.

The reciprocal quantity to k is called *compressibility* κ,

$$\kappa = \frac{1}{k} \quad .$$

Young modulus E and Poisson's ratio σ. Let us consider a rod oriented along the axis x_1 and pulled in this direction, i.e., $\tau_{11} \neq 0$ and all the other components of the stress tensor are zero. From the form of the Hooke law in which e_{ij} is expressed in terms of τ_{ij}, we get

$$e_{11} = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}\tau_{11} \,, \quad e_{22} = e_{33} = -\frac{\lambda\tau_{11}}{2\mu(3\lambda + 2\mu)} \,, \quad e_{12} = e_{13} = e_{23} = 0 \quad .$$

Instead of λ and μ, we can now introduce parameters E and σ in the following way:

$$\tau_{11} = Ee_{11} \quad , \quad \sigma = -\frac{e_{22}}{e_{11}} = -\frac{e_{33}}{e_{11}} \quad .$$

From these definitions, we get

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad , \quad \sigma = \frac{1}{2}\frac{\lambda}{\lambda + \mu} \quad .$$

The quantity E is the Young modulus. It represents a ratio of the tension to the relative extension caused by the tension. The quantity σ is the ratio of relative contraction of the thickness of the rod to its relative extension. From the definition, both E and σ are positive, $E > 0$, $\sigma > 0$. Moreover, for λ large and/or μ small, σ has an upper limit so that $0 < \sigma < \frac{1}{2}$. This condition also implies that $\lambda > 0$.

2.4.6 Units, numerical values

It follows from the Hooke law that the elastic parameters for anisotropic media, c_{ijkl} and for isotropic media, λ and μ are measured in the same units as stress, i.e. in pascals ($1Pa = kgm^{-1}s^{-2}$). The parameters k and E are measured in the same units. The Poisson's ratio is dimensionless, the compressibility κ is measured in $(Pa)^{-1}$.

The Earth's material is often modeled as *Poisson's solid*, i.e., the material in which $\lambda = \mu$ and thus $\sigma = 1/4$. We shall see later that in such a case the ratio os squares of P and S wave velocities is 3.

The parameters λ, μ, E and k are usually of the order $10^{10} - 10^{12} Pa$ in the Earth's interior.

2.5 Elastodynamic equations

In the following, we shall derive several important and useful forms of the elastodynamic equations expressed in displacements. All the previous theory concerned wave propagation in elastic solids. In seismology, however, we often meet with necessity to consider wave propagation through elastic fluids, examples being the oceans and the outer core of the Earth. Even propagation in solid materials is sometimes considered as propagation in fluids (so-called *acoustic case* in reflection seismology). We could derive the equations of motion for fluids from the equations of hydromechanics. Another possibility, leading to the same results, is to specify the above derived equations for elastic solids for fluids by letting $\mu \to 0$. We shall follow the latter approach.

2.5.1 Elastodynamic equations for solid media

Let is insert the general form of the generalized Hooke law,

$$\tau_{ij} = c_{ijkl} e_{kl} = c_{ijkl} \frac{\partial u_k}{\partial x_l}$$

into the equation of motion

$$\tau_{ji,j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad .$$

We get the most general form of the elastodynamic equation

$$\frac{\partial}{\partial x_j} \left(c_{ijkl} \frac{\partial u_k}{\partial x_l} \right) + f_i = \rho \frac{\partial^2 u_i}{\partial t^2} \quad .$$

Since $c_{ijkl} = c_{ijkl}(x_m)$ and $\rho = \rho(x_m)$, the equation describes a *general inhomogeneous anisotropic medium*. The equation is a linear vectorial partial differential equation of the second order with varying coefficients. Complete solution of such an equation cannot be written in a closed form. Numerical methods, such as FD or FE (finite difference or finite element) methods must be used to solve it. Another option is to apply high-frequency asymptotic methods such as the ray method to solve the elastodynamic equation.

For a *homogeneous anisotropic medium*, in which elastic parameters and density are constant, i.e. $c_{ijkl,m} = 0$, $\rho_{,m} = 0$, the above equation yields

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + f_i = \rho\frac{\partial^2 u_i}{\partial t^2} \quad .$$

This is again a linear vectorial partial differential equation of the second order but now with constant coefficients. For such an equation, solution in the form of plane waves can be found. To find such solutions is our next task.

The elastodynamic equation for an *inhomogeneous isotropic* medium can be obtained by inserting

$$c_{ijkl} = \lambda\delta_{kl}\delta_{ij} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) \quad ,$$

where $c_{ijkl} = c_{ijkl}(x_m)$, $\lambda = \lambda(x_m)$, $\mu = \mu(x_m)$ into the equation of motion for inhomogeneous anisotropic medium. We rewrite it in the following notation

$$(c_{ijkl}u_{k,l})_{,j} + f_i = \rho u_{i,tt} \quad .$$

Then we get

$(\nabla\mu \times rot u)_i + \mu_{,k} u_{i,k}$ $(\lambda+\mu)\nabla(div\, u)$

$\nabla\lambda \cdot div\, u$ $\mu\Delta u$

$$\lambda_{,i}u_{k,k} + \mu_{,l}u_{i,l} + \mu_{,k}u_{k,i} + \lambda u_{l,li} + \mu u_{i,ll} + \mu u_{j,ij} + f_i = \rho u_{i,tt} \quad .$$

This can be rewritten in the vector notation if we take into account the identity

$$(\nabla\mu \times rot\vec{u})_i = \mu_{,k}u_{k,i} - \mu_{,k}u_{i,k} \quad .$$

We get

$$(\lambda+\mu)\nabla(\text{div } \vec{u}) + \mu\Delta\vec{u} + \nabla\lambda \text{ div } \vec{u} + \nabla\mu \times \text{rot } \vec{u} + 2(\nabla\mu\nabla)\vec{u} + \vec{f} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \,.$$

Specifying λ, μ and ρ constant, i.e. $\lambda_{,i} = 0$, $\mu_{,i} = 0$ and $\rho_{,i} = 0$, we arrive to the equation of motion for *homogeneous isotropic media*,

$$(\lambda+\mu)u_{k,ki} + \mu u_{i,kk} + f_i = \rho u_{i,tt} \quad ,$$

which has in the vector notation the form

$$(\lambda+\mu)\nabla(\text{div } \vec{u}) + \mu\Delta\vec{u} + \vec{f} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \,.$$

This can be rewritten into another useful form if we consider the identity

$$\Delta\vec{u} = \nabla(\text{div } \vec{u}) - \text{ rot (rot } \vec{u}) \quad .$$

Then

$$(\lambda+2\mu)\nabla(\text{div } \vec{u}) - \mu \text{ rot (rot } \vec{u}) + \vec{f} = \rho\frac{\partial^2\vec{u}}{\partial t^2} \,.$$

2.5.2 Equation of motion for fluids - acoustic case

As we said above, we shall specify the equations of motion of elastic solids for fluids so that we put $\mu = 0$. In this case we shall get $E = 0$, $\sigma = 1/2$ and $k = \lambda$.

In fluids, the stress is hydrostatic and the mean stress, $\tau_{ii}/3$ is usually negative. It is denoted by $-p$, where p is called *the pressure*. The Hooke law

$$\tau_{ij} = \lambda\theta\delta_{ij} = k\theta\delta_{ij} \quad ,$$

see Sec. 2.4.5, can then be written

$$p = -k\theta \quad \text{or} \quad \theta = -\kappa p \quad .$$

It is common in acoustics to work with the pressure p and the particle velocity $v_i = \partial u_i/\partial t$ rather than with the particle displacement u_i. We, therefore, rewrite the elastodynamic equation to these quantities. From the elastodynamic equation

$$\tau_{ji,j} + f_i = \rho\frac{\partial^2 u_i}{\partial t^2} \quad ,$$

we immediately get

$$-p_{,i} + f_i = \rho\frac{\partial v_i}{\partial t} \quad ,$$

where we took into account that $\tau_{ij} = -p\delta_{ij}$. Instead of three unknown components of u_i, we have now three unknown components v_i and the pressure p is fourth unknown. We, therefore, need one more equation. We get it by differentiating, with respect to time, the Hooke law for fluids. The complete system of equations will then consist of four equations for four unknown quantities, v_i, p:

$$\left\{ \begin{aligned} & p_{,i} + \rho\frac{\partial v_i}{\partial t} = f_i \quad , \\ & v_{k,k} + \kappa\frac{\partial p}{\partial t} = 0 \quad . \end{aligned} \right.$$

$\tau_{ij} = c_{ijkl}\, e_{ij}$

$\frac{\partial \tau_{ij}}{\partial t} = c_{ijkl}\, \lambda\, \partial \dot{v}_{ij}$

2.6 Initial and boundary conditions

Our aim now is to solve the above equations of motion in order to find the spatial and temporal distribution of displacement vector (in elastodynamic case) or particle velocity vector and pressure (in acoustic case) in an investigated region. For this purpose, we need yet to know the values of the displacement vector $u_i(x_j, t)$ and of its first derivative $\dot{u}_i(x_j, t)$

for a time t_o at any point of the investigated region. This information represents *initial conditions*. In addition we need to know the values of the displacement vector and/or traction on a boundary surrounding the investigated region for any time t greater than t_o, $t \geq t_o$. This information represents *boundary conditions*. Final information, which we need to know is spatial distribution of body forces in the investigated region for any $t \geq t_o$. The effect of gravitational forces on wave propagation is negligible in comparison with the effects of surface forces. We shall, therefore, from now on, consider the contribution of gravitational forces to voluminal forces f_i zero (let us note that there are elastic motions, in which gravitational forces cannot be neglected - e.g. some modes of free oscillations).

2.6.1 Initial conditions

Common initial conditions in seismology are such that $u_i(x_j, t)$ and $\dot{u}_i(x_j, t)$ are zero at time $t = t_o$. At this moment, i.e. at $t = t_o$, or later, forces concentrated to a certain point (or surface or volume) start to act. For the description of such a situation, we can use two approaches.

In the first approach we specify the voluminal forces f_i in the equation of motion in such a way that they represent seismic sources. This approach is, for example, used when the elastodynamic equations are solved by finite difference or finite element methods. We shall use this approach when we shall look for solutions of elastodynamic equations using integral transform methods or Fourier method of separation of variables.

In the second approach, we consider the elastodynamic equation only outside the region of the source. Then the term f_i representing voluminal forces in the elastodynamic equation is zero and the elastodynamic equation becomes homogeneous. The effect of the source is specified by boundary conditions, which are given on a surface surrounding the source. Such a specification is known as *radiation pattern* of the source. This approach is typical for the ray method.

2.6.2 Boundary conditions

The above derived various forms of equations of motion can be applied to media in which the elastic parameters or compressibility and density together with their derivatives vary continuously so that the coefficients of equations of motion are defined. There are, however, surfaces in the medium, on which the above continuity is violated (Earth's surface, Moho discontinuity, etc.). Such surfaces are called *boundaries* or *interfaces*.

The displacement vector and traction must satisfy on the boundaries and interfaces certain conditions which are called *boundary conditions*. Sometimes, boundary conditions imposed on displacement vector are called *kinematic*, the conditions imposed on traction are called *dynamic*.

In the following we shall consider the boundary conditions resulting from *the welded contact* of both parts of the medium separated by an interface. Welded contact prevents creation of cavities at interfaces or difusion of the material from one side of the interface to the other side. It also prevents sliding of two solid media along the separating interface.

We shall consider two situations:

1. interface between two media;
2. free surface.

Although we used in acoustic case particle velocity vector and pressure instead of the displacement vector and traction, here we use the latter quantities. To rewrite the corresponding conditions into former quantities is straightforward.

1. *Interface between two media*

   We distinguish three situations:

   (a) *Interface between two solids*
   We require continuity of the displacement vector and traction across an interface.

   (b) *Interface between fluid and solid*
   Since the particles of the fluid can slide arbitrarily along the interface, we cannot impose any condition on the tangential components of the displacement vector. Normal components of the displacement vector, however, must change continuously across an interface in order to avoid formation of cavities or difusion of fluid into solid. Since there are no tangential stresses (with respect to the interface) in the fluid, we require tangential components of traction zero and normal components continuous across an interface.

   (c) *Interface between two fluids*
   From similar reasons as above, only normal components of displacement vector and traction must change continuously across an interface. In the acoustic case, these conditions result in continuity of the normal component of the particle velocity and pressure across the interface.

2. *Free surface*

   By free surface, we understand the boundary between the solid or fluid on one side and vacuum on the other side. This approximates the Earth's continental or ocean surface.

   (a) *Free surface of the solid*

   We cannot impose any condition on the displacement vector but we require zero traction on the boundary.

   (b) *Free surface of the fluid*

   We cannot impose any condition on the displacement vector and tangential components of traction. Normal components of traction, however, must be zero. In the acoustic case this results in requirement of zero pressure at the free surface.

Finally, let us note that all the above special cases of boundary conditions could be also obtained in a more *formal* way from the boundary conditions for an interface separating two solid media. In this formal way fluid is specified by putting $\mu = 0$ in expressions for an isotropic solid. The vacuum is specified by putting $c_{ijkl} = 0$ or $\lambda = \mu = 0$ and $\rho = 0$ in the free space.

2.7 Separation of the elastodynamic equation in a homogeneous isotropic medium

Let us consider the elastodynamic equation for homogeneous isotropic media in component notation,

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,kk} + f_i = \rho u_{i,tt} \quad .$$

Let us sucessively apply on this equation two operations, operation divergence and rotation. Applying divergence, we get

$$(\lambda + \mu)u_{k,kii} + \mu u_{i,kki} + f_{i,i} = \rho u_{i,itt} \quad .$$

Using the notation $\theta = u_{k,k}$, we can rewrite this equation as follows

$$(\lambda + \mu)\theta_{,ii} + \mu\theta_{,kk} + f_{i,i} = \rho\theta_{,tt} \quad ,$$

which yields

$$\frac{\lambda + 2\mu}{\rho}\theta_{,ii} + \rho^{-1} f_{i,i} = \theta_{,tt} \quad .$$

Applying rotation, we get

$$(\lambda + \mu)\varepsilon_{jli}u_{k,kil} + \mu\varepsilon_{jli}u_{i,kkl} + \varepsilon_{jli}f_{i,l} = \rho\varepsilon_{jli}u_{i,ltt} \quad .$$

Using the notation $\Omega_j = \varepsilon_{jli} u_{i,l}$ and taking into account that $\varepsilon_{jli} u_{k,kil} = 0$, we get

$$\frac{\mu}{\rho} \Omega_{j,kk} + \rho^{-1} \varepsilon_{jli} f_{i,l} = \Omega_{j,tt} \quad .$$

Thus, we got two equations with a similar structure. They can be rewritten in the vectorial forms as follows:

$$\alpha^2 \Delta\theta + \rho^{-1} \operatorname{div} \vec{f} = \frac{\partial^2 \theta}{\partial t^2}, \quad \text{where} \quad \alpha^2 = \frac{\lambda + 2\mu}{\rho} \quad ,$$

$$\beta^2 \Delta\vec{\Omega} + \rho^{-1} \operatorname{rot} \vec{f} = \frac{\partial^2 \vec{\Omega}}{\partial t^2}, \quad \text{where} \quad \beta^2 = \frac{\mu}{\rho} \quad .$$

The above equations are *scalar wave equation* for θ and *vectorial wave equation* for $\vec{\Omega}$.

Let us consider a one-dimensional (1-D) case in which all the quantities depend on only one spatial coordinate denoted by x. Let us also neglect the effects of body forces, i.e. let us put $f_i = 0$. Then the above wave equations reduce to the form

$$\alpha^2 \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial^2 \theta}{\partial t^2} = 0\,, \quad \theta = \frac{\partial u_1}{\partial x}\,,$$

$$\beta^2 \frac{\partial^2 \vec{\Omega}}{\partial x^2} - \frac{\partial^2 \vec{\Omega}}{\partial t^2} = 0\,, \quad \vec{\Omega} = \left(0, -\frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x}\right) \,.$$

Let us look for the solution of the scalar wave equation for θ in the following form,

$$\theta = \theta(at + bx)$$

with a and b constants to be determined. Let us insert the above solution into the reduced wave equation. We get

$$\alpha^2 \theta'' b^2 - a^2 \theta'' = 0 \quad ,$$

from which $b^2 = a^2/\alpha^2$. If we put $a = 1$, we get the most general solution of the wave equation,

$$\theta = \theta\left(t \pm \frac{x}{\alpha}\right) \quad .$$

Similarly for the vectorial wave equation we get

$$\vec{\Omega} = \vec{\Omega}\left(t \pm \frac{x}{\beta}\right) \quad .$$

Each of these solutions describes two waves propagating in the opposite direction along the x-axis with velocities α and β.

Thus the wave equations describe wave processes characterized by the velocities α and β. The velocities are, therefore, called the *wave propagation velocities*. In a homogeneous isotropic medium, we have thus two independent wave processes. Both wave processes not only differ by their wave propagation velocities, $\alpha > \beta$, but they also differ by their properties. The *faster* process connected with the velocity α is described by the *dilatation* θ, which means that the process is characterized by voluminal changes. The waves propagating with the velocity α are called *compressional*, *dilatational* or *longitudinal* waves. Very often, they are simply called *P waves*, where P comes from the word "primary" since P waves arrive prior to the other type of waves. The slower wave process connected with the velocity β is described by the *rotation*. The waves propagating with the velocity β are called *shear*, *rotational* or *transverse* waves. Very often, they are called *S waves*, where S comes from the word "secondary" since they arrive after the P waves.

Applying operations div and rot to the elastodynamic equation for a *homogeneous* isotropic solid, we found that the elastodynamic equation describes two fully *separated* wave processes. If we apply the same operators to the elastodynamic equation for *inhomogeneous* isotropic medium, we shall not succeed in separating the above two waves. In an inhomogeneous medium, both waves are *coupled together*. This means that the wave processes in inhomogeneous isotropic media are neither purely dilatational nor purely rotational. Because of this, the P waves and the S waves in inhomogeneous media should *not* be called dilatational and rotational but compressional and shear.

From our "formal" definition of the fluid as a medium in which $\mu = 0$ follows immediately that in fluids *only one wave* can propagate, namely the P wave. The S wave velocity in a fluid is $\beta = 0$.

The wave equations which we have found above are not the final solution of our problem. We must solve them for θ and $\vec{\Omega}$ and then to determine the displacement vector $\vec{u}$ from

$$\theta = \operatorname{div} \vec{u} \quad , \quad \vec{\Omega} = \operatorname{rot} \vec{u}\,.$$

This complicated procedure can be avoided by the use of *the Lamé theorem*. The principal idea is to express already the displacement vector satisfying the elastodynamic equation as *two* separate terms corresponding to P and S waves. According to the theorem, the displacement vector $\vec{u}$ can be expressed in terms of the *Helmholtz potentials* φ and $\vec{\psi}$,

$$\vec{u} = \operatorname{grad} \varphi + \operatorname{rot} \vec{\psi} \quad , \quad \operatorname{div} \vec{\psi} = 0 \quad ,$$

with φ and $\vec{\psi}$ satisfying scalar and vectorial wave equations. This holds if the body force $\vec{f}$ can also be expressed in terms of the Helmholtz potentials

$$\vec{f} = \operatorname{grad} \Phi + \operatorname{rot} \vec{\Psi} \quad , \quad \operatorname{div} \vec{\Psi} = 0$$

and if certain additional conditions are satisfied, see Aki & Richards (1980).

We shall show that a sufficient condition for the elastodynamic equation to hold is that φ and $\vec{\psi}$ satisfy corresponding wave equations. Let us insert $\vec{u}$ expressed in terms of the Helmholtz potentials into the elastodynamic equation

$$(\lambda+\mu)u_{k,ki}+\mu u_{i,kk}+f_i=\rho u_{i,tt} \quad .$$

We get

$$(\lambda+\mu)(\varphi_{,kki}+\varepsilon_{klm}\psi_{m,lki})+\mu(\varphi_{,ikk}+\varepsilon_{ilm}\psi_{m,lkk})+\Phi_{,i}+\varepsilon_{ilm}\Psi_{m,l}=\rho(\varphi_{,itt}+\varepsilon_{ilm}\psi_{m,ltt}) ,$$

which can be arranged

$$[(\lambda+2\mu)\varphi_{,kk}+\Phi-\rho\varphi_{,tt}]_{,i}+\varepsilon_{ilm}[\mu\psi_{m,kk}+\Psi_m-\rho\psi_{m,tt}]_{,l}=0 .$$

Sufficient but not necessary conditions for this equation to hold are

$$(\lambda+2\mu)\varphi_{,kk}+\Phi-\rho\varphi_{,tt}=0 \quad , \quad \mu\psi_{m,kk}+\Psi_m-\rho\psi_{m,tt}=0 \quad ,$$

which in the vectorial form read

$$\alpha^2\Delta\varphi+\rho^{-1}\Phi=\frac{\partial^2\varphi}{\partial t^2} \quad , \quad \beta^2\Delta\vec{\psi}+\rho^{-1}\vec{\Psi}=\frac{\partial^2\vec{\psi}}{\partial t^2} \quad .$$

These are wave equations for the Helmholtz potentials φ (P waves) and $\vec{\psi}$ (S waves). As soon as they are found, the displacement vector can be simply determined from the equation

$$\vec{u}=\text{ grad }\varphi+\text{ rot }\vec{\psi} \quad .$$

CHAPTER 3

PLANE WAVES

We shall study the properties of plane waves propagating in *homogeneous* acoustic or perfectly elastic, anisotropic or isotropic media. Plane waves represent one of the possible solutions of elastodynamic and/or wave equations for homogeneous media. They do not exist in real media but they are a good approximation of waves generated by a distant source. One of the advantages of working with plane waves is that they can be studied without considering their source.

Study of plane waves is important from several aspects:

a) The behavior of plane waves is simple and still it is possible to demonstrate on them the most important effects of the wave propagation.

b) Plane waves play an important role in the theory of wave propagation of spherical and cylindrical waves. These waves are studied as expansions into plane waves. The wave is expanded into plane waves and each plane wave is propagated through the investigated region to the receiver. At the receiver, all the plane wave contributions are synthesized to form the originally studied wave at the receiver.

c) High-frequency asymptotic methods such as the ray method are based on a *local* application of plane wave solutions. The propagation of a general wave in a slightly inhomogeneous medium can be then understood as propagation of perturbed plane waves, their perturbation being caused by the deviations of the medium from homogeneity.

In the following, we shall mostly investigate the plane waves in the *time domain*, i.e., we shall study so-called *transient plane waves*. We shall, however, also study some solutions in the *frequency domain*, i.e. we shall also study *time-harmonic plane waves* since they play an important role in seismology.

3.1 Properties of plane waves

3.1.1 Time-harmonic plane waves

We shall seek the solutions of homogeneous acoustic and elastodynamic equations in the following form

$$w(x_m, t) = W \exp\left[-i\omega(t - T(x_m))\right].$$

Here w may be either a scalar (in case that it represents pressure) or a component of a vector (in case that it represents a component of the particle velocity or displacement vector).

The physically meaningful expression is either Re(w) or Im(w). The use of complex w above is a standard approach in solving wave propagation problems. Since the equation of motion is linear and it does not contain complex operations, it is satisfied by Re(w), Im(w) as well as by w. Because it is simpler to handle complex exponential function than to work with trigonometric functions, we shall work with complex w and we shall specify only the final results for Re (w) (or Im (w)). This procedure will fail as soon as we shall start to work with quantities nonlinear in w. This will happen, for example, when we start to deal with energy. Then, we shall have to work with Re (w) or Im (w) from the beginning.

Let us return to the expression for $w(x_m, t)$. Symbol W is a constant, possibly complex-valued quantity, called the *scalar* or *vectorial amplitude* depending on the meaning of w. Symbol ω denotes the *circular frequency*, $\omega = 2\pi f$. Here f is the *frequency*, i.e. the number of oscillations of the wave in a second. Frequency f is related to the *period* T, i.e. the time length of one oscillation, by the relation $f = 1/T$. The period T is measured in seconds, f in Hz (Hertz) $= s^{-1}$. Symbol t denotes time and $T(x_m)$ is a real linear homogeneous function of x_i,

$$T(x_m) = p_i x_i \quad ,$$

where p_i are real constant coefficients. The equation

$$t - T(x_m) = t - p_i x_i = \text{ const}$$

for t fixed is an equation of a plane and simultaneously an equation of the constant phase of the exponential function in the expression for w. Note that w is constant on this plane. Due to this, the plane $p_i x_i = t+$const is called the *phase front* and the wave associated with it, the *plane wave*. With varying time t, the phase front moves so that the normal N_i to it, called the *phase normal*, does not change its direction. Differentiation of the equation of the constant phase with respect to a length parameter ξ along a line perpendicular to the phase front yields

$$\frac{dt}{d\xi} = p_i \frac{dx_i}{d\xi} = p_i N_i \, .$$

Here $\frac{dt}{d\xi} = c^{-1}$, where c is the velocity with which the phase front moves and is, therefore, called the *phase velocity*. Thus, the vector p_i can be expressed as

$$p_i = N_i/c\,, \quad \text{i.e.} \quad p_k p_k = c^{-2}\,.$$

We have already obtained similar result in Sec.2.7. Since the size of the vector p_i is inversely proportional to the phase velocity c, the vector p_i is called the *slowness vector*. The vector $\vec{c} = c\vec{N}$ is called the *phase velocity vector*. We can now introduce the term the *wavelength* λ as a length passed by a plane wave with the speed c within the period T,

$$\lambda = cT = c/f = \frac{2\pi c}{\omega} = \frac{2\pi}{k}\,.$$

Here $k = \omega/c$ is called the *wave number*. In case of time harmonic waves, the slowness vector is sometimes substituted by the *wave vector* k_i,

$$k_i = \omega p_i\,.$$

We can see that the size of this vector is k.

3.1.2 Transient plane waves

There are two possible ways how to obtain solutions for transient plane waves:

1. Application of the Fourier transform to the time harmonic solutions.
2. Direct use of transient signals in the trial solution of the elastodynamic equation.

The first approach is straightforward and clear. Here we shall use the second approach. This approach is connected with the term *analytic signal*, which we have introduced in Sec.1.3. We shall seek the solution of the acoustic and/or elastodynamic equation in the form

$$w(x_m, t) = WF(t - T(x_m)) \quad .$$

Here w and T have the same meaning as in Sec.3.1.1, W is a constant, possibly complex-valued quantity. The function $F(\xi)$ represents the analytic signal, see Sec.1.3 and it is defined as follows,

$$F(\xi) = g(\xi) + ih(\xi) \quad .$$

Here $g(\xi)$ is the actual transient signal for which the analytic signal is constructed, $h(\xi)$ is the Hilbert transform of $g(\xi)$,

$$h(\xi) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\sigma)}{\sigma - \xi}d\sigma \quad .$$

Similarly as in the case of harmonic waves, where we preferred to work with complex exponential function $\exp(-i2\pi f\xi)$ and not with $\cos 2\pi f\xi$, in the case of transient signals it is more convenient to work with the analytic signal

$$F(t) = 2\int_0^\infty g(f)e^{-i2\pi ft}df$$

than with the real signal $g(t)$. The form of the above plane wave expression is especially convenient when A becomes complex.

We have shown that the functions $g(\xi)$, $h(\xi)$ and $F(\xi)$ can be specified as $\cos\xi$, $-\sin\xi$ and $\exp(-i\xi)$, respectively. This shows that the time-harmonic plane waves represent a special case of the transient plane waves.

Using the analytic signal $F(t)$, it is possible to construct an envelope of the real signal $g(t)$. The envelope is given as

$$|F(t)| = \sqrt{g^2(t) + h^2(t)} \quad .$$

The envelope of the transient plane wave $w = WF$ can be thus expressed as

$$|w(t)| = |W|\sqrt{g^2(t) + h^2(t)} \quad .$$

We can see that the form of the envelope is the same everywhere, no matter if W is a complex-valued or real-valued constant. The form of the transient plane wave $w(t)$, however, may change if W is complex-valued:

$$Re(w(t)) = Re(WF(t)) = (Re(W)g(t) - Im(W)h(t) \quad .$$

For real-valued W this reduces to

$$Re(w(t)) = Wg(t) \quad ,$$

which means that in case of real-valued W the form of the transient plane wave $w(t)$ is preserved.

In the following, we shall mostly work with the solutions $WF(\xi)$ but physically meaningful solutions are either $Wg(\xi)$ or $Wh(\xi)$.

3.2 Transient acoustic plane waves

Let us insert the trial solution

$$p(x_m, t) = PF(t - T(x_m)) \, ,$$

$$v_i(x_m, t) = V_i F(t - T(x_m))$$

into the acoustic equations specified for $f_i = 0$:

$$p_{,i} + \rho \frac{\partial v_i}{\partial t} = 0 \quad , \quad v_{k,k} + \kappa \frac{\partial p}{\partial t} = 0 \, .$$

(Please, do not mix $p_{,i}$ - the derivative of pressure with respect to x_i with p_i – the i-th component of the slowness vector.) We get two important results

$$c = (\rho\kappa)^{-1/2} \quad , \quad V_i = \frac{P}{\rho c} N_i \quad .$$

The first equation express the phase velocity in terms of parameters describing the acoustic medium, the second equation specifies the direction and magnitude of the vectorial quantity V_i. We can see that the particle velocity vector is parallel to the phase normal.

The plane wave solution for the acoustic case has thus the form

$$p(x_m, t) = PF(t - T(x_m)) \quad , \quad v_i(x_m, t) = \frac{PN_i}{\rho c} F(t - T(x_m)) \, ,$$

where

$$T(x_m) = \frac{x_i N_i}{c} \quad , \quad c = (\rho\kappa)^{-1/2} \, .$$

We can see that the solution represents a *single wave* propagating with the phase velocity c and polarized perpendicularly to the phase front. Let us add that we would get the same results if we worked in the frequency domain.

3.3 Transient elastic plane waves

We start our considerations from *homogeneous* anisotropic media. Results for homogeneous isotropic media can be obtained as a specification of results obtained for anisotropic media.

3.3.1 Homogeneous anisotropic medium

Let us insert the trial solution

$$u_i(x_m, t) = U_i F(t - T(x_m)) \quad , \quad T = \frac{N_i x_i}{c}$$

into the elastodynamic equation for a homogeneous anisotropic medium with $f_i = 0$,

$$a_{ijkl} u_{k,lj} = u_{i,tt} \quad .$$

Here we used the following notation

$$a_{ijkl} = c_{ijkl}/\rho \quad .$$

The parameters a_{ijkl} are called the *density normalized elastic parameters.* Their dimension is $(m/s)^2$.

We get

$$a_{ijkl}U_k p_l p_j - U_i = 0 \quad .$$

The above equation can be rewritten using $p_i = N_i/c$:

$$(\Gamma_{ik} - c^2\delta_{ik})U_k = 0 \quad ,$$

where

$$\Gamma_{ik} = a_{ijkl}N_l N_j \quad .$$

is *the Christoffel matrix.* The above equation including Γ_{ik} is called *the Christoffel equation.* Let us note that in what follows, we shall also use the name the Christoffel matrix for the matrix $\bar{\Gamma}_{ik} = a_{ijkl}p_l p_j$. We can see that the Christoffel equation has a form of the equation for solving eigenvalue problem for the matrix Γ_{ik}: to find the eigenvalues of Γ_{ik} which equal c^2 and the eigenvectors g_j ($g_j g_j = 1$) proportional to the constant vector U_j, $U_j = Ag_j$.

Let us mention first several important properties of the Christoffel matrix Γ_{ik}. From the definition of the matrix Γ_{ik} we can immediately see that it is a 3 x 3 real-valued matrix, which is *symmetric*:

$$\Gamma_{ik} = \Gamma_{ki} \quad .$$

The Christoffel matrix Γ_{ik} is also *positively definite.* To proof this, we must show that for any vector d_i

$$\Gamma_{ik}d_i d_k > 0 \quad .$$

The expression on the left-hand side can be rewritten as follows

$$\Gamma_{ik}d_i d_k = a_{ijkl}p_j p_l d_i d_k = a_{ijkl}b_{ij}b_{kl} \quad ,$$

where

$$b_{ij} = d_i p_j \quad .$$

In Sec.2.4.3, we have shown that the strain energy W is always positive,

$$\frac{1}{2}c_{ijkl}e_{ij}e_{kl} > 0 \quad ,$$

which automatically implies

$$a_{ijkl}e_{ij}e_{lk} > 0 \quad .$$

We can see that the inequality is satisfied for an arbitrary symmetric tensor of small strain. Let us therefore write

$$b_{ij} = \frac{1}{2}(b_{ij} + b_{ji}) + \frac{1}{2}(b_{ij} - b_{ji}) = b_{ij}^S + b_{ij}^A \quad .$$

The above expression for $a_{ijkl}b_{ij}b_{kl}$ can be thus rewritten as

$$a_{ijkl}(b_{ij}^S + b_{ij}^A)(b_{kl}^S + b_{kl}^A) = a_{ijkl}b_{ij}^S b_{kl}^S > 0 \quad .$$

All the terms containing b_{ij}^A are zero due to the symmetry of a_{ijkl} and antisymmetry of b_{ij}^A. The Christoffel matrix is thus possitively definite.

It follows from the above properties of Γ_{ik} that it has three real and positive eigenvalues and to each of them corresponding eigenvector can be found. Since the eigenvalues c^2 are positive, it means that we have three couples of real values $\pm c$, which correpond to three couples of waves propagating in an anisotropic medium. Each couple consists of two waves propagating with the same phase velocity c into opposite directions. From now on, we are going to consider only the propagation into positive direction, with c positive. Thus, in an anisotropic homogeneous medium, in a direction specified by the phase normal N_i, three, generally independent, waves can propagate. Their phase velocities can be found explicitely from the condition of solvability of the Christoffel equation, which reads

$$\det(\Gamma_{ik} - c^2\delta_{ik}) = 0 \quad .$$

This is a cubic equation for c^2. For a given phase normal N_i, it will yield three, generally different, phase velocities $c^{(m)}$, $m = 1, 2, 3$. The velocities depend on the same variables as Γ_{ik}, i.e. on a_{ijkl} and N_i. The dependence on N_i implies the dependence of the phase velocity on the direction of wave propagation.

As we know from Sec.1.1.4, one eigenvector corresponds to each eigenvalue of the matrix Γ_{ik}. If all the three eigenvalues are different, the corresponding eigenvectors can be determined uniquely. In Sec.1.1.4, we found that they are mutually perpendicular if the corresponding matrix is real and symmetric. This means that the three different waves propagating in an anisotropic medium differ not only by the phase velocity $c^{(m)}$ but also by their *polarization*, i.e. by the orientation of the eigenvectors $g_i^{(m)}$, which specify the directions of the particle motion vector u_i. Particles move along *lines* specified by $g_i^{(m)}$. This type of the polarization is therefore called the *linear polarization*. If all the three waves propagate in the same direction, i.e. their phase fronts are parallel, then the particle motion vectors corresponding to these waves are mutually perpendicular.

The situation is different when two of the eigenvalues of the matrix Γ_{ik} coincide. Such a case is called *degenerate*. Then the eigenvectors corresponding to the coinciding eigenvalues cannot be determined uniquely. They are situated in the plane perpendicular to the third eigenvector which is determined uniquely. Any two mutually perpendicular vectors in this plane can be chosen as eigenvectors. The directions N_i, for which two eigenvalues of Γ_{jk} coincide are called *singular*. In an isotropic medium, this occurs for any direction.

The plane wave solution of the elastodynamic equation for homogeneous anisotropic medium has the form

$$u_i(x_m,t) = A^{(l)} g_i^{(l)} F(t - T(x_m)) \,, \quad T = \frac{N_i x_i}{c^{(l)}} \,, \quad l = 1,2,3.$$

Here $A^{(l)}$ is an arbitrary, generally complex-valued constant, $c^{(l)}$ is the phase velocity and $g_i^{(l)}$ normalized particle motion vector of the l-th of the three waves which can propagate in a homogeneous anisotropic medium. Vector $g_i^{(l)}$ is generally neither parallel nor perpendicular to N_i. Usually, the plane wave with greatest phase velocity and polarization closest to the direction of the phase normal N_i is called *quasi-compressional* or *qP wave*. The other two waves are called *quasi-shear* or *qS waves* (qS1 and qS2). Let us again add that the same results as above we would obtain for time-harmonic plane waves.

3.3.2 Homogeneous isotropic medium

Here we can proceed in two ways. We can seek the coefficients of the trial plane wave solution by inserting it into the elastodynamic equation for homogeneous isotropic media or we can specify results of Sec.3.3.1 for isotropy. We shall follow the first approach.

Let us insert the trial solution

$$u_i(x_m,t) = U_i F(t - T(x_m)) \,, \quad T = \frac{N_i x_i}{c}$$

into the elastodynamic equation for a homogeneous isotropic medium with $f_i = 0$:

$$(\lambda + \mu) u_{k,ki} + \mu u_{i,kk} = \rho u_{i,tt} \quad .$$

We get

$$(\lambda + \mu) U_k p_k p_i + \mu U_i p_k p_k - \rho U_i = 0 \quad .$$

Inserting $p_i = N_i/c$ into it, the equation can be rewritten

$$\left[\frac{\lambda + \mu}{\rho} N_k N_i + \left(\frac{\mu}{\rho} - c^2 \right) \delta_{ik} \right] U_k = 0 \quad .$$

This is the Christoffel equation for an isotropic homogeneous medium. The condition of its solvability is

$$\begin{vmatrix} \frac{\lambda+\mu}{\rho}N_1^2 + \left(\frac{\mu}{\rho} - c^2\right) & \frac{\lambda+\mu}{\rho}N_1N_2 & \frac{\lambda+\mu}{\rho}N_1N_3 \\ \frac{\lambda+\mu}{\rho}N_1N_2 & \frac{\lambda+\mu}{\rho}N_2^2 + \left(\frac{\mu}{\rho} - c^2\right) & \frac{\lambda+\mu}{\rho}N_2N_3 \\ \frac{\lambda+\mu}{\rho}N_1N_3 & \frac{\lambda+\mu}{\rho}N_2N_3 & \frac{\lambda+\mu}{\rho}N_3^2 + \left(\frac{\mu}{\rho} - c^2\right) \end{vmatrix} = 0$$

This is a cubic equation for c^2. To determine all the three roots c^2, we could develop the above determinant in which N_i is chosen quite arbitrarily. Since the investigated medium is isotropic, the solution which we get for one specific choice of N_i will hold for any other specification of N_i. Let us choose the phase normal as follows

$$\vec{N} = (1,0,0) \quad .$$

The above determinant then reduces to

$$\left(\frac{\lambda+2\mu}{\rho} - c^2\right)\left(\frac{\mu}{\rho} - c^2\right)^2 = 0 \quad .$$

From the above equation, we can see that two of the three eigenvalues c^2 are identical. The above equation factorizes so well that we can write the expressions for phase velocity explicitely. We again choose the waves propagating in a positive direction. Then we have

$$c^{(1)} = c^{(2)} = \sqrt{\frac{\mu}{\rho}}, \quad c^{(3)} = \sqrt{\frac{\lambda+2\mu}{\rho}} \quad .$$

Thus, in an isotropic medium, only two different waves can propagate: the faster one with the phase velocity $c = \alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}$, the other one with the phase velocity $c = \beta = \sqrt{\frac{\mu}{\rho}}$. We called the faster wave in Sec.2.7 the *compressional* or *P wave*, the slower one the *shear* or *S wave*.

Let us determine the polarization of both waves, i.e. let us find eigenvectors corresponding to the eigenvalues α^2 and β^2. For this purpose let us multiply the Christoffel equation by the vector U_i to get

$$\frac{\lambda+\mu}{\rho}(N_kU_k)(N_iU_i) + \left(\frac{\mu}{\rho} - c^2\right)(U_iU_i) = 0 \quad .$$

Let us first investigate the shear wave. In this case we have two coinciding eigenvalues β^2 and thus the eigenvectors cannot be determined uniquely. For $c^2 = \beta^2 = \mu/\rho$, the above equation gives

$$\frac{\lambda+\mu}{\rho}(N_kU_k)^2 = 0 \quad .$$

Since $(\lambda+\mu)/\rho > 0$, this implies that $N_k U_k = 0$, which means that the polarization of the S wave is perpendicular to the direction of the propagation specified by the phase normal N_i. Thus the vector U_k is situated in the plane of the phase front. Any two mutually perpendicular unit vectors in the plane of the phase front can be chosen as the eigenvectors $g_i^{(1)}$, $g_i^{(2)}$.

From the last sentence of the above paragraph, we can conclude that the eigenvector $g_i^{(3)}$ corresponding to the P wave propagating in the direction N_i is perpendicular to the phase front and thus parallel to N_i. This conclusion can be confirmed by inserting $c^2 = \alpha^2 = (\lambda+2\mu)/\rho$ into the Christoffel equation multiplied by U_i. We get

$$\frac{\lambda+\mu}{\rho}[(N_k U_k)^2 - (U_k U_k)] = 0 \quad .$$

Since $(\lambda+\mu)/\rho > 0$, this implies that

$$(N_k U_k)^2 = (U_k U_k) \quad .$$

This holds only when $U_k \parallel N_k$.

The P plane wave solutions of the elastodynamic equation for a homogeneous isotropic medium can be written as

$$u_i(x_m,t) = A N_i F\left(t - \frac{N_i x_i}{\alpha}\right) \quad ,$$

where A is an arbitrary, possibly complex-valued, constant, N_i is the phase normal and α the P-wave phase velocity.

The S plane wave solution has a more complicated form:

$$u_i(x_m,t) = (B g_i^{(1)} + C g_i^{(2)}) F\left(t - \frac{N_i x_i}{\beta}\right) \quad ,$$

where B and C are arbitrary, possibly complex-valued, constants and β is the S-wave phase velocity. The vectors $g_i^{(1)}$ and $g_i^{(2)}$ are mutually perpendicular unit vectors arbitrary chosen in the plane of the phase front.

From the first of the above two expressions we can see that the P wave particles always move along a line parallel to N_i, i.e. along the line perpendicular to the phase front. The P wave polarization is thus *linear*. The S wave particle motion is more complicated. It is confined to the plane perpendicular to N_i. Let us, for a while, consider the analytic signal $F(\xi)$ to be $\exp(i\omega\xi)$. Then the solution for a plane S wave has a form

$$u_i(x_m,t) = (B g_i^{(1)} + C g_i^{(2)}) exp\left[-i\omega\left(t - \frac{N_i x_i}{\beta}\right)\right] \quad .$$

Let us look for the path of the particle specified by the vector $\mathrm{Re}(u_i)$. We denote the real parts of the components of the vector u_i into the vectors $g_i^{(1)}$, $g_i^{(2)}$ by u_1 and u_2. For u_1 and u_2 we get

$$u_1(x_m, t) = |B| \cos\left[\omega(t-T) - \varphi_B\right] \quad ,$$
$$u_2(x_m, t) = |C| \cos\left[\omega(t-T) - \varphi_C\right] \quad ,$$

where we used for the generally complex constants B and C the following expressions

$$B = |B| exp(i\varphi_B)\,, \quad C = |C| exp(i\varphi_C) \quad .$$

For simpler manipulation, we denote

$$\varphi = \omega(t-T) = \omega\left(t - \frac{N_i x_i}{\beta}\right), \quad \Delta\varphi = \varphi_B - \varphi_C \quad .$$

Then

$$u_1 = |B| \cos(\varphi - \varphi_B)\,, \quad u_2 = |C| \cos(\varphi - \varphi_B + \Delta\varphi) \quad .$$

We can write

$$\frac{u_2}{|C|} = \cos(\varphi - \varphi_B)\cos\Delta\varphi - \sin(\varphi - \varphi_B)\sin\Delta\varphi \quad .$$

Multiplying $u_1/|B|$ by $\cos\Delta\varphi$ and subtracting it from the above equation, we get

$$\frac{u_2}{|C|} - \frac{u_1}{|B|}\cos\Delta\varphi = -\sin(\varphi - \varphi_B)\sin\Delta\varphi \quad .$$

Now from the equation for u_1 we get

$$\sin^2(\varphi - \varphi_B) = 1 - \frac{u_1^2}{|B|^2} \quad ,$$

which we insert into the squared previous equation,

$$\left(\frac{u_2}{|C|} - \frac{u_1}{|B|}\cos\Delta\varphi\right)^2 = \left(1 - \left(\frac{u_1}{|B|}\right)^2\right)\sin^2\Delta\varphi \quad .$$

After an arrangement of the terms,

$$\left(\frac{u_2}{|C|}\right)^2 + \left(\frac{u_1}{|B|}\right)^2 - 2\frac{u_1 u_2}{|B||C|}\cos\Delta\varphi = \sin^2\Delta\varphi \quad .$$

This is an equation of an ellipses. It means that the displacement vector for $\Delta\varphi \neq k\pi$ traces an ellipses in the plane of the phase front. Therefore, this type of the polarization of an S wave is called *elliptical polarization*. For $\Delta\varphi = \pm k\pi$, where k is an integer, the above equation of ellipses reduces to

$$u_2 = \pm\frac{|C|}{|B|}u_1 \quad ,$$

which is an equation of a line. Thus in special situations, one of them being the case of real constants B and C (then $\varphi_B = \varphi_C = 0$ and thus $\Delta\varphi = 0$), the S wave can also be *linearly polarized* in the plane of the phase front. If the constants B and C have the same magnitude $|B| = |C|$ but phases $\Delta\varphi \neq \pm k\pi$ the resulting polarization is called *circular* polarization.

Due to the change of the form of the signal in the case of transient waves, the S wave polarization is not any more elliptical but it forms complicated spiral-like forms. Such a polarization is called *quasielliptical* .

3.4 Energy considerations

In Secs.2.4.1 and 2.4.2 we introduced strain energy W, kinetic energy K, elastic energy $E = K + W$ and the vector of energy flux S_i (remember that all the above quantities are in fact energy densities!). In this section, we shall specify them for plane waves propagating in homogeneous acoustic, anisotropic and isotropic media. The above quantities are defined as instantaneous quantities, i.e. as functions of time. In practice, we often prefer to work with these quantities time-independent. Their time independency can be reached by *time-averaging* or by *time integration.* The former approach is convenient for applications to time-harmonic waves, the latter one for transient waves, i.e for seismic signals.

The time averaging for harmonic waves is done over one period. The time averaged quantities $\overline{W}$, $\overline{K}$, $\overline{E}$ and $\overline{S}_i$ are determined from their instantaneous counterparts W, K, E and S_i by the following rule applied here, e.g., to strain energy W

$$\overline{W}(x_m) = T^{-1} \int_t^{t+T} W(x_m, t) dt \quad .$$

The time integrated quantities $\hat{W}$, $\hat{K}$, $\hat{E}$ and $\hat{S}_i$ for seismic signals are determined from W, K, E and S_i by the following rule

$$\hat{W}(x_m) = \int_{-\infty}^{+\infty} W(x_m, t) dt \quad .$$

Since the seismic signal lasts only within a limited time interval, only this interval contributes to the integral, outside the interval the contributions are zero.

Since the energy quantities are nonlinear in pressure, particle velocity or displacement, we must work with physically meaningful parts of the complex solutions of the equation of motion. In the following we shall, therefore, consider only the real part of the complex solution

$$w(x_m, t) = WF(t - T(x_m)) \quad .$$

For simplicity, we shall also denote it $w(x_m, t)$ and we shall define it as

$$w(x_m, t) = \frac{1}{2}(WF + W^*F^*) \quad ,$$

where symbol * denotes complex conjugacy. Please, do not confuse the above symbol W with strain energy.

3.4.1 Acoustic medium

Real parts of the complex solutions for pressure and particle velocity can be written as (we consider p_i real)

$$p = \frac{1}{2}(PF + P^*F^*), \quad v_i = \frac{1}{2}\frac{p_i}{\rho}(PF + P^*F^*) \quad .$$

We did not write the argument of the analytic signal $F(t - (N_i x_i)/c)$). Let us first express W, K, E and S_i in terms of the pressure and particle velocity given above. Keeping in mind that in the acoustic case,

$$p = -k\theta, \quad \tau_{ij} = -p\delta_{ij} \quad ,$$

we can write

$$\begin{aligned}
W &= \frac{1}{2}\tau_{ij}u_{i,j} = -\frac{1}{2}pu_{i,i} = \frac{1}{2k}p^2 = \frac{1}{2}\kappa p^2 = \frac{1}{8}\kappa(PF + P^*F^*)^2, \\
K &= \frac{1}{2}\rho\dot{u}_i\dot{u}_i = \frac{1}{2}\rho v_i v_i = \frac{1}{8}(\rho c^2)^{-1}(PF + P^*F^*)^2, \\
S_i &= -\tau_{ij}\dot{u}_j = p\delta_{ij}v_j = pv_i = \frac{1}{4}p_i\rho^{-1}(PF + P^*F^*)^2,
\end{aligned}$$

where we use the notation

$$\dot{u}_i = \frac{\partial u_i}{\partial t} \quad .$$

Since $c^2 = (\kappa\rho)^{-1}$, the above equations yield an important result

$$W = K \quad .$$

Thus, the strain energy of a plane acoustic wave equals at any time to its kinetic energy. Thus, for elastic energy E we can write

$$E = \frac{1}{4}(\rho c^2)^{-1}(PF + P^*F^*)^2 = \frac{1}{4}\kappa(PF + P^*F^*)^2 .$$

We can also immediately derive the expression for the velocity of the energy flux, see Sec. 2.4.2. This velocity is also known as *the group velocity* $v_i^{(g)}$,

$$v_i^{(g)} = S_i/E = c^2 p_i = cN_i = c_i \quad .$$

Thus, in a homogeneous acoustic medium, the velocity of the energy flux has the same direction and magnitude as the phase velocity.

Let us now determine the time averaged or time integrated quantities. In case of time integrated quantities, we shall deal with the following integrals

$$\int_{-\infty}^{+\infty} F^2(t)dt\,, \quad \int_{-\infty}^{+\infty} (F^*(t))^2 dt\,, \quad \int_{-\infty}^{+\infty} F(t)F^*(t)dt \quad .$$

The above integrals can be simply evaluated if we take into account the following properties of a quadratically integrable function $g(t)$,

$$\int_{-\infty}^{+\infty} g^2(t)dt = \int_{-\infty}^{+\infty} h^2(t)dt\,, \quad \int_{-\infty}^{+\infty} g(t)h(t)dt = 0 \quad ,$$

where $h(t)$ again denotes the Hilbert transform of $g(t)$. Using the above integrals, we get

$$\int_{-\infty}^{+\infty} F^2(t)dt \;=\; \int_{-\infty}^{+\infty} [g(t) + ih(t)]^2 dt = \int_{-\infty}^{+\infty} (g^2(t) - h^2(t))dt + 2i\int_{-\infty}^{+\infty} g(t)h(t)dt = 0$$

similarly, we get

$$\int_{-\infty}^{+\infty} (F^*(t))^2 dt = 0\,, \quad \int_{-\infty}^{+\infty} F(t)F^*(t)dt = 2\int_{-\infty}^{+\infty} g^2(t)dt\,.$$

Using these identities and the above expressions for W, K, E and S_i, we can find the time integrated quantities $\hat{W}$, $\hat{K}$, $\hat{E}$ and $\hat{S}_i$ to be

$$\hat{W} = \int_{-\infty}^{+\infty} W dt = \frac{\kappa}{8}\int_{-\infty}^{+\infty} (PF + P^*F^*)^2 dt = \frac{\kappa}{2} PP^* f_a \quad ,$$

where

$$f_a = \int_{-\infty}^{+\infty} g^2(t)dt \quad .$$

Similarly, we get

$$\hat{K} = \frac{1}{2}\kappa PP^* f_a \quad , \quad \hat{S}_i = \frac{p_i}{\rho} PP^* f_a \quad .$$

We can see that, as expected, the time integrated strain energy and time integrated kinetic energy are again equal. Thus, for $\hat{E}$ we can write

$$\hat{E} = \kappa PP^* f_a \quad .$$

In addition, we can see that the group velocity $v_i^{(g)}$ defined above as an instantaneous quantity, can be also defined as a velocity of the time integrated energy flux,

$$v_i^{(g)} = \hat{S}_i / \hat{E} = c^2 p_i = cN_i \quad .$$

Let us note that similar expressions could be also obtained for the averaged quantities $\bar{W}$, $\bar{K}$, $\bar{E}$ and $\bar{S}_i$. Only the integral f_a must be substituted by

$$f_A = T^{-1}\int_t^{t+T} g^2(t)dt \quad .$$

3.4.2 Homogeneous anisotropic medium

Now we shall deal with the displacement vector in the form

$$u_i = \frac{1}{2}(U_i F + U_i^* F^*) = \frac{1}{2}(AF + A^* F^*) g_i \quad ,$$

where A is a scalar, constant, possibly complex-valued factor and g_i is a unit vector specifying the polarization of the considered wave. (Note again that we are using the same symbol u_i to denote the real part of the complex displacement vector u_i.) The quantities W, K, E and S_i can be expressed in terms of A, F, F' ($F'(\xi) = dF/d\xi$; as was shown in Sec.1.3, F' is also an analytic signal) and g_i as follows

$$\begin{aligned}
W &= \frac{1}{2}\rho a_{ijkl} u_{i,j} u_{k,l} = \frac{1}{8}\rho a_{ijkl} c^{-2} (U_i N_j F' + U_i^* N_j F^{*\prime})(U_k N_l F' + U_k^* N_l F^{*\prime}) \\
&= \frac{1}{8}\rho c^{-2} \Gamma_{ik} (AF' + A^* F^{*\prime})^2 g_i g_k \, , \\
K &= \frac{1}{2}\rho \dot{u}_i \dot{u}_i = \frac{1}{8}\rho (AF' + A^* F^{*\prime})^2 \, , \\
S_i &= -\rho a_{ijkl} u_{k,l} \dot{u}_j = \frac{1}{4}\rho a_{ijkl} c^{-1} N_l (AF' + A^* F^{*\prime})^2 g_j g_k \, .
\end{aligned}$$

If we consider the Christoffel equation, see Sec.3.3.1, multiplied by the vector g_i, we get

$$\Gamma_{ik} g_i g_k - c^2 = 0 \quad ,$$

from which we can see that $c^{-2}\Gamma_{ik} g_i g_k = 1$. If we use this identity in the expression for W, we can see that as in the case of acoustic medium

$$W = K \quad ,$$

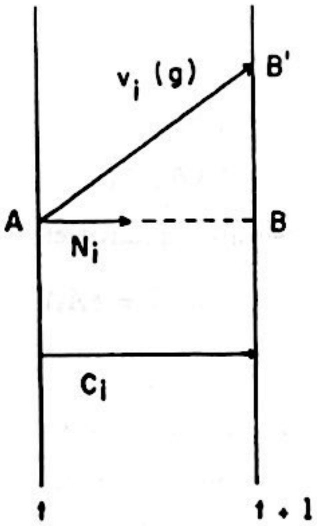
i.e. that for a plane wave propagating in a homogeneous anisotropic medium the strain energy W equals at any time to the kinetic energy K. For the elastic energy E we can thus write

$$E = \frac{1}{4}\rho (AF' + A^* F^{*\prime})^2 \quad .$$

The expression for the group velocity $v_i^{(g)}$ has now the form

$$v_i^{(g)} = S_i / E = c^{-1} a_{ijkl} N_l g_j g_k = a_{ijkl} p_l g_j g_k \quad .$$

We can see that in contrast to the acoustic case, the above expression for the group velocity does not indicate coincidence with the phase velocity. Moreover, due to the dependence on N_i, we can expect that the value and the direction of the group velocity will vary with varying direction of N_i. We can thus conclude that the energy of a plane wave propagating



in a homogeneous anisotropic medium propagates generally in a direction different from the direction of propagation of phase front and the velocity of energy propagation is different from phase velocity. To illuminate the relation between group and phase velocity, let us multiply the expression for $v_i^{(g)}$ by $p_i = N_i/c$. We get

$$v_i^{(g)} p_i = a_{ijkl} p_l p_i g_j g_k = c^{-2} \Gamma_{jk} g_j g_k \quad .$$

From the Christoffel equation $(\Gamma_{jk} - c^{-2}\delta_{jk})g_k = 0$ multiplied by g_j we get

$$\Gamma_{jk} g_j g_k = c^2 \quad .$$

If we use this identity in the expression for $v_i^{(g)} p_i$, it will yield

$$v_i^{(g)} p_i = 1 \quad \Leftrightarrow \quad v_i^{(g)} N_i = c \quad .$$

From this equation, we can see that the group velocity equals phase velocity only if $\vec{v}^{(g)} \parallel \vec{N}$. Another consequence of the above equation is that *the group velocity is always higher or equal to the phase velocity.* This can also be seen in the picture. During a unit time interval the phase front shifts from the position t to the position $t+1$. Corresponding phase velocity vector $\vec{c}$ is perpendicular to the phase front and is just equal to the distance between both phase fronts. Group velocity vector is generally not perpendicular to the phase front and must be thus larger. Since $v_i^{(g)}$ describes flow of energy, the above picture shows that the particle oscillating at the point A will not affect the particle at the point B but at the point B'.

We can now evaluate time integrated quantities $\hat{W}$, $\hat{K}$, $\hat{E}$ and $\hat{S}_i$. We shall take into account that F', $F^{*\prime}$ have the same properties as F, F^*. We then get

$$\hat{W} = \int_{-\infty}^{+\infty} W dt = \frac{1}{8}\rho c^{-2} \Gamma_{ik} g_i g_k \int_{-\infty}^{+\infty} (AF' + A^* F^{*\prime})^2 dt = \frac{1}{2}\rho c^{-2} \Gamma_{ik} g_i g_k A A^* f_e \ ,$$

where

$$f_e = \int_{-\infty}^{+\infty} \dot{g}(t)^2 dt \quad .$$

For remaining quantities we get

$$\hat{K} = \frac{1}{2}\rho A A^* f_e \quad , \quad \hat{S}_i = c^{-1}\rho a_{ijkl} N_l g_j g_k A A^* f_e \, .$$

We again see that also time integrated quantities yield $\hat{W} = \hat{K}$ and thus

$$\hat{E} = \rho A A^* f_e$$

and $v_i^{(g)} = \hat{S}_i / \hat{E}$.

The expressions for the time averaged quantities $\bar{W}$, $\bar{K}$, $\bar{E}$ and $\bar{S}_i$ are again the same as for the time integrated ones, only the integral f_e must be substituted by f_B,

$$f_B = \frac{1}{T}\int_t^{t+T} \dot{g}^2(t) dt \, .$$

3.4.3 Homogeneous isotropic medium

The expressions for instantaneous values of W, K, E and S_i and their time integrated and time averaged values can be derived in exactly the same way as in Sections 3.4.1 and 3.4.2. Another possibility how to derive them is to specify the corresponding expressions derived in Sec.3.4.2 for the isotropic case. It can be done with the following expression

$$a_{ijkl} = (\alpha^2 - 2\beta^2)\delta_{ij}\delta_{kl} + \beta^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \, .$$

We shall consider the displacement vector separately for the P wave,

$$u_i = \frac{1}{2}(AF + A^* F^*) N_i$$

and for the S wave

$$u_i = \frac{1}{2}[(BF + B^* F^*) g_i^{(1)} + (CF + C^* F^*) g_i^{(2)}] \, .$$

We present the results without derivation. Indices P denote that the corresponding quantity is related to the P wave, indices S that it relates to the S wave:

$$\begin{aligned}
W_P &= \frac{1}{8}\rho(AF' + A^* F^{*\prime})^2 \, , \qquad W_S = \frac{1}{8}\rho[(BF' + B^* F^{*\prime})^2 + (CF' + C^* F^{*\prime})^2] \, , \\
K_P &= \frac{1}{8}\rho(AF' + A^* F^{*\prime})^2 \, , \qquad K_S = \frac{1}{8}\rho[(BF' + B^* F^{*\prime})^2 + (CF' + C^* F^{*\prime})^2] \, , \\
S_{Pi} &= \frac{1}{4}\rho\alpha N_i(AF' + A^* F^{*\prime})^2 \, , \quad S_{Si} = \frac{1}{4}\rho\beta N_i[(BF' + B^* F^{*\prime})^2 + (CF' + C^* F^{*\prime})^2] \, , \\
E_P &= \frac{1}{4}\rho(AF' + A^* F^{*\prime})^2 \, , \qquad K_S = \frac{1}{4}\rho[(BF' + B^* F^{*\prime})^2 + (CF' + C^* F^{*\prime})^2]
\end{aligned}$$

The time integrated quantities are now

$$\begin{aligned}
\hat{W}_P &= \frac{1}{2}\rho AA^* f_e\,, \quad \hat{W}_S = \frac{1}{2}\rho(CC^* + BB^*)f_e \\
\hat{K}_P &= \hat{W}_P\,,\ \hat{K}_S = \hat{W}_S \quad \Rightarrow \quad \hat{E}_P = \rho AA^* f_e\,,\ \hat{E}_S = \rho(BB^* + CC^*)f_e \\
\hat{S}_{Pi} &= \rho\alpha N_i AA^* f_e\,,\ \hat{S}_{Si} = \rho\beta N_i(BB^* + C^*C)f_e\,.
\end{aligned}$$

As a consequence of the above expressions for instantaneous and time integrated quantities, we get for the group velocities of P and S waves

$$v_{Pi}^{(g)} = S_{Pi}/E_P = \hat{S}_{Pi}/\hat{E}_P = \alpha N_i\,, \quad v_{Si}^{(g)} = S_{Si}/E_S = \hat{S}_{Si}/\hat{E}_S = \beta N_i\,.$$

Thus in a homogeneous isotropic medium, similarly as in the acoustic case, the group velocity equals the phase velocity, both in size and direction. Finally, let us add that the formulae for time averaged quantities $\bar{W}$, $\bar{E}$, $\bar{K}$ and $\bar{S}_i$ can be deduced from the above formulae for time integrated quantities by substitution of the integral f_e by f_B.

3.5 Comparison of wave propagation in anisotropic, isotropic and acoustic media

The wave propagation in anisotropic and isotropic homogeneous media differs in the following features:

1. For a specified direction of the phase normal there are generally three independent plane waves with different phase velocities in an anisotropic medium, two independent plane waves in an isotropic medium and one plane wave in an acoustic medium.

2. Phase and group velocity vectors are generally different in an anisotropic medium, the group velocity being greater than the phase velocity. The group velocity and the phase velocity have the same size and direction in isotropic and acoustic media.

3. Phase and group velocity vectors are generally angularly dependent in anisotropic media. Specifically, they both depend on the phase normal N_i. The phase velocity (and thus also the group velocity) is the same in all directions in isotropic and acoustic media.

4. The polarization of waves in anisotropic media is generally linear with the direction of polarization generally different from the direction of the phase normal, of tangent to the wavefront or of the group velocity vector. In isotropic media, the polarization direction is either parallel to the phase normal (P waves) or in the plane of the phase

front (S waves). The polarization of P waves is linear, of S waves is generally elliptical or quasielliptical. The polarization of the wave in the acoustic case is the same as of the P wave in an isotropic case.

5. Shear wave singularities occur in certain directions in anisotropic media. In these directions, qS_1 and qS_2 waves propagate with the same phase velocity and have complicated behavior. These singularities usually do not exist for P waves in anisotropic media as well as for all the wave types in isotropic and acoustic media.

3.6 Wave surfaces

The directional dependence of wave propagation in anisotropic media is often demonstrated on the following three types of surfaces: slowness surface, phase velocity surface and group velocity or wave surface.

3.6.1 Slowness surface

Slowness surface is probably the most important of the three surfaces. It plays an important role, e.g. in the problems of the determination of the slowness vector of waves generated by an incidence of a plane wave at a plane interface.

The slowness surface is a surface specified by the endpoints of the radius vector r_i,

$$r_i = p_i = c^{-1} N_i \ ,$$

where N_i is the phase normal and c is the phase velocity. Since there are three different waves propagating in anisotropic media, the slowness surface consists of three sheets. The inner one corresponding to the wave with the highest phase velocity and thus corresponding to the qP wave is in most cases separated from the remaining two outer ones which belong to the qS_1 and qS_2 waves. The latter two surfaces have common points, i.e. they intersect or touch each other. The directions corresponding to these points are the singular directions along which the phase velocities of both qS waves coincide and corresponding polarization vectors cannot be uniquely determined. The equation of the slowness surface can be derived from the condition of solvability of the Christoffel equation,

$$\det(a_{ijkl} N_i N_l - c^2 \delta_{jk}) = 0 \quad .$$

If we divide the above equation by c^6, we get an equation in which, in addition to the constants a_{ijkl}, only the components of the slowness vector appear,

$$\det(a_{ijkl} p_i p_l - \delta_{jk}) = 0 \quad .$$

This is an equation of the surface of the sixth degree in the space of p_k. Thus, any straight line intersects the slowness surface in 6 points, real or complex. Since the slowness surface consists of three sheets, the inner one (corresponding to the qP wave) must always be *convex*. If it would not, then it would be possible that, in addition to 4 intersections of a line with outer sheets of the slowness surface there could be 4 more intersections with the inner sheet which contradicts the total number of 6 intersections. One of the qS slowness surface sheets, however, may be *concave*.

We can introduce a matrix $\bar{\Gamma}_{ik}$ by substituting N_l in the matrix Γ_{ik} by components of the slowness vector p_l,

$$\bar{\Gamma}_{ik} = a_{ijkl}p_jp_l \quad .$$

The Christoffel equation with the matrix $\bar{\Gamma}_{ik}$ has the form

$$(\bar{\Gamma}_{ik} - \delta_{ik})U_k = 0 \quad .$$

Its solution can be again found by determining the eigenvalues $G^{(m)}$ and eigenvectors $g_i^{(m)}$ of the matrix $\bar{\Gamma}_{ik}$, i.e. by solving

$$(\bar{\Gamma}_{ik} - G\delta_{ik})g_k = 0 \quad .$$

The eigenvalues G must be subject to the condition $G = 1$.

The eigenvectors g_k specify the orientation of U_k. Note that multiplication of the above equation by g_i yields

$$G = \bar{\Gamma}_{ik}g_ig_k \quad .$$

The eigenvalues of the matrix $\bar{\Gamma}_{ik}$ can be again found from the condition of solvability of the Christoffel equation, which has now form

$$\det(\bar{\Gamma}_{ik} - \delta_{ik}) = 0 \quad .$$

But this is exactly the equation of the slowness surface. It can be simplified if we diagonalize the Christoffel matrix $\bar{\Gamma}_{ik}$. Then it reads

$$(G^{(1)} - 1)(G^{(2)} - 1)(G^{(3)} - 1) = 0 \quad .$$

Let us assume that all the three eigenvalues are different, i.e., we are considering a nondegenerate case. Then if one of the eigenvalues, say $G^{(m)}$, is put equal 1, the other two must be different from 1. In such a case, the equation of the slowness surface reduces to

$$G^{(m)} = \bar{\Gamma}_{ik}g_i^{(m)}g_k^{(m)} = 1 \quad .$$

Normal to the slowness surface will be a vector proportional to $\partial G^{(m)}/\partial p_j$. For $\partial G^{(m)}/\partial p_j$, we can write

$$\frac{\partial G^{(m)}}{\partial p_j} = 2a_{ijkl}p_l g_i g_k = 2v_j^{(g)} \quad .$$

Thus, we can immediately deduce that the normal to the slowness surface is parallel to the group velocity vector.

To determine the form of the slowness surface in an isotropic medium we can either start from the condition of solvability of the Christoffel equation or from the equation for a separated slowness sheet

$$G^{(m)} = a_{ijkl}p_j p_l g_i^{(m)} g_k^{(m)} = 1 \quad .$$

In an isotropic medium this reduces to

$$(\alpha^2 - \beta^2)p_j p_k g_j^{(m)} g_k^{(m)} + (\beta^2 p_i p_i - 1) = 0 \quad .$$

For P waves, for which $g_j^{(m)} \parallel p_j$ we get

$$p_i p_i = \alpha^{-2} \quad ,$$

for S waves, for which $g_j^{(m)} \perp p_j$, we get

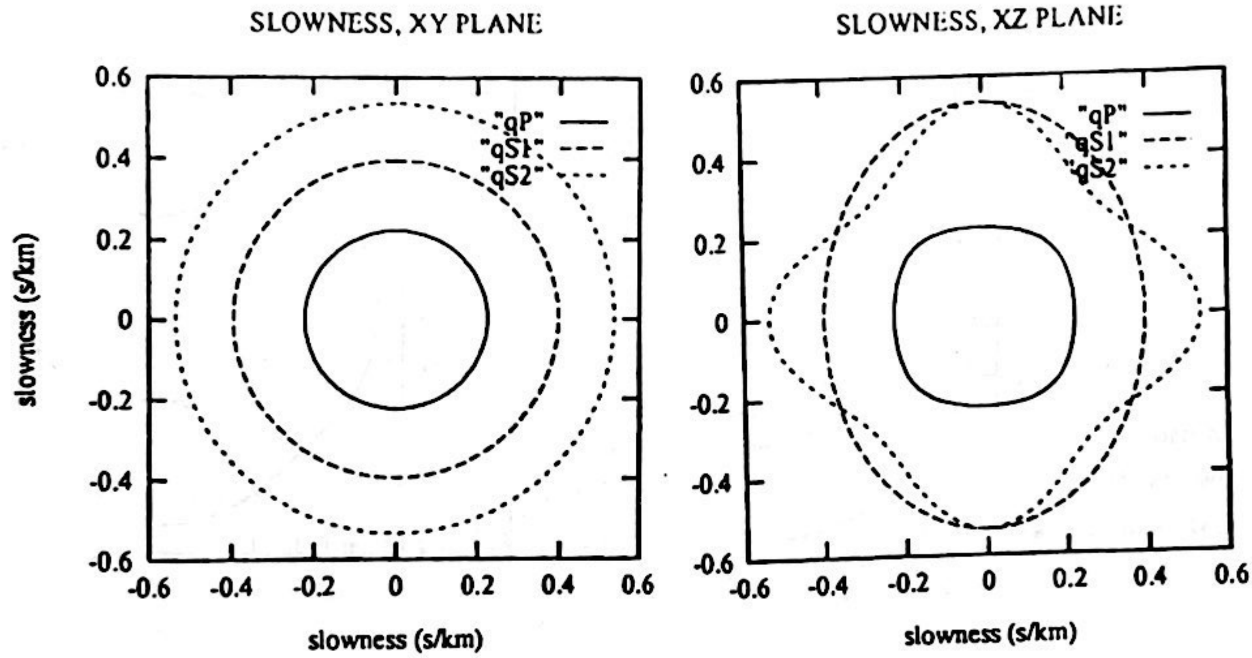
$$p_i p_i = \beta^{-2} \quad .$$

The above equations are well known eikonal equations. They are equations of spheres, one with the radius α^{-1}, the other with the radius β^{-1}. As expected, the slowness surface in an isotropic medium consists of two concentric spheres corresponding to P and S waves.

As an illustration, we show horizontal (XY) and vertical (XZ) sections of the *slowness surface* corresponding to a transversely isotropic medium (hexagonal symmetry with vertical axis of symmetry) described by the matrix of elastic parameters

$$\begin{pmatrix} 20.16 & 7.40 & 7.26 & 0.00 & 0.00 & 0.00 \\ & 20.16 & 7.26 & 0.00 & 0.00 & 0.00 \\ & & 19.63 & 0.00 & 0.00 & 0.00 \\ & & & 3.48 & 0.00 & 0.00 \\ & & & & 3.48 & 0.00 \\ & & & & & 6.38 \end{pmatrix} \quad .$$

Note concave parts of the qS_2 surface, which are responsible for the loops in the group velocity surface of the qS_2 wave, see Sec.3.6.3. The surface qS_2 corresponds to the qS wave with *in plane polarization* while the surface qS_1 corresponds to the qS wave with *horizontal polarization* (SH wave).



3.6.2 Phase velocity surface

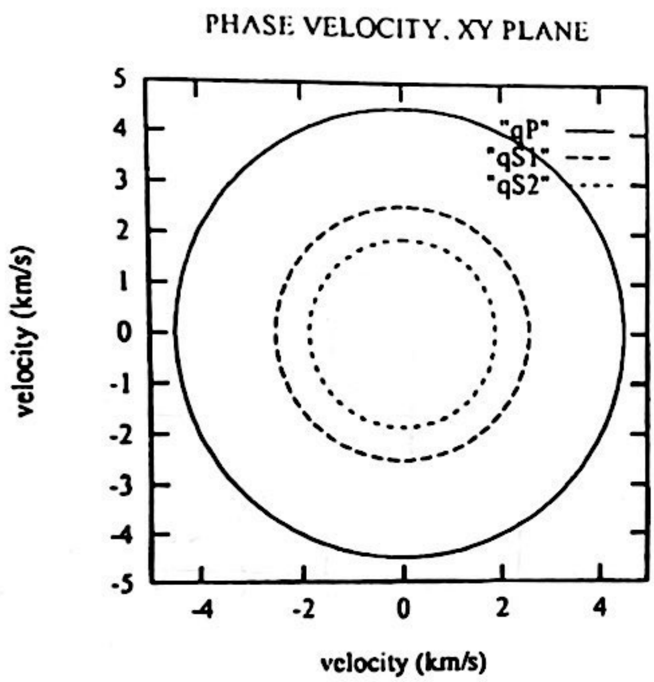
Phase velocity surface is a surface specified by the endpoints of the radius vector r_i,

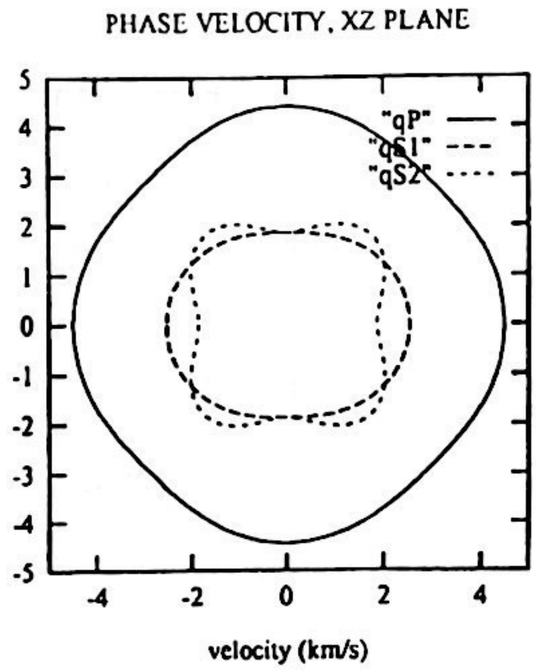
$$r_i = c_i = cN_i \quad .$$

Its lenght is inversely proportional to the lenght of the radius vector of the slowness surface. The phase velocity surface consists again of three sheets corresponding to qP, qS_1 and qS_2 waves. Now, the sheet corresponding to the qP wave represents the outer sheet.

In case of an isotropic medium, the phase velocity surface reduces to two concentric spheres corresponding to qP and qS waves.

As an illustration, we show horizontal (XY) and vertical (XZ) sections of the *phase velocity surface* corresponding to the transversely isotropic medium described in Sec.3.6.1.





Note that for the transversely isotropic medium:

$$c_{qP\,hor} = \sqrt{C_{11}}, \quad c_{qP\,ver} = \sqrt{C_{33}}, \quad c_{qS_1\,hor} = \sqrt{C_{66}},$$
$$c_{qS_1\,ver} = \sqrt{C_{44}}, \quad c_{qS_2\,hor} = \sqrt{C_{44}}, \quad c_{qS_2\,ver} = \sqrt{C_{44}}\,.$$

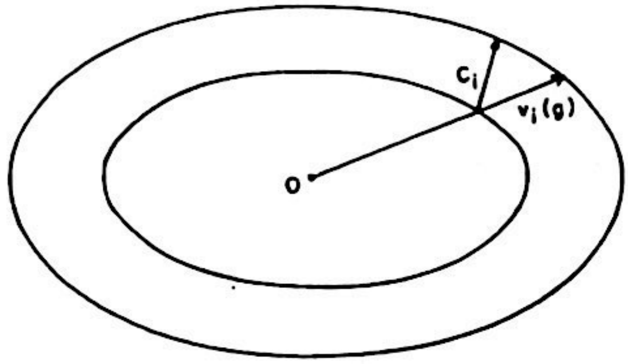
3.6.3 Group velocity surface

This surface itself is often called wave surface. It again consists of three sheets corresponding to qP, qS_1 and qS_2 waves. This surface is specified by the endpoints of the radius vector r_i,

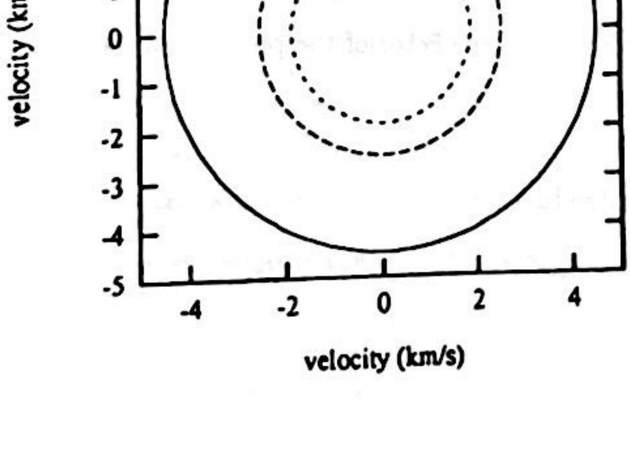
$$r_i = v_i^{(g)} \quad ,$$

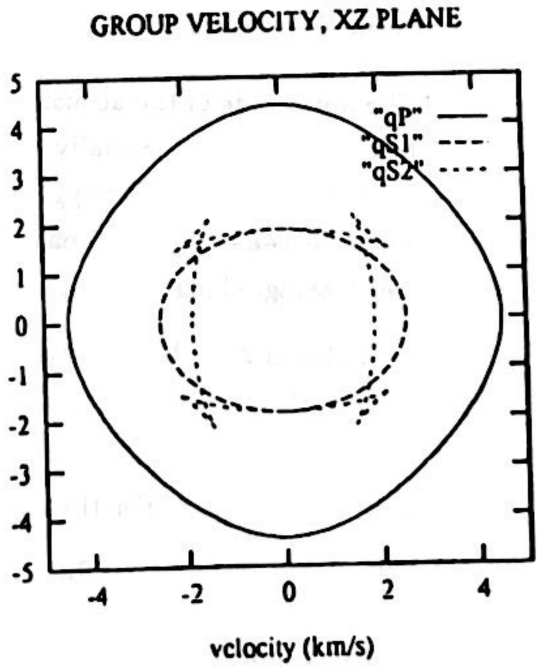
where $v_i^{(g)}$ is the group velocity. Thus the group velocity surface represents a surface to which a disturbance generated at the origin arrives in a unit time. All the points of this surface are in the same phase, thus the normal to it is the phase normal. The following picture shows a separate sheet of the group velocity surface (inner curve). It shows the relation between the group and phase velocity. Phase velocity vector c_i is oriented along the normal to the group velocity surface, the group velocity vector $v_i^{(g)}$ is oriented along the radius from the origin 0.

The wave surfaces can have a rather complicated form for the qS sheets. They may contain loops, cusppoints. This is a consequence of possibly concave form of the qS slowness



surface sheets. As we have shown above, the group velocity vector is parallel to the normal to the slowness surface. As shown in the examples below, in case of concave qS sheet, we can find three different directions of the phase normal N_i for which three parallelly oriented group velocity vectors of generally different size can be determined. This means that the group velocity corresponding to a separate qS wave can along one radius vector attain three, generally different, values. In other words, the group velocity surface may be multivalued, as shown on the picture below. This can never happen to qP waves since their slowness surface is always convex. It is possible to show that the degree of the wave surface can be upto 150, i.e., the degree of the equation of the group velocity surface may reach 150.





In the isotropic case, the group velocity surface reduces to two concentric spheres with radii α and β, i.e. the group velocity surface coincides with the phase velocity surface.

As na illustration, we show horizontal (XY) and vertical (XZ) sections of the *group velocity surface* corresponding to the transversely isotropic medium described in Sec.3.6.1. The loops on the surface qS_2 are due to the concave elements of the corresponding slowness surface, see Sec.3.6.1.

3.7 Inhomogeneous plane waves

We have been assuming until now that the slowness vector p_i is a real-valued quantity. But equations of motion can be satisfied even by plane waves with complex slowness vector p_i. We shall consider such plane waves now. We can write

$$p_l = p_l^R + ip_l^I \quad .$$

Let us consider a time-harmonic acoustic plane wave with the above complex slowness vector. It reads

$$\begin{aligned} p(x_m, t) &= P \exp\left(-\omega p_m^I x_m\right) \exp\left[-i\omega(t - p_m^R x_m)\right], \\ v_i(x_m, t) &= \rho^{-1} P p_i \exp\left(-\omega p_m^I x_m\right) \exp\left[-i\omega(t - p_m^R x_m)\right]. \end{aligned}$$

The amplitude of the acoustic wave with the complex-valued slowness vector is not constant but it decays exponentially in the direction of the vector p_m^I. The rate of the decay depends on the size of the vector p_m^I and the circular frequency ω. Thus, in addition to the *plane of constant phase* $p_m^R x_m =$ const, we can now define the *plane of constant amplitudes*, $p_m^I x_m =$ const along which the amplitudes do not vary.

In Sec.3.2, we have shown that the slowness vector of the plane acoustic wave must satisfy the equation

$$p_i p_i = c^{-2}$$

with $c = (\rho\kappa)^{-1/2}$. For the above introduced complex-valued slowness vector, this yields,

$$p_i^R p_i^R - p_i^I p_i^I = Re(c^{-2}) \quad , \quad 2p_i^R p_i^I = Im(c^{-2}) \, .$$

In the following, we shall exclude the case in which $Im(c^{-2}) \neq 0$, which corresponds to an absorbing medium. Thus, we have $Im(c^{-2}) = 0$ which implies

$$p_i^R p_i^I = 0 \quad ,$$

i.e. the real and imaginary parts of the slowness vector are mutually perpendicular. This means that the planes of constant phase and constant amplitude are also perpendicular. Such a wave is called the *inhomogeneous wave* in contrast to the *homogeneous wave* in which the planes of constant phase coincide with the planes of constant amplitude. All the plane waves which we have considered upto now were homogeneous plane waves. Let us note that if we consider absorption, and the slowness vector is complex-valued only due to the complex-valued phase velocity we would get a slowness vector with p_i^R and p_i^I parallel. Thus, even in that case we would deal with homogeneous waves.

Inhomogeneous plane waves cannot exist in unbounded media without sources. They can arise during the process of reflection/transmission of a plane wave at a plane interface. They play also an important role in the expansion of a spherical wave into plane waves.

We have found above that the amplitude of a time harmonic inhomogeneous plane wave varies along the phase front. The phase front $p_m^R x_m = \text{const}$ moves with a phase velocity c_R which, according to the above equations, satisfies

$$\frac{1}{c_R^2} = p_i^R p_i^R = \frac{1}{c^2} + p_i^I p_i^I > \frac{1}{c^2} \quad .$$

This means that the phase velocity of an inhomogeneous plane wave is less than the phase velocity of a homogeneous wave. Moreover, it is smaller for waves whose amplitudes decrease faster with depth.

Another interesting feature of the plane wave propagation with complex slowness vector is the fact that the particle velocity vectorial amplitude is complicated. Physically meaningful part of the complex particle velocity vector reads

$$\rho^{-1} P \exp\left(-\omega p_i^I x_i\right)[p_k^R \cos\left(\omega(t - p_m^R x_m)\right) + p_k^I \sin(\omega(t - p_m^R x_m))] \, .$$

This equation has important consequences for particle motion of an inhomogeneous acoustic wave. Since the vectors p_i^R and p_i^I are mutually perpendicular, the above expression describes an elliptical polarization in a plane perpendicular to the phase front.

All the above considerations can be also extended to the case of transient inhomogeneous acoustic plane waves. It is only necessary to generalize the analytic signal $F(t-T)$ for the case of complex T.

In homogeneous *anisotropic* media, similarly as in the acoustic case, the particle motion vectors g_k are complex, $g_i = g_i^R + i g_i^I$. The complexity of g_i follows from the complexity of the Christoffel matrix Γ_{ik}. It is, therefore, natural to require the following normalization condition for the particle motion vectors,

$$g_k g_k^* = 1 \quad .$$

Multiplying the Christoffel equation

$$(\Gamma_{ik} - c^2\delta_{ik})g_i = 0$$

by g_k^*, we thus get

$$\Gamma_{ik}g_i g_k^* = c^2 \Leftrightarrow a_{ijkl}p_j p_l g_i g_k^* = 1 \quad .$$

If we insert into this equation $p_i = p_i^R + ip_i^I$ and take into account that the terms containing $Im(g_i g_k^*)$ are zero due to the symmetry of the tensor a_{ijkl}, we get

$$a_{ijkl}(p_j^R p_l^R - p_j^I p_l^I)g_i g_k^* = 1 \, ,$$
$$a_{ijkl}(p_j^R p_l^I + p_j^I p_l^R)g_i g_k^* = 0 \, .$$

These equations are an anisotropic equivalent of the conditions on p_i^R and p_i^I, which we have got for the acoustic case above. We can see that the vectors p_i^R, p_i^I need not necessarily be mutually perpendicular. In fact, it is possible to show that the vector p_i^I is perpendicular to the vector of the group velocity and not to p_i^R. Thus, in a homogeneous nonabsorbing anisotropic medium, the planes of constant phase and constant amplitude of an inhomogeneous plane wave are not perpendicular. All the other conclusions are similar to those obtained for the acoustic inhomogeneous plane wave.

In an isotropic medium, we get the same conditions for p_i^R and p_i^I as in the acoustic case, i.e. that they are mutually perpendicular, separately for P and S wave. We found earlier that the polarization vector of the P wave is parallel to the corresponding slowness vector. This has the same consequences concerning the polarization as in the acoustic case. In case of the S wave, the particle motion vector satisfies the condition $p_i g_i = 0$. From it, we can find that the particle motion of an inhomogeneous plane S wave has a complicated spatial form. All the other conclusions made for the acoustic waves hold also for P and S waves.

3.8 Reflection/transmission of plane waves

Similarly as the plane wave propagation in an unbounded medium, the plane wave reflection/transmission at a plane interface is an important phenomenon not only from tutorial reasons but it has also important applications, e.g. in the reflectivity method or the ray method.

We shall consider a plane interface Σ and we put the origin of Cartesian coordinates at an arbitrary point of Σ. The choice of the axes of the Cartesian coordinate system can be arbitrary. If we denote the normal to Σ by n_i, the equation of the interface can be written as $n_i x_i = 0$.

The interface divides the space in two halfspaces, which are assumed to be in *welded* contact. In the halfspace, which we denote halfspace 1, we consider a plane wave propagating towards the interface - the *incident wave*. We choose the orientation of the normal to the interface Σ such that it points into the halfspace, where the incident wave propagates. In order to satisfy the boundary conditions discussed in Sec.2.6 at the interface Σ , we must introduce two types of waves in addition to the incident wave: *reflected waves* propagating in the halfspace 1 and *transmited waves* propagating in the halfspace 2. The number of generated waves depends on the type of medium, in which the generated wave propagates. In fluids (acoustic case), only one reflected or transmitted wave will be generated. In isotropic media, two waves will be generated and in anisotropic media, three waves will be generated in each halfspace.

We start the investigation of reflection/transmission process in acoustic media since there the problem is simplest.

3.8.1 Acoustic medium

We consider two fluid halfspaces separated by the plane interface Σ. In this case, the boundary conditions reduce to the requirements of continuity of the pressure p and normal component of the particle velocity v_i across the interface, see Sec.2.6. We denote the phase velocity and density in the halfspace 1 by c_1 and ρ_1 and in the halfspace 2 by c_2 and ρ_2. We denote by index r the parameters corresponding to the reflected wave, by t those corresponding to the transmitted wave. No indices will be used for the incident wave. We thus have

$$\begin{aligned}
p(x_m,t) &= PF(t-p_kx_k)\,, & v_i(x_m,t) &= \rho_1^{-1}Pp_iF(t-p_kx_k)\,,\\
p^r(x_m,t) &= P^rF^r(t-p_k^rx_k-\varphi^r)\,, & v_i^r(x_m,t) &= \rho_1^{-1}P^rp_i^rF^r(t-p_k^rx_k-\varphi^r)\,,\\
p^t(x_m,t) &= P^tF^t(t-p_k^tx_k-\varphi^t)\,, & v_i^t(x_m,t) &= \rho_2^{-1}P^tp_i^tF^t(t-p_k^tx_k-\varphi^t)\,.
\end{aligned}$$

Here φ^r and φ^t denote possible time lags of reflected and transmitted waves with respect to the incident wave. In the above equations, ρ_1, ρ_2, c_1, c_2 are known parameters describing the medium, P, F and p_k are known parameters of the incident wave. The quantities P^r, P^t, F^r, F^t, p_k^r, p_k^t, φ^r, φ^t are to be determined. The two boundary conditions can be now written as

$$PF(t-p_kx_k)+P^rF^r(t-p_k^rx_k-\varphi^r)=P^tF^t(t-p_k^tx_k-\varphi^t)\,,$$

$$\rho_1^{-1}PF(t-p_kx_k)p_ln_l+\rho_1^{-1}P^rF^r(t-p_k^rx_k-\varphi^r)p_l^rn_l=\rho_2^{-1}P^tF^t(t-p_k^tx_k-\varphi^t)p_l^tn_l\,.$$

3.8.1.1 Transformation of the slowness vector across an interface

Because our model is invariant with respect to the translation parallel to the interface Σ, the above conditions must be the same at any point of Σ and for any time. In other words, they cannot depend on the coordinates x_k and time t. Thus at any point of Σ and t, we require

$$F(t - p_k x_k) = F^r(t - p^r_k x_k - \varphi^r) = F^t(t - p^t_k x_k - \varphi^t) \quad .$$

We can see that the analytic signals corresponding to the generated waves are the same as the analytic signal of the incident wave. Moreover, the arguments of the above functions must be equal for any point on Σ,

$$p_k x_k = p^r_k x_k + \varphi^r = p^t_k x_k + \varphi^t \quad .$$

If we consider two points of Σ, x_{1i}, x_{2i}, we get from the above relations

$$p_k(x_{1k} - x_{2k}) = p^r_k(x_{1k} - x_{2k}) = p^t_k(x_{1k} - x_{2k}) \quad .$$

From this, we can immediately conclude that the tangential components of the slowness vectors of the incident and generated waves into the interface Σ are equal. This also implies that $\varphi^r = \varphi^t = 0$ since $p_k x_k = p^r_k x_k = p^t_k x_k$. The slowness vectors can thus be written as

$$p_k = a_k + (p_m n_m) n_k \,, \quad p^r_k = a^r_k + (p^r_m n_m) n_k \,, \quad p^t_k = a^t_k + (p^t_m n_m) n_k \,,$$

where

$$a_k = a^t_k = a^r_k \quad ,$$

a_k, a^t_k, a^r_k being the vectorial tangential components of the slowness vectors p_k, p^r_k and p^t_k into the interface Σ. This can be expressed as

$$p_k - (p_m n_m) n_k = p^t_k - (p^t_m n_m) n_k = p^r_k - (p^r_m n_m) n_k \,.$$

We could determine p^r_k and p^t_k from the above equality if the components of these vectors into the normal to Σ are known. For this purpose, we can use the eikonal equation

$$p_i p_i = c^{-2} \quad ,$$

which must be satisfied on both sides of the interface for all the three waves. For the incident and transmitted waves, we get

$$\begin{aligned} p_i p_i &= a_i a_i + (p_m n_m)^2 = c_1^{-2} \quad , \\ p^t_i p^t_i &= a_i a_i + (p^t_m n_m)^2 = c_2^{-2} \quad . \end{aligned}$$

From the second equation we obtain

$$p^t_m n_m = -(c_2^{-2} - a_i a_i)^{1/2} = -[c_2^{-2} - c_1^{-2} + (p_m n_m)^2]^{1/2} \quad .$$

The minus sign in front of the square root corresponds to our choice of the orientation of the normal n_i to the interface Σ; n_i points against p_i and p^t_i. Now we can use the equality of the tangential components of the slowness vector and combine it with the above equation to get

$$p^t_k = p_k - \{(p_m n_m) + [c_2^{-2} - c_1^{-2} + (p_m n_m)^2]^{1/2}\} n_k \quad .$$

Similarly, for the reflected wave we obtain

$$p^r_m n_m = +(c_1^{-2} - a_i a_i)^{1/2} = [(p_m n_m)^2]^{1/2} = |p_m n_m| = -p_m n_m$$

and thus

$$p^r_k = p_k - 2(p_m n_m) n_k \quad .$$

The expressions for p^t_k and p^r_k, which we found, are very general, they can be used even in the study of wave propagation in slightly inhomogeneous media with curved interfaces by the zero-order approximation of the ray method, see Chap.7. Let us note that the formula for the slowness vectors of the transmitted wave yields real slowness vector only if

$$c_2^{-2} - c_1^{-2} + (p_m n_m)^2 \geq 0 \quad .$$

We shall show later that this is the condition of subcritical incidence of the plane wave at the interface.

Let us discuss some interesting consequences of the above derivations. We found that the components of the slowness vectors tangent to the interface are equal,

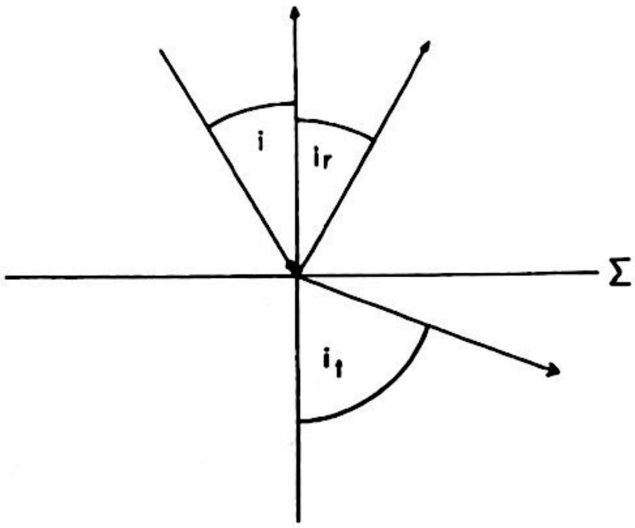
$$p_k - (p_m n_m) n_k = p^g_k - (p^g_m n_m) n_k \quad ,$$

where g stands either for r (reflected wave) or t (transmitted wave). If we denote by i the acute angle between the normal n_i to Σ and the slowness vector p_i of the incident wave and by i_g the acute angle between n_i and the slowness vector of the generated wave (see picture) and take squares on both sides of the above equation, we get

$$c_1^{-2}(1 - \cos_i^2) = c_g^{-2}(1 - \cos^2 i_g) \quad .$$

Since i and i_g are acute angles, this yields

$$\frac{\sin i}{c_1} = \frac{\sin i_g}{c_g} \quad .$$



We can see that the formulae which we derived above automatically incorporate the *Snell's law.* The angles i and i_g are the *angles of incidence* and of *reflection/transmission.* For the case of reflection, the Snell's law yields $i_r = i$.

We can now rewrite the condition for the existence of real-valued slowness vector of the transmitted wave in terms of the angles of incidence and trasnmission i and i_t. The condition reads

$$c_2^{-2} - c_1^{-2} + c_1^{-2} \cos^2 i = c_2^{-2} \cos^2 i_t \geq 0 \quad .$$

This can be rewritten as

$$\frac{c_2}{c_1} \sin i = \sin i_t \leq 1 \, .$$

This inequality shows what we could expect. It guarantees not only the existence of the real slowness vector of the transmitted wave but also the real angle of transmission i_t.

For $\sin i_t = 1$, which implies $c_1 \leq c_2$, we have

$$\sin i^* = \frac{c_1}{c_2} \, .$$

The angle of incidence i^* is called the *critical angle.* Corresponding angle of transmission is $i_t = \frac{1}{2}\pi$, i.e. the corresponding transmitted wave propagates parallelly with the interface Σ. The incidence of a plane wave at the interface Σ with an angle $i < i^*$ is referred to as the *subcritical incidence* while for $i > i^*$, the *overcritical incidence.*

For

$$c_2^{-2} - c_1^{-2} + c_1^{-2} \cos^2 i = c_2^{-2} \cos^2 i_t < 0 \quad ,$$

which is equivalent to

$$\sin i > \frac{c_1}{c_2} \quad ,$$

the slowness vector of the transmitted wave as well as the angle of transmission become complex. We can see from the above definition that this situation occurs in case of the overcritical incidence. The slowness vector of the transmitted wave can be now written in the following form

$$p_k^t = p_k^{tR} + ip_k^{tI} \quad ,$$

where

$$\begin{aligned} p_k^{tR} &= p_k - (p_m n_m) n_k \quad , \\ p_k^{tI} &= \pm - [c_1^{-2} - c_2^{-2} - (p_m n_m)^2]^{1/2} n_k \quad . \end{aligned}$$

We have shown in Sec.3.7 that such a slowness vector belongs to an inhomogeneous wave. Before we discuss its properties, let us make a choice of the proper sign in front of the square root. The sign must be chosen such that the amplitude of the transmitted inhomogeneous plane wave decreases with increasing distance from the interface. We make the choice of the sign for simplicity for a time harmonic inhomogeneous acoustic plane wave of pressure. We can then extend the result to transient waves if we take into account that a transient plane wave can be expressed in terms of an integral over time harmonic waves. The transmitted inhomogeneous acoustic plane wave of pressure has the form, see Sec.3.7,

$$p^t(x_m, t) = P^t \exp\left(-\omega p_m^{tI} x_m\right) \exp\left[-i\omega(t - p_m^{tR} x_m)\right] \quad .$$

This expression describes physically acceptable inhomogeneous wave if

$$p_m^{tI} x_m > 0$$

in the halfspace 2, i.e., if the term

$$\pm[c_1^{-2} - c_2^{-2}(p_m n_m)^2]^{1/2} n_k x_k$$

is positive. Since in the halfspace 2,

$$n_k x_k < 0 \quad ,$$

(remember, n_k points into the halfspace 1) $p_m^{tI} x_m$ is positive if we choose the *negative sign* in front of the square root.

From the expressions for p_k^{tR} we immediately see that the vector p_k^{tR} is parallel to the interface Σ, i.e. the inhomogeneous wave propagates along Σ with the phase front perpendicular to Σ. For its phase velocity we can now find

$$c_R^{-2} = p_i^{tR} p_i^{tR} = c_2^{-2} + p_i^{tI} p_i^{tI} = c_2^{-2} + c_1^{-2} - c_2^{-2} - c_1^{-2} \cos^2 i = \frac{\sin^2 i}{c_1^2} \, .$$

Thus,

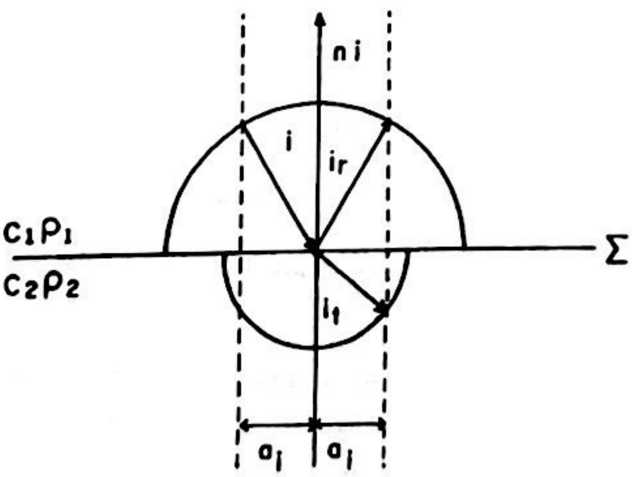
$$c_R = c_1 / \sin i \quad .$$

The phase velocity c_R of the transmitted inhomogeneous wave corresponds to the apparent velocity of the phase front of the incident wave with which it moves along Σ. It decreases with increasing i, from $i = i^*$, where it is equal to the velocity of the halfspace 2, $c_R = c_2$, to $i = \frac{1}{2}\pi$, for which it approaches the velocity of the first halfspace, $c_R = c_1$. Thus, the phase velocity c_R of the inhomogeneous wave can attain any value of the velocity from the interval (c_1, c_2) in dependence on the angle of incidence.

The vector p_i^{tI} is perpendicular to the interface, i.e. the direction of the largest decay of amplitudes is against the normal n_i to the interface Σ. For overcritically incident waves the decay increases with increasing angle of incidence. For $i = i^*$, $p_k^{tI} = 0$, i.e. there is no decay and we are dealing with the homogeneous transmitted wave propagating along the interface Σ with phase velocity $c_R = c_2$. For $i = \frac{1}{2}\pi$, $p_k^{tI} = -(c_1^{-2} - c_2^{-2})^{(1/2)} n_k$.

When we check the equations for the determination of the slowness vectors of the generated waves, we can see that all the slowness vectors are situated in a plane perpendicular to the interface Σ and specified by the vectors n_i and p_i. This plane is called the *plane of incidence*. The derivations connected with the reflection/transmission are usually performed in this plane. We have derived reflection/transmission formulae without using the concept of the plane of incidence.

The above analytic derivations are often performed using geometrical reasoning. It is especially common in case of interfaces between anisotropic media. For acoustic case the derivation would be as shown in the picture. The slowness vectors of reflected and transmit-



ted waves are sought as slowness vectors having the same projection into Σ as the slowness vector of the incident wave.

3.8.1.2 Coefficients of reflection/transmission

Due to the equality $F(\xi) = F^r(\xi) = F^t(\xi)$ along the interface Σ, the boundary conditions yield

$$P + P^r = P^t \quad , \quad \rho_1^{-1} P p_l n_l + \rho_1^{-1} P^r p_l^r n_l = \rho_2^{-1} p_l^t n_l P^t \, .$$

These are two inhomogeneous algebraic equations for two unknowns P^r and P^t. All the remaining quantities are known or can be determined in the way shown in the preceeding section.

Since $p_k^r = p_k - 2(p_m n_m) n_k$,

$$p_k^r n_k = -p_k n_k \quad .$$

This can be introduced into the second equation above. Instead of P^r and P^t, we introduce

$$R_P^r = P^r / P \quad , \quad R_P^t = P^t / P \quad .$$

We call the quantities R_P^r and R_P^t the *reflection and transmission coefficients* for pressure. The above two equations can be thus rewritten

$$R_P^r - R_P^t = -1$$
$$\rho_1^{-1} p_l n_l R_P^r + \rho_2^{-1} p_l^t n_l R_P^t = \rho_1^{-1} p_l n_l \, .$$

Their solution is

$$R_P^r = \frac{\rho_2 p_l n_l - \rho_1 p_l^t n_l}{\rho_2 p_l n_l + \rho_1 p_l^t n_l} \quad , \quad R_P^t = \frac{2\rho_2 p_l n_l}{\rho_2 p_l n_l + \rho_1 p_l^t n_l} \quad .$$

If we use the angles of incidence and reflection/transmission introduced above, we get

$$R_P^r = \frac{\rho_2 c_2 \cos i - \rho_1 c_1 \cos i_t}{\rho_2 c_2 \cos i + \rho_1 c_1 \cos i_t} \quad , \quad R_P^t = \frac{2\rho_2 c_2 \cos i}{\rho_2 c_2 \cos i + \rho_1 c_1 \cos i_t} \quad .$$

Let us note that the reflection/transmission coefficients introduced above relate pressure of the reflected/transmitted wave to the pressure of the incident wave

$$p^r = R_P^r p \quad , \quad p^t = R_P^t p \quad ,$$

and are, therefore, called the *pressure reflection/transmission coefficients*. In a similar way we could also introduce *particle velocity coefficients* R^r, R^t by requiring

$$A^r = R^r A \quad , \quad A^t = R^t A \quad ,$$

where A, A^r and A^t are defined as follows

$$v_i = A N_i F(t - p_k x_k) \, , \quad A = \rho_1^{-1} P c_1^{-1} \, ,$$
$$v_i^r = A^r N_i^r F(t - p_k^r x_k) \, , \quad A^r = \rho_1^{-1} R_P^r P c_1^{-1} \, ,$$
$$v_i^t = A^t N_i^t F(t - p_k^t x_k) \, , \quad A^t = \rho_1^{-1} R_P^t P c_2^{-1} \, .$$

We can simply find that

$$R^r = A^r/A = \rho_1^{-1} R_P^r P c_1^{-1} / \rho_1^{-1} P c_1^{-1} = R_P^r ,$$
$$R^t = A^t/A = \rho_2^{-1} R_P^t P c_2^{-1} / \rho_1^{-1} P c_1^{-1} = \rho_2^{-1} c_2^{-1} \rho_1 c_1 R_P^t .$$

Thus the particle velocity reflection coefficient R^r is the same as the pressure reflection coefficient R_P^r. The particle velocity transmission coefficient is

$$R^t = \frac{2\rho_1 c_1 \cos i}{\rho_2 c_2 \cos i + \rho_1 c_1 \cos i_t} \quad .$$

In the following, we shall briefly investigate the properties of the R^r and R^t coefficients. For this purpose, it is useful to introduce the following notation:

$$n = c_1/c_2 \text{ (refraction index)}, \quad m = \rho_2/\rho_1 \quad .$$

It is also useful to expres the R^r and R^t coefficients only in terms of the angle of incidence i. Then the coefficients read

$$R^r = \frac{m \cos i - (n^2 - \sin^2 i)^{1/2}}{m \cos i + (n^2 - \sin^2)^{1/2}} \quad , \quad R^t = \frac{2n \cos i}{m \cos i + (n^2 - \sin^2)^{1/2}} .$$

3.8.1.3 Properties of the reflection/transmission coefficients

In case of incidence of a wave on an interface under the angle $i = 0$, we speak about *normal incidence*. The Snell law yields $i_t = i_r = 0$, the laws of the transmission/reflection of the slowness vectors yield

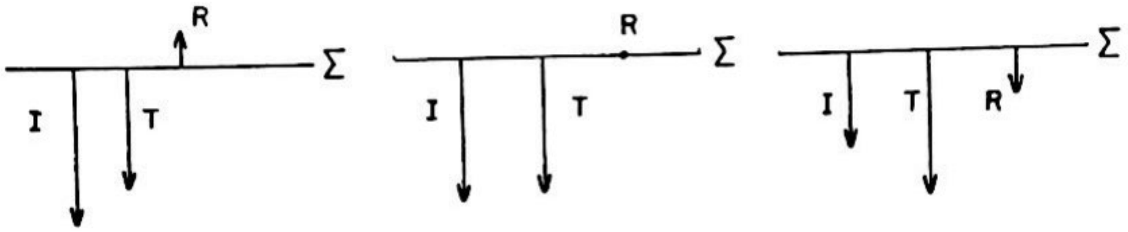
$$p_k^r = -p_k \quad , \quad p_k^t = -n_k/c_2 \quad .$$

The R^r, R^t coefficients attain the form:

$$R^r = \frac{m - n}{m + n} = \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} \quad ,$$

$$R^t = \frac{2n}{m + n} = \frac{2\rho_1 c_1}{\rho_2 c_2 + \rho_1 c_1} \quad .$$

The coefficients depend on the products of ρ and c only, which are called the *characteristic impedancies*. The following picture shows schematically particle velocity vectors for the following three situations (I - incident, T - transmitted, R - reflected wave):



a) $\rho_2 c_2 > \rho_1 c_1$ $R^r > 0$ b) $\rho_2 c_2 = \rho_1 c_1$ $R^r = 0$ c) $\rho_2 c_2 < \rho_1 c_1$ $R^r < 0$

Let us note that exactly the same reflection/transmission coefficients will be obtained for normal incidence of P wave in an isotropic medium.

In case $c_1 = c_2$ but $\rho_1 \neq \rho_2$ $(n = 1,\ m \neq 1)$, the reflection/transmission coefficients attain very simple form (since $i_t = i$)

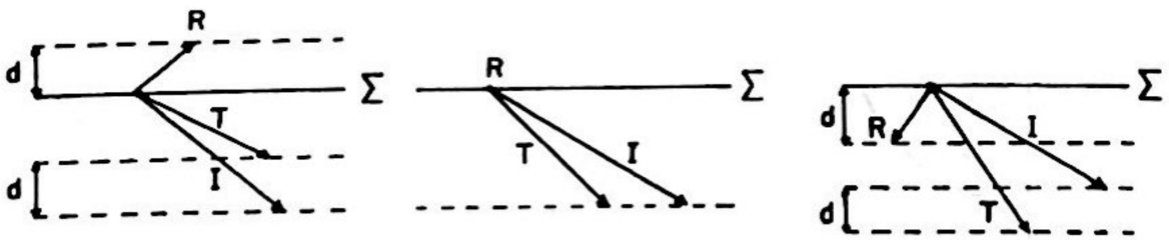
$$R^r = \frac{m-1}{m+1} \quad , \quad R^t = \frac{2}{m+1} \quad .$$

In this case, the coefficients do not depend on the angle of incidence.

In case that

$$m \cos i - n \cos i_t = m \cos i - (n^2 - \sin^2 i)^{1/2} = 0 \quad ,$$

the reflection coefficient becomes zero (case *b)* for normal incidence was a special case). This phenomenon is called the *total transparency of the interface*. The following picture shows schematically the particle velocity vectors in three possible situations (notation as before):



a) $m \cos i > n \cos i_t$ $R^r > 0$ b) $m \cos i = n \cos i_t$ $R^r = 0$ c) $m \cos i < n \cos i_t$ $R^r < 0$

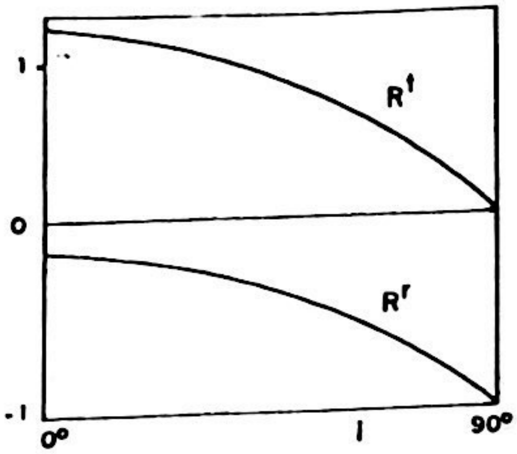
From the above equation, we get the angle i for which the total transparency can occur:

$$\tan^2 i = (m^2 - n^2)/(n^2 - 1) \quad .$$

Thus the angle of the total transparency is real when either $m > n > 1$ or $m < n < 1$.

Differentiating the expressions for the R/T coefficients with respect to the angle of incidence we would find:

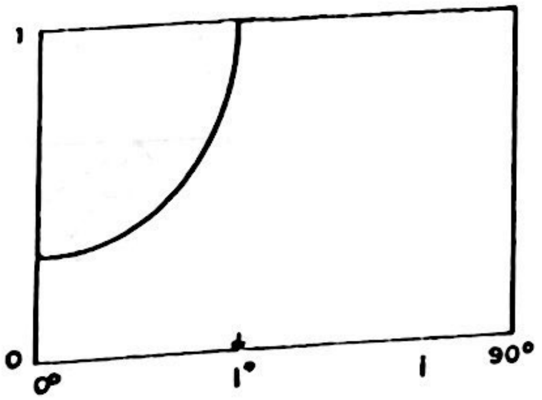
a) for $n > 1 (c_1 > c_2)$: the R^r and R^t coefficients are monotonously decreasing functions of the incident angle i; see the bottom schematic picture.

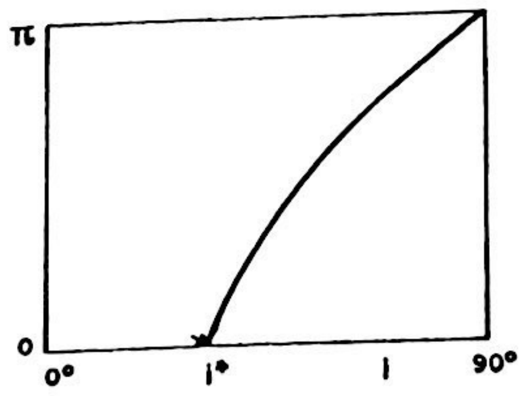


b) for $n < 1 (c_1 < c_2)$: the coefficients are monotonously increasing, but only in the interval $0 \le i \le i^*$. For $i > i^*$, the coefficients of the reflection/transmission become complex-valued since i_t becomes complex. The reflection coefficient R^r for $i > i^*$ attains the form

$$R^r = \frac{m \cos i - i(\sin^2 i - n^2)^{1/2}}{m \cos i + i(\sin^2 i - n^2)^{1/2}} \quad ,$$

see the bottom schematic pictures.





We took the sign in front of the square root in accordance with its previous choice when we determined the slowness vector of transmitted inhomogeneous wave. For $i > i^*$, i.e. for i_t complex, we get

$$\cos i_t = -c_2(p_k^t n_k) = -ic_2(p_k^{tI} n_k) = ic_2(c_1^{-2} - c_2^{-2} - c_1^{-2} \cos^2 i)^{1/2} = in^{-1}(\sin^2 i - n^2)^{1/2} .$$

The reflection coefficient can also be rewritten in the following form:

$$R^r = |R^r|e^{i\varphi_r} \quad .$$

It can be easily shown that

$$|R^r| = 1 \quad , \quad \varphi_r = -2 \text{ atan} \left[\frac{(\sin^2 i - n^2)^{1/2}}{m\cos i}\right] \, .$$

We can see that *the overcritical reflection is total* since the modulus of the reflection coefficient is equal to 1. Due to the minus sign in the expression for φ_r, the phase shift caused by the complex-valued reflection coefficient is the *phase delay*.

The transmission coefficient is also complex. Its modulus varies from $2n/m$ for $i = i^*$ to zero for $i \to \pi/2$. The phase shift φ_t represents half of the phase shift of the reflection coefficiet, $\varphi_t = \frac{1}{2}\varphi_r$.

3.8.1.4 Energetic considerations

As was shown in Sec.3.4.1, the time integrated energy flux of an acoustic plane wave is

$$\hat{S}_i = \rho^{-1} p_i PP^* f_a \quad ,$$

where $f_a = \int_{-\infty}^{+\infty} g^2 dt$. We shall now consider also situations, in which p_i becomes complex (overcritical incidence). Then the above equation reads

$$\hat{S}_i = \rho^{-1} p_i^R PP^* f_a \quad ,$$

where p_i^R is the real part of the slowness vector p_i, $p_i = p_i^R + ip_i^I$. The energy flux trough a unit element of the interface Σ with the unit normal n_i is thus

$$|\hat{S}_i n_i| = \rho^{-1}|p_i n_i| PP^* f_a = \rho^{-1} c^{-1} \cos i PP^* f_a \quad \text{if } Im\, p_i = 0 \, ,$$

$$|\hat{S}_i n_i| = \rho^{-1}|p_i^R n_i| PP^* f_a = 0 \quad \text{if } Im\, p_i \neq 0 \, .$$

The second relation results from the fact that $p_i^R \perp n_i$ see Sec.3.8.1.1.

We can now introduce the *energy reflection/transmision coefficients* R_B by

$$R_B^r R_B^{r*} = |\hat{S}_i^r n_i| / |\hat{S}_i n_i| \quad , \quad R_B^t R_B^{t*} = |\hat{S}_i^t n_i| / |\hat{S}_i n_i| \, .$$

This immediately yields

$$\begin{aligned}
R_B^r R_B^{r*} &= \rho_1^{-1} c_1^{-1} \cos i (P^r P^{r*}) f_a / \rho_1^{-1} c_1^{-1} \cos i (PP^*) f_a = R_P^r R_P^{r*} \\
R_B^t R_B^{t*} &= \rho_2^{-1} c_2^{-1} \cos i (P^t P^{t*}) f_a / \rho_1^{-1} c_1^{-1} \cos i (PP^*) f_a \\
&= R_P^t R_P^{t*} (\rho_1 c_1 \cos i_t)/(\rho_2 c_2 \cos i) \qquad \text{if } Im\, i_t = 0 \, , \\
R_B^t R_B^{t*} &= 0 \qquad \text{if } Im\, i_t \neq 0 \, .
\end{aligned}$$

In this way,

$$R_E^r = R_P^r = R^r = \frac{\rho_2 c_2 \cos i - \rho_1 c_1 \cos i_t}{\rho_2 c_2 \cos i + \rho_1 c_1 \cos i_t} ,$$

$$R_E^t = R_P^t \left(\frac{\rho_1 c_1 \cos i_t}{\rho_2 c_2 \cos i} \right)^{1/2} = \frac{2(\rho_1 c_1 \rho_2 c_2 \cos i \cos i_t)^{1/2}}{\rho_2 c_2 \cos i + \rho_1 c_1 \cos i_t} \quad \text{if } Im\, i_t = 0 ,$$

$$R_E^t = 0 \quad \text{if } Im\, i_t \neq 0 .$$

The coefficients R_E^r, R_E^t have very convenient property. If we interchange the incident and generated wave, we get the same R_E^r and R_E^t coefficients. This is true even for R_P^r and R^r but not for R_P^t and R^t coefficients.

Let us now determine the energy flux through a unit element of the interface Σ of the reflected and transmitted wave. We get

$$|\hat{S}_i^r n_i| + |\hat{S}_i^t n_i| = [(R_E^r R_E^{r*} + (R_E^t R_E^{t*})]|\hat{S}_i n_i| =$$

$$= |\hat{S}_i n_i| \frac{(\rho_2 c_2 \cos i + \rho_1 c_1 \cos i_t)^2}{(\rho_2 c_2 \cos i + \rho_1 c_1 \cos i_t)^2} = |\hat{S}_i n_i| \quad \text{for } Im\, i_t = 0 ,$$

$$|\hat{S}_i^r n_i| + |\hat{S}_i^t n_i| = (R_E^r R_E^{r*})|\hat{S}_i n_i| = |\hat{S}_i n_i| \quad \text{for } Im\, i_t \neq 0 .$$

The result $(R_E^r R_E^{r*}) = 1$ for $Im\, i_t \neq 0$ simply follows from our observation that $|R_E^r| = |R^r| = 1$. The above equations yield

$$|\hat{S}_i^r n_i| + |\hat{S}_i^t n_i| = |\hat{S}_i n_i| , \quad (R_E^r R_E^{r*}) + (R_E^t R_E^{t*}) = 1 .$$

We proved energy conservation during the process of reflection/transmission. The first equation above shows that the total energy flowing to the interface Σ is carried away from the interface by reflected and transmitted waves.

In case of the overcritical incidence, the energy flux connected with inhomogeneous transmitted wave is zero because the vector $\hat{S}_i$ is parallel to Σ, see above. This means that in this case, the trasmitted wave does not carry energy away from the interface. Energy of the incident wave is now totally reflected, $(R_E^r R_E^{r*}) = 1$. Due to the existence of the inhomogeneous transmitted wave in the lower halfspace, the reflection coefficient R^r contains the phase shift. The energy flux parallel to the interface may differ in both halfspaces.

3.8.1.5 Waveform changes of generated waves

The complex-valued reflection and transmission coefficients cause waveform changes of the studied waves. As explained before, only real parts of the calculated expressions for pressure or particle velocity have physical meaning. For the particle velocity of reflected wave we got

$$v_i^r(x_m, t) = \rho_1^{-1} R^r P p_i^r F(t - p_k^r x_k) \quad .$$

Its real part is given by the expression

$$Re[v_i^r(x_m, t)] = \rho_1^{-1} P p_i^r [Re(R^r) g(t - p_k^r x_k) - Im(R^r) h(t - p_k^r x_k)] \, .$$

In case of subcritical reflection, when R^r is real-valued, the above formula yields

$$Re[v_i^r(x_m, t)] = \rho_1^{-1} R^r P p_i^r g(t - p_k^r x_k) \quad ,$$

and we can see that the particle motion of subcritically reflected wave has the same form as the particle motion of the incident wave.

In case of overcritical reflection when R^r is complex, the form of the reflected wave differs from the form of the incident wave. In fact, the form is given by a linear combination of the incident wave and its Hilbert transform.

The situation is even more complicated in case of an inhomogeneous transmitted wave. In Sec.3.7, we have shown that such a wave has an elliptical or quasielliptical polarization in the plane perpendicular to the interface and the phase front. Moreover, due to the complexity of p_i^t, also the argument of the analytic signal is complex-valued. Thus, a generalized form of the analytic signal with a complex-valued argument must be used.

3.8.2 Isotropic medium

Now consider two *homogeneous isotropic halfspaces* separated by a plane interface Σ. The orientation of the normal to Σ is again specified so that it points into the halfspace, where incident wave propagates. We consider an arbitrary coordinate system with its origin situated at the interface Σ. The density and P and S wave velocities in the halfspace 1 are ρ_1, α_1, β_2 and in halfspace 2, ρ_2, α_2, β_2. Two vectorial (it is six scalar) boundary conditions are to be satisfied: continuity of the displacement vector and continuity of the traction (stress vector) across the interface Σ. To satisfy these conditions, we consider two reflected (P and S) and two transmitted (P and S) waves generated by an incidence of P or S plane wave at Σ. We call the generated waves which are of the same type as the incident wave *unconverted waves*, the other waves are *converted waves*. The displacement of the incident P wave is

$$u_i(x_m, t) = A N_i F(t - p_k x_k) \quad ,$$

the displacement of the incident S wave is

$$u_i(x_m, t) = (B g_i^{(1)} + C g_i^{(2)}) F(t - p_k x_k) \quad .$$

The vectors $g_i^{(1)}$, $g_i^{(2)}$, N_i are three mutually perpendicular unit vectors, N_i is perpendicular to the phase front, $g_i^{(1)}$ and $g_i^{(2)}$ are situated in the plane of the phase front. A characterizes the size of the displacement vector of the incident P wave, B and C characterize the sizes of the projections of the incident S wave displacement vector into the vectors $g_i^{(1)}$ and $g_i^{(2)}$. We use the same expressions as above for corresponding generated waves. We denote again the parameters belonging to the reflected waves by upper index r and those of the transmitted wave by t. Thus, quantities A or B, C, F and p_k are supposed to be known, the quantities A^r, B^r, C^r, A^t, B^t, C^t, F^r, F^t, p_k^r and p_k^t are to be determined.

The expression for the traction can be written as follows, see Sec.2.3:

$$T_i = \tau_{ij}n_j = \lambda n_i u_{k,k} + \mu n_j(u_{i,j} + u_{j,i}) \quad .$$

For the incident P wave we get in this way

$$T_i = -A(\lambda_1 n_i N_k p_k + 2\mu_1 n_j p_j N_i)\dot{F}(t - p_k x_k)$$

and for the incident S wave we get

$$T_i = -[B(g_i^{(1)}p_j + g_j^{(1)}p_i) + C(g_i^{(2)}p_j + g_j^{(2)}p_i)]\mu_1 n_j \dot{F}(t - p_k x_k)\,.$$

We can now write the six boundary conditions as follows

$$\begin{aligned} A^t N_i^t F_p^t + B^t g_i^{(1)t} F_s^t + C^t g_i^{(2)t} F_s^t - A^r N_i^r F_p^r - B^r g_i^{(1)r} F_s^r - C^r g_i^{(2)r} F_s^r &= D_i\,, \\ A^t X_i^t \dot{F}_p^t + B^t Y_i^t \dot{F}_s^t + C^t Z_i^t \dot{F}_s^t - A^r X_i^r \dot{F}_p^r - B^r Y_i^r \dot{F}_s^r - C^r Z_i^r \dot{F}_s^r &= E_i\,. \end{aligned}$$

The first equation results from the continuity of the displacement, the second from the continuity of the traction. The following notation was used:

$$\begin{aligned} X_i &= \lambda n_i N_k p_k + 2\mu n_j p_j N_i \quad , \\ Y_i &= \mu n_j(g_i^{(1)}p_j + g_j^{(1)}p_i) \quad , \\ Z_i &= \mu n_j(g_i^{(2)}p_j + g_j^{(2)}p_i) \quad . \end{aligned}$$

The terms D_i and E_i are given as follows:

$$D_i = AN_iF_p \quad , \quad E_i = AX_i\dot{F}_p$$

in case of the incident P wave and

$$D_i = BF_s g_i^{(1)} + CF_s g_i^{(2)} \quad , \quad E_i = BY_i\dot{F}_s + CZ_i\dot{F}_s$$

in case of the incident S wave.

3.8.2.1 Transformation of slowness vectors across an interface

By the same argument as in the acoustic case, we can find that analytic signals and their derivatives corresponding to the generated waves are the same as the analytic signal and its derivative corresponding to the incident wave at any point of the interface Σ. In exactly the same way as in the acoustic case, we can also find the equation for the determination of the slowness vector $\tilde{p}_i$ of any generated wave

$$\tilde{p}_i = p_i - \{(p_m n_m) \pm [\tilde{V}^{-2} - V^{-2} + (p_m n_m)^2]^{1/2}\} n_i \quad ,$$

see Sec.3.8.1.1. Here "+" corresponds to the transmitted wave, "−" to the reflected wave. The symbols V and $\tilde{V}$ denote the velocity α or β of the incident wave and of the generated waves, respectively. The above equation holds for any reflected/transmitted wave. In case of the reflection of an unconverted wave, when $\tilde{V} = V$, we get simpler formula

$$\tilde{p}_k = p_k - 2(p_m n_m) n_k \quad ,$$

see also Sec.3.8.1.1.

The only condition which must be satisfied by the vectors $g_i^{(1)r}$, $g_i^{(2)r}$, $g_i^{(1)t}$, $g_i^{(2)t}$, is the condition of mutual perpendicularity of the vectors $g_i^{(1)}$, $g_i^{(2)}$ and p_i. In the next section, we show that for a special choice of $g_i^{(1)r}$, $g_i^{(2)r}$, $g_i^{(1)t}$, $g_i^{(2)t}$, the boundary conditions yield relatively simple system of equations which can be separeted into two indenpendent systems and the corresponding reflection/transmission coefficients can be expressed analytically.

We can introduce the angles of incidence, transmission and reflection to derive the Snell law for the case of isotropic media. The above formulae now yield

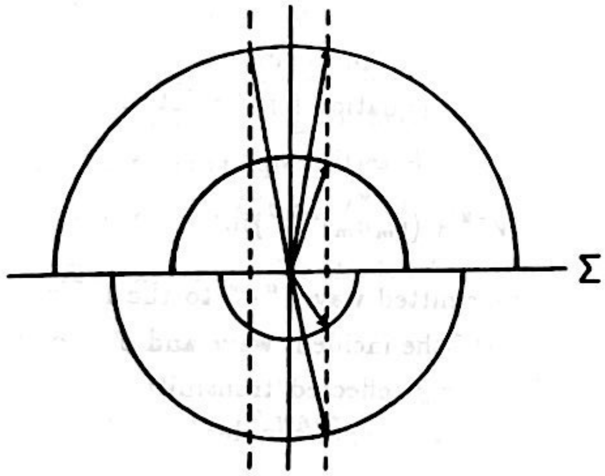
$$\frac{\sin i_p^r}{\alpha_1} = \frac{\sin i_s^r}{\beta_1} = \frac{\sin i_p^t}{\alpha_2} = \frac{\sin i_s^t}{\beta_2} = \frac{\sin i}{V} \quad .$$

If $\alpha_1 < \beta_2$ and $\alpha_1 < \alpha_2$, for an incident P wave, two critical angles will exist,

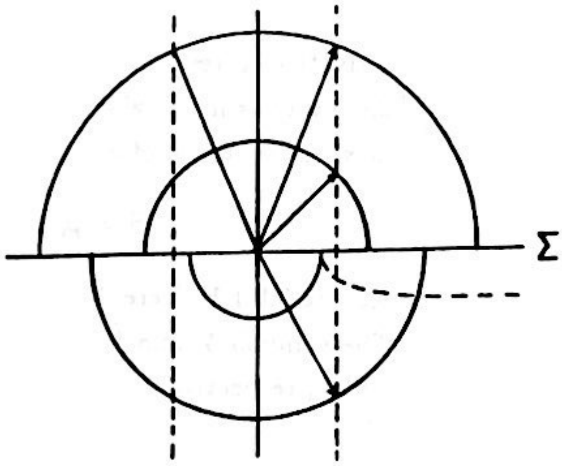
$$\sin i_1^* = \alpha_1/\alpha_2 \quad , \sin i_2^* = \alpha_1/\beta_2 \quad .$$

For an incident S wave even three critical angles can exist if $\beta_1 < \alpha_1, \beta_1 < \beta_2, \beta_1 < \alpha_2$. Overcritical incidence is connected with generation of inhomogeneous transmitted and also reflected waves. Their behaviour is similar to the behaviour of inhomogeneous acoustic waves, see also Sec.3.7. The situation with the three critical angles in case of incidence of an S wave at an interface separating two halfspaces with velocities $\beta_1 < \beta_2 < \alpha_1 < \alpha_2$ is shown in the following pictures. The pictures show slowness surfaces of the generated waves for angles of incidence in different ranges.

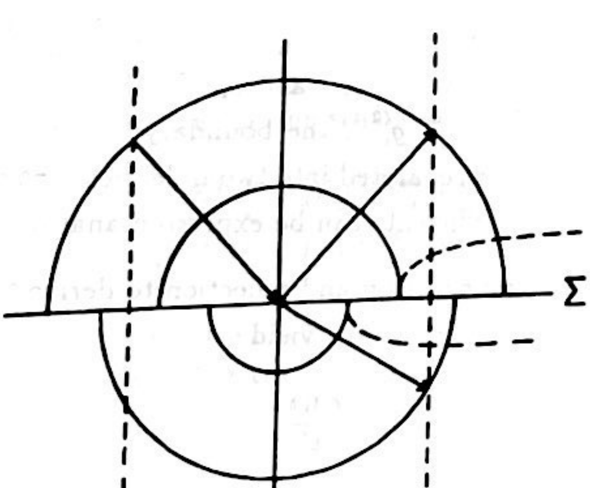
S wave incidence: $\beta_1 < \beta_2 < \alpha_1 < \alpha_2$



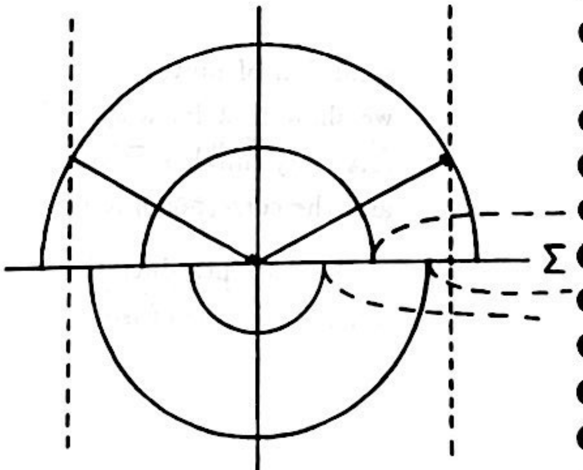
$\sin i < \beta_1/\alpha_2$



$\sin i > \beta_1/\alpha_2, \quad \sin i < \beta_1/\alpha_1$



$\sin i > \beta_1/\alpha_1, \quad \sin i < \beta_1/\beta_2$



$\sin i > \beta_1/\beta_2$

Dotted line in the above figures shows the magnitude of the vector p_k^I (perpendicular to Σ) as a function of p_k^R.

3.8.2.2 Coefficients of reflection/transmission

Due to the equality of analytic signals of incident and generated waves along Σ, the boundary conditions reduce to the following system of six inhomogeneous linear equations for six unknowns, A^r, B^r, C^r, A^t, B^t and C^t,

$$A^t N_i^t + B^t g_i^{(1)t} + C^t g_i^{(2)t} - A^r N_i^r - B^r g_i^{(1)r} - C^r g_i^{(2)r} = \bar{D}_i \ ,$$
$$A^t X_i^t + B^t Y_i^t + C^t Z_i^t - A^r X_i^r - B^r Y_i^r - C^r Z_i^r = \bar{E}_i \ ,$$

where

$$\bar{D}_i = AN_i \quad , \quad \bar{E}_i = AX_i$$

for the P wave incident at Σ and

$$\bar{D}_i = Bg_i^{(1)} + Cg_i^{(2)} \quad , \quad \bar{E}_i = BY_i + CZ_i$$

for the S wave incident at Σ.

We can introduce the *displacement reflection/transmission coefficients* R^r_{mn}, R^t_{mn}, $m,n = 1,2,3$, where the index m specifies the type of the incident wave and the index n the type of the generated wave in the following way:

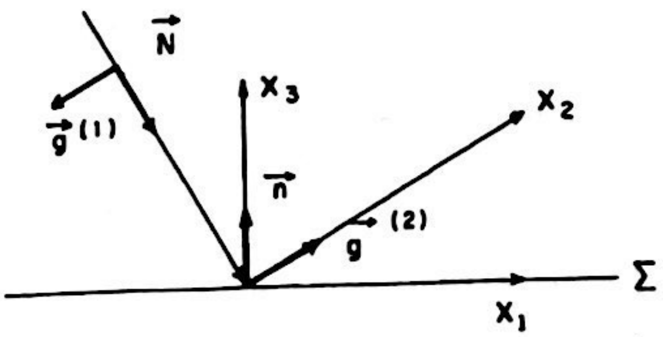
$m,n = 1$: $S1$ component of the S wave;

$m,n = 2$: $S2$ component of the S wave;

$m,n = 3$: P wave.

By $S1$ and $S2$ components of the S wave we understand the components of the S wave into $g_i^{(1)}$ and $g_i^{(2)}$ vectors, respectively.

Using the above notation, the reflection coefficient $R^r_{31} = B^r/A$ corresponds to the incident P wave and reflected $S1$ component of the S wave. In a whole, we have 9 reflection and 9 transmission coefficients. Rather than to look for analytic expressions of the coefficients, it is reasonable to solve the system of the above 6 equations for the coefficients of reflection and transmission numerically.



The system of equations can be, however, simplified if we choose the coordinate system in a special way and if we also choose a special orientation of the vectors $g_i^{(1)}$, $g_i^{(2)}$. Let us choose the coordinate axes x_1, x_2 in the plane Σ, x_1 axis in the plane of incidence and the vectors $g_i^{(2)}$ for all incident and generated waves perpendicular to the plane of incidence. Then we have P wave displacement vector and the $S1$ component of the S wave situated in the plane of incidence, i.e. in the plane (x_1, x_3). The $S2$ component of the S wave is perpendicular to the plane of incidence, i.e. parallel to the axis x_2. This specification yields

$$\vec{N} \equiv (N_1, 0, N_3)\,,\; \vec{n} \equiv (0,0,1)\,,\; \vec{g}^{(1)} \equiv (g_1^{(1)}, 0, g_3^{(1)})\,,\; \vec{g}^{(2)} \equiv (0,1,0)\,.$$

If we insert the above specifications into the equations of continuity, the equations of continuity will split into two systems. The first one contains generated P waves and S waves with $S1$ component and has the form:

$$\begin{aligned}
A^t N_1^t + B^t g_1^{(1)t} - A^r N_1^r - B^r g_1^{(1)r} &= \bar{D}_1\,,\\
A^t N_3^t + B^t g_3^{(1)t} - A^r N_3^r - B^r g_3^{(1)r} &= \bar{D}_3\,,\\
A^t X_1^t + B^t Y_1^t - A^r X_1^r - B^r Y_1^r &= \bar{E}_1\,,\\
A^t X_3^t + B^t Y_3^t - A^r X_3^r - B^r Y_3^r &= \bar{E}_3\,.
\end{aligned}$$

The second system contains generated S waves with the components $S2$:

$$\begin{aligned}
C^t - C^r &= \bar{D}_2 \quad ,\\
C^t Z_2^t - C^r Z_2^r &= \bar{E}_2 \quad .
\end{aligned}$$

The vectors X_i, Y_i, Z_i have now the following form:

$$\begin{aligned}
&X_1 = 2\mu p_3 N_1\,, && X_2 = 0\,, && X_3 = \lambda(p_1 N_1 + p_3 N_3) + 2\mu p_3 N_3\,,\\
&Y_1 = \mu(g_1^{(1)} p_3 + g_3^{(1)} p_1)\,, && Y_2 = 0\,, && Y_3 = 2\mu g_3^{(1)} p_3\,,\\
&Z_1 = 0\,, && Z_2 = \mu p_3\,, && Z_3 = 0\,.
\end{aligned}$$

For an incident P wave, we have

$$\bar{D}_2 = \bar{E}_2 = 0 \quad .$$

Similar result can be obtained for an incident S wave with $S1$ component. In this case the right-hand sides of the second system of equations are zero and its solution is also zero. This means that an incident P wave or S wave with the $S1$ component does not generate an S wave with $S2$ component. In other words, the wave polarized in the plane of incidence does not generate waves polarized perpendicularly to the plane of incidence.

In case of incidence of an S wave with $S2$ component

$$\bar{D}_1 = \bar{D}_3 = \bar{E}_1 = \bar{E}_3 = 0 \quad .$$

In this case, the first system of equations has zero solution, which means that the wave polarized perpendicularly to the plane of incidence generates only shear wave polarized in the same way.

We can thus see that we can investigate independently the process of reflection/transmission of waves polarized in the plane of incidence (P, $S1$) and waves polarized perpendicularly to the plane of incidence ($S2$).

By introducing the above special Cartesian coordinate system and by the special choice of the polarization vectors corresponding to incident and generated waves, we reduced the number of reflection and transmission coefficients, each from 9 to 5 (R_{pp}, R_{ps1}, R_{s1p}, R_{s1s1}, R_{s2s2}). These coefficients can be simply obtained in an analytic form by solving the above separated systems of equations. The analytic expressions, known as Zöpritz coefficients or Fresnel coefficients are given in many textbooks.

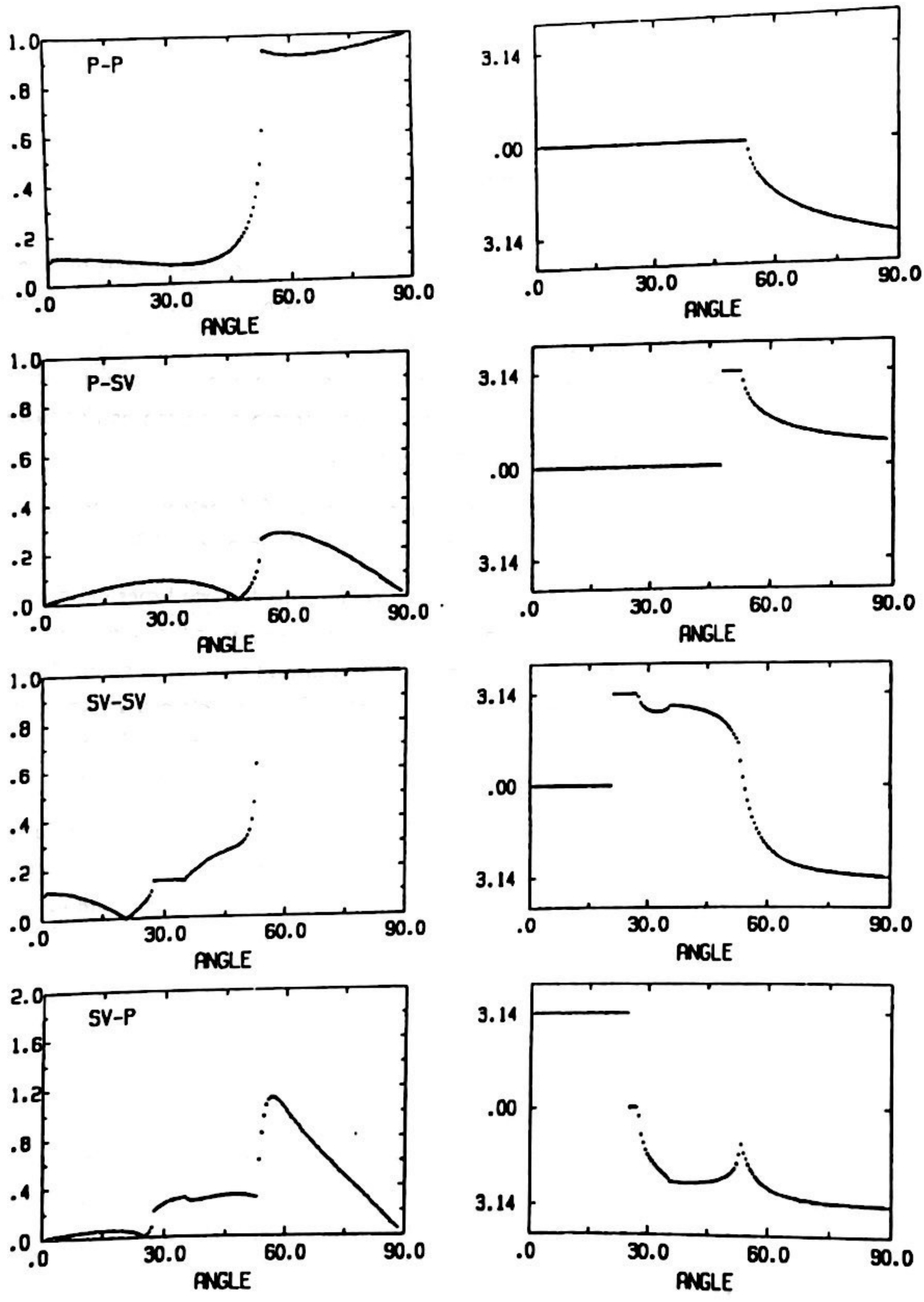
The number of the coefficients can be reduced to 4 if instead of the above *displacement* coefficients R we introduce the *energy* coefficients R_B, see Sec. 3.8.1.4.

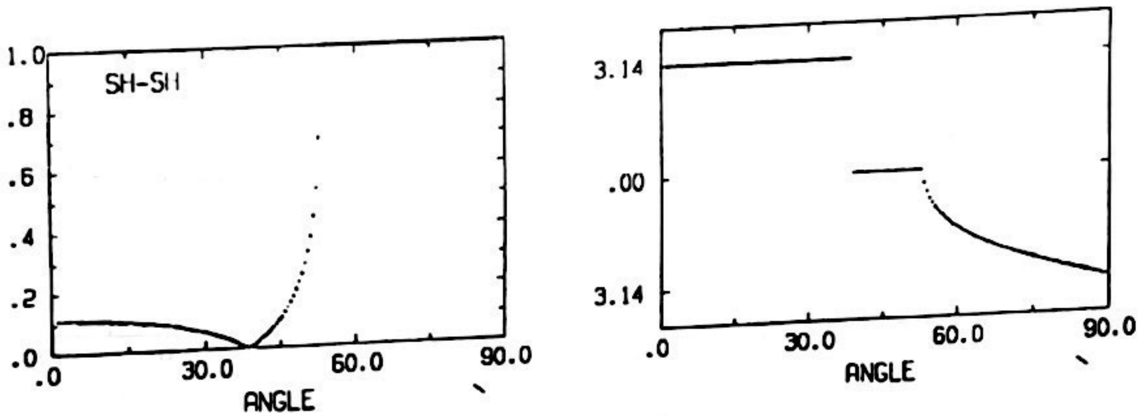
For illustration, reflection/transmission coefficients for two types of velocity contrast are shown. The first is for the P wave velocity ratio $n = \alpha_1/\alpha_2 = 0.8$, the other for $n = 0.4$. The first interface is called *weak interface* since the jump in velocity is small, the other interface is called *strong interface* because of the great difference between P wave velocities in the 1st and 2nd medium. The ratio α/β is considered constant everywhere, $\alpha/\beta = \sqrt{3}$ (Poisson solid), the ratio of densities $m = \rho_2/\rho_1$ is $m = 1$.

In the following, we use the commonly used notation for shear waves, in which the shear wave polarized in the plane of incidence ($S1$) is called SV (its polarization is in a vertical (V) plane) and the shear wave polarized perpendicularly to the plane of incidence ($S2$) is called SH (its polarization is in a horizontal (H) plane).

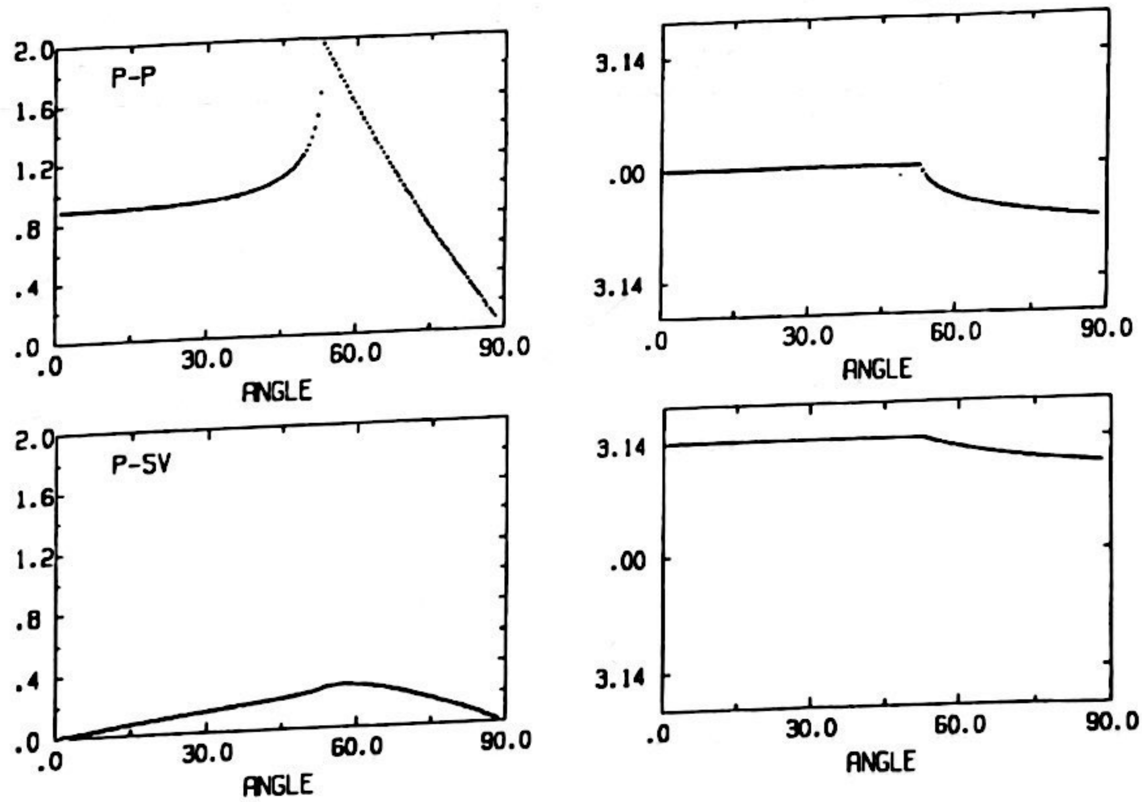
For each type of the velocity contrast, five reflection and five transmission coefficients are presented in the order: $P-P$, $P-SV$, $SV-SV$, $SV-P$, $SH-SH$. Each coefficient is presented in two frames. The left-hand side frame contains the modulus of the coefficient and the right-hand side its phase.

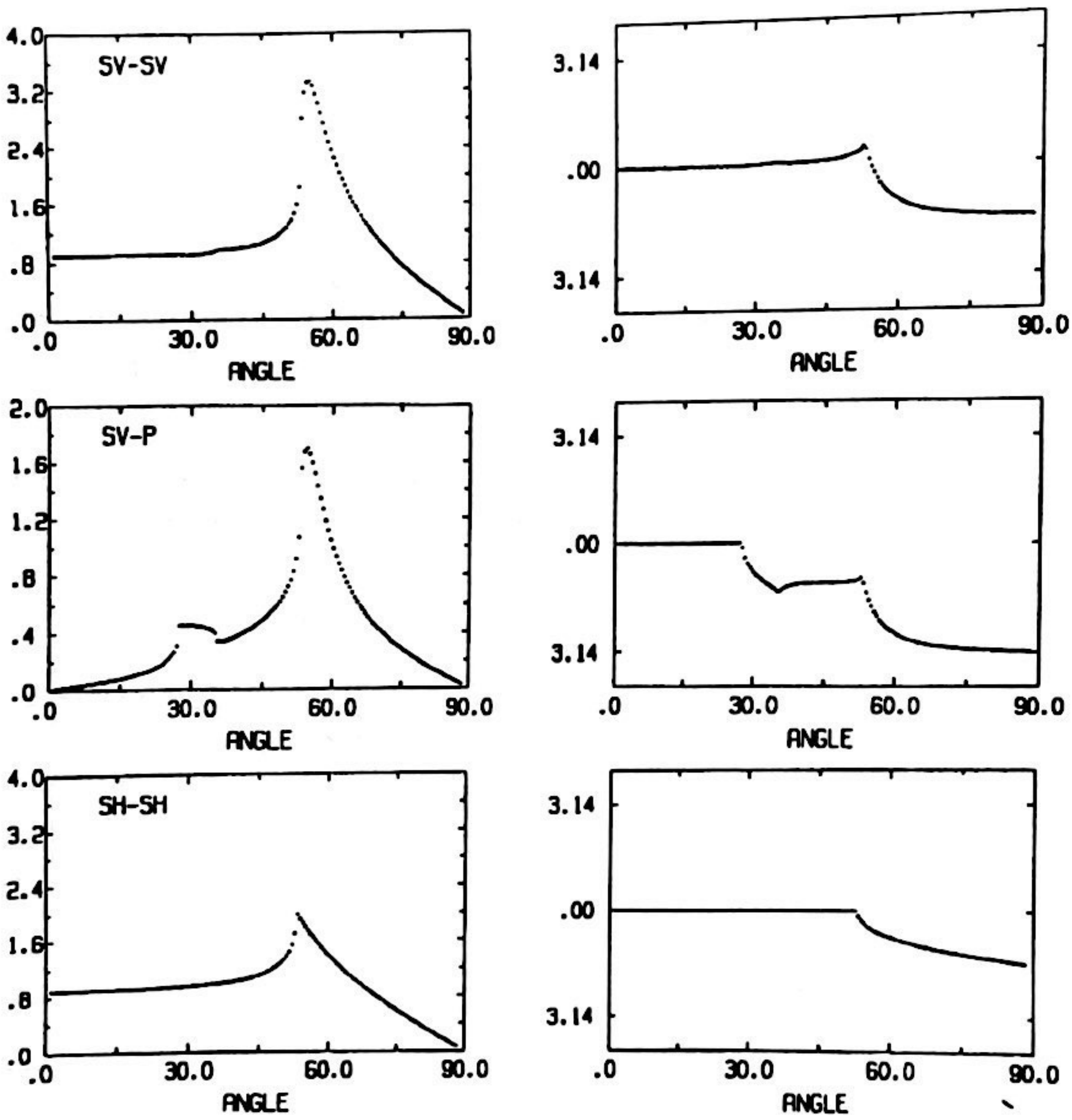
The first set contains reflection coefficients from the weak interface. Three critical angles can appear: $i_1^* = sin^{-1}(\beta_1/\alpha_2) = 27.51°$, $i_2^* = sin^{-1}(\beta_1/\alpha_1) = 35.26°$, $i_3^* = sin^{-1}(\alpha_1/\alpha_2) = sin^{-1}(\beta_1/\beta_2) = 53.13°$. In case of $P-P$, $SH-SH$ and $P-SV$ reflection coefficients, only i_3^* occurs, in case of $SV-SV$ and $SV-P$, all the three critical angles occur.


P-P
P-SV
SV-SV
SV-P
ANGLE
3.14
.00

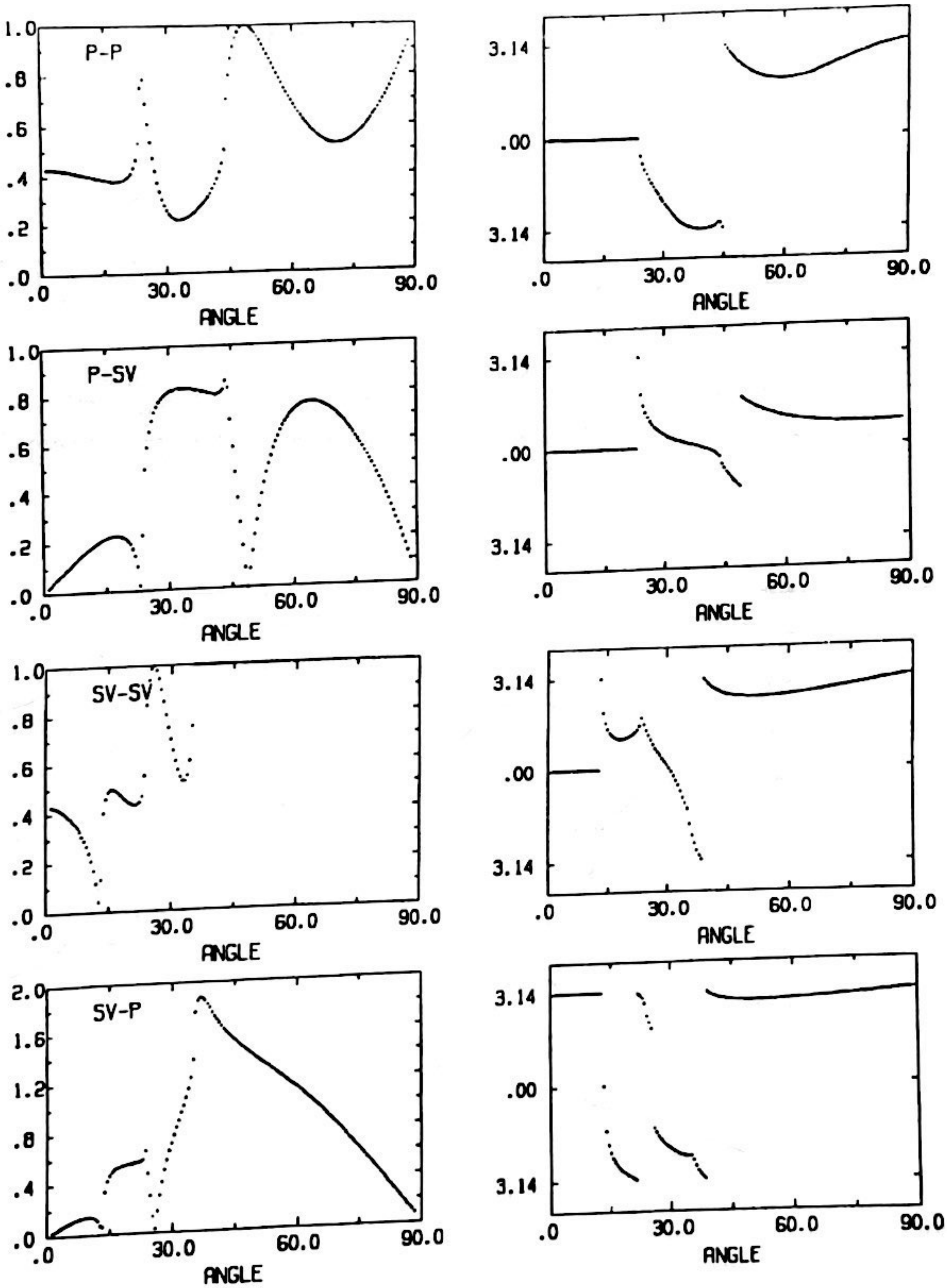


The second set contains transmission coefficients from the weak interface. The number and position of the critical angles are the same as in the above case of reflection.

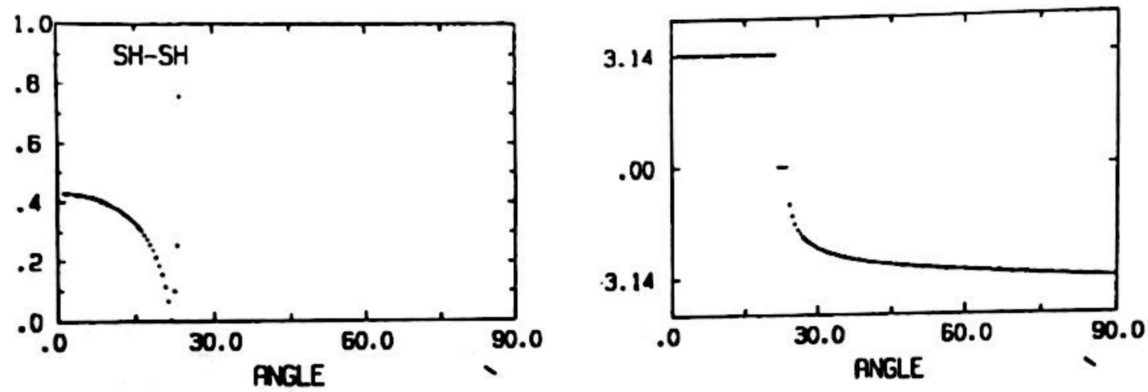




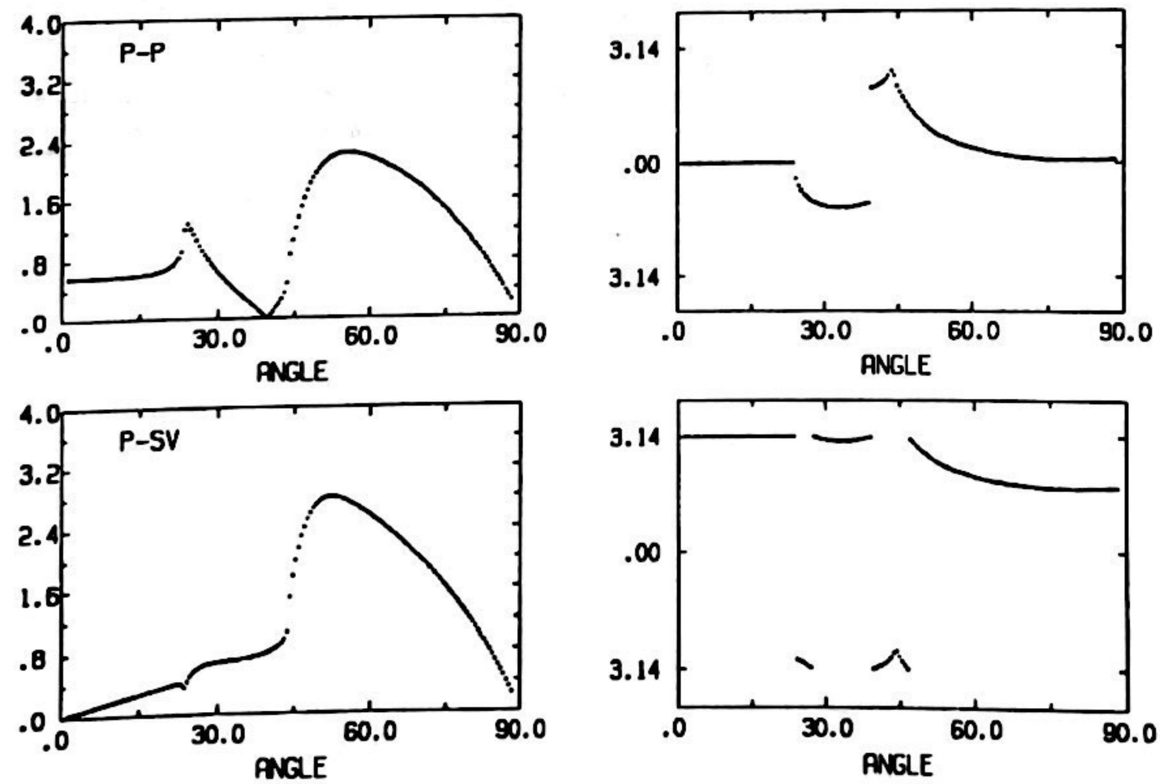
The third set contains reflection coefficients from the strong interface. Four critical angles can appear: $i_1^* = sin^{-1}(\beta_1/\alpha_2) = 13.35°$, $i_2^* = sin^{-1}(\alpha_1/\alpha_2) = sin^{-1}(\beta_1/\beta_2) = 23.59°$, $i_3^* = sin^{-1}(\beta_1/\alpha_1) = 35.26°$ and $i_4^* = sin^{-1}(\alpha_1/\beta_2) = 43.84°$. In case of $P-P$ and $P-SV$ reflection coefficients, i_2^* and i_4^* occur, in case of $SH-SH$ reflection coefficient, only i_2^* occurs and in case of $SV-SV$ and $SV-P$, i_1^*, i_2^* and i_3^* occur.

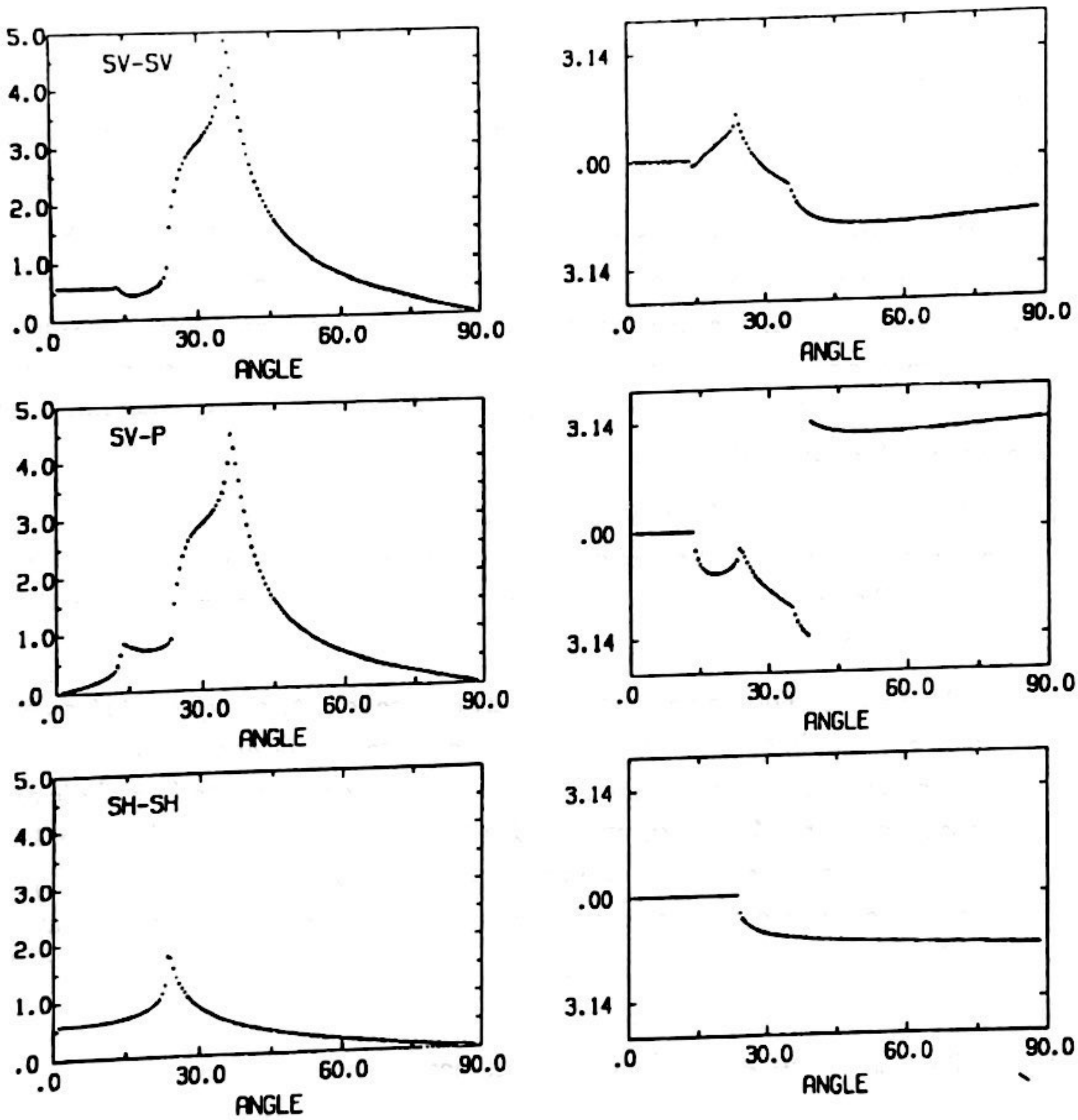


P-P
P-SV
SV-SV
SV-P
ANGLE
.0
30.0
60.0
90.0
1.0
.8
.6
.4
.2
2.0
1.6
1.2
3.14
.00



The fourth set contains transmission coefficients from the strong interface. The number and position of the critical angles are again the same as in the previous case of reflection.





There are several interesting features in the above figures. For $i = 0°$ and $i = 90°$, all the coefficients of unconverted waves, i.e., $P - P$, $SV - SV$, $SH - SH$, satisfy the relation

$$g_i^I + Rg_i^R = Tg_i^T \quad ,$$

where g_i^I, g_i^R, g_i^T denote the polarization vectors of the considered incident, reflected and transmitted waves respectivelly and R and T are the corresponding reflection and transmission coefficients. The above relation implies that all the incident ennergy is distributed only to the reflected and transmitted waves of the same type as the incident wave. Let us note that the same phenomenon holds in the acoustic case and in the case of SH wave reflection/transmission for any angle of incidence.

As a consequence of the above phenomenon, the coefficients of converted waves, i.e., $P-SV$ and $SV-P$ are zero for $i=0°$. As was mentioned earlier, an incident SH wave does not generate P or SV waves and vice versa. Therefore, corresponding coefficients are identically zero.

Other interesting phenomenon is missing total reflection of P wave both for $n=0.8$ and 0.4. The explanation is in the transfer of part of energy of the incident P wave into generated S waves. In case of the weak interface, the transfer of energy into S waves is small and, therefore, the phenomenon is not as prominent as in the case of strong interface. The total reflection occurs for shear waves since their energy cannot be trasferred into another type of a wave in the overcritical region.

Another interesting feature are relatively high values of transmission coefficents in some regions, note the different scales on diagrams of moduli. Similar phenomenon can also occur for $SV-P$ reflection. It is caused by higher velocity of the generated wave than that of the incident wave.

There are two noticeable differences between the reflection coefficients for weak and strong interface. First, in case of the weak interface, there is only *one* critical point for the $P-P$ coefficient, given by $\sin i_3^* = \alpha_1/\alpha_2$ while in case of the strong interface, there are *two* critical points given by $\sin i_2^* = \alpha_1/\alpha_2$ and $\sin i_4^* = \alpha_1/\beta_2$. Second, in case of the weak interface, the subcritical reflections are very weak while in case of the strong interface they are relatively strong. The coefficients for stronger interfaces have generally more complicated forms for P and SV waves.

3.8.2.3 Free surface reflection coefficients

Free surface reflection coefficients can be obtained from the above coefficients by setting $\alpha_2 = \beta_2 = \rho_2 = 0$, see Sec. 2.6.2.

As expected, for the incident S wave with $S2$ component, the reflection coefficient is equal 1 since all the incident energy is transferred into the only one reflected wave with the same polarization as the incident wave. There is *no critical incidence* in this case.

The P wave incident on a free surface also *does not have critical incidence* since the velocity of the incident wave is equal or larger than any velocity of the generated waves. *Critical incidence can occur* in case of incidence of an S wave with the $S1$ component.

The condition for the critical angle i^* is

$$\sin i^* = \beta_1/\alpha_1 \quad .$$

For a Poisson solid, see Sec.2.4.6, i.e. when $\alpha_1/\beta_1 = \sqrt{3}$, the above formula yields $i^* \sim 35°$. Since the Earth's material close to the Earth's surface usually does not deviate much from the Poisson solid, $i^* \sim 35°$ can be taken as a critical angle of the S wave polarized in the plane of incidence and incident on the Earth's surface. Let us remind that for $i > i^*$, the reflected S wave changes its form.

When the measurements are performed at the Earth's surface, we in fact, do not record only an incident wave but also the waves generated by this incidence. Let us note that the same phenomena can be observed at interfaces inside a medium. In case of incidence of an S wave with the $S2$ component, we record two waves of the same amplitude, incident and reflected (coefficient of reflection is equal 1), both polarized in the same way. The effect of the free surface is thus that we observe doubled amplitudes of the incident S wave with $S2$ component. In case of incidence of a P wave or an S wave with the $S1$ component, we record incident wave and unconverted and converted reflected waves.

To obtain the total displacement recorded on the Earth's surface, the amplitude of the displacement vector of the incident wave must be multiplied by the so-called *conversion coefficients*, which incorporate the effects of generated waves and give required component of the displacement vector. For an S wave with the $S2$ component, there is only one conversion coefficient and its value is 2. For an incident P wave or S wave with the $S1$ component, there are coefficients for the horizontal and the vertical component. For $i < i^*$, the conversion coefficients are real-valued and the waveform of the recorded wave is the same as the waveform of the incident wave (only multiplied by a constant). For $i > i^*$, however, the coefficients become complex-valued and the waveform of the recorded wave changes. For $i > i^*$ is, therefore, difficult to reconstruct the incident waveform from observations. From this reason, the local studies of shear waves are usually limited to the region called the *shear wave window*, i.e. the region specified by angles of incidence $i < i^*$, in case of the Poisson solid for angles $i < 35^0$.

3.8.3 Anisotropic medium

Let us consider two *homogeneous anisotropic halfspaces* separated by a plane interface Σ. The orientation of the normal n_k to the interface Σ is specified so that it points into the halfspace where the incident wave propagates. The coordinate system has its origin on Σ, otherwise it is arbitrary. The density ρ and the density normalized elastic parameters in the halfspace 1 are ρ_1 and $a^{(1)}_{ijkl}$, in the halfspace 2, ρ_2 and $a^{(2)}_{ijkl}$. The boundary conditions are the same as in the isotropic case, i.e. the continuity of the displacement and traction vectors across Σ. These conditions are satisfied by three reflected and three transmitted plane waves generated by an incident plane wave. The three waves are one quasicompressional (qP) and two quasishear ($qS1$, $qS2$) waves. The incident wave can also be one of these three wave types and can be written as

$$u_i(x_m, t) = Ag_i F(t - p_k x_k) \quad .$$

Quantity A again characterizes the size of the displacement vector of the incident wave, g_i is the polarization vector. The displacement vectors of the six generated waves are given by the expressions

$$u^{(j)}_i(x_m, t) = A^{(j)} g^{(j)}_i F^{(j)}(t - p^{(j)}_k x_k - \varphi^{(j)}) \, , \quad j = 1, \ldots, 6 \quad .$$

Quantities with $j = 1, 2, 3$ correspond to the reflected waves, with $j = 4, 5, 6$ to the transmitted waves ($qS1$, $qS2$, qP). Our aim is thus to determine the quantities $A^{(j)}$, $g^{(j)}_i$, $F^{(j)}$, $p^{(j)}_i$ $\varphi^{(j)}$ for $j = 1, \ldots, 6$ from the known quantities A, g_i, F and p_i.

The expression for the traction is, see Sec.2.3:

$$T_i = \tau_{ij} n_j = \rho a_{ijkl} n_j u_{k,l} \quad .$$

For the incident wave we thus have

$$T_i = -A\rho_1 a^{(1)}_{ijkl} n_j g_k p_l \dot{F}(t - p_m x_m) \quad .$$

Analogous expressions can be written for tractions of the generated waves. The boundary conditions can thus be written as follows:

$$-A^{(1)} g^{(1)}_i F^{(1)} - A^{(2)} g^{(2)}_i F^{(2)} - A^{(3)} g^{(3)}_i F^{(3)} + A^{(4)} g^{(4)}_i F^{(4)} + A^{(5)} g^{(5)}_i F^{(5)} + A^{(6)} g^{(6)}_i F^{(6)} = Ag_i F$$
$$-A^{(1)} X^{(1)}_i \dot{F}^{(1)} - A^{(2)} X^{(2)}_i \dot{F}^{(2)} - A^{(3)} X^{(3)}_i \dot{F}^{(3)} + A^{(4)} X^{(4)}_i \dot{F}^{(4)} + A^{(5)} X^{(5)}_i \dot{F}^{(5)} + A^{(6)} X^{(6)}_i \dot{F}^{(6)} = AX_i \dot{F} .$$

Here

$$X_i = \rho a_{ijkl} n_j g_k p_l \quad .$$

3.8.3.1 Transformation of slowness vectors across an interface

By the same argument as in the acoustic case, we can show that the analytic signals and their derivatives are the same for all the generated waves and are equal to the analytic signal of the incident wave and its derivative. In the same way as in the acoustic case, we can also deduce that tangent components of the slowness vectors of the incident and generated waves to the interface Σ are the same, i.e.

$$p_i - n_i(p_k n_k) = p_i^{(j)} - n_i(p_k^{(j)} n_k) \quad .$$

To determine the components of the slowness vectors of generated waves into the normal to the interface, we must use the equation of the slowness surface in an anisotropic medium, see Sec.3.6.1,

$$\det(\bar{\Gamma}_{ik} - \delta_{ik}) = \det(a_{ijkl} p_j p_l - \delta_{ik}) = 0 \quad ,$$

which substitutes the role of the eikonal equation in an isotropic medium. For reflected waves, we must use this equation with $a_{ijkl}^{(1)}$, for transmitted waves with $a_{ijkl}^{(2)}$. Since the slowness vector of a generated wave can be written as

$$p_i^{(m)} = b_i + \xi^{(m)} n_i \quad ,$$

where

$$b_i = p_i - n_i(p_k n_k)$$

and $\xi^{(m)}$ is the sought projection of the slowness vector $p_i^{(m)}$ on the normal n_i, we must solve the equation

$$\det\left[a_{ijkl}(b_j + \xi^{(m)} n_j)(b_l + \xi^{(m)} n_l) - \delta_{ik}\right] = 0 \quad .$$

This is a sixth order polynomial equation for $\xi^{(m)}$. Let us again emphasize that for reflected waves, we must consider $a_{ijkl}^{(1)}$, for transmited waves $a_{ijkl}^{(2)}$. Since in each medium, maximum 3 waves can be generated, the above equation gives more roots than are required and their selection must be done. The roots of the equation which are real correspond to homogeneous waves. Complex-valued solutions correspond to inhomogeneous waves. Since the coefficients of the above polynomial equation are real, complex roots appear in pairs of two complex conjugate numbers.

From the complex roots, only those are acceptable which give decrese of amplitudes of the inhomogeneous waves with increasing distance from the interface, see e.g. Sec.3.8.1.1. Imaginary parts of such roots must satisfy the condition

$$p_m^I x_m = Im(\xi) n_m x_m > 0 \quad .$$

This condition yields for the halfspace 1, where the *reflected waves* propagate, $Im(\xi) > 0$ since $n_m x_m > 0$. In the halfspace 2, where the *transmitted waves* propagate, the above condition yields $Im(\xi) < 0$ since $n_m x_m < 0$ there.

From the real roots, only those are acceptable, for which the energy flux vectors (parallel to the group velocity vectors) point into the halfspace, in which the sought wave is to propagate. If we take into account the expression for the group velocity, see Sec.3.4.2, this condition can be written as

$$n_i v_i^{(g)} = n_i a_{ijkl}^{(2)} p_l g_j g_k \leq 0 \quad ,$$

or

$$n_i v_i^{(g)} = n_i a_{ijkl}^{(1)} p_l g_j g_k \geq 0 \quad ,$$

where the upper inequality corresponds to the transmitted wave, the lower inequality to the reflected wave. The limiting case, in which $n_i v_i^{(g)} = 0$, corresponds to the critical incidence. Since the directions of the energy flux and corresponding slowness vector generally differ in an anisotropic medium, it can happen that the slowness vector of the generated wave points into the other halfspace than that into which energy flux vector points.

If we introduce the angles of incidence, transmission and reflection as the acute angles between the corresponding slowness vectors and the normal to the interface, we can derive the Snell law for anisotropic media by multiplying $p_i^{(m)} = b_i + \xi^{(m)} n_i$ by a unit vector tangent to the interface. We get

$$\frac{\sin i^{(m)}}{c_m(i^{(m)})} = \frac{\sin i}{c(i)} \quad , \quad m = 1, 2, \ldots 6 \, .$$

Here c denotes the phase velocity of the incident wave, c_m of the generated wave. Although the form of the Snell law is formally the same as for isotropic media, there is a substantial difference between both. In the above Snell law, the phase velocities are functions of the angle of reflection/transmission.

3.8.3.2 Coefficients of reflection/transmission

Due to the equality of the analytic signals of the incident and generated waves, the boundary conditions reduce to the following system of six inhomogeneous linear algebraic equations for six unknowns $A^{(m)}$, $m = 1, \ldots 6$

$$\begin{aligned} -A^{(1)} g_i^{(1)} - A^{(2)} g_i^{(2)} - A^{(3)} g_i^{(3)} + A^{(4)} g_i^{(4)} + A^{(5)} g_i^{(5)} + A^{(6)} g_i^{(6)} &= A g_i \, , \\ -A^{(1)} X_i^{(1)} - A^{(2)} X_i^{(2)} - A^{(3)} X_i^{(3)} + A^{(4)} X_i^{(4)} + A^{(5)} X_i^{(5)} + A^{(6)} X_i^{(6)} &= A X_i \end{aligned}$$

with

$$X_i = \rho a_{ijkl} n_j g_k p_l \quad .$$

As in isotropic case, we can introduce the *displacement reflection/transmission coefficients* $R_{mn}^A = A^{(m)}/A$. Here $m = 1, \ldots 6$ corresponds to the generated waves (R_{mn}^A thus represents both reflection and transmission coefficients), $n = 1, 2, 3$ correspond to the incident waves $qS1$, $qS2$ and qP. In special situations, when e.g. an axis of symmetry of an anisotropic medium is situated in the plane of incidence, the above system can be simplified or even factorized. For generally anisotropic media, however, it must be solved numerically.

The polarization vectors of the generated waves can be simply determined from the Christoffel equation

$$(\Gamma_{ik} - \delta_{ik})g_i = (a_{ijkl}p_j p_l - \delta_{ik})g_i = 0 \quad ,$$

where we sucessively insert slowness vectors of the generated waves.

In Sec.3.4.2, we have shown that the time integrated energy flux of an elastic plane wave is

$$\hat{S}_i = \rho a_{ijkl} p_l g_j g_k AA^* f_e = v_i^{(g)} \rho AA^* f_e \quad .$$

Here $f_e = \int_{-\infty}^{+\infty} \dot{g}^2 dt$. Similarly as in Sec.3.8.1.4, we can rewrite this equation for the case of the complex-valued group velocity (overcritical incidence) as follows

$$\hat{S}_i = \rho Re(v_i^*(g)) AA^* f_e \quad .$$

The energy flux trough a unit element of the interface Σ characterized by the unit normal n_i is thus

$$|\hat{S}_i n_i| = \rho |v_i^{(g)} n_i| AA^* f_e \quad \text{if } Im\, v_i^{(g)} = 0 \,,$$

$$|\hat{S}_i n_i| = 0 \quad \text{if } Im\, v_i^{(g)} \neq 0 \,.$$

The second relation results from the fact that $Re(v_i^*(g)) \perp n_i$.

Similarly as in Sec.3.9.1.4, we can introduce the *energy reflection/transmision coefficients* R_E as

$$R_E^r R_E^{r*} = |\hat{S}_i^r n_i| / |\hat{S}_i n_i| \quad , \quad R_E^t R_E^{t*} = |\hat{S}_i^t n_i| / |\hat{S}_i n_i| \,.$$

This yields the following relations between the energy (R_E^r, R_E^t) and the displacement (R^r, R^t) reflection/transmission coefficients

$$\begin{aligned} R_E^r R_E^{r*} &= R^r R^{r*} |v_i^{(g)r} n_i| / |v_i^{(g)} n_i| \\ R_E^t R_E^{t*} &= R^t R^{t*} (\rho_2 |v_i^{(g)t} n_i| / (\rho_1 |v_i^{(g)} n_i| && \text{if } Im(v_i^{(g)} = 0 \,, \\ R_E^r R_E^{r*} &= 0 \,, \quad R_E^t R_E^{t*} = 0 && \text{if } Im\, i_t \neq 0 \,. \end{aligned}$$

Similarly as in the isotropic case, the coefficients are reciprocal. This has been proved numerically.

Introduction of the free surface reflection coefficients is similar to the way in which it was done for isotropic media.

3.9 Absorption and dispersion

The wave processes, which we have considered upto now could continue forever without losses of elastic energy. In reality, however, the elastic energy is transformed into other types of energy and the amplitudes of waves propagating through real media decrease. Without specifying individual causes of energy losses, we can explain them formally by an *internal friction*.

A characteristics often used to describe the internal friction is the *quality factor* Q. For harmonically varying stress applied to a volume of a medium, the quality factor is defined as a nondimensional quantity

$$Q^{-1}(\omega) = -\Delta E/2\pi\bar{E}\,.$$

Here $\bar{E}$ is the amount of time averaged elastic energy in a volume of the medium, ΔE is the energy loss in the volume during one cycle. Symbol ω denotes the circular frequency. In a perfectly elastic medium, there are no energy losses and, therefore, $\Delta E = 0$, which implies $Q(\omega) \to \infty$. With increasing ΔE, the value of the quality factor decreases. It was observed that for frequencies used in seismology, i.e. for 0.001-100 Hz, the quality factor Q remains practically constant with respect to ω.

As was shown in Sec.3.4, the elastic energy is proportional to the square of the amplitude of a harmonic plane wave. Let us consider an acoustic plane wave propagating with a velocity c in the direction N_i

$$p(x_m, t) = P \exp\left[-i\omega\left(t - \frac{N_m x_m}{c}\right)\right] \quad ,$$

entering a region of absorbing acoustic medium. We can now rewrite the above equation for $Q^{-1}(\omega)$ in terms of the amplitude P,

$$Q^{-1}(\omega) = -\Delta P/\pi P \quad .$$

We consider that the attenuation is isotropic, i.e. the change of the amplitude P due to the attenuation is independent of the direction of propagation of the wave. Then we can write for ΔP

$$\Delta P = \frac{dP}{ds}\lambda = (2\pi c/\omega)\frac{dP}{ds} \quad ,$$

where ds is a length element in the direction N_i. After inserting ΔP into the equation for $Q^{-1}(\omega)$, we get

$$P^{-1}\frac{dP}{ds} = -\frac{\omega}{2cQ(\omega)} \quad ,$$

which yields

$$P = P_o \exp\left[-\frac{\omega s}{2cQ(\omega)}\right] = P_o \exp\left[-\frac{\omega N_i x_i}{2cQ(\omega)}\right]$$

Here s is a distance from the origin of coordinates and P_o is the amplitude of the wave at the origin. After inserting the above expression for P into the equation for the acoustic plane wave $p(x_m, t)$, we get

$$p(x_m,t) = P_o \exp\left[-i\omega\left(t - \frac{N_m x_m}{c_a}\right)\right] \quad ,$$

where

$$\frac{1}{c_a} = \frac{1}{c} + \frac{i}{2cQ(\omega)} \quad .$$

We can see that the expression for the plane wave is formally the same as in case of the perfect elasticity. The real-valued phase velocity c was, however, in the anelastic case substituted by a complex-valued and weakly (due to the weak frequency dependence of Q) frequency-dependent velocity c_a.

We can rewrite the expression for $p(x_m,t)$ in yet another form

$$p(x_m,t) = P_o \exp\left[-i\omega(t - p_m^{(a)} x_m)\right] = P_o \exp\left(-\omega p_m^I x_m\right) \exp\left[-i\omega(t - p_m^R x_m)\right] .$$

Here

$$p_i^{(a)} = \frac{N_i}{c_a} = p_i^R + ip_i^I \quad .$$

The above expression for the homogeneous acoustic plane wave resembles the one which described an inhomogeneous plane wave in a *perfectly* elastic medium. There is, however, a substantial difference: the vectors p_i^R and p_i^I are parallel, $p_i^R = N_i/c$, $p_i^I = N_i/2cQ$! In case of an inhomogeneous plane wave propagating in an absorbing acoustic medium, the vector p_i^I can be expanded into a part parallel to N_i, which describes the attenuation effects, and a part perpendicular to N_i, which is connected with the exponential decrease of the amplitudes of the inhomogeneous wave in the direction perpendicular to the direction of propagation.

We have shown that the above formulae describe propagation of a homogeneous plane wave in an absorbing medium in the direction N_i with the phase velocity c. The amplitude of this wave decreases in the same direction, i.e. the planes of constant phase and constant amplitude are *parallel*. The decrease of amplitudes is controlled by the term $\omega/2cQ(\omega)$.

Attenuation effects can be described also by another quantity, the *coefficient of absorption* α_{ABS}. It is defined as

$$\alpha_{ABS}(\omega) = \omega/2cQ(\omega) \quad .$$

For $Q \to \infty$, i.e. for a perfectly elastic medium, $\alpha_{ABS} \to 0$. For the frequencies used in seismology, α_{ABS} is thus a linear function of the circular frequency ω. The expresssion for $p(x_m,t)$ can be rewritten using α_{ABS} instead of Q,

$$P(x_m,t) = P_o \exp\left(-\alpha_{ABS}(\omega) N_m x_m\right) \exp\left[-i\omega\left(t - \frac{N_m x_m}{c}\right)\right] \quad .$$

The higher the frequency ω the stronger is the attenuation of the wave.

Until now, we have considered effects of attenuation on harmonic waves. Let us now consider an impulse wave and let us consider its waveform to be the δ-function,

$$p(x_m, t) = P\delta(t - p_m x_m) \quad .$$

The spectrum of the function $\delta(t - p_m x_m)$ is as follows

$$\delta(\omega) = \int_{-\infty}^{+\infty} \delta(t - p_m x_m) e^{i\omega t} dt = e^{i\omega p_m x_m}$$

thus the impulse plane wave given above can be expressed as

$$p(x_m, t) = (2\pi)^{-1} P \int_{-\infty}^{+\infty} e^{-i\omega(t - p_m x_m)} d\omega \quad .$$

We can see that the δ-impulse wave can be obtained by a synthesis of harmonic waves of equal amplitudes. Let us now extend this synthesis to anelastic media, for which we have found that a harmonic plane wave with frequency $\omega = 2\pi f$ has the form

$$p(x_m, t) = P_o \exp\left[-\frac{1}{2Q(\omega)}|\omega p_m x_m|\right] \exp\left[-i\omega(t - p_m x_m)\right]$$

and let us see how is the equality of amplitudes in the synthesis affected by attenuation. For simplicity, but in accordance with what was stated above, let us consider Q independent of the frequency. We can now write

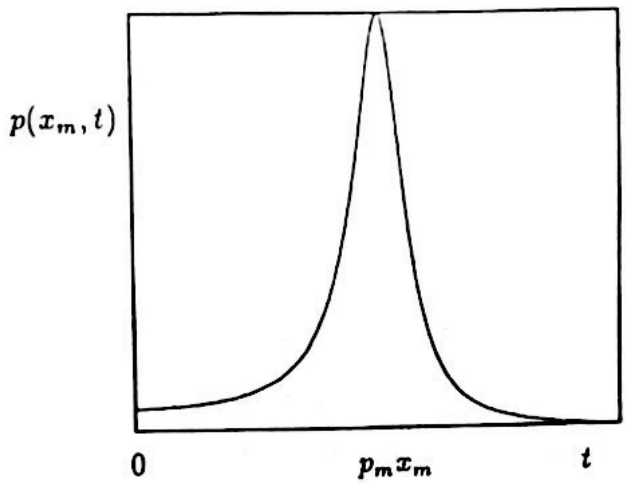
$$p(x_m, t) = \frac{P_o}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega(t - p_m x_m)} e^{-\frac{|\omega p_m x_m|}{2Q}} d\omega \quad .$$

We consider $|\omega(p_m x_m)|$ in order to avoid infinitely increasing amplitudes of terms with negative frequency and $p_m x_m > 0$ (or positive ω and $p_m x_m < 0$). We get

$$\begin{aligned} p(x_m, t) &= \frac{P_o}{2\pi} \int_{-\infty}^{0} e^{-i\omega(t - p_m x_m)} e^{\frac{-|\omega|}{2Q}|p_m x_m|} d\omega \\ &\quad + \frac{P_o}{2\pi} \int_{0}^{+\infty} e^{-i\omega(t - p_m x_m)} e^{\frac{-\omega}{2Q}|p_m x_m|} d\omega \, . \end{aligned}$$

This yields

$$\begin{aligned} p(x_m, t) &= \frac{P_o}{2\pi} \int_{0}^{\infty} e^{i\omega(t - p_m x_m)} e^{-\frac{\omega}{2Q}|p_m x_m|} d\omega + \frac{P_o}{2\pi} \int_{0}^{\infty} e^{-i\omega(t - p_m x_m)} e^{-\frac{\omega}{2Q}|p_m x_m|} d\omega \\ &= -\frac{P_o}{i2\pi(t - p_m x_m) - \pi|p_m x_m|Q^{-1}} + \frac{P_o}{i2\pi(t - p_m x_m) + \pi|p_m x_m|Q^{-1}} \\ &= \frac{2Q|p_m x_m|P_o}{4\pi(t - p_m x_m)^2 Q^2 + \pi(p_m x_m)^2} \, . \end{aligned}$$



The found signal has the form shown in the picture. The result, which we have obtained violates observations from several aspects. The most serious violation is the violation of the *causality principle*. The signal is non-zero even for $t < 0$, i.e. for times before it was generated. Our signal is symmetric while observations indicate short "rise time" of the signal followed by a slower decay. There are several possible explanations of the misfit of our "simplified" theory and observations. The basic cause of our problems is neglection of the *dispersion* of waves. In other words, the velocity c must be considered as a function of frequency, $c = c(\omega)$. In this way we remove the symmetry of the signal. To guarantee causality of the signal, i.e. $p(x_m, t) = 0$ for $t < 0$, the velocity $c(\omega)$ and the quality factor $Q(\omega)$ *cannot be independent* quantities. They are connected by the so-called Kramers-Krönig relations. These relations are a consequence of the fact that the real and imaginary parts of the spectrum of a causal function form a Hilbert pair.

There are various models of absorption which better or worse satisfy the above conditions. One well-fitting model gives the following relation of phase velocities at two different frequencies

$$\frac{c(\omega_1)}{c(\omega_2)} = 1 + \frac{1}{\pi Q} \ln\left(\frac{\omega_1}{\omega_2}\right) \quad .$$

Here Q depends on ω very weakly, it is practically constant. If we use a reference frequency, say $f = 1$ Hz, we can write for c_a approximately (keeping only terms of order Q^{-1} and neglecting terms of order Q^{-2})

$$\frac{1}{c_a} = \frac{1}{c_1}\left(1 + \frac{1}{\pi Q} ln\left(\frac{2\pi}{\omega}\right) + \frac{i}{2Q}\right) \quad .$$

Here c_1 is the phase velocity $c(\omega)$ specified for $f = 1$ Hz. The higher is the frequency , the higher is the velocity c_a.

Similar derivations as for the acoustic case could also be done for isotropic and anisotropic case. Again, it is possible to consider formally the same equations as those which govern the propagation in perfectly elastic media. Only parameters of the medium must be considered complex-valued. In this way, for example, the Hooke law for an anisotropic solid can be written as

$$\tau_{ij} = a^{(a)}_{ijkl} e_{kl}$$

where

$$a^{(a)}_{ijkl} = a_{ijkl} + i a^{I}_{ijkl} \quad .$$

Here a^{I}_{ijkl} controls absorption effects, which, in this case, are directionally dependent. Both a_{ijkl} and a^{I}_{ijkl} are, of course, considered frequency-dependent.

Due to the frequency dependence of elastic parameters, also phase velocity depends on frequency and we thus deal again with dispersive waves. The frequency dependence of the phase velocity of a plane wave propagating in an anelastic anisotropic homogeneous medium has the same form as the dependence of c_a above. Higher frequency components of signals thus propagate faster than the low frequency ones causing a sharp rise time of the signals.

Similarly as the anelasticity affects wave propagation in an unbounded medium, it also affects reflection and transmission at interfaces separating anelastic media. Instead of real-valued parameters of the medium used in the case of wave propagation in an elastic medium, complex-valued parameters must be used.

Let us now investigate the dispersion effects. For simplicity, we consider again an acoustic plane wave

$$p(x_m, t) = PF\left(t - \frac{N_m x_m}{c}\right)$$

propagating in a *nonabsorbing* acoustic medium. Let us now, however, consider frequency-dependent phase velocity, $c = c(\omega)$. The analytic signal has now the form

$$F\left(t - \frac{N_m x_m}{c(\omega)}\right) = \frac{1}{\pi}\int_0^\infty g(\omega) e^{-i\omega\left(t - \frac{N_m x_m}{c(\omega)}\right)} d\omega \, ,$$

where $g(\omega)$ is the spectrum of the considered real signal. The expression for the acoustic plane wave thus attains a form

$$p(x_m, t) = \frac{P}{\pi}\int_0^\infty g(\omega) e^{-i\omega\left(t - \frac{N_m x_m}{c(\omega)}\right)} d\omega \, .$$

For c independent of frequency, the above formula reduces to the formula for a harmonic transient plane wave

$$p(x_m, t) = PF\left(t - \frac{N_m x_m}{c}\right) \quad .$$

We, however, consider frequency-dependent velocity $c = c(\omega)$. For simplicity, we assume that $g(\omega)$ is effectively non-zero for only a narrow band of frequencies, $\omega_o - \Delta\omega < \omega < \omega_o + \Delta\omega$, i.e., we are considering a quasiharmonic signal. We can rewrite the above integral expression as

$$p(x_m, t) = \frac{P}{\pi} \int_{\omega_o - \Delta\omega}^{\omega_o + \Delta\omega} g(\omega) e^{-i(\omega t - k(\omega) N_m x_m)} d\omega \quad .$$

Here $k(\omega) = \omega / c(\omega)$ is the wave number introduced in Sec.3.1.1. In the vicinity of ω_o, we can expand the wave number as follows

$$k(\omega) = k(\omega_o) + \left.\frac{dk}{d\omega}\right|_{\omega_o} (\omega - \omega_o) + \ldots$$

We introduce a quantity U,

$$U = \frac{d\omega}{dk} \quad .$$

From the definition of the wave number k, we have $\omega = kc(k)$ which yields

$$U = c(k) + k\frac{dc(k)}{dk} \quad .$$

We can see that the quantity U has dimension of velocity and we are going to show that U is the *group velocity*, i.e. velocity, with which elastic energy propagates. It can be expressed in several other forms, one of them being

$$U(\omega) = \frac{c(\omega)}{1 - \frac{\omega}{c(\omega)} \frac{dc(\omega)}{d\omega}} \quad .$$

If c does not depend on the frequency ω, then $U = c$. In an acoustic medium, the group velocity U and the phase velocity c differ only if both depend on the frequency. If we assume that the spectrum $g(\omega)$ does not vary too much in the interval $(\omega_o - \Delta\omega, \omega_o + \Delta\omega)$, we can approximate $g(\omega)$ as $g(\omega) \sim g(\omega_o)$ and the expression for $p(x_m, t)$ attains the form

$$p(x_m, t) \quad \sim \quad \frac{P}{\pi} e^{-i(\omega_o t - k(\omega_o) N_m x_m)} g(\omega_o) \int_{\omega_o - \Delta\omega}^{\omega_o + \Delta\omega} e^{-i(\omega - \omega_o)\left(t - \frac{N_m x_m}{U(\omega_o)}\right)} d\omega \, .$$

The evaluation of the integral is now simple:

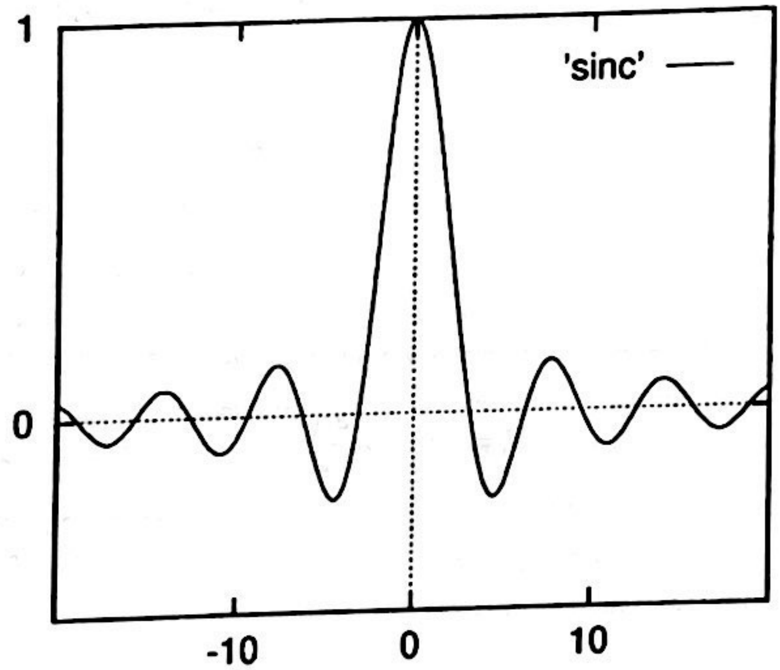
$$\int_{\omega_o - \Delta\omega}^{\omega_o + \Delta\omega} e^{-i(\omega - \omega_o)\left(t - \frac{N_m x_m}{U(\omega_o)}\right)} d\omega \; = \; \left[- \frac{e^{-i(\omega - \omega_o)\left(t - \frac{N_m x_m}{U(\omega_o)}\right)}}{i\left(t - \frac{N_m x_m}{U(\omega_o)}\right)} \right]_{\omega_o - \Delta\omega}^{\omega_o + \Delta\omega}$$

$$= \; \frac{e^{-i\Delta\omega\left(t - \frac{N_m x_m}{U(\omega_o)}\right)} + e^{i\Delta\omega\left(t - \frac{N_m x_m}{U(\omega_o)}\right)}}{i\left(t - \frac{N_m x_m}{U(\omega_o)}\right)} = 2\Delta\omega sinc\left[\Delta\omega\left(t - \frac{N_m x_m}{U(\omega_o)}\right)\right] \quad .$$

The function $sinc(t)$ is defined as

$$sinc(t) = \frac{\sin t}{t}$$

and has the form shown in the picture. It has its maximum value equal 1 at $t = 0$. For $t \to \pm\infty$, it approaches zero.



We can now write the final expression for $p(x_m, t)$. It reads

$$p(x_m, t) \sim C sinc\left[\Delta\omega\left(t - \frac{N_k x_k}{U(\omega_o)}\right)\right] exp\left[-i\omega_o\left(t - \frac{N_m x_m}{c(\omega_o)}\right) + i\varphi_o\right] ,$$

where

$$\begin{aligned} C &= 4P|g(\omega_o)|\Delta\omega\,, \quad g(\omega_o) = |g(\omega_o)|e^{i\varphi_o}\,, \quad k(\omega_o) = \frac{\omega_o}{c(\omega_o)}\,, \\ U(\omega_o) &= c(\omega_o)/\left[1 - \frac{\omega_o}{c(\omega_o)}\frac{dc(\omega_o)}{d\omega_o}\right] \quad . \end{aligned}$$

The above formula describes a harmonic plane wave propagating in the direction N_i with the phase velocity $c(\omega_o) = \omega_o/k_o$ and frequency ω_o. The amplitude of the plane wave is modulated by the function *sinc*. The function $sinc(t)$ forms thus an envelope of the plane wave and this envelope also propagates in the direction N_i but with the velocity $U(\omega_o)$. Since the elastic energy is proportional to the square of the amplitude of the plane wave, i.e. in our case to function *sinc*, it means that energy moves with the velocity U and thus U is the group velocity.

We could arrive to the same conclusions even if we did not make simplifying assumptions. The derivation would be, however, more complicated. Dispersion is not connected only with anelastic media. Dispersion is typical for various interference waves like, e.g. surface waves. In such a case we speak about *geometrical dispersion*. In case of dispersion due to anelastic behavior of the medium, we speak about *material dispersion*. Let us note that the formula relating the group velocity U and the phase velocity c,

$$U = c/\left(1 - \frac{\omega}{c}\frac{dc}{d\omega}\right)$$

is universal for geometrical as well as material dispersion. In the former case, usually $dc/d\omega < 0$ and thus $U \leq c$. Then we speak about *normal dispersion*. In case of material dispersion, we have found from

$$\frac{c(\omega_1)}{c(\omega_2)} = 1 + \frac{1}{\pi Q} ln\left(\frac{\omega_1}{\omega_2}\right)$$

that $dc/d\omega > 0$ and thus $U \geq c$. In this case, we speak about *anomalous dispersion*.

Let us now briefly discuss the difference between the phase and the group velocities, which we studied in *elastic anisotropic* media and in *dispersive isotropic* structures. In elastic anisotropic structure which we have studied there was *no frequency dispersion*. The difference between the group and the phase velocities raised from the fact that the parameters of the medium were *direction dependent*. The phase and group velocity vectors differred both in *size and direction*. In dispersive isotropic media, the difference between the group and the phase velocity is caused by the frequency dependence of velocity, i.e by *frequency dispersion*. The size of both velocities is different but their direction is the same. In a *dispersive anisotropic* medium both effects, of course, combine.

CHAPTER 4

POINT AND LINE SOURCE SOLUTIONS OF ELASTODYNAMIC AND ACOUSTIC EQUATIONS

In Chap.3, we studied plane wave solutions of the elastodynamic and acoustic equations. In this section, we are going to study other types of solutions of these equations. Specifically, we concentrate on the solutions due to point or line sources. To seek such solutions, it is often more convenient to use other coordinate systems than Cartesian, such as spherical or cylindrical coordinates. First we shall seek solutions independently of the type of the point or line source. The solutions will describe transport of energy from a source to another point. We shall seek the solutions, which describe response of the medium to the application of a specific type of the source in the next chapter.

4.1 Equations of motion

4.1.1 Acoustic equation

In Sec.2, we derived acoustic equations as one scalar equation for pressure and one vectorial equation for the particle velocity

$$\frac{\partial p}{\partial x_i} + \rho \frac{\partial v_i}{\partial t} = f_i \quad , \quad \frac{\partial v_i}{\partial x_i} + \kappa \frac{\partial p}{\partial t} = q \quad .$$

Here p is pressure, $v_i = \partial u_i / \partial t$ is the particle velocity vector, ρ is density, $\kappa = (\rho c^2)^{-1}$ is compressibility. The quantity f_i is a body force per unit volume and q represents the density of the rate of voluminal changes and is called the *volume injection rate density*. We shall transform the above two equations into one differential equation of second order for pressure p. By differentiating the first equation with respect to x_i and the second one with respect to t, we get

$$\frac{\partial^2 p}{\partial x_i^2} + \frac{\partial \rho}{\partial x_i}\frac{\partial v_i}{\partial t} + \rho \frac{\partial^2 v_i}{\partial x_i \partial t} = \frac{\partial f_i}{\partial x_i} \quad , \quad \frac{\partial^2 v_i}{\partial x_i \partial t} + \kappa \frac{\partial^2 p}{\partial t^2} = \frac{\partial q}{\partial t} \quad .$$

From this

$$\frac{\partial^2 p}{\partial x_i^2} + \rho^{-1}\frac{\partial \rho}{\partial x_i}\left(f_i - \frac{\partial p}{\partial x_i}\right) - \rho\kappa \frac{\partial^2 p}{\partial t^2} = \frac{\partial f_i}{\partial x_i} - \rho \frac{\partial q}{\partial t}$$

Keeping the derivatives of p on the left hand side, we get

$$\frac{\partial^2 p}{\partial x_i^2} - \frac{1}{\rho}\frac{\partial \rho}{\partial x_i}\frac{\partial p}{\partial x_i} - c^{-2}\frac{\partial^2 p}{\partial t^2} = F$$

or

$$\frac{\partial}{\partial x_i}\left(\rho^{-1}\frac{\partial p}{\partial x_i}\right) - \kappa\frac{\partial^2 p}{\partial t^2} = \rho^{-1}F \quad ,$$

where the "source" term F is given by

$$F = \frac{\partial f_i}{\partial x_i} - \frac{1}{\rho}\frac{\partial \rho}{\partial x_i}f_i - \rho\frac{\partial q}{\partial t} \, .$$

For the constant density, we get

$$\frac{\partial^2 p}{\partial x_i^2} - c^{-2}\frac{\partial^2 p}{\partial t^2} = \bar{F} \quad ,$$

which is the wave equation with the source term $\bar{F}$,

$$\bar{F} = \frac{\partial f_i}{\partial x_i} - \rho\frac{\partial q}{\partial t} \quad .$$

From the last equation we can see the relation between both types of sources, we are considering, i.e. f_i and q. We can see that the time derivative of the volume injection rate density multiplied by ρ can be substituted by the divergence of the force f_i.

From the acoustic equations of the first order, we could also derive second order differential equation for particle velocity. If we differentiate the first differential equation of the first order with respect to time, the second one with respect to spatial coordinates and relate both equations through the term $\partial^2 p/\partial x_i \partial t$, we would get

$$\frac{\partial}{\partial x_i}(\rho c^2 \frac{\partial v_k}{\partial x_k}) - \rho\frac{\partial^2 v_i}{\partial t^2} = -\frac{\partial f_i}{\partial t} + \frac{\partial}{\partial x_i}(\rho c^2 q) \quad .$$

We can easily recognize that the left-hand side represents an elastodynamic equation for an inhomogeneous isotropic medium, which we have derived in Sec.2.5.1, specified for $\mu = 0$ (acoustic case) and differentiated with respect to time. For ρ and c constant, the above equation reduces to

$$\rho c^2\frac{\partial^2 v_k}{\partial x_k \partial x_i} - \rho\frac{\partial^2 v_i}{\partial t^2} = -\frac{\partial f_i}{\partial t} + \rho c^2\frac{\partial q}{\partial x_i} \quad .$$

From this equation we can see that the time derivative of the force f_i can be substituted by the gradient of the volume injection rate density multiplied by density and squared velocity.

4.1.2 Elastodynamic equation

Basic forms of elastodynamic equations in various types of elastic media were given in Sec.2.5.1. The elastodynamic equation in terms of the stress tensor and displacement vector has the form

$$\tau_{ij,j} + f_i = \rho u_{i,tt} \quad ,$$

where τ_{ij} is the stress tensor, u_i is the displacement vector. For a general inhomogeneous anisotropic medium, in which the generalized Hooke law has a form

$$\tau_{ij} = c_{ijkl} u_{k,l} \quad ,$$

we get the elastodynamic equation for the displacement as

$$(c_{ijkl} u_{k,l})_{,j} + f_i = \rho u_{i,tt} \quad .$$

In the following, we shall sometimes write this equation in the form

$$L_i(u_m) + f_i = \rho u_{i,tt} \quad ,$$

in which L_i is a linear vectorial differential operator containing only the spatial derivatives.

4.2 Acoustic equation in curvilinear orthogonal coordinates

Using results of Sec.1.2.2, we can write the acoustic equation with constant density in curvilinear orthogonal coordinates as

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \gamma_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial p}{\partial \gamma_1}\right) + \frac{\partial}{\partial \gamma_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial p}{\partial \gamma_2}\right) + \frac{\partial}{\partial \gamma_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial p}{\partial \gamma_3}\right)\right] - c^{-2}\frac{\partial^2 p}{\partial t^2} = \bar{F} \, ,$$

where

$$\bar{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial \gamma_1}(h_2 h_3 f_1) + \frac{\partial}{\partial \gamma_2}(h_1 h_3 f_2) + \frac{\partial}{\partial \gamma_3}(h_1 h_2 f_3)\right] - \rho \frac{\partial q}{\partial t} \, .$$

4.2.1 Spherical coordinates

We introduce spherical coordinates r, θ, φ as follows

$$\begin{array}{lll} x_1 = r\sin\theta\cos\varphi & h_r = 1 & \theta - \text{ colatitude} \\ x_2 = r\sin\theta\sin\varphi & h_\theta = 1 & \varphi - \text{ longitude} \\ x_3 = r\cos\theta & h_\varphi = r\sin\theta & \end{array}$$

The acoustic equation with constant density has now the form

$$\frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial p}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial p}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\sin\theta}\frac{\partial p}{\partial \varphi}\right)\right] - c^{-2}(r,\theta,\varphi)\frac{\partial^2 p}{\partial t^2} = \bar{F}(r,\theta,\varphi,t).$$

After an arrangement

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial p}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 p}{\partial \varphi^2} - c^{-2}(r,\theta,\varphi)\frac{\partial^2 p}{\partial t^2} = \bar{F}(r,\theta,\varphi,t).$$

Let us now consider a homogenous medium $c = c(r,\theta,\varphi) = \text{const}$. At the point $r = 0$ in this medium, we consider a spherically symmetric *point source* of pressure waves so that $p = p(r,t)$. Then, outside $r = 0$, the above equation reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p(r,t)}{\partial r}\right) - c^{-2}\frac{\partial^2 p(r,t)}{\partial t^2} = 0 \quad .$$

This equation can be rewritten in several forms. The obvious one is

$$\frac{2}{r}\frac{\partial p(r,t)}{\partial r} + \frac{\partial^2 p(r,t)}{\partial r^2} - c^{-2}\frac{\partial^2 p(r,t)}{\partial t^2} = 0 \quad .$$

We can see that this equation differs from the 1-D equation for plane waves by the first term. This term will vanish for $r \to \infty$. Thus, far away from the source situated at $r = 0$, we can expect propagation of nearly plane waves.

Instead of variable $p(r,t)$, we can introduce a new independent variable $rp(r,t)$. Then the above equation takes the following form

$$\frac{\partial^2 rp(r,t)}{\partial r^2} - c^{-2}\frac{\partial^2 rp(r,t)}{\partial t^2} = 0 \quad .$$

This is 1-D wave equation whose solution can be written as

$$rp(r,t) = P_1 F\left(t - \frac{r}{c}\right) + P_2 F\left(t + \frac{r}{c}\right) \quad .$$

The expression for $rp(r,t)$ yields

$$p(r,t) = \frac{P_1}{r} F\left(t - \frac{r}{c}\right) + \frac{P_2}{r} F\left(t + \frac{r}{c}\right) \quad .$$

The solution is a wave with concentric phase fronts - the *spherical wave*. At $r = 0$, it has an infinite amplitude which decreases with increasing r. Note that when we investigated plane wave propagation in homogeneous media, we considered only the solution represented by the first term on the right-hand side of the above equation. This term describes a wave propagating away from the source, which we call the *outgoing wave*. The second term describes a wave propagating towards the source and we call it the *ingoing wave*. In the following sections, we shall mostly work with outgoing waves, i.e. with $P_2 = 0$. We shall thus try to find the constant P_1 for each specific situation.

4.2.2 Cylindrical coordinates

We introduce cylindrical coordinates r, φ, z as follows

$$\begin{aligned} x_1 &= r\cos\varphi & h_r &= 1 \\ x_2 &= r\sin\varphi & h_\varphi &= r \\ x_3 &= z & h_z &= 1 \end{aligned}$$

Then, the acoustic equation has a form

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{r}\frac{\partial p}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial p}{\partial z}\right)\right] - c^{-2}(r,\varphi,z)\frac{\partial^2 p}{\partial t^2} = \bar{F}(r,\varphi,z)\,.$$

After an arrangement

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \varphi^2} + \frac{\partial^2 p}{\partial z^2} - c^{-2}(r,\varphi,z)\frac{\partial^2 p}{\partial t^2} = \bar{F}(r,\varphi,z)\,.$$

In a similar way as in the case of spherical coordinates, we could look for the solution of the above equation in a homogeneous medium. This time, we would consider a cylindrical symetric source of pressure so that $p = p(r,t)$. Now the source is concentrated to the line $r = 0$, therefore we speak about the *line source*. The solution has more complicated form than in the spherical coordinates. The solutions are the *cylindrical waves*.

4.3 Fourier method of separation of variables

Fourier method of separation of variables is applied to linear partial differential equations and is based on the assumption that a particular solution of such a differential equation, which is a function of several variables, can be expressed as a product of functions, each of them being function of only one variable,

$$A = A(b,c) = B(b)C(c) \quad .$$

Linear partial differential equations can be substantially simplified by this assumption. They can be reduced to a system of *ordinary* differential equations which are much easier to solve. By linearly combining the particular solutions of the ordinary differential equations, we can form more general solutions. The coefficients of the linear combination are then chosen such that they satisfy given boundary and/or initial conditions.

A special case of the separation of variables is separation of *time* in space-time partial differential equations, which leads to the transformation of the original equation to the *frequency domain*. We are going to discuss it in the next section both for the acoustic and the elastodynamic equations. In the following sections, we apply the Fourier method of separation of variables to the separation of the *spatial* variables in the acoustic equation in Cartesian, spherical and cylindrical coordinates.

4.3.1 Separation of time

First we consider the acoustic equation with constant density and with the zero source term in the Cartesian coordinates (i.e. we investigate the equation outside the source region)

$$\frac{\partial^2 p}{\partial x_i^2} - c^{-2}(x_m)\frac{\partial^2 p}{\partial t^2} = 0 \quad .$$

We assume that pressure $p(x_m, t)$ can be sought in the form

$$p(x_m, t) = P(x_m)T(t) \quad .$$

Inserting this into the acoustic equation, we get

$$\frac{\partial^2 P(x_m)}{\partial x_i^2}T(t) - c^{-2}(x_m)P(x_m)\frac{d^2T}{dt^2} = 0 \quad ,$$

which can be rewritten

$$c^2(x_m)\frac{\partial^2 P(x_m)}{\partial x_i^2}P^{-1}(x_m) = \frac{d^2T(t)}{dt^2}T^{-1}(t) \quad .$$

The expression on the LHS (left-hand side) depends only on x_m, the RHS depends only on t. For an arbitrary choice of x_m and t, the LHS and RHS are equal, which implies that they must be constant. The constant can be either positive or negative. We shall choose it negative, equal $-\omega^2$. For positive choice, we would get physically unacceptable solution. We thus have

$$\frac{\partial^2 P}{\partial x_i^2} + k^2 P = 0\,, \quad \frac{d^2T}{dt^2} + \omega^2 T = 0\,, \quad k = \frac{\omega}{c}\,.$$

The equation for P is the frequency domain form of the wave equation which is called the *Helmholtz equation*. We can write its solution as $P = P(x_m, \omega)$. The equation for time is *the equation for the harmonic oscillator* and has a simple solution

$$T(t) = T_1(\omega)e^{i\omega t} + T_2(\omega)e^{-i\omega t} \quad .$$

A particular solution of the acoustic equation (wave equation) is thus

$$p(x_m, t) = P(x_i, \omega)[(T_1(\omega)e^{i\omega t} + T_2(\omega)e^{-i\omega t}] \quad .$$

Due to the linearity of the acoustic equation, a general solution can be written in the form of an integral superposition

$$p(x_m, t) = \int_{-\infty}^{\infty} S(\omega) P(x_i, \omega) e^{-i\omega t} d\omega \quad ,$$

where $S(\omega)$ is a frequency dependent coefficient to be determined from the initial and/or boundary conditions.

Let us now consider the elastodynamic equation with the zero source term

$$L_i(u_k) = \rho u_{i,tt} \quad .$$

We assume again that $u_k(x_m, t)$ can be written as

$$u_k(x_m, t) = U_k(x_m) T(t) \quad .$$

Inserting this into the above equation, we get

$$T L_i(U_k) = \rho U_i \frac{d^2 T}{dt^2} \quad ,$$

which can be rewritten as
$$\frac{L_i(U_k) U_i}{\rho U_m U_m} = \frac{d^2 T}{dt^2} T^{-1} = \text{const} \ .$$

After the same considerations as in the case of the acoustic equation, we can split the elastodynamic equation into two equations

$$L_i(U_k) + \rho\omega^2 U_i = 0 \quad , \quad \frac{d^2 T}{dt^2} + \omega^2 T = 0 \quad .$$

A particular solution can thus be written in the form

$$u_k(x_m, t) = U_k(x_m, \omega)[T_1(\omega) e^{i\omega t} + T_2(\omega) e^{-i\omega t}] \quad .$$

The general solution can be again obtained in the form of an integral

$$u_k(x_m, t) = \int_{-\infty}^{\infty} S(\omega) U_k(x_m, \omega) e^{-i\omega t} d\omega$$

with the coefficient $S(\omega)$ to be determined from the boundary and/or initial conditions.

Thus, in cases of acoustic and elastodynamic equations, we can separate the time variable and in this way we can transform the equation into the frequency domain, where it can be solved. A general solution is then obtained as a sum or integral over individual frequency contributions, see above.

4.3.2 Acoustic equation in Cartesian coordinates

Let us consider the Helmholtz equation

$$\frac{\partial^2 P}{\partial x_i^2} + k^2 P = 0 \quad ,$$

in which we assume that the wave number k is a function of only one spatial coordinate, say z, i.e. $k = k(z) = \omega/c(z)$. We now apply the Fourier method of separation of variables to the spatial variables and we assume that

$$P(x_i, \omega) = X(x,\omega)Y(y,\omega)Z(z,\omega) \quad .$$

Here we used the following notation: $x = x_1$, $y = x_2$, $z = x_3$. We insert the above expression into the Helmholtz equation:

$$\frac{d^2X}{dx^2}YZ + \frac{d^2Y}{dy^2}XZ + \frac{d^2Z}{dz^2}XY + k^2(z)XYZ = 0 \quad .$$

This can be rewritten as follows

$$X^{-1}\frac{d^2X}{dx^2} + Y^{-1}\frac{d^2Y}{dy^2} + Z^{-1}\frac{d^2Z}{dz^2} + k^2(z) = 0 \quad .$$

As before, we can see that the first term on the LHS is function of x only, the second of y only and the third and fourth terms of the z only. We can introduce constants of separation $-k_x^2$, $-k_y^2$ so that

$$\frac{d^2X}{dx^2} + k_x^2 X = 0\,, \quad \frac{d^2Y}{dy^2} + k_y^2 Y = 0\,, \quad \frac{d^2Z}{dz^2} + (k^2(z) - k_x^2 - k_y^2)Z = 0 \quad .$$

With the notation

$$k_z^2(z) = k^2(z) - k_x^2 - k_y^2 = \frac{\omega^2}{c^2(z)} - k_x^2 - k_y^2$$

we can rewrite the last equation in the form similar to that of the former two equations,

$$\frac{d^2Z}{dz^2} + k_z^2(z)Z = 0 \quad .$$

Note, however, that the parameter k_z is varying with z. From the definition of $k_z^2(z)$, we get

$$k_x^2 + k_y^2 + k_z^2 = k^2 \quad .$$

From this, we can see that k_x, k_y and k_z are x, y and z components of the wave vector $k_i = \omega p_i$, see Sec.3.1.1. Here p_i is the slowness vector.

Equations for X and Y are again equations for a harmonic oscillator, which have the solutions

$$X(x, k_x, \omega) = X_1 e^{ik_x x} + X_2 e^{-ik_x x}, \quad Y(y, k_y, \omega) = Y_1 e^{ik_y y} + Y_2 e^{-ik_y y}.$$

where $X_I = X_I(k_x, \omega)$, $Y_I = Y_I(k_y, \omega)$, $I = 1, 2$.

The ordinary linear differential equation of the second order for Z has two linearly independent solutions:

$$Z_1 = Z_1(z, k_x, k_y, \omega), \quad Z_2 = Z_2(z, k_x, k_y, \omega) \quad .$$

For some simple velocity-depth functions $c = c(z)$, the solutions Z_1 and Z_2 can be found in an analytical form. For the determination of others, a numerical procedure must be used.

If we take into account that k_x and k_y may be any real numbers from the interval $(-\infty, \infty)$, we can write the particular solution of the Helmholtz equation in the form

$$\begin{aligned} P(x_i, \omega) &= W_1(k_x, k_y, \omega) e^{ik_x x + ik_y y} Z_1(z, k_x, k_y, \omega) \\ &+ W_2(k_x, k_y, \omega) e^{ik_x x + ik_y y} Z_2(z, k_x, k_y, \omega). \end{aligned}$$

When we take into account that ω can attain any real value form the interval $(-\infty, \infty)$, the particular solution of the *acoustic equation* for pressure $p(x_m, t)$ can be written as

$$\begin{aligned} p(x_m, t) &= W_1(k_x, k_y, \omega) e^{-i\omega t + ik_x x + ik_y y} Z_1(z, k_x, k_y, \omega) \\ &+ W_2(k_x, k_y, \omega) e^{-i\omega t + ik_x x + ik_y y} Z_2(z, k_x, k_y, \omega). \end{aligned}$$

The choice of the exponent $(-i\omega t)$ in this form will become clear later.

We can now write the general solution of the homogeneous acoustic equation with constant density as follows

$$\begin{aligned} p(x_m, t) &= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; W_1 Z_1 e^{-i\omega t + ik_x x + ik_y y} \\ &+ \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; W_2 Z_2 e^{-i\omega t + ik_x x + ik_y y}. \end{aligned}$$

This is a general solution for the case of an arbitrary velocity-depth distribution of velocity $c = c(z)$. We can see that the solution of the acoustic equation was reduced to the solution of a linear ordinary diferential equation of the second order for z and to the subsequent 3-D integration.

In case of constant velocity, $c = \text{const}$, the ordinary differential equation for z reduces to the equation for the harmonic oscillator, the two linearly indepedent solutions of which are

$$Z_1 = e^{ik_z z} \quad , \quad Z_2 = e^{-ik_z z} \quad .$$

Here k_z is given by the formula

$$k_z = \left(\frac{\omega^2}{c^2} - k_x^2 - k_y^2\right)^{1/2} \quad .$$

The general solutions in this case reads

$$\begin{aligned} p(x_m, t) \; = \; & \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; W_1(k_x, k_y, \omega) e^{-i\omega t + ik_x x + ik_y y + ik_z z} \\ + & \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; W_2(k_x, k_y, \omega) e^{-i\omega t + ik_x x + ik_y y - ik_z z} . \end{aligned}$$

If we realize that the exponential terms can be rewritten as follows

$$e^{-i2\pi f(t - p_x x - p_y y - p_z z)} \quad , \quad e^{-i2\pi f(t - p_x x - p_y y + p_z z)}$$

we can see that the above formula represents an expansion of the solution of the acoustic equation into plane waves. The first term contains plane waves propagating in the positive direction of the z-axis and the second term the plane waves propagating against the positive direction of the z-axis. In the following, we shall consider that the positve orientation of the z-axis is downward, i.e. the positive z-axis represents the depth axis. Then the first term describes *downgoing* waves, the second *upgoing* waves. In the following we shall study the above integrals in the halfspace $z > 0$. From the mathematical point of view, the above formula has a form of the inverse 3-D Fourier transform (there are only missing some scaling factors in front of integrals), which transforms the variables ω, k_x and k_y into the variables t, x and y, respectively. The quantity k_z is not transformation variable. There is no integration over k_z and, in fact, k_z is not an independent variable, it is a function of k_x and k_y. Now it is also clear why we chose the exponent $(-i\omega t)$ in the exponential function in the general solution. It corresponds to the inverse Fourier transform introduced earlier.

The plane waves in the above expansion are homogeneous waves when

$$k_x^2 + k_y^2 \leq k^2 \Longleftrightarrow p_x^2 + p_y^2 \leq c^{-2}$$

and inhomogeneous waves when

$$k_x^2 + k_y^2 > k^2 \Longleftrightarrow p_x^2 + p_y^2 > c^{-2} .$$

In the latter case, k_z must be considered in the form

$$k_z = \pm i(k_x^2 + k_y^2 - k^2)^{1/2} \quad .$$

The choice of the sign must be done so that the amplitudes of inhomogeneous waves decay with increasing distance from the source. This requirement is satisfied when the sign "+" is selected for downgoing waves and the sign "- " for upgoing waves.

The above expansion formula has various important applications. We are going to consider two of them.

4.3.2.1 Wavefield extrapolation

Let us assume that we know the wavefield at some depth $z = z_o$, i.e. that we know $p(x, y, z_o, t)$ and let us extrapolate the wavefield from the depth z_o to a depth z_1, where $z_1 > z_o$. If we use *downgoing* waves, we speak about *forward extrapolation.* If we use the *upgoing* waves, we speak about *backward extrapolation.* Both these concepts are very important in the so-called $(\omega - k)$ migration.

Let us consider the forward extrapolation, i.e. let us consider only downgoing waves in the expansion formula. The expansion formula yields at $z = z_o$

$$p(x, y, z_o, t) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y W_1(k_x, k_y, \omega) e^{-i\omega t + ik_x x + ik_y y + ik_z z_o} \quad .$$

We can also express the wavefield at $z = z_o$ in the form of 3-D inverse Fourier transform

$$p(x, y, z_o, t) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y S(k_x, k_y, \omega, z_o) e^{-i\omega t + ik_x x + ik_y y} \quad .$$

(The factor $(2\pi)^3$ is a consequence of the use of the Fourier transform with the variable ω instead of f.) By comparing the above expressions for $p(x, y, z_o, t)$, we get

$$S(k_x, k_y, \omega, z_o) = (2\pi)^3 W_1(k_x, k_y, \omega) e^{ik_z z_o} \quad ,$$

from which

$$W_1(k_x, k_y, \omega) = \frac{1}{(2\pi)^3} S(k_x, k_y, \omega, z_o) e^{-ik_z z_o} \quad .$$

In this way, we determined $W_1(k_x, y_x, \omega)$ since $S(k_x, k_y, \omega, z_o)$ is the Fourier transform of the known wavefield at $z = z_o$,

$$S(k_x, k_y, \omega, z_o) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, p(x, y, z_o, t) e^{i\omega t - ik_x x - ik_y y} \quad .$$

The wavefield at any depth z_1 can now be determined from the expansion formula (by inserting the above expression for W_1):

$$p(x,y,z_1,t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y\, S(k_x,k_y,\omega,z_o)e^{-i\omega t+ik_x x+ik_y y+ik_z(z_1-z_o)} .$$

Thus, the above formula represents the extrapolation formula for the wavefield from depth $z = z_o$ to the depth $z = z_1$. The extrapolation in the $\omega - k$ domain is very simple. If the spectrum of the wavefield is known at the depth $z = z_o$, then the spectrum $S(k_x,k_y,\omega,z_1)$ at the depth $z = z_1$ is obtained simply by the multiplication of $S(k_x,k_y,\omega,z_o)$ by $e^{ik_z(z_1-z_o)}$, i.e., it is obtained by the phase shift $k_z(z_1 - z_o)$,

$$S(k_x,k_y,\omega,z_1) = S(k_x,k_y,\omega,z_o)e^{ik_z(z_1-z_o)} \quad .$$

In exactly the same way, we could proceed in the case of the backward extrapolation of an upgoing wave. The multiplicative factor would then be $e^{-ik_z(z_1-z_o)}$ for $z_1 > z_0$.

4.3.2.2 Expansion of a spherical wave into plane waves

In a way similar to the preceding section, we shall determine the coefficient $W_1(k_x,k_y,\omega)$ in the case of a downgoing spherical wave. In Sec.4.2.1, we found that the outgoing spherical wave can be expressed as

$$p(x_m,t) = \frac{P_1}{r}F\left(t - \frac{r}{c}\right) \quad .$$

We shall now consider a harmonic wave with the unit amplitude $P_1 = 1$. Then, we can write

$$p(x_m,\omega,t) = \frac{e^{-i\omega(t-r/c)}}{r} \quad .$$

The expansion formula for a downgoing harmonic wave reads

$$p(x_m,\omega,t) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y W_1(k_x,k_y,\omega)e^{-i\omega t+ik_x x+ik_y y+ik_z z} ,$$

where

$$k_z = \left(\frac{\omega^2}{c^2} - k_x^2 - k_y^2\right)^{1/2} \quad .$$

Since ω is not an integration variable, this can be rewritten

$$p(x_m,\omega,t) = e^{-i\omega t}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y W_1(k_x,k_y,\omega)e^{ik_x x+ik_y y+ik_z z} .$$

We can evaluate $W_1(k_x,k_y,\omega)$ at an arbitrary depth z. The simplest way is to do it at $z = 0$. There the spherical wave has the form

$$p(x,y,0,\omega,t) = \frac{e^{-i\omega(t-\sqrt{x^2+y^2}/c)}}{\sqrt{x^2+y^2}} \quad .$$

The expansion formula has for $z=0$ the form

$$p(x,y,0,\omega,t) = e^{-i\omega t}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y W_1(k_x,k_y,\omega) e^{ik_x x + ik_y y} \quad .$$

As in Sec.4.3.2.1, we shall find W_1 from its relation to the spectrum of the spherical wave. The spherical wave at $z=0$ can be written as

$$p(x,y,0,t) = \frac{e^{-i\omega(t-\sqrt{x^2+y^2}/c)}}{\sqrt{x^2+y^2}} = e^{-i\omega t}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y S(k_x,k_y,\omega,z=0) e^{ik_x x + ik_y y} \quad .$$

Here $S(k_x,k_y,\omega,0)$ is the 2-D Fourier transform of the expression for the harmonic spherical wave

$$S(k_x,k_y,\omega,0) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{e^{i\omega\frac{\sqrt{x^2+y^2}}{c}}}{\sqrt{x^2+y^2}} e^{-ik_x x - ik_y y} \quad .$$

By comparing the expansion formula with the above inverse Fourier transform expression, we can find that

$$W_1(k_x,k_y,\omega) = \frac{1}{(2\pi)^2} S(k_x,k_y,\omega,0) \quad .$$

Thus, to determine W_1, we need to evaluate the expression for $S(k_x,k_y,\omega,0)$. We introduce polar coordinates in the plane (x,y) so that

$$k_x = q\cos\psi\ , \quad k_y = q\sin\psi\ , \quad q^2 = k_x^2 + k_y^2\ ,$$
$$x = \rho\cos\varphi\ , \quad y = \rho\sin\varphi\ , \quad dxdy = \rho d\rho d\varphi \quad .$$

The expression for $S(k_x,k_y,\omega,0)$ now reads

$$S(k_x,k_y,\omega,0) = \int_0^{2\pi} d\varphi \int_0^{\infty} d\rho e^{i\rho\left[\frac{\omega}{c} - q\cos(\psi-\varphi)\right]} \quad .$$

We can simply evaluate the integral with respect to ρ. This yields

$$S(k_x,k_y,\omega,0) = \int_0^{2\pi} d\varphi \left[\frac{e^{i\rho\left[k-q\cos(\psi-\varphi)\right]}}{i[k-q\cos(\psi-\varphi)]}\right]_0^{\infty} \quad .$$

We assume that the medium is very slightly absorbing, i.e. the velocity c is complex-valued and c^{-1} has a small positive imaginary part. Then the limit $\rho\to\infty$ yields zero and we have

$$S(k_x,k_y,\omega,0) = \int_0^{2\pi} d\varphi \frac{i}{[k-q\cos(\psi-\varphi)]} = \int_0^{2\pi} \frac{id\delta}{k-q\cos\delta} \quad .$$

For the above integral, we can find

$$\int_0^{2\pi} \frac{id\delta}{k-q\cos\delta} = \frac{2\pi i}{\sqrt{k^2-q^2}} \quad \text{for } q^2 < k^2$$

and

$$\int_0^{2\pi} \frac{id\delta}{k - q\cos\delta} = \frac{i(-2\pi i)}{\sqrt{q^2 - k^2}} \quad \text{for } q^2 > k^2 \quad .$$

Thus, for $S(k_x, k_y, \omega, 0)$, we can write

$$S(k_x, k_y, \omega, 0) = \int_0^{2\pi} \frac{id\delta}{k - q\cos\delta} = \frac{2\pi i}{k_z} \quad ,$$

where

$$k_z = \sqrt{k^2 - q^2} \quad \text{for } q^2 < k^2 \quad ,$$
$$k_z = i\sqrt{q^2 - k^2} \quad \text{for } q^2 > k^2 \quad .$$

For $W_1(k_x, k_y, \omega)$ this yields

$$W_1(k_x, k_y, \omega) = \frac{i}{2\pi k_z} \quad .$$

If we insert this into the general expansion formula for the downgoing wave, we get

$$r^{-1}e^{-i\omega(t-r/c)} = \frac{i}{2\pi}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y k_z^{-1} e^{-i\omega t + ik_x x + ik_y y + ik_z z} \quad .$$

This is the *Weyl integral* which offers an expansion of a spherical wave into plane waves. The Weyl integral is known in many alternative forms. Let us show some of them. If we use the relation between the wave vector and the slowness vector which has been used earlier in this chapter,

$$(k_x, k_y, k_z) = \omega(p_x, p_y, p_z) \quad ,$$

then

$$r^{-1}e^{-i\omega(t-r/c)} = \frac{i\omega}{2\pi}\int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y\, p_z^{-1} e^{-i\omega(t - p_x x - p_y y - p_z z)}$$

or

$$r^{-1}e^{-i2\pi f(t-r/c)} = if\int_{-\infty}^{\infty} dp_x \int_{-\infty}^{\infty} dp_y\, p_z^{-1} e^{-i2\pi f(t - p_x x - p_y y - p_z z)} \quad .$$

Here

$$p_z = (c^{-2} - p_x^2 - p_y^2)^{1/2} \quad .$$

Instead of p_x and p_y, we can introduce polar coordinates p and ϕ ($p \geq 0$, $0 \leq \phi \leq 2\pi$),

$$p_x = p\cos\phi \quad , \quad p_y = p\sin\phi \quad .$$

For this specification, we have

$$dp_x dp_y = p dp d\phi \quad , \quad p_z = \sqrt{c^{-2} - p^2} \quad .$$

Then the Weyl integral attains the form

$$r^{-1}e^{-i\omega(t-r/c)} = \frac{i\omega}{2\pi}e^{-i\omega t}\int_0^\infty \left[\frac{pe^{i\omega\sqrt{c^{-2}-p^2}z}}{\sqrt{c^{-2}-p^2}}\int_0^{2\pi} e^{i\omega p(x\cos\phi+y\sin\phi)}d\phi\right] dp\,.$$

If we express also x and y in the polar coordinates

$$x = \rho\cos\varphi \quad , \quad y = \rho\sin\varphi \quad ,$$

the integral with ϕ can be rewritten as follows

$$\int_0^{2\pi} e^{i\omega p\rho\cos(\phi-\varphi)}d\phi = 2\pi\mathcal{J}_o(\omega p\rho)\,,$$

where

$$\mathcal{J}_o(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\cos t}dt$$

is the Bessel function of the zero order and the first kind. The 2-D Weyl integral thus reduces to the single integral over p,

$$r^{-1}e^{-i\omega(t-r/c)} = i\omega e^{-i\omega t}\int_0^\infty \frac{pe^{i\omega\sqrt{c^{-2}-p^2}z}}{\sqrt{c^{-2}-p^2}}\mathcal{J}_o(\omega p\rho)dp\,.$$

This is the well known *Sommerfeld integral.* It represents the expansion of a spherical wave into cylindrical waves. We shall obtain it again in Sec.4.3.4 when we shall solve the acoustic equation in cylindrical coordinates.

Let us return to the last form of the Weyl integral. As we noticed before, the expansion integral contains the homogeneous and inhomogeneous waves. The wave under the above integral is homogeneous when

$$p_x^2 + p_y^2 < c^{-2} \Leftrightarrow p < c^{-1} \quad .$$

The wave is inhomogeneous when

$$p_x^2 + p_y^2 > c^{-2} \Leftrightarrow p > c^{-1} \quad .$$

In the former case

$$p_z = \sqrt{c^{-2} - p^2} \quad ,$$

while in the case of an inhomogeneous wave

$$p_z = i\sqrt{p^2 - c^{-2}} \quad .$$

The choice of the positive sign in front of i guarantees decay of amplitudes with increasing depth, i.e. with increasing distance from the source. The Weyl integral can be thus expressed as a sum of two integral superpositions, the first being the superposition of homogeneous waves and the second, the superposition of inhomogeneous waves.

$$r^{-1}e^{-\imath\omega(t-r/c)} = \frac{\imath\omega}{2\pi}e^{-\imath\omega t}\int_0^{c^{-1}}\left[\frac{pe^{\imath\omega\sqrt{c^{-2}-p^2}\,z}}{\sqrt{c^{-2}-p^2}}\int_0^{2\pi}e^{\imath\omega p(x\cos\phi+y\sin\phi)}d\phi\right]dp$$
$$+\ \frac{\omega}{2\pi}e^{-\imath\omega t}\int_{c^{-1}}^{\infty}\left[\frac{pe^{-\omega\sqrt{p^2-c^{-2}}\,z}}{\sqrt{p^2-c^{-2}}}\int_0^{2\pi}e^{\imath\omega p(x\cos\phi+y\sin\phi)}d\phi\right]dp\,.$$

The superposition of only homogeneous plane waves, would alone give finite value of the wavefield at the source. The existence of inhomogeneous plane waves causes the necessary singularity at the source.

Let us note that the above expansion formulae are often used for the investigation of an important problem of incidence of a spherical wave at a plane interface. The spherical wave incident at the interface is expanded into homogeneous and inhomogeneous plane waves. These individual plane waves interact with the interface and generate waves, which are again superposed using the above expansion formulae. This problem is referred in literature as the Lamb problem.

4.3.3 Acoustic equation in spherical coordinates

We consider again the Helmholtz equation, this time specified in spherical coordinates. For this case the equations of Sec.4.2.1 yield

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 P}{\partial\varphi^2}+k^2(r)P=0\,,$$

where $P = P(r,\theta,\varphi)$. As shown in the above equation, we consider that $k = k(r)$ and $c = c(r)$, i.e. that the velocity distribution varies only with the radius. We perform the Fourier separation of variables by assuming that

$$P(r,\theta,\varphi) = R(r)\Theta(\theta)\phi(\varphi)\quad .$$

Inserting this into the Helmholtz equation, we get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)\Theta\phi+\frac{R\phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{R\Theta}{r^2\sin^2\theta}\frac{d^2\phi}{d\varphi^2}+k^2R\Theta\phi=0\,.$$

Multiplying this equation by $r^2(R\Theta\phi)^{-1}$, we get

$$R^{-1}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+k^2(r)r^2+\frac{\Theta^{-1}}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{\phi^{-1}}{\sin^2\theta}\frac{d^2\phi}{d\varphi^2}=0\,.$$

The first two terms depend on r only while the remaining two terms depend on θ and φ. We can thus separate the two first terms and denote the separation constant as α. We thus get

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+[k^2(r)r^2-\alpha]R=0 \quad ,$$

$$\Theta^{-1}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\alpha\sin^2\theta+\phi^{-1}\frac{d^2\phi}{d\varphi^2}=0 \quad .$$

We can see that the second equation can be further separated. If we denote the separation variable as β, we get

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(\alpha-\frac{\beta}{\sin^2\theta}\right)\Theta=0 \quad ,$$

$$\frac{d^2\phi}{d\varphi^2}+\beta\phi=0 \quad .$$

The first equation resembles the equation for the associated Legendre functions $P_l^m(\cos\theta)$, which has the form

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_l^m}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]P_l^m(\cos\theta)=0 \quad .$$

Here l and m are integers. By comparison of our equation for Θ with the above Legendre equation, we can see that we must specify the constants of separation and $\Theta(\theta)$ as follows:

$$\alpha=l(l+1)\,,\quad \beta=m^2 \quad \text{and} \quad \Theta(\theta)=P_l^m(\cos\theta)\,.$$

In this way we separated the acoustic equation in the frequency domain into the above three ordinary differential equations.

The equation for Θ is the equation for the associated Legendre functions $P_l^m(\cos\theta)$, which can be expressed as follows

$$P_l^m(x)=(1-x^2)^{m/2}\frac{1}{2^l l!}\frac{d^{l+m}}{dx^{l+m}}[(x^2-1)^l] \quad .$$

We can see that the associated Legendre polynomials are nonzero when $m\le l$ (the polynomial of the degree $2l$ in the square brackets becomes after the differentiation of the degree $l-m$; it will be of zero or positive degree when $m\le l$). Let us write explicitely several Legendre functions for low values of l and m:

$$\begin{aligned}
P_o^o &= 1 \\
P_1^o &= x \qquad\qquad P_1^1=\sqrt{1-x^2} \\
P_2^o &= \frac{1}{2}(3x^2-1) \quad P_2^1=3\sqrt{1-x^2}x \quad P_2^2=3(1-x^2)
\end{aligned}$$

The equations for $R(r)$ and $\phi(\varphi)$ have the final form

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + [r^2k^2(r) - l(l+1)]R = 0 \quad ,$$

$$\frac{d^2\phi}{d\varphi^2} + m^2\phi = 0 \quad .$$

The last equation is again equation of the harmonic oscillator and its particular solution is

$$\phi(\varphi) = \phi_1 e^{\imath m\varphi} + \phi_2 e^{-\imath m\varphi} \quad .$$

The equation for R can be rewritten as follows. We shall assume a homogeneous medium so that the velocity c as well as the wave number k are constants. Then we get

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(k^2 - \frac{l(l+1)}{r^2}\right)R = 0 \quad .$$

Let us introduce the substitution $R = r^{-1/2}J(r)$. If we also substitute the argument r by $\xi = kr$, we get

$$\frac{d^2J(\xi)}{d\xi^2} + \frac{1}{\xi}\frac{dJ(\xi)}{d\xi} + \left(1 - \frac{(l+1/2)^2}{\xi^2}\right)J(\xi) = 0 \quad .$$

This is the *Bessel differential equation*, the solutions of which are the *Bessel spherical functions of the first kind* $J_{l+1/2}(kr)$, $J_{-l-1/2}(kr)$. A particular solution of the equation for R can thus be written

$$R(r) = r^{-1/2}[R_1 J_{l+1/2}(kr) + R_2 J_{-l-1/2}(kr)] \quad ,$$

where R_1, R_2 are constants. For $l = 0$, the Bessel spherical functions are

$$J_{1/2}(kr) = \sqrt{\frac{2}{\pi kr}}\sin kr \quad , \quad J_{-1/2}(kr) = \sqrt{\frac{2}{\pi kr}}\cos kr \quad .$$

The above functions can, possibly, be substituted by the *Hankel functions of the first and second kind* $H^{(1)}_{-l-1/2}(kr)$, $H^{(2)}_{-l-1/2}(kr)$. For $l = 0$, they read

$$H^{(1)}_{-1/2}(kr) = \sqrt{\frac{2}{\pi kr}}e^{ikr} \quad , \quad H^{(2)}_{-1/2}(kr) = \sqrt{\frac{2}{\pi kr}}e^{-ikr} \quad .$$

A particular solution of the acoustic equation in spherical coordinates can be now written as follows:

$$p(r,\theta,\varphi,t) = r^{-1/2}[R_1 J_{l+1/2}(kr) + R_2 J_{-l-1/2}(kr)]Y_l(\theta,\varphi)e^{-i\omega t} \quad ,$$

where

$$Y_l(\theta,\varphi) = \sum_{m=0}^{l} P_l^m(\cos\theta)(\phi_1 \sin m\varphi + \phi_2 \cos m\varphi)$$

is the *surface spherical harmonics of the degree l.* For $l = 0$, i.e. for total spherical symmetry, the particular solution reduces to

$$p(r,t) = \sqrt{\frac{2}{\pi k}}\frac{1}{r}(\bar{R}_1 e^{ikr} + \bar{R}_2 e^{-ikr})\phi_2 e^{-i\omega t} \quad ,$$

which, if we consider only outgoing wave (i.e. $\bar{R}_2 = 0$) and concentrate all the constants into one, denoted C, gives

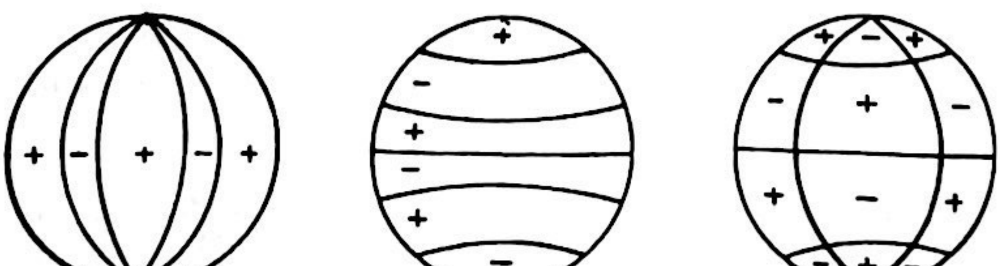
$$p(r,t) = \frac{C}{r} e^{-i\omega t + ikr} \quad ,$$

which is the spherical wave derived in Sec.4.2.1. It represents a special case of the particular solution. The general solution of the acoustic equation in spherical coordinates must be sought as a linear combination of the above particular solutions.

Let us briefly comment the properties of the surface spherical harmonics $Y_l(\theta, \varphi)$ and specifically of the *harmonics of the degree l and the order m*,

$$P_l^m(\cos\theta)(\phi_1 \sin m\varphi + \phi_2 \cos m\varphi) \, .$$

In the interval $(0, 2\pi)$ of the variable φ, the functions $\sin m\varphi$ and $\cos m\varphi$ have $2m$ nodal points, i.e., there are $2m$ values of φ for which $\sin m\varphi$ (or $\cos m\varphi$) is zero. Thus, if we move along a parallel on a sphere with given r, we shall observe $2m$ changes of the sign of the surface harmonics during one cycle, see picture A. Such a kind of harmonics is called the *sectorial harmonics*. From the definition of the associate Legendre function $P_l^m(\cos\theta)$ we can see that it is a polynomial of the degree $(l - m)$, which means that it has $(l - m)$ nodal points at which it is zero. Thus moving along the meridian from pole to pole on the sphere with given r, we observe $(l - m)$ changes of the sign of the surface harmonics, see picture B. Such a kind of harmonics is called the *zonal harmonics*. For each harmonics, the whole surface of the sphere is thus divided into sectors with different signs of the surface harmonics. Combination of sectorial and zonal harmonics shown in picture C is called the *tesseral harmonics*.



A - sectorial harmonics B - zonal harmonics C - tesseral harmonics.

4.3.4 Acoustic equation in cylindrical coordinates

The Helmholtz equation in cylindrical coordinates r, φ, z can be derived from the equations of Sec.4.2.2. It reads

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 P}{\partial\varphi^2}+\frac{\partial^2 P}{\partial z^2}+k^2(z)P=0 \quad ,$$

where $P = P(r,\varphi,z)$. We consider the wave number and the velocity to depend on the z-coordinate only. We perform the Fourier separation of variables assuming that

$$P(r,\varphi,z)=R(r)\phi(\varphi)Z(z) \quad .$$

Inserting this into the Helmholtz equation, yields

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)\phi Z+\frac{1}{r^2}\frac{d^2\phi}{d\varphi^2}RZ+\frac{d^2 Z}{dz^2}R\phi+k^2(z)R\phi Z=0 \quad .$$

If we divide the equation by $R\phi Z$, we get

$$\frac{1}{rR}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{1}{r^2\phi}\frac{d^2\phi}{d\varphi^2}+\frac{1}{Z}\frac{d^2 Z}{dz^2}+k^2(z)=0 \quad .$$

The third and fourth terms depend only on z and can thus be separated from the rest of the equation. If α is the constant of separation, we can write

$$\frac{d^2 Z}{dz^2}+(k^2(z)-\alpha)Z=0 \quad .$$

The rest of the equation yields

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{1}{\phi}\frac{d^2\phi}{d\varphi^2}+\alpha r^2=0 \quad .$$

We can see that the last equation can be further separated. If we denote the constant of separation β, we get

$$\frac{d^2\phi}{d\varphi^2}+\beta\phi=0 \quad ,$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)+(\alpha r^2-\beta)R=0 \quad .$$

From the natural requirement on the function $\phi(l)$ that it must be periodic,

$$\phi(\varphi+2\pi)=\phi(\varphi) \quad ,$$

we can deduce that the constant β must be chosen such that the differential equation for ϕ reads

$$\frac{d^2\phi}{d\varphi^2} + l^2\phi = 0 \quad ,$$

where l is an integer constant. The solution of this equation can be written in the well known form,

$$\phi = \phi_1 e^{il\varphi} + \phi_2 e^{-il\varphi} \quad .$$

From the equation for Z, when we compare it with the similar equation which we obtained in Cartesian coordinates, we can see that we can put

$$k^2(z) - \alpha = k_z^2 \quad .$$

Here $k_z(z)$ represents the z-component of the wave vector. The symbol α thus represents a square of the component of the wave vector into the plane perpendicular to the axis z. In the previous section, we denoted this component q. We can thus write

$$\alpha = k^2(z) - k_z^2 = q^2 = \omega^2 p^2 \quad .$$

The equation for z can now be rewritten as

$$\frac{d^2 Z}{dz^2} + \omega^2(c^{-2}(z) - p^2)Z = 0 \quad .$$

This equation has two linearly independent solutions $Z_1(z,p,\omega)$ and $Z_2(z,p,\omega)$. If we assume that the velocity c is constant, then the above equation becomes the equation of the harmonic oscillator and its solution can be written as

$$Z(z,p,\omega) = Z_1 e^{i\omega\sqrt{c^{-2}-p^2}\,z} + Z_2 e^{-i\omega\sqrt{c^{-2}-p^2}\,z} \quad .$$

The equation for R, after the specification of α and β, reads

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\omega^2p^2 - \frac{l^2}{r^2}\right)R = 0 \quad .$$

This equation resembles the *Bessel differential equation*, which has the form

$$\frac{d^2w}{d\xi^2} + \frac{1}{\xi}\frac{dw}{d\xi} + \left(1 - \frac{l^2}{\xi^2}\right)w = 0 \quad .$$

We can get the equation for R into the form of the Bessel equation, if we introduce a new variable ξ instead of r, $\xi = \omega p r$. Then the equation for R reads

$$\frac{d^2R}{d\xi^2} + \frac{1}{\xi}\frac{dR}{d\xi} + \left(1 - \frac{l^2}{\xi^2}\right)R = 0 \quad .$$

The solution of this equation is the Bessel function of the first kind and the order l,

$$R(\xi) = J_l(\xi) \quad .$$

For $\xi \to \infty$, the function has an asymptotics

$$J_l(\xi) \sim \sqrt{\frac{2\pi}{\xi}} \cos\left(\xi - \frac{l\pi}{2} - \frac{\pi}{4}\right) \quad .$$

We can now write a particular solution of the acoustic equation with constant velocity c in cylindrical coordinates,

$$\begin{aligned} p(r,\varphi,z,t) &= J_l(\omega pr)\left[Z_1 e^{i\omega\sqrt{c^{-2}-p^2}z} + Z_2 e^{-i\omega\sqrt{c^{-2}-p^2}z}\right] \\ &\times(\phi_1 e^{il\varphi} + \phi_2 e^{-il\varphi})e^{-i\omega t} \quad . \end{aligned}$$

The general solution can be obtained from the particular solutions by *integration over p and summation over l.*

Let us now consider a particular solution representing a *cylindrical wave.* Such solution must be symmetric arround the axis z, in other words, it must be independent of the angle φ. This can be satisfied when we put $l = 0$. Then, a cylindrical wave can be written as

$$p(r,z,t) = J_o(\omega pr)\left[Z_1 e^{i\omega\sqrt{c^{-2}-p^2}z} + Z_2 e^{-i\omega\sqrt{c^{-2}-p^2}z}\right] e^{-i\omega t} \quad .$$

From the asymptotic behavior of the function J_o for $\omega pr \to \infty$, we can see that the cylindrical waves can propagate in all directions. The first term represents a downgoing cylindrical wave, the second term upgoing cylindrical wave. The phase fronts of these waves form cones around the axis z. For $p < c^{-1}$ we deal with homogeneous cylindrical waves, for $p > c^{-1}$ we deal with inhomogeneous waves. In the latter case, the sign of the expression $\pm i\sqrt{p^2 - c^{-2}}$ must be chosen so that the solution has a physical meaning. The limiting case, $p = c^{-1}$ corresponds to a wave, which has cylindrical phase font. It can be considered as being generated by a line source along the z-axis. From the asymptotic behavior of the function J_o, we can deduce that the cylindrical wave generated by a line source decays as $1/\sqrt{r}$ in contrast to the spherical wave, which decays as $1/r$.

For the solutions of the acoustic equation, which are symmetric around the z-axis, we can write an expansion into cylindrical waves, which reads

$$p(r,z,t) = \int_0^\infty J_o(\omega pr)\left[Z_1(p,\omega)e^{i\omega\sqrt{c^{-2}-p^2}z} + Z_2(p,\omega)e^{-i\omega\sqrt{c^{-2}-p^2}z}\right] e^{-i\omega t}dp \quad .$$

Since the spherical wave is symmetric around the z-axis, we can use the above formula for its expansion into cylindrical waves. Since we consider only downgoing part of the spherical wave generated at r, $z = 0$, we use only the downgoing terms in the expansion , i.e., we put $Z_2 = 0$. Then we get

$$r^{-1}e^{-i\omega(t-\frac{r}{c})} = \int_0^{\infty} Z_1(p,\omega)J_o(\omega pr)e^{-i\omega(t-\sqrt{c^{-2}-p^2}z)}dp \quad .$$

Fortunately, we do not need any complicated procedure for determining the weighting function $Z_1(\omega,p)$. We can determine it by comparing the above formula with the Sommerfeld integral, derived in Sec.4.3.2.2. We get

$$Z_1(p,\omega) = i\omega p/\sqrt{c^{-2}-p^2} \quad .$$

The Sommerfeld integral, similarly as the Weyl integral, could be splitted into two parts, the first containing only homogeneous cylindrical waves and the second only the inhomogeneous cylindrical waves. It is obvious that in the latter case, the square root must be considered in the form $+i\sqrt{p^2-c^{-2}}$.

4.4 Method of integral transform

Method of integral transform is another method applicable to solving linear partial differential equations. Various integral transforms such as the Fourier transform, Laplace transform can be used. In the case of the Fourier transform, the method represents an alternative to the transformation of an equation into the frequency domain using the Fourier method of separation of variables.

Let us again consider the elastodynamic equation from Sec.4.1.2,

$$L_i[u_k(x_m,t)] - \rho(x_m)\partial^2 u_i(x_m,t)/\partial t^2 = f_i(x_m,t) \quad .$$

Application of the Fourier transform to the equation consists of multiplication of the equation by $e^{i\omega t}$ and then its integration from $-\infty$ to $+\infty$ over time t. We get

$$L_i[U_k(x_m,\omega)] + \rho(x_m)\omega^2 U_i(x_m,\omega) = F_i(x_m,\omega) \quad ,$$

where

$$U_k(x_m,\omega) = \int_{-\infty}^{+\infty} u_k(x_m,t)e^{i\omega t}dt\,, \quad F_k(x_m,\omega) = \int_{-\infty}^{+\infty} f_k(x_m,t)e^{i\omega t}dt \quad .$$

We took into account the well-known fact that the Fourier transform of the time derivative represents multiplication by $(-i\omega)$ in the frequency domain

$$\int_{-\infty}^{+\infty} \frac{\partial^2 u_i(x_m,t)}{\partial t^2}e^{i\omega t}dt = -\omega^2\int_{-\infty}^{+\infty} u_i(x_m,t)e^{i\omega t}dt \quad .$$

The equation

$$L_i(U_k) + \rho\omega^2 U_i = F_i$$

is the Fourier transformed equation of motion and it is fully equivalent to the frequency domain equation derived in Sec.4.3.1 (with exception of the source term, which was not considered in Sec.4.3.1). After solving the Fourier transformed equation, we can return to the time domain using the inverse Fourier transform

$$u_i(x_m, \omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U_i(x_m, t)e^{-i\omega t}d\omega \quad .$$

CHAPTER 5

GREEN FUNCTIONS

In the previous chapter, we have shown how to extrapolate solutions of wave and elastodynamic equation from the region, where the wavefield is known either theoretically or by measurements, into another region. We were not interested how the wavefield was generated. In this chapter, we are going to study wavefields generated by special types of point sources. Corresponding solutions of the acoustic or elastodynamic equations are known as the *Green functions*. The Green functions play an important role in many branches of physics. In seismology, for example, they play a basic role in theoretical modelling of earthquake sources. Let us note that all the types of point sources commonly used in seismology like single force, explosive source, double couple force can be determined from the corresponding Green function. In seismic prospecting, the Green functions are used in migration procedures. Another important application of Green functions is, for example, in the boundary integral methods, in which the wavefield is expressed as a superposition of Green functions distributed with different weights over the internal interfaces and the free surface of a model.

Under the Green function in the acoustic case, we understand a response of the acoustic medium to a spherically symmetric point source radiating a δ pulse. In the elastodynamic case, the Green function represents a response of an isotropic or anisotropic medium to a unit point force also radiating the δ pulse.

5.1 Acoustic Green function

We consider the acoustic equation for pressure with constant density (but variable velocity) and with the source term of the following form

$$\frac{\partial^2 G}{\partial x_i^2} - c^{-2}(x_m)\frac{\partial^2 G}{\partial t^2} = -\delta(x_m - x_{om})\delta(t - t_o) \quad .$$

The function $G = G(x_m, t; x_{om}, t_o)$ is the sought acoustic Green function. The first two parameters in its argument belong to the receiver, x_m are coordinates of the receiver and t is the time at the receiver (current time, not arrival time). The second two parameters in the argument of the Green function characterize the source, x_{om} are coordinates of the source and t_o is the time at which the source generates the δ pulse. Since t and t_o represent current time, we can arbitrarily shift it. If we shift the time counting by $-t_o$, we get

$$G(x_m, t - t_o; x_{om}, 0) = G(x_m, t; x_{om}, t_o) \quad ,$$

where $t - t_o$ is the arrival time. Similarly, we can get

$$G(x_m, -t_o; x_{om}, -t) = G(x_m, t; x_{om}, t_o) \quad ,$$

the relation which expresses the *reciprocity of the Green function* with respect to the source and receiver time.

Let us now introduce the *time harmonic Green function* $G_\omega(x_m, x_{om}, \omega, t_o)$ by the following relation

$$G(x_m, x_{om}, \omega, t_o) = \int_{-\infty}^{+\infty} G(x_m, t, x_{om}, t_o) e^{i\omega t} dt \quad ,$$

The time-harmonic Green function is the solution of the Fourier transformed acoustic equation ($t - \omega$ Fourier transform) shown above

$$\frac{\partial^2 G_\omega}{\partial x_i^2} + k^2(x_m) G_\omega = -\delta(x_m - x_{om}) e^{i\omega t_o} \quad ,$$

where

$$k^2(x_m) = \omega^2 / c^2(x_m) \quad .$$

The expression on the RHS of the above equation was obtained using the obvious relation

$$\int_{-\infty}^{+\infty} \delta(t - t_o) e^{i\omega t} dt = e^{i\omega t_o} \quad .$$

Let us now solve the above time-harmonic acoustic equation under the assumption of the *constant velocity* of the medium. The solution will be a spherical wave, which we have studied in Sec.4.2.1. We shall proceed in the same way as we did there. The only difference is that in Sec.4.2.1, we sought the solution in the time domain while here in the frequency domain. We rewrite the above equation in spherical coordinates r, θ, φ and take into account that our problem is spherically symmetric and, therefore, the Green function must depend on radius r only,

$$G_\omega = G_\omega(r, \omega, t_o) \quad ,$$

where

$$r = [(x_m - x_{om})(x_m - x_{om})]^{1/2}$$

is the distance between the source and the receiver. The time-harmonic acoustic equation in spherical coordinates reads

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dG_\omega}{dr} \right) + k^2 G_\omega = -\delta(r) e^{i\omega t_o} \quad .$$

As in Sec.4.2.1, we rewrite the equation as follows

$$\frac{1}{r}\frac{d^2rG_\omega}{dr^2}+k^2G_\omega=-\delta(r)e^{i\omega t_o} \quad .$$

The solution of this equation for $r \neq 0$, i.e. the solution of the homogeneous equation was found in Sec.4.2.1 and it is

$$G_\omega(r,\omega,t_o)=\frac{G_1}{r}e^{ikr}+\frac{G_2}{r}e^{-ikr} \quad .$$

We shall now consider only the outgoing wave so that we put $G_2 = 0$. Thus, the full specification of the Green function requires now only the determination of the value of the constant G_1. The value of G_1 cannot be determined for $r \neq 0$ since then any constant satisfies homogeneous Helmholtz equation. On the other hand, at $r = 0$, the solution has a singularity and it is again impossible to determine G_1 at the point $r = 0$ alone. We can get rid of the δ-function on the right hand side of the Helmholtz equation by integrating the equation over a sphere with its centre at $r = 0$. We shall then determine G_1 by inserting the solution of the homogeneous equation into the resulting formula and by reducing the radius of the sphere to zero.

Let us integrate the Helmholtz equation in Cartesian coordinates over the sphere with the radius ϵ and the centre at $r = 0$. We get

$$\iiint\limits_{V_\epsilon}\frac{\partial^2G_\omega}{\partial x_i^2}dV+k^2\iiint\limits_{V_\epsilon}G_\omega dV=-e^{i\omega t_o} \quad .$$

Applying the Gauss theorem on the first integral on the left hand side, we get

$$\iiint\limits_{V_\epsilon}\frac{\partial^2G_\omega}{\partial x_i^2}dV=\iint\limits_{S_\epsilon}\frac{\partial G_\omega}{\partial x_i}n_i dS=\iint\limits_{S_\epsilon}\frac{dG_\omega}{dr}dS \quad .$$

After inserting

$$G_\omega=\frac{G_1}{r}e^{ikr}$$

into the above integral, we get

$$\iint\limits_{S_\epsilon}\frac{dG_\omega}{dr}dS = \iint\limits_{S_\epsilon}\left(-\frac{G_1}{\epsilon^2}+\frac{G_1}{\epsilon}ik\right)e^{ik\epsilon}dS = 4\pi\epsilon^2\left(-\frac{G_1}{\epsilon^2}+\frac{G_1}{\epsilon}ik\right)e^{ik\epsilon}$$

$$= 4\pi(-G_1+ik\epsilon G_1)e^{ik\epsilon} \quad .$$

The Helmholtz equation thus attains the form

$$4\pi(-G_1+ik\epsilon G_1)e^{ik\epsilon}+k^2\iiint\limits_{V_\epsilon}\frac{G_1}{r}e^{ikr}dV=-e^{i\omega t_o} \quad .$$

Taking the limit $\epsilon \to 0$, the volume integral goes to zero and the equation yields

$$G_1 = \frac{1}{4\pi} e^{i\omega t_o} \quad .$$

Thus the harmonic Green function for the Helmholtz equation reads

$$G_\omega(r, \omega, t_o) = \frac{1}{4\pi r} e^{i\omega(t_o + \frac{r}{c})} \quad .$$

The Green function in time domain, $G(x_m, t; x_{om}, t_o)$ can be obtained from the harmonic Green function by the inverse Fourier transform:

$$\begin{aligned} G(x_m, t; x_{om}, t_o) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} G_\omega(r, \omega, t_o) e^{-i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{4\pi r} e^{-i\omega(t - t_o - \frac{r}{c})} d\omega = \frac{1}{4\pi r} \delta(t - t_o - \frac{r}{c}) \quad , \end{aligned}$$

where we took into account that

$$\delta(t - t_o) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega(t - t_o)} d\omega$$

and

$$r = [(x_i - x_{oi})(x_i - x_{oi})]^{1/2} \quad .$$

From the above expression for the Green function we can immediately see that the Green function is not only *reciprocal* in time, as we have shown before, but also in the *position of the source* x_{om} *and the receiver* x_m,

$$G(x_m, t; x_{om}, t_o) = G(x_{om}, t; x_m, t_o) \quad .$$

Let us note that the form of the Green function depends on the form, in which we consider the acoustic equation. If we, for instance, consider the acoustic equation in the form

$$c^2 \frac{\partial^2 G}{\partial x_i^2} - \frac{\partial^2 G}{\partial t^2} = -\delta(x_m - x_{om}) \delta(t - t_o) \quad ,$$

i.e., with the left hand side of the original acoustic equation multiplied by c^2, the corresponding Green function will be different from the previous one because, in fact, we are using a different source term (which is not equally scaled as the LHS of the equation). The source term in the above equation is c^2 times weaker than in the previous equation and, therefore, the resulting Green function is also c^2 times weaker,

$$G(x_m, t; x_{om}, t_o) = \frac{1}{4\pi r c^2} \delta\left(t - t_o - \frac{r}{c}\right) \quad .$$

We can also show that the above Green function gives us possibility to construct solutions of the acoustic equation with more complicated source terms. For showing this, we shall use an important property of the acoustic equation, its *linearity*. It guarantees that if we know two solutions of the acoustic equation, their linear combination is also a solution. Let us first consider the acoustic equation in the form

$$\frac{\partial^2 \bar{f}}{\partial x_i^2} - c^{-2}\frac{\partial^2 \bar{f}}{\partial t^2} = -\delta(x_m - x_{om})f(t - t_o) \quad ,$$

where f represents a source-time function. We can express it as

$$f(t - t_o) = \int_{-\infty}^{+\infty} f(\tau)\delta(t - t_o - \tau)d\tau \quad .$$

The solution of the above acoustic equation with source-time function $\delta(t - t_o - \tau)$ simply follows from the expression for the Green function,

$$\frac{1}{4\pi r}\delta(t - t_o - \tau - \frac{r}{c}) \quad .$$

The solution $\bar{f}(x_m, t)$ can thus written as the superposition

$$\bar{f}(x_m, t) = \frac{1}{4\pi r}\int_{-\infty}^{+\infty} f(\tau)\delta\left(t - t_o - \tau - \frac{r}{c}\right) dt = \frac{f\left(t - t_o - \frac{r}{c}\right)}{4\pi r} \quad .$$

Let us now consider the following source term in the acoustic equation

$$\frac{\partial^2 \bar{\bar{f}}}{\partial x_i^2} - c^{-2}\frac{\partial^2 \bar{\bar{f}}}{\partial t^2} = -f(x_m, t - t_o) \quad ,$$

where $f(x_m, t-t_o)$ is now spatially dependent source-time function. It represents a continuous distribution of sources in some volume V. For one of these sources, we can write the solution of the acoustic equation in the form, which we found for the source term $-\delta(x_m - x_{om})f(t-t_o)$. The complete solution will be a superposition of solutions for all the sources in the volume V, i.e.

$$\bar{\bar{f}}(x_m, t) = \frac{1}{4\pi}\iiint\limits_V \frac{f(x'_m, t - t_o - \frac{r'}{c})}{r'} dV(x'_m) \quad , \quad r' = |x'_m - x_m| \, .$$

In the beginning of this section, we excluded from our considerations the ingoing wave by putting $G_2 = 0$. In exactly the same way as above for the outgoing wave, we could also derive expression for the ingoing Green function. It reads

$$G(x_m, t, x_{om}, t_o) = \frac{1}{4\pi r}\delta\left(t - t_o + \frac{r}{c}\right) \quad .$$

The ingoing Green function is often used in seismic prospecting, specifically in migration. The outgoing Green function is often called the *exploding* or *causal* Green function, while the ingoing one is called *imploding* or *anticausal*. Let us note that the latter notation does not mean that the ingoing Green function is noncausal.

5.2 Elastodynamic Green function

The elastodynamic Green function $G_{ij}(x_m, t; x_{om}, t_o)$ is defined as a solution of the equation

$$(c_{ijkl}G_{kn,l})_{,j} - \rho G_{in,tt} = -\delta_{in}\delta(x_m - x_{om})\delta(t - t_o) \quad .$$

This is the elastodynamic equation with the source term $f_i = \delta_{in}\delta(x_m - x_{om})\delta(t - t_o)$. The Green function G_{in} is thus i-th Cartesian component of the displacement vector recorded at the point x_m at time t and generated by a unit point force oriented along the n-th Cartesian axis at the point x_{om} at time t_o. The form of the generated signal is, as in acoustic case, the δ pulse.

The solution of the above equation is much more complicated than in the acoustic case. Complications are connected with the tensorial form of the solution and vectorial form of the source as well as with the fact that the above elastodynamic equation describes more than one wave. Only in the case of a homogeneous isotropic medium and/or in some special cases of homogeneous anisotropic media, the analytic closed form expressions can be found for the Green function. Otherwise, a numerical solution is necessary. In the following, we shall, therefore, describe basic steps of the determination of the Green function for a homogeneous isotropic medium.

In a homogeneous isotropic medium, the equation for elastodynamic Green function reduces to

$$\rho(\alpha^2 - \beta^2)G_{jn,ij} + \rho\beta^2 G_{in,jj} - \rho G_{in,tt} = -\delta_{in}\delta(x_m - x_{om})\delta(t - t_o) \, .$$

We consider index n (specifying the coordinate axis, along which the point force is oriented) as a given constant. We shall solve the above equation using the Helmholtz potentials and the Lamé's theorem, see Sec.2.7. For detailed derivation see Aki & Richards (1980).

Using the Lamé's theorem, we shall look for the Green function in the following way:

$$G_{in}(x_m, t; x_{om}, t_o) = \varphi_{,i}(x_m, t) + \epsilon_{ijk}\psi_{k,j}(x_m, t) \quad .$$

Here φ and ψ_k are scalar and vectorial potentials, the latter satisfying the condition

$$\psi_{k,k}(x_m, t) = 0 \quad .$$

According to the Lamé's theorem, we obtain the potentials φ and ψ_k by solving the wave equations, see Sec.2.7,

$$\rho\alpha^2\varphi_{,jj} - \rho\varphi_{,tt} = -\Phi\delta(t - t_o), \quad \rho\beta^2\psi_{k,jj} - \rho\psi_{k,tt} = -\Psi_k\delta(t - t_o) \, .$$

The quantities Φ and Ψ_k are the scalar and vectorial potentials ($\Psi_{k,k} = 0$) in terms of which the single force term in the elastodynamic equation can be expressed

$$f_i(x_m) = \delta_{in}\delta(x_m - x_{om}) = \Phi_{,i}(x_m) + \epsilon_{ijk}\Psi_{k,j}(x_m) \quad .$$

Thus, before we can solve the above wave equations, it is necessary to determine Φ and Ψ_k. Let us express $\Phi(x_m)$ and $\Psi_k(x_m)$ in terms of an unknown vectorial function $W_k(x_m)$ so that

$$\Phi(x_m) = W_{k,k}(x_m)\,, \quad \Psi_k(x_m) = -\epsilon_{klj}W_{j,l}(x_m) \quad .$$

With this definition, we have automatically

$$\Psi_{k,k}(x_m) = 0 \quad .$$

The equation for f_i can now be rewriten as

$$f_i(x_m) = W_{k,ki}(x_m) - \epsilon_{ijk}\epsilon_{klp}W_{p,lj}(x_m) = W_{i,kk}(x_m) \quad .$$

Equation

$$W_{i,kk}(x_m) = f_i(x_m)$$

is vectorial form of the *Poisson equation*. Its solution can be found using the solution of the acoustic equation

$$\overline{\overline{f}}_{,kk} - c^{-2}\overline{\overline{f}}_{,tt} = -f(x_m, t - t_o) \quad .$$

As a solution of this equation we found

$$\overline{\overline{f}}(x_m, t) = \frac{1}{4\pi}\iiint\limits_V \frac{f(x'_m, t - t_o - \frac{r'}{c})}{r'}dV(x'_m)\,, \quad r' = |x'_m - x_m|\,.$$

Comparing the Poisson equation with the acoustic one, we can see that the acoustic equation will reduce to the Poisson equation if we suppres time dependence in it. Then we find easily for $W_i(x_m)$,

$$W_i(x_m) = -\frac{1}{4\pi}\iiint\limits_V \frac{f_i(x'_m)}{r'}dV(x'_m) = -\frac{1}{4\pi}\iiint\limits_V \delta_{in}\frac{\delta(x'_m - x_{om})}{r'}dV(x'_m) \quad .$$

This yields immediately

$$W_i(x_m) = -\frac{\delta_{in}}{4\pi r}\,, \quad r = |x_m - x_{om}| \quad .$$

Using the above relations between Φ, Ψ_k and W_k, we can find that

$$\Phi(x_m) = \frac{N_n}{4\pi r^2}\,, \quad \Psi_k(x_m) = -\frac{\epsilon_{kln}N_l}{4\pi r^2} \quad .$$

Here N_i is a unit vector in the direction of radius vector, $N_i = \partial r/\partial x_i$. With potentials Φ and Ψ_k known, we can solve the wave equations for φ and ψ_k. We can again use the similarity between the above wave equations and the acoustic equations, which we have solved in the previous section. Using this similarity, we can write, for example for φ,

$$\varphi = \frac{1}{4\pi\rho\alpha^2} \iiint \frac{N_n'}{4\pi r'^2} \delta\left(t - t_o - \frac{r'}{c}\right) dV(x_m') \quad .$$

We would obtain a similar expression for ψ_k. Aki & Richards (1980) give solutions of the above integrals, which read

$$\varphi(x_m, t) = \frac{N_n}{4\pi\rho r^2} \int_0^{r/\alpha} \tau\delta(t - t_o - \tau) d\tau \quad ,$$

$$\psi_k(x_m, t) = -\frac{\epsilon_{kln} N_l}{4\pi\rho r^2} \int_0^{r/\beta} \tau\delta(t - t_o - \tau) d\tau \quad .$$

Because of several different definitions, our equations slightly differ from those derived in Aki & Richards (1980).

With the potentials φ and ψ_k, we can now evaluate the Green function. Using

$$G_{in} = \varphi_{,i} + \epsilon_{ijk}\psi_{k,j} \quad ,$$

we get

$$\begin{aligned} G_{in}(x_m, t; x_{om}, t_o) &= \frac{3N_iN_n - \delta_{in}}{4\pi\rho r^3} \int_{r/\alpha}^{r/\beta} \tau\delta(t - t_o - \tau) d\tau \\ &+ \frac{N_iN_n\delta(t - t_o - \frac{r}{\alpha})}{4\pi\rho\alpha^2 r} - \frac{(N_iN_n - \delta_{in})\delta(t - t_o - \frac{r}{\beta})}{4\pi\rho\beta^2 r} \quad . \end{aligned}$$

We can immediately see that the last two terms decay much slower (as $1/r$) with increasing distance from the source than the first term (which decays as $1/r^3$). The last two terms will thus dominate in regions far from the source. Because of this property, they are called *far field terms.*

We can identify the term

$$\frac{1}{4\pi\rho\alpha^2 r} N_iN_n\delta\left(t - t_o - \frac{r}{\alpha}\right)$$

as *the far-field elastodynamic P wave Green function.* We can see that it has spherical phase front and propagates away from the source with the compressional velocity $\alpha = \sqrt{(\lambda + 2\mu)/\rho}$ and decays as $1/r$. It preserves the form of the initial signal as it propagates. Its polarization is specified by the vector N_i and is, therefore, radial. Although the phase front is symmetric around the source, the distribution of amplitudes on it is not. The directional dependence of amplitudes is controlled by the term N_n, which is the only quantity which varies if we move along the phase front. N_n is, in fact, cosine of the angle γ made by the positive direction of the force and the radius r_i of the observer. Thus, the amplitude will be maximum in the direction, to which the force points. It will have the same but negative value in the opposite direction. It will be zero in the direction perpendicular to the force.

Similarly, we can identify the term

$$\frac{1}{4\pi\rho\beta^2 r}(\delta_{in} - N_i N_n)\delta\left(t - t_o - \frac{r}{\beta}\right)$$

as *the far-field elastodynamic S wave Green function.* It again has spherical phase front and propagates with the shear wave velocity and decays as $1/r$. Similarly to the P wave far-field Green function, it does not change its form when it propagates. Its polarization is perpendicular to N_i, i.e. to the radial direction. We can simply check it by multiplying the above term by N_i, which will yield zero. We can, therefore, speak about transverse polarization. Although the phase front is symmetric around the source, the amplitude distribution on it is not. In this case, we can simply find that the amplitude varies as $\sqrt{1 - N_n^2}$, i.e. as $\sin\gamma$. Thus, where we observed maximum amplitudes of the P wave far-field Green function, we shall find zero amplitudes for the S wave far-field Green function and vice versa. By making scalar product of the vectorial amplitude, proportional to $\delta_{in} - N_i N_n$ with the force f_i, we find that the projection of the amplitude vector into the force, $(1 - N_n^2)$ is always nonnegative.

Using the above information, we can present the distribution of the amplitude vectors of the far-field elastodynamic P-wave and S-wave Green functions in the following form. The big arrow in the midle of each picture shows the considered orientation of the unit force.

The arrows along the circles (phase fronts) are vectorial amplitudes (amplitudes multiplied by the polarization vectors).

By comparing both far-field terms, we can see that the maximum amplitudes of the

S-wave Green function are α^2/β^2 times stronger than maximum amplitude of the P-wave Green function, i.e. approximately 3 times for materials close to the Poisson solid, for which $\alpha^2/\beta^2 = 3$.

Let us finally note that in the far-field, where the spherical waves become similar to plane waves, the wavefield consists of two distinct waves, P and S wave, similarly as in case of plane wave propagation.

The first term in the expression for the Green function decays more rapidly than the above discussed terms and is important only near the source. Therefore, it is called the *near-field term.* Most of the seismological data are collected in the far field. In the recent years, however, interest in the near field increased mostly among earthquake seismologists. The branch of seismology connected with the measurements in the near-field regions is called *strong motion seismology.*

The near-field term is only nonzero between P and S wave arrivals. Its polarization is neither purely radial nor transversal. This can be checked by observing that the near-field term is not proportional to N_i and that the multiplication of it by N_i does not yield zero.

The near field term represents again spherically symmetric wave with the amplitude varying with direction. Let us investigate separately its radial and transverse components. The former one can be obtained by multiplying the near-field term by N_i, the latter one by subtracting the radial component from the total near-field term. For the radial component, we get

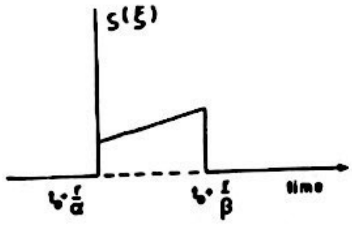
$$\frac{2N_nN_i}{4\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\tau\delta(t-t_o-\tau)d\tau = \frac{N_iN_n}{2\pi\rho r^3}(t-t_o)$$

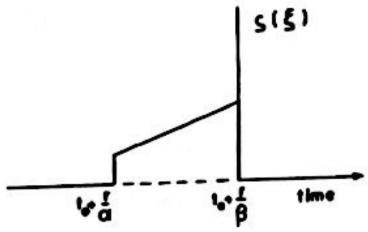
and for the tranverse component

$$\frac{N_iN_n-\delta_{in}}{4\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\tau\delta(t-t_o-\tau)d\tau = \frac{N_iN_n-\delta_{in}}{4\pi\rho r^3}(t-t_o) \quad .$$

In both cases the components attain the above values only for $(t-t_o)$ from the interval $< r/\alpha, r/\beta >$. For $(t-t_o)$ outside this interval, the components are zero. We can see that the directional dependence of amplitudes of the radial component is similar to the far-field P wave Green function, and of the transverse component to the far-field S wave Green function.

Using the above information, we can present the radial and transverse components of the elastodynamic Green function for a homogeneous isotropic medium at the distance r in the near field in the following form





The integral in the near-field term will in frequency domain transform into terms containing coefficients $1/i\omega$ and $(1/i\omega)^2$. It means that the importance of the near-field term will decrease with increasing frequency considered. It also means that the most important terms in the high-frequency approximation are the far-field terms.

From the formula for the elastodynamic Green function, we can see that, similarly as in the acoustic case, the Green function is reciprocal in times at the source and the receiver,

$$G_{in}(x_m, t; x_{om}, t_o) = G_{in}(x_m, -t_o; x_{om}, -t) \quad .$$

The Green function is also reciprocal in the position of the source and the receiver, namely

$$G_{in}(x_m, t; x_{om}, t_o) = G_{ni}(x_{om}, t; x_m, t_o) \quad .$$

CHAPTER 6

BASIC THEOREMS OF ACOUSTIC AND ELASTODYNAMIC THEORY

As in most previous considerations, we shall deal with acoustic or perfectly elastic media. For these media, we shall show that for specified sources, initial and boundary conditions there is only one solution of the acoustic or elastodynamic equation. This is the consequence of the *uniqueness theorem*. From the equations of the acoustic or elastodynamic theory, we shall also derive important reciprocity relations, whose consequences we have observed when we studied Green functions. The reciprocity relations are formulated as the *reciprocity theorem*. Using these theorems and the definition of the Green function, we shall find expressions for the computed field at an arbitrary point of a studied region in terms of the sources distribution and the boundary conditions specified on the boundary surrounding the studied region. These expressions are known as the *representation theorems* in elastodynamics. In acoustic, they are often called the *Kirchhoff representation* or the *Kirchhoff formula*.

6.1 Uniqueness theorem

Let us consider a volume V surrounded by a closed surface Σ. The displacement anywhere within V must satisfy the elastodynamic equation

$$\rho\ddot{u}_i - \tau_{ij,j} = f_i \quad ,$$

where $\ddot{u}_i = \partial^2 u_i/\partial t^2$. The displacement in V can be caused by a body force acting anywhere in V and/or by distribution of the tractions or displacements along the surface Σ. In fact, the distribution of tractions can be substituted by the distribution of spatial derivatives of displacement along Σ or along its parts and vice versa. This is because the traction and the spatial derivatives of the displacement are related by the formula

$$T_i = c_{ijkl}u_{k,l}n_j \quad .$$

The tractions and the displacements must be each specified on a different part of Σ. They cannot be specified at one point both because they are not independent. If we know the initial values of the displacement $u_i(t_o)$ and the particle velocity $\dot{u}_i(t_o)$ inside V at a moment t_o, and if we know for all times $t \geq t_o$ the values of

a) the body forces f_i in V and the heat supplied to V,

b) the displacement and/or its spatial derivatives (which is equivalent to specifying the tractions) along Σ,

then the displacement $u_i(x_m, t)$ is determined *uniquely* at any point x_m of V and for any time $t \geq t_o$.

The proof is as follows. Let us consider two displacement distributions, $u_i^{(1)}$ and $u_i^{(2)}$, which satisfy the same initial conditions and are caused by the same distribution of body forces in V and by the same heat supply to V, and by the same distribution of displacement or its derivatives on Σ. For the displacement $U_i = u_i^{(1)} - u_i^{(2)}$, the above conditions imply: at $t = t_o$, the displacement U_i satisfies zero initial conditions, $U_i(t_o) = \dot{U}_i(t_o) = 0$ everywhere in V. The body forces and the heat supply causing U_i are zero and the distribution of U_i and/or $U_{i,k}$ on Σ is also zero. These are the initial and boundary conditions for U_i. Let us now determine U_i for $t > t_o$ anywhere inside V.

In Sec.2.4.2, we have shown that under the assumption of adiabatic processes (no heat exchange with surrounding medium), the time rate of elastic energy ϵ is given by the formula

$$\begin{aligned}\frac{d\epsilon}{dt} &= \iiint\limits_V \dot{u}_i f_i dV + \iiint\limits_V (\tau_{ij}\dot{u}_i)_{,j} dV \\ &= \iiint\limits_V \dot{u}_i f_i dV + \iint\limits_\Sigma c_{ijkl} u_{k,l} n_j \dot{u}_i dS \quad .\end{aligned}$$

The elastic energy ϵ itself is given by the sum of the kinetic and strain energy in the volume V,

$$\begin{aligned}\epsilon &= \frac{1}{2}\iiint\limits_V (\rho \dot{u}_i \dot{u}_i + \tau_{ij} e_{ij}) dV \\ &= \frac{1}{2}\iiint\limits_V (\rho \dot{u}_i \dot{u}_i + c_{ijkl} u_{k,l} u_{i,j}) dV \quad .\end{aligned}$$

If we apply the first equation to the displacement U_i, for which $f_i = 0$ and either $u_i = 0$ or $u_{i,k} = 0$ on Σ, we get:

$$\epsilon = \text{constant} \quad .$$

Due to the zero initial conditions, we have

$$\epsilon = 0 \quad .$$

Thus elastic energy corresponding to the displacement U_i is zero. This, however, implies that also kinetic and strain energy must be zero because both of them are represented by positively definite quadratic forms. The kinetic energy is zero if

$$\dot{U}_i(x_m, t) = 0$$

everywhere in V for $t \geq t_o$. Due to the initial conditions $U_i(x_m, t_o) = 0$, this yields

$$U_i(x_m, t) = 0 \Leftrightarrow u_i^{(1)}(x_m, t) = u_i^{(2)}(x_m, t)$$

for any x_m in V and $t \geq t_o$.

Let us note that the proof of the uniqueness theorem for the acoustic case follows from the just derived theorem when we specify c_{ijkl} for an isotropic medium and take $\mu = 0$.

6.2 Reciprocity theorem

The reciprocity property is an important tool for the derivation of theoretical results like the representation theorems to be derived in the next section. It also plays an important role in the solution of various practical problems. We derive first the reciprocity relations for the acoustic and then for the elastodynamic case.

6.2.1 Acoustic case

We start again from the first order inhomogeneous acoustic equations for pressure p and the particle velocity v_i

$$\frac{\partial p}{\partial x_i} + \rho \frac{\partial v_i}{\partial t} = f_i \quad , \quad \frac{\partial v_i}{\partial x_i} + \kappa \frac{\partial p}{\partial t} = q \quad .$$

As before, f_i is the body force density. The quantity q represents again the *volume injection rate density* or briefly the *volume velocity*. Let us again derive the second order differential equation for p from the above two equations. We get

$$\kappa \frac{\partial^2 p}{\partial t^2} - \frac{\partial}{\partial x_i} \left(\rho^{-1} \frac{\partial p}{\partial x_i} \right) = \frac{\partial q}{\partial t} - \frac{\partial}{\partial x_i} (\rho^{-1} f_i) \quad .$$

We now rewrite both first order and second order equations into the frequency domain. We use the following notation

$$P(x_m, \omega) = \int_{-\infty}^{+\infty} p(x_m, t) e^{i\omega t} dt \quad , \quad Q(x_m, \omega) = \int_{-\infty}^{+\infty} q(x_m, t) e^{i\omega t} dt \quad ,$$

$$V_i(x_m, \omega) = \int_{-\infty}^{+\infty} v_i(x_m, t) e^{i\omega t} dt \quad , \quad F_i(x_m, \omega) = \int_{-\infty}^{+\infty} f_i(x_m, t) e^{i\omega t} dt \quad .$$

The first order equations will yield

$$P_{,i} - i\omega\rho V_i = F_i \quad , \quad V_{i,i} - i\omega\kappa P = Q \quad .$$

The second order equation in the frequency domain reads

$$\omega^2\kappa P + (\rho^{-1}P_{,i})_{,i} = i\omega Q + (\rho^{-1}F_i)_{,i} \quad .$$

The above equations hold for *arbitrary* variation of velocity ($\kappa = \rho^{-1}c^{-2}$) and density. We shall now consider their two different solutions, $P^{(1)}$ and $P^{(2)}$ ($V_i^{(1)}$ and $V_i^{(2)}$), which correspond to the distribution of sources $Q^{(1)}$, $F_i^{(1)}$ and $Q^{(2)}$, $F_i^{(2)}$ respectively. The quantities $P^{(1)}$, $Q^{(1)}$, $F_i^{(1)}$ and $P^{(2)}$, $Q^{(2)}$, $F_i^{(2)}$ must satisfy the above second order differential harmonic acoustic equation. We shall multiply the above second order differential equation for $P^{(1)}$ by $P^{(2)}$ and the equation for $P^{(2)}$ by $P^{(1)}$ and then subtract the resulting equations. The result will be

$$[\rho^{-1}(P_{,i}^{(1)} - F_i^{(1)})]_{,i}P^{(2)} - [\rho^{-1}(P_{,i}^{(2)} - F_i^{(2)})]_{,i}P^{(1)} = i\omega(Q^{(1)}P^{(2)} - Q^{(2)}P^{(1)}) \quad .$$

This can be rewritten

$$\{\rho^{-1}[(P_{,i}^{(1)} - F_i^{(1)})P^{(2)} - (P_{,i}^{(2)} - F_i^{(2)})P^{(1)}]\}_{,i}$$

$$= \rho^{-1}[(P_{,i}^{(1)} - F_i^{(1)})P_{,i}^{(2)} - (P_{,i}^{(2)} - F_i^{(2)})P_{,i}^{(1)}] + i\omega(Q^{(1)}P^{(2)} - Q^{(2)}P^{(1)}) \quad .$$

As we can see, the RHS can be further simplified,

$$\{\rho^{-1}[(P_{,i}^{(1)} - F_i^{(1)})P^{(2)} - (P_{,i}^{(2)} - F_i^{(2)})P^{(1)}]\}_{,i}$$

$$= \rho^{-1}(F_i^{(2)}P_{,i}^{(1)} - F_i^{(1)}P_{,i}^{(2)}) + i\omega(Q^{(1)}P^{(2)} - Q^{(2)}P^{(1)}) \quad .$$

We now use the relations derived from the first order differential harmonic acoustic equations,

$$P_{,i}^{(k)} - i\omega\rho V_i^{(k)} = F_i^{(k)}$$

to substitute $P_{,i}^{(1)}$ and $P_{,i}^{(2)}$ in the above equation by V_i and F_i so that we get an expression for the *fundamental interaction quantity* between the states (1) and (2), $(V_i^{(1)}P^{(2)} - V_i^{(2)}P^{(1)})_{,i}$

$$(V_i^{(1)}P^{(2)} - V_i^{(2)}P^{(1)})_{,i} = (F_i^{(2)}V_i^{(1)} - F_i^{(1)}V_i^{(2)}) + (Q^{(1)}P^{(2)} - Q^{(2)}P^{(1)}).$$

Let us now integrate this equation in the volume V surrounded by a surface Σ containing all the sources and observation points. This will yield

$$\iiint_V (V_i^{(1)}P^{(2)} - V_i^{(2)}P^{(1)})_{,i}dV$$

$$= \iiint_V (F_i^{(2)}V_i^{(1)} - F_i^{(1)}V_i^{(2)})dV + \iiint_V (Q^{(1)}P^{(2)} - Q^{(2)}P^{(1)})dV \quad .$$

The volume integral on the LHS can be transformed into the surface integral using the Gauss theorem

$$\iiint_V (V_i^{(1)}P^{(2)} - V_i^{(2)}P^{(1)})_{,i}dV = \iint_\Sigma (V_i^{(1)}P^{(2)} - V_i^{(2)}P^{(1)})n_i d\Sigma \quad .$$

Here n_i is the unit outward normal to Σ. If we insert the above identity into the previous integral equation, we get the *acoustic relation of reciprocity in the integral form*, which, through the surface integral, also incorporates boundary conditions.

A special case occurs when the surface integral vanishes. It can vanish for several reasons. For example, if boundary conditions are specified so that either $V_i n_i$ or P are zero on the surface Σ (homogeneous boundary conditions) or if the P and V_i wave fields decay sufficiently rapidly when the surface Σ is expanded to infinity, i.e., if the *Sommerfeld radiation conditions* are satisfied. Sommerfeld radiation conditions substitute boundary conditions in case of an unlimited volume V. They require that the considered wavefields have *outgoing* character, see the discussion in Sec.5.1, and that the wavefield decays with distance as $1/r$, where r is the distance from the source. The former condition can be expressed in the following way (we shall return to it in the end of Sec. 6.3.1.):

$$\lim_{r\to\infty} r\left(\frac{\partial u_i}{\partial r} - \frac{i\omega}{c}u_i\right) = 0 \quad .$$

The latter condition can be expressed as

$$u_i \sim 1/r \quad ,$$

which means that u_i behaves like $1/r$.

If the above boundary conditions are specified or if the volume V is expanded to infinity, the above integral equation can be reduced to

$$\iiint_V (V_i^{(1)}F_i^{(2)} - P^{(1)}Q^{(2)})dV = \iiint_V (V_i^{(2)}F_i^{(1)} - P^{(2)}Q^{(1)})dV \quad .$$

The symbol V stands either for finite volume on the closure of which homogeneous boundary conditions are satisfied or it represents the whole space with the wave field satisfying the Sommerfeld radiation conditions.

In the above integral equation we gathered the solutions $V_i^{(1)}$, $P^{(1)}$ on the LHS and $V_i^{(2)}$, $P^{(2)}$ on the RHS. We shall further simplify it. We shall specify it for two point sources situated at points $x_m^{(1)}$ and $x_m^{(2)}$ in V generating the pressure wavefields $P^{(1)}$ and $P^{(2)}$ and particle velocity wavefields $V_i^{(1)}$ and $V_i^{(2)}$. Thus, the body forces and volume velocities $F_i^k(x_m,\omega)$, $Q^{(k)}(x_m,\omega)$, $k=1,2$ can be expressed as follows

$$F_i^{(k)}(x_m,\omega) = \hat{F}_i^{(k)}(\omega)\delta(x_m - x_m^{(k)}), \quad Q^{(k)}(x_m,\omega) = \hat{Q}^{(k)}(\omega)\delta(x_m - x_m^{(k)})\ ,$$

where $\bar{F}_i^{(k)}(\omega)$ are the *vectorial source signatures* and $\bar{Q}^{(k)}(\omega)$ the *scalar source signatures.* Inserting this into the integral form of the reciprocity relation, we get

$$\iiint\limits_V [V_i^{(1)}(x_m,\omega)\hat{F}_i^{(2)}(\omega) - P^{(1)}(x_m,\omega)\hat{Q}^{(2)}(\omega)]\delta(x_m - x_m^{(2)})dV$$

$$= \iiint\limits_V [V_i^{(2)}(x_m,\omega)\hat{F}_i^{(1)}(\omega) - P^{(2)}(x_m,\omega)\hat{Q}^{(1)}(\omega)]\delta(x_m - x_m^{(1)})dV \quad ,$$

which after integration yields

$$V_i^{(1)}(x_m^{(2)},\omega)\hat{F}_i^{(2)}(\omega) - P^{(1)}(x_m^{(2)},\omega)\hat{Q}^{(2)}(\omega) = V_i^{(2)}(x_m^{(1)},\omega)\hat{F}_i^{(1)}(\omega) - P^{(2)}(x_m^{(1)},\omega)\hat{Q}^{(1)}(\omega) \quad .$$

This is *the acoustic reciprocity relation in the local form.* It can be interpreted in several ways according to the specification of the sources.

a) *The sources of voluminal velocity at $x_m^{(1)}$ and $x_m^{(2)}$ ($\hat{F}_i^{(1)} = \hat{F}_i^{(2)} = 0$)*

The above local reciprocity principle yields in this case

$$P^{(1)}(x_m^{(2)},\omega)\hat{Q}^{(2)}(\omega) = P^{(2)}(x_m^{(1)},\omega)\hat{Q}^{(1)}(\omega) \quad .$$

This is the principle of reciprocity for the pressure and the volume velocity source and it can be interpreted as follows. If the sources at $x_m^{(1)}$ and $x_m^{(2)}$ are related as follows, $\hat{Q}^{(2)}(\omega) = k\hat{Q}^{(1)}(\omega)$, the pressure wave field $P^{(1)}(x_m^{(2)},\omega)$ due to the source at $x_m^{(1)}$ is k-times weaker than the pressure wave field $P^{(2)}(x_m^{(1)},\omega)$ due to the source at $x_m^{(2)}$.

Let us note that for $\hat{Q}^{(1)} = \hat{Q}^{(2)} = 1$, i.e., for $Q^{(1)}(x_m,\omega) = \delta(x_m - x_m^{(1)})$ and $Q^{(2)}(x_m,\omega) = \delta(x_m - x_m^{(2)})$, the above pressure wave fields represent Green functions

$$G(x_m^{(2)},x_m^{(1)},\omega) = P^{(1)}(x_m^{(2)},\omega), \quad G(x_m^{(1)},x_m^{(2)},\omega) = P^{(2)}(x_m^{(1)},\omega) \quad .$$

Note that we do not use the fourth argument of the Green function, t_o, as we did in Sec.5.1 because our original acoustic equations are specified with $t_o = 0$.

The local reciprocity principle implies for the Green functions

$$G(x_m^{(2)},x_m^{(1)},\omega) = G(x_m^{(1)},x_m^{(2)},\omega) \quad ,$$

i.e., the acoustic time harmonic Green function is reciprocal. This we have already observed in Sec.5.1 when we studied properties of the acoustic Green function in a homogeneous medium. Here, we found that it holds for generally inhomogeneous media.

b) *The point force sources at* $x_m^{(1)}$ *and* $x_m^{(2)}$ $(\hat{Q}^{(1)} = \hat{Q}^{(2)} = 0)$

In this case, the local reciprocity principle yields

$$V_i^{(1)}(x_m^{(2)},\omega)\hat{F}_i^{(2)}(\omega) = V_i^{(2)}(x_m^{(1)},\omega)\hat{F}_i^{(1)}(\omega) \quad .$$

This is the reciprocity relation for the particle velocity and the force source. It says that the quantity, which is invariant with respect to the interchange of the source and the receiver is the scalar product of particle velocity due to one body force with the other body force. We can simplify the situation and consider the two forces, acting as a source, of the same magnitude but of different orientation. The force $\hat{F}_i^{(1)}(\omega)$ is specified along the unit vector $n_i^{(1)}$, the force $\hat{F}_i^{(2)}(\omega)$ along the unit vector $n_i^{(2)}$. Then the above relation yields

$$V_i^{(1)}(x_m^{(2)},\omega)n_i^{(2)} = V_i^{(2)}(x_m^{(1)},\omega)n_i^{(1)} \quad .$$

For $n_i^{(1)}$ along x_k axis and $n_i^{(2)}$ along x_l axis we have

$$V_l^{(1)}(x_m^{(2)},\omega) = V_k^{(2)}(x_m^{(1)},\omega) \quad .$$

The above situation can be simply generalized for the original situation with forces of different magnitude. In the next section, we shall see that the acoustic reciprocity principle for point forces and particle velocity corresponds to the elastodynamic reciprocity principle.

c) *The voluminal velocity source at* $x_m^{(1)}$ *and point force source at* $x_m^{(2)}$ $(\hat{F}_i^{(1)} = 0,\ \hat{Q}^{(2)} = 0)$

This is a kind of mixed problem. Now the local reciprocity principle yields

$$V_i^{(1)}(x_m^{(2)},\omega)\hat{F}_i^{(2)}(\omega) = -P^{(2)}(x_m^{(1)},\omega)\hat{Q}^{(1)}(\omega) \quad .$$

In this case, the negative scalar product of the particle velocity at $x_m^{(2)}$ due to the voluminal velocity source at $x_m^{(1)}$ with the force at $x_m^{(2)}$, which generates the pressure field at $x_m^{(1)}$ is equivalent to the product of the pressure and voluminal velocity at $x_m^{(1)}$. For unit voluminal velocity $\hat{Q}^{(1)}(\omega)$ and unit force $\hat{F}_i^{(2)}(\omega)$ oriented along a unit vector $n_i^{(2)}$, the above equation gives

$$V_i^{(1)}(x_m^{(2)},\omega)n_i^{(2)} = -P^{(2)}(x_m^{(1)},\omega) \quad .$$

For $n_i^{(2)}$ alonf the x_k-axis, this further reduces to

$$V_k^{(1)}(x_m^{(2)},\omega) = -P^{(2)}(x_m^{(1)},\omega) \quad .$$

6.2.2 Elastodynamic case. Betti's theorem

We shall now look for similar relations to those which we found in the acoustic case between displacements and point forces. Our problem is more general since it includes both P and S waves while in the acoustic case, it included only P waves. In contrast to the acoustic case, we shall now work in the time domain.

Let us consider a volume V of an inhomogeneous, generally anisotropic, medium, surrounded by a closed surface Σ. Inside V, we consider two displacement vectors, $u_i^{(1)}(x_m, t)$ and $u_i^{(2)}(x_m, t)$. The displacements are considered to be caused by generally different distributions of body forces $f_i^{(1)}(x_m, t)$ and $f_i^{(2)}(x_m, t)$ in V and by generally different boundary conditions specified on Σ. The boundary conditions can be again specified in terms of displacements or their spatial derivatives (the latter being equivalent to the specification by tractions) over Σ. The Betti's theorem has then the following form

$$\iiint_V (f_i^{(1)} - \rho u_{i,tt}^{(1)})u_i^{(2)} dV + \iint_\Sigma c_{ijkl} n_j u_{k,l}^{(1)} u_i^{(2)} dS = \iiint_V (f_i^{(2)} - \rho u_{i,tt}^{(2)})u_i^{(1)} dV + \iint_\Sigma c_{ijkl} n_j u_{k,l}^{(2)} u_i^{(1)} dS \, .$$

Let us note that the presented form slightly differs from the traditionally used form, in which tractions are used in the surface integrals, see Aki & Richards (1980). Let us also note that the Betti's theorem does not contain initial conditions. Moreover, the Betti's theorem does not require that the quantities indexed (1) are specified at the same time as the quantities indexed (2). Thus, it holds if the quantities $f_i^{(1)}$, $u_i^{(1)}$ and spatial and time derivatives of $u_i^{(1)}$ are evaluated at a time t_1 different from a time t_2, at which $f^{(2)}$, $u_i^{(2)}$ and spatial and time derivatives of $u_i^{(2)}$ are evaluated.

The proof of the Betti's theorem is elementary. Let us recall the form of the elastodynamic equation for inhomogeneous anisotropic media. It reads

$$(c_{ijkl} u_{k,l})_{,j} + f_i = \rho u_{i,tt} \quad .$$

Then we can rewrite the voluminal integrals in the Betti's theorem as follows (for the sake of brevity, we write explicitely only the time dependence of individual quantities although they are also space dependent)

$$\iiint_V (f_i^{(r)}(t_r) - \rho u_{i,tt}^{(r)}(t_r)) u_i^{(s)}(t_s) dV = - \iiint_V (c_{ijkl} u_{k,l}^{(r)}(t_r))_{,j} u_i^{(s)}(t_s) dV \, ,$$

with $r \neq s$, $r, s = 1, 2$. The surface integrals in the Betti's theorem can be transformed into voluminal ones using the Gauss theorem. We get

$$\iiint_\Sigma (c_{ijkl} u_{k,l}^{(r)}(t_r) u_i^{(s)}(t_s)) n_j dS = \iiint_V (c_{ijkl} u_{k,l}^{(r)}(t_r) u_i^{(s)}(t_s))_{,j} dV$$

with $r \neq s$, $r,s = 1,2$. Inserting the above integrals into the Betti's theorem, we get

$$\iiint_V c_{ijkl} u_{k,l}^{(1)}(t_1) u_{i,j}^{(2)}(t_2) dV = \iiint_V c_{ijkl} u_{k,l}^{(2)}(t_2) u_{i,j}^{(1)}(t_1) dV \quad ,$$

which is due to the symmetry of the elastic tensor, $c_{ijkl} = c_{klij}$, an identity. This proves the Betti's theorem.

Let us choose the times t_1 and t_2 in the following way

$$t_1 = t \quad , \quad t_2 = \tau - t \quad ,$$

and let us integrate in this way specified Betti's relation in the time interval from 0 to τ. We get

$$\int_0^\tau dt \left\{ \iiint_V [f_i^{(1)}(t) - \rho u_{i,tt}^{(1)}(t)] u_i^{(2)}(\tau - t) dV + \iint_\Sigma c_{ijkl} n_j u_{k,l}^{(1)}(t) u_i^{(2)}(\tau - t) dS \right\}$$

$$= \int_0^\tau dt \left\{ \iiint_V [f_i^{(2)}(\tau - t) - \rho u_{i,tt}^{(2)}(\tau - t)] u_i^{1)}(t) dV + \iint_\Sigma c_{ijkl} n_j u_{k,l}^{(2)}(\tau - t) u_i^{(1)}(t) dS \right\} \quad .$$

Let us consider the part of this equation, which contains the time derivatives. It reads

$$\iiint_V dV \int_0^\tau \rho [u_{i,tt}^{(1)}(t) u_i^{(2)}(\tau - t) - u_{i,tt}^{(2)}(\tau - t) u_i^{(1)}(t)] dt$$

$$= \iiint_V dV \rho [u_{i,t}^{(1)}(t) u_i^{(2)}(\tau - t) + u_{i,t}^{(2)}(\tau - t) u_i^{(1)}(t)]_0^\tau$$

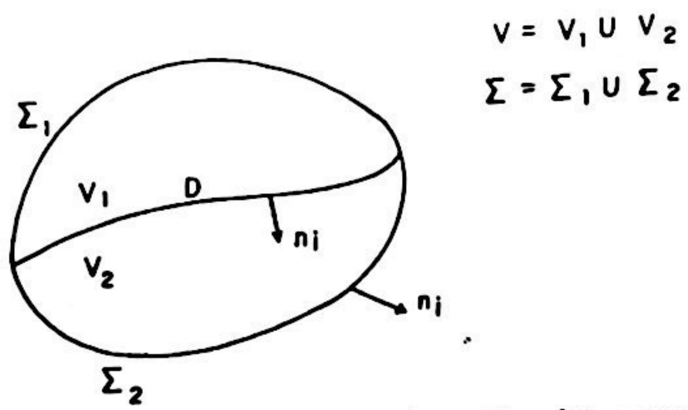
$$= \iiint_V dV \rho [u_{i,t}^{(1)}(\tau) u_i^{(2)}(0) - u_{i,t}^{(1)}(0) u_i^{(2)}(\tau) + u_{i,t}^{(2)}(0) u_i^{(1)}(\tau) - u_{i,t}^{(2)}(\tau) u_i^{(1)}(0)] \quad .$$

If there is time τ_o such that for $t \leq \tau_o$, $u_i^{(1)}(t)$, $u_i^{(2)}(t)$, $u_{i,t}^{(1)}(t)$, $u_{i,t}^{(2)}(t)$ are zero (we then speak about the *quiescent past*), then the integral above, in which the limits $(0,\tau)$ are expanded to $(-\infty, \infty)$, vanishes and the time integral of the Betti's relation becomes

$$\int_{-\infty}^{+\infty} dt \left\{ \iiint_V [f_i^{(1)}(t) u_i^{(2)}(\tau - t) - f_i^{(2)}(\tau - t) u_i^{(1)}(t)] dV \right\}$$

$$= \int_{-\infty}^{+\infty} dt \left\{ \iint_\Sigma c_{ijkl} n_j [u_{k,l}^{(2)}(\tau - t) u_i^{(1)}(t) - u_{k,l}^{(1)}(t) u_i^{(2)}(\tau - t)] dS \right\} \quad .$$

The equation represents *the elastodynamic reciprocity relation in the integral form*. The surface integral contains the corresponding boundary conditions. This equation will be used in Sec.6.3.2 for the derivation of the elastodynamic representation theorem.



Before we make specifications in the above integral equation, let us consider the following situation, see the picture. We split the volume V into two parts V_1 and V_2 bounded by parts of the outer surface of the volume V, Σ_1 and Σ_2 and by an internal surface D. Let us denote the elastic parameters in the volume V_1 by $c^{(1)}_{ijkl}$ and in V_2 by $c^{(2)}_{ijkl}$. The unit normal to D points into the volume V_2, the unit normal to $\Sigma = \Sigma_1 \cup \Sigma_2$ is, as before, pointing out of V. We can now consider various distributions of sources. Let us consider that the sources generating the wavefield $u^{(1)}_i$ are all concentrated in V_1, the sources generating $u^{(2)}_i$ are all concentrated in V_2. Then, using the above reciprocity relation, we can write for V_1 and V_2

$$\int_{-\infty}^{+\infty} dt \left\{ \iiint\limits_{V_1} f^{(1)}_i(t) u^{(2)}_i(\tau - t) dV - \iint\limits_{\Sigma_1} c^{(1)}_{ijkl} n_j D_{ikl} dS - \iint\limits_{D} c^{(1)}_{ijkl} n_j D_{ikl} dS \right\} = 0 ,$$

$$\int_{-\infty}^{+\infty} dt \left\{ \iiint\limits_{V_2} -f^{(2)}_i(\tau - t) u^{(1)}_i(t) dV - \iint\limits_{\Sigma_2} c^{(2)}_{ijkl} n_j D_{ikl} dS + \iint\limits_{D} c^{(2)}_{ijkl} n_j D_{ikl} dS \right\} = 0 .$$

Here

$$D_{ikl} = u^{(2)}_{k,l}(\tau - t) u^{(1)}_i(t) - u^{(1)}_{k,l}(t) u^{(2)}_i(\tau - t) \quad .$$

The plus sign in front of the last integral in the second equation (for V_2) is due to the inward character of the normal to D in this case.

If we recall the boundary conditions for a discontinuity between two elastic solids, we can rewrite them for the considered case in the following way:

Continuity of the displacement: $u_i(D^-) = u_i(D^+)$,

where D^+, D^- denote opposite faces of the surface D.

Continuity of the traction: $\tau_{ij}(D^-) n_j = \tau_{ij}(D^+) n_j$,

which can be rewritten as

$$c^{(1)}_{ijkl} u_{k,l}(D^-) n_j = c^{(2)}_{ijkl} u_{k,l}(D^+) n_j \quad .$$

We can thus see that both the above integral equations are intercoupled. Intercoupling terms are the surface integrals over D. If we use the boundary conditions in these integrals, we find that they can be rewritten into one equation

$$\int_{-\infty}^{+\infty} dt \left\{ \iiint\limits_{V_1} f_i^{(1)}(t)u_i^{(2)}(\tau - t)dV - \iiint\limits_{V_2} f_i^{(2)}(\tau - t)u_i^{(1)}(t)dV \right.$$
$$- \iint\limits_{\Sigma_1} c_{ijkl}^{(1)} n_j [u_{k,l}^{(2)}(\tau - t)u_i^{(1)}(t) - u_{k,l}^{(1)}(t)u_i^{(2)}(\tau - t)]dS$$
$$\left. - \iint\limits_{\Sigma_2} c_{ijkl}^{(2)} n_j [u_{k,l}^{(2)}(\tau - t)u_i^{(1)}(t) - u_{k,l}^{(1)}(t)u_i^{(2)}(\tau - t)]dS \right\} = 0 .$$

If we take into account that without any change we can consider both voluminal integrals as one integral over V and that the last two surface integrals can be written as one over the surface $\Sigma = \Sigma_1 \cup \Sigma_2$, we arrive to exactly the same equation of elastodynamic reciprocity as before, when we considered the volume V filled with smoothly varying inhomogeneous medium. Thus, we have shown that the above *elastodynamic reciprocity relation can be extended even to media with discontinuities.*

Similarly as in Sec.6.2.1, we shall consider special situations, in which the surface integral vanishes. This can occur when, for example, homogeneous boundary conditions are specified on Σ, i.e., when $u_i(x_m, t) = 0$ and/or $u_{i,j}(x_m, t) = 0$ for $t \geq \tau_o$ for x_m situated on Σ. Another possibility is to expand V to infinity and then, due to the Sommerfeld radiation conditions, the surface integral also vanishes. Thus we get

$$\int_{-\infty}^{+\infty} dt \iiint\limits_{V} [f_i^{(1)}(x_m, t)u_i^{(2)}(x_m, \tau - t) - f_i^{(2)}(x_m, \tau - t)u_i^{(1)}(x_m, t)]dV = 0 .$$

Here again V stands either for the finite volume of the closure on which homogeneous boundary conditions are specified or for the whole space when the boundary conditions are substituted by the radiation conditions. Further simplification of the above integral formula can be reached by the specification of two point forces situated at points $x_m^{(1)}$ and $x_m^{(2)}$ generating displacements $u_i^{(1)}(x_m, t)$ and $u_i^{(2)}(x_m, t)$. The forces are specified as

$$f_i^{(1)}(x_m, t) = f^{(1)} n_i^{(1)} \delta(x_m - x_m^{(1)})\delta(t - t^{(1)}) ,$$

$$f_i^{(2)}(x_m, t) = f^{(2)} n_i^{(2)} \delta(x_m - x_m^{(2)})\delta(t - t^{(2)}) .$$

Here $t^{(1)}$ and $t^{(2)}$ are times at which the first and the second source act, $f^{(1)}$ and $f^{(2)}$ are strengths of these sources and the unit vectors $n_i^{(1)}$ and $n_i^{(2)}$ specify their orientation. If we insert the expressions for point forces into the above integral equation, we get

$$f^{(1)}n_i^{(1)}u_i^{(2)}(x_m^{(1)},\tau - t^{(1)}) - f^{(2)}n_i^{(2)}u_i^{(1)}(x_m^{(2)},\tau - t^{(2)}) = 0\,.$$

This is the *elastodynamic reciprocity relation in the local form.* It differs from the one derived in Sec.6.2.1 in two aspects. It is given in the time domain and thus also allows investigation of the temporal reciprocity. Second, since it is derived from the elastodynamic equation, it allows to investigate together compressional and shear waves in the reciprocity relations. For $f^{(1)} = f^{(2)} = 1$ and for $n_i^{(1)}$ situated along x_k-axis and $n_i^{(2)}$ along x_l-axis, the above local reciprocity relation reduces to the following simple form

$$u_k^{(2)}(x_m^{(1)},\tau - t^{(1)}) = u_l^{(1)}(x_m^{(2)},\tau - t^{(2)}) \quad .$$

If we consider two point forces of the unit strength, then the one oriented along the x_k-axis produces at the point, where the other force, situated along the x_l-axis, is located the displacement whose component into x_l-axis equals the component into the x_k-axis of the displacement generated by the force along x_l-axis at the point, where the first force acts.

Since we considered above the sources with unit forces, the displacements $u_i^{(1)}$, $u_i^{(2)}$ represent the Green functions

$$u_l^{(1)}(x_m^{(2)},\tau - t^{(2)}) = G_{lk}(x_m^{(2)},\tau - t^{(2)}; x_m^{(1)}, t^{(1)})\,,$$

$$u_k^{(2)}(x_m^{(1)},\tau - t^{(1)}) = G_{kl}(x_m^{(1)},\tau - t^{(1)}; x_m^{(2)}, t^{(2)})\,.$$

The local reciprocity relation immediately yields

$$G_{kl}(x_m^{(1)},\tau - t^{(1)}; x_m^{(2)}, t^{(2)}) = G_{lk}(x_m^{(2)},\tau - t^{(2)}; x_m^{(1)}, t^{(1)}) \quad .$$

This equation confirms the time-spatial reciprocity of the elastodynamic Green function, which we have already observed in Sec.5.2. In Sec.5.2, however, we found this relation under the assumption of homogeneity the medium. Here, generally inhomogeneous anisotropic medium is considered. The medium can contain even discontinuities and thus the reciprocity holds for reflected and transmitted waves and also for converted waves.

If we specify $t_1 = t_2 = 0$, this yields

$$G_{kl}(x_m^{(1)},\tau; x_m^{(2)},0) = G_{lk}(x_m^{(2)},\tau; x_m^{(1)},0) \quad ,$$

which is another frequently used reciprocity relation.

6.3 Representation theorems

6.3.1 Kirchhoff formula

We are first going to derive the representation theorem for acoustic media. We shall start from the equations derived in Sec.6.2.1 in the frequency domain:

$$\begin{aligned}\iint\limits_{\Sigma}(V_i^{(1)}P^{(2)} - V_i^{(2)}P^{(1)})n_i d\Sigma &= \iiint\limits_{V}(F_i^{(2)}V_i^{(1)} - F_i^{(1)}V_i^{(2)})dV \\ &+ \iiint\limits_{V}(Q^{(1)}P^{(2)} - Q^{(2)}P^{(1)})dV\,.\end{aligned}$$

Here again, V denotes a finite volume of space surrounded by the closed surface Σ. The quantities under the integrals depend on spatial coordinates. We shall now consider only the volume velocity sources, i.e., we assume $F_i^{(1)} = F_i^{(2)} = 0$. Thus the first integral on the RHS of the above equation will vanish. Moreover, if we recall the form of the Fourier transformed vectorial linear acoustic equation connecting pressure P and the particle velocity V_i, the specification of zero body forces implies

$$V_i = P_{,i}/i\omega\rho \quad .$$

Taking this into account, the above integral equation can be rewritten as follows

$$(i\omega)^{-1}\iint\limits_{\Sigma}\rho^{-1}(P_{,i}^{(1)}P^{(2)} - P_{,i}^{(2)}P^{(1)})n_i d\Sigma = \iiint\limits_{V}(Q^{(1)}P^{(2)} - Q^{(2)}P^{(1)})dV \quad .$$

To remove the term $i\omega$, which complicates the above equation we shall make the following substitution. Instead of $Q^{(1)}$ and $Q^{(2)}$, which are Fourier trasformed quantities $q^{(1)}$, $q^{(2)}$, we shall use quantities $\bar{Q}^{(1)}$, $\bar{Q}^{(2)}$, which are Fourier transforms of $\partial q^{(1)}/\partial t$, $\partial q^{(2)}/\partial t$. $\bar{Q}^{(I)}$ and $Q^{(I)}$ are then related by the obvious equation

$$\bar{Q}^{(I)} = -i\omega Q^{(I)} \quad .$$

Thus, we now have

$$\iint\limits_{\Sigma}\rho^{-1}(P_{,i}^{(1)}P^{(2)} - P_{,i}^{(2)}P^{(1)})n_i dS = \iiint\limits_{V}(\bar{Q}^{(2)}P^{(1)} - \bar{Q}^{(1)}P^{(2)})dV \quad .$$

Let us now consider that the wavefield $P^{(2)}$ is generated by a voluminal velocity point source

$$\bar{Q}^{(2)}(x_m,\omega) = \delta(x_m - x_m^s) \quad ,$$

situated at the point x_m^s *inside* the volume V. This means that $P^{(2)}(x_m,\omega)$ is harmonic Green function,

$$P^{(2)}(x_m,\omega) = G(x_m, x_m^s, \omega) \quad .$$

Since we have uniquely specified one of the wavefields, we can omit all the upper indices in the formulae to simplify the notation. Inserting the above expressions for the point source and the Green function into the integral equation, we get

$$\iint_\Sigma \rho^{-1}[n_i P_{,i}(x_m,\omega)G(x_m,x_m^s,\omega) - n_i G_{,i}(x_m,x_m^s,\omega)P(x_m,\omega)]dS$$

$$= P(x_m^s,\omega) - \iiint_V (\bar{Q}(x_m,\omega)G(x_m,x_m^s,\omega)dV \quad .$$

The first term on the RHS of the equation was obtained under the assumption that $x_m \in V$. If it were situated outside V, the term would be zero. We can rewrite the equation into its final form known as the *acoustic representation theorem:*

$$P(x_m^s,\omega) = \iiint_V \bar{Q}(x_m,\omega)G(x_m,x_m^s,\omega)dV$$

$$+ \iint_\Sigma \rho^{-1}[P_{,i}(x_m,\omega)n_i G(x_m,x_m^s,\omega) - P(x_m,\omega)G_{,i}(x_m,x_m^s,\omega)n_i]dS \quad .$$

The pressure wavefield at an arbitrary point $x_m^{(s)}$ of the volume V can be uniquely determined from the above equation. The equation relates $P(x_m^s,\omega)$ to the distribution of sources $\bar{Q}(x_m,\omega)$ in V and to the boundary conditions on Σ. The above formula without the voluminal integral is known as the *Kirchhoff formula*. This name is, however, often used for the above representation theorem, too.

If $P(x_m,\omega)$ is specified on Σ, we speak about the *Dirichlet boundary conditions*. If $\partial P(x_m,\omega)/\partial n = P_{,i}(x_m,\omega)n_i$ is specified on Σ, we speak about the *Neuman boundary conditions*. If the combination of the both conditions is specified, we speak about the *mixed boundary conditions*.

The above Kirchhoff formula is also sometimes called the *Kirchhoff inner formula* (because of the location of the point x_m inside the volume V). It can be rewritten into more natural form if the Green function is reciprocal

$$G(x_m, x_m^{(s)}, \omega) = G(x_m^{(s)}, x_m, \omega) \, .$$

Since the Kirchhoff formula holds for arbitrarily specified Green function, we can always specify the Green function so that it is reciprocal. If the Green function satisfies, for example, the *homogeneous* Dirichlet boundary conditions on Σ (the free surface conditions), the

Kirchhoff formula attains the form

$$P(x_m^s,\omega) = \iiint_V \bar{Q}(x_m,\omega)G(x_m^s,x_m,\omega)dV - \iint_\Sigma \rho^{-1}P(x_m,\omega)\frac{\partial G(x_m^s,x_m,\omega)}{\partial x_i}n_i dS \,.$$

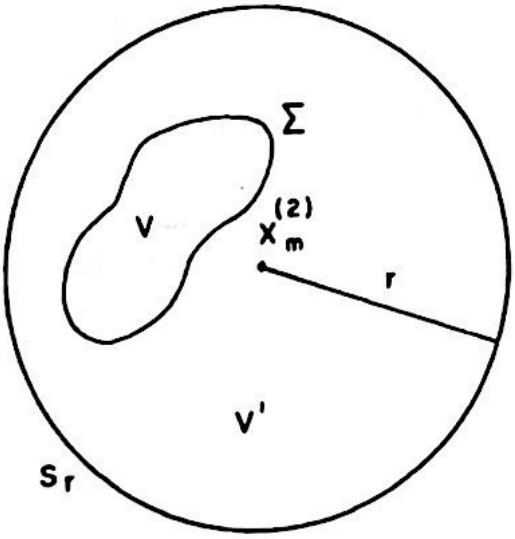
The inner formula can also be rewritten into the time domain. Using the inverse Fourier transform, we get

$$p(x_m^s,t) = \iiint_V q_{,t}(x_m,t) * G(x_m^s,t;x_m,0)dV$$

$$+ \iint_\Sigma \rho^{-1}\left[\frac{\partial P}{\partial n}(x_m,t) * G(x_m^s,t;x_m,0) - P(x_m,\omega) * \frac{\partial G}{\partial n}(x_m^s,t;x_m,0)\right] dS \quad .$$

Here the symbol "$*$" stands for the time convolution and $\frac{\partial}{\partial n} = n_i\frac{\partial}{\partial x_i}$.

In a similar way as the inner formula, we can derive the *Kirchhoff outer formula* for a point x_m^s situated outside the volume V. In this case, we substitute the volume V by a volume V' defined as follows. We consider a sphere with the surface S_r such that it contains



the whole volume V and also the point x_m^s (in the picture denoted by $x_m^{(2)}$) at its center. Volume V' is then defined as the volume of the sphere outside the volume V. Similarly as we wrote Kirchhoff inner formula for the volume V with the surface Σ, we can write the expression for the volume V' limited by the surfaces S_r and Σ,

$$P(x_m^s,\omega) = \iiint_{V'} \bar{Q}(x_m,\omega)G(x_m,x_m^s,\omega)dV$$

$$+ \iint_\Sigma \rho^{-1}[P_{,n'}(x_m,\omega)G(x_m,x_m^s,\omega) - P(x_m,\omega)G_{,n'}(x_m,x_m^s,\omega)]dS$$

$$+ \iint_{S_r} \rho^{-1}[P_{,n}(x_m,\omega)G(x_m,x_m^s,\omega) - P(x_m,\omega)G_{,n}(x_m,x_m^s,\omega)]dS \,.$$

Here we denoted by n' the inward normal to Σ, which plays the role of outward normal to the surface limiting the volume V'.

We shall now expand the sphere S_r to infinity. Since we require that the pressure wave field satisfies the Sommerfeld radiation conditions, the last integral over surface $S_r \to \infty$ will vanish. We shall prove it for the case of a *homogeneous medium inside* V', i.e. for c and ρ constant. If we take into account that

$$G_{,n} = \partial G/\partial n = \partial G/\partial r \text{ and } G = \frac{\rho}{4\pi r} e^{i\omega \frac{r}{c}} \quad ,$$

we can rewrite the integral over S_r in the following way

$$\frac{1}{4\pi} \iint\limits_{S_r} \left[P_{,n} \frac{e^{i\omega r/c}}{r} - P\left(-\frac{1}{r^2} + \frac{i\omega}{rc}\right) e^{i\omega r/c} \right] r^2 \sin\theta d\theta d\varphi \,.$$

We can see that this integral vanishes when the following two conditions are satisfied

$$\lim_{r\to\infty} r\left(\partial P/\partial r - \frac{i\omega}{cr} P\right) = 0\,; \quad |P| \le k\frac{1}{r} \quad k > 0,\, r \to \infty\,.$$

But these conditions are nothing else than the Sommerfeld radiation conditions. If we take a trial solution of the acoustic wave equation

$$P \sim \frac{e^{-i\omega(t-r/c)}}{r} \quad ,$$

which represents the *outgoing wave* and insert it into the first condition, we get

$$r\left[\left(\frac{i\omega}{rc} - \frac{1}{r^2}\right) e^{-i\omega(t-r/c)} - \frac{i\omega}{cr} e^{-i\omega(t-r/c)}\right] = -\frac{1}{r} e^{-i\omega(t-r/c)} \quad .$$

The resulting expression satisfies the first Sommerfeld condition. If we take as a trial solution the *ingoing wave*,

$$P \sim \frac{e^{-i\omega(t+r/c)}}{r} \quad ,$$

we get

$$r\left[-\left(\frac{i\omega}{rc} + \frac{1}{r^2}\right) e^{-i\omega(t+r/c)} - \frac{i\omega}{cr} e^{-i\omega(t+r/c)}\right] = -\left(\frac{1}{r} + \frac{2i\omega}{c}\right) e^{-i\omega(t+r/c)} \quad .$$

This expression does not satisfy the first condition. We can see that the Sommerfeld radiation conditions are satisfied only by outgoing waves. For the pressure wave field satisfying the Sommerfeld radiation conditions, we can write the *Kirchhoff outer formula* in the following form

$$\begin{aligned} P(x_m^{(s)}, \omega) = & \iiint\limits_{V'} \bar{Q}(x_m, \omega) G(x_m, x_m^{(s)}, \omega) dV \\ & - \iint\limits_{\Sigma} \rho^{-1} [P_{,n}(x_m, \omega) G(x_m, x_m^{(s)}, \omega) - P(x_m, \omega) G_{,n}(x_m, x_m^{(s)}, \omega)] dS \,. \end{aligned}$$

Here V' denotes the whole space with exception of the volume V. The point $x_m^{(s)}$ is supposed to be situated outside the volume V, i.e. in V'.

Similarly as in the case of the inner formula, we could rewrite the outer formula for the reciprocal Green function into a more natural form, and we could also transform it into the time domain.

6.3.2 Elastodynamic representation theorem

We start from the equation derived in Sec.6.2.2 from the Betti's theorem

$$\int_{-\infty}^{+\infty} dt \left\{ \iiint_V [f_i^{(1)}(t)u_i^{(2)}(\tau - t) - f_i^{(2)}(\tau - t)u_i^{(1)}(t)]dV \right\}$$
$$= \int_{-\infty}^{+\infty} dt \left\{ \iint_\Sigma c_{ijkl}n_j[u_{k,l}^{(2)}(\tau - t)u_i^{(1)}(t) - u_{k,l}^{(1)}(t)u_i^{(2)}(\tau - t)]dS \right\} .$$

Let us now consider that the wavefield (2) is generated by a unit point force situated at a point $x_m^{(s)}$ inside the volume V, and the force is oriented along the x_n axis and acting at time t_s,

$$f_i^{(2)}(x_m, t) = \delta_{in}\delta(x_m - x_m^{(s)})\delta(t - t_s) \quad .$$

Then $u_i^{(2)}(x_m, t)$ represents the Green function

$$u_i^{(2)}(x_m, t) = G_{in}(x_m, t; x_m^{(s)}, t_s) \quad .$$

As in Sec.5.2, we consider the index n (specifying the coordinate axis, along which the point force is oriented) fixed. Thus the only free indices in the above equations are i and the equations are correct. Inserting the above expressions into the previous integral equation, and omitting the indices (1) and (2) from the same reasons as in the acoustic case, we get

$$\int_{-\infty}^{+\infty} dt \iiint_V f_i(t)G_{in}(x_m, \tau - t; x_m^{(s)}, t_s)dV - u_n(x_m^{(s)}, \tau - t_s)$$
$$= \int_{-\infty}^{+\infty} dt \{ \iint_\Sigma c_{ijkl}n_j[G_{kn,l}(x_m, \tau - t; x_m^{(s)}, t_s)u_i(x_m, t)$$
$$-u_{k,l}(x_m, t)G_{in}(x_m, \tau - t; x_m^{(s)}, t_s)]dS\} \quad ,$$

where

$$G_{in,l}(x_m, \tau - t; x_m^{(s)}, t_s) = \partial G_{in}(x_m, \tau - t; x_m^{(s)}, t_s)/\partial x_l \quad .$$

In the final form, we can thus write

$$u_n(x_m^{(s)}, \tau - t_s) = \int_{-\infty}^{+\infty} dt \iiint_V f_i(t) G_{in}(x_m, \tau - t; x_m^{(s)}, t_s) dV$$

$$+ \int_{-\infty}^{+\infty} dt \iint_\Sigma c_{ijkl} n_j [u_{k,l}(x_m, t) G_{in}(x_m, \tau - t; x_m^{(s)}, t_s)$$

$$-G_{kn,l}(x_m, \tau - t; x_m^{(s)}, t_s) u_i(x_m, t)] dS\} \quad .$$

This is *the elastodynamic representation theorem.* The displacement at an arbitrary point $x_m^{(s)}$ of the volume V and at an arbitrary moment $\tau - t_s$ is expressed in terms of the distribution of body forces inside the volume V (the first integral) and in terms of spatial derivatives of the displacement (tractions) and the displacement itself along the surface Σ (the second integral).

If the Green function is reciprocal, i.e. if it is chosen so that the surface integral in the reciprocity theorem vanishes, the elastodynamic representation theorem can be rewritten in a more natural way. If the Green function satisfies, for example, *homogeneous* boundary conditions for the displacement on Σ (Σ - rigid boundary), then the Green function is reciprocal and the elastodynamic representation theorem reads

$$u_n(x_m^{(s)}), \tau - t_s) = \int_{-\infty}^{+\infty} dt \iiint_V f_i(t) G_{ni}(x_m^{(s)}, \tau - t_s; x_m, t) dV$$

$$- \int_{-\infty}^{+\infty} dt \iint_\Sigma c_{ijkl} n_j u_i(x_m, t) \frac{\partial G_{nk}}{\partial x_l}(x_m^{(s)}, \tau - t_s; x_m, t) dS \, .$$

Similarly as in the acoustic case, we could call the above formula - the *inner formula*, and we could derive also the *outer formula.*

6.4 Applications of the Kirchhoff formula

Let us consider a volume V surrounded by a surface Σ filled by an inhomogeneous acoustic medium with constant density $\rho = 1$, and in it a distribution of voluminal sources $q(x_m, t)$. The Helmholtz equation can thus be written as follows

$$P_{,ii}(x_m, \omega) + \omega^2 c^{-2}(x_m) P(x_m, \omega) = -\bar{Q}(x_m, \omega) \quad ,$$

where

$$\bar{Q}(x_m, \omega) = \int_{-\infty}^{+\infty} q_{,t}(x_m, t) e^{i\omega t} dt \quad .$$

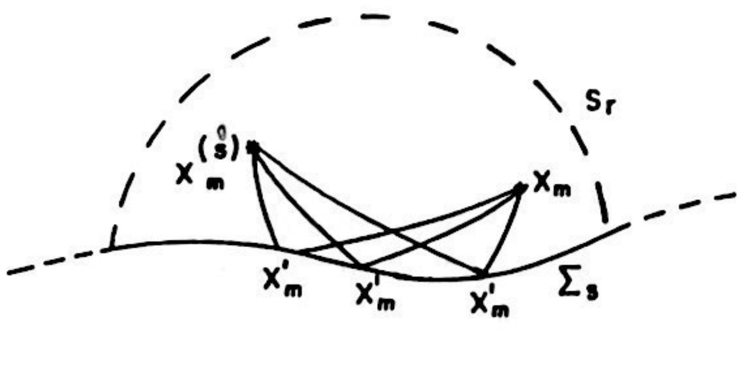
The solution of this equation can be expressed in terms of the Kirchhoff formula. In the following, we present applications of this formula to solving *forward* and *inverse problems*. We shall not go into details, only main ideas are presented.

By the *forward problem* we understand such a problem, in which we assume that we know the velocity distribution $c(x_m)$ and positions and properties of volume velocity sources, and we want to determine the pressure wave field at a chosen point x_m. There might be a variety of *inverse problems*. Most frequently considered inverse problems are such, in which we know the positions and properties of sources and we also know the response of the structure to these sources at a system of points (receivers). Our aim is to determine the velocity distribution. In slightly different inverse problem, the velocity distribution is assumed to be known and we are seeking the form and reflectivity of reflectors. In still another inverse problem formulation, we may have knowledge of complete structure, of positions of sources and of responses of the structure at a system of points and we may try to determine the spectrum of the source signal, i.e. we may perform the so-called *wavelet estimation*.

In the following, we consider an application of the Kirchhoff formula to the problems of surface and voluminal scattering. There are many other applications of this important formula, e.g. to downward continuation or wavelet estimation, which we are not presenting here. See, e.g. Weglein & Stolt (1992).

6.4.1 Surface scatterer - Kirchhoff approximation

We consider the situation shown in the picture. Above the scatterer Σ_s, we consider a volume velocity point source situated at the point $x_m^{(0)}$:



$$\bar{Q}(x_m, \omega) = A(\omega)\delta(x_m - x_m^{(0)})$$

and an observer at the point x_m. For the application of the Kirchhoff inner formula, the source and the receiver are considered to be inside the volume bounded by a closed surface formed partially by the scatterer Σ_s and by a sphere S_r, which we are going to expand to infinity. The Kirchhoff inner formula for this specification reads

$$P(x_m,\omega) = A(\omega)G(x_m,x_m^{(0)},\omega)$$

$$+ \iint\limits_{\Sigma_s+S_r} [P_{,i}(x'_m,\omega)n_i G(x_m,x'_m,\omega) - P(x'_m,\omega)G_{,i}(x_m,x'_m,\omega)n_i]dS(x'_m) .$$

As the Green function, we consider the free space Green function, which is reciprocal.

Let us now decompose the field $P(x_m,\omega)$ into the field of the incident wave $P_B(x_m,\omega)$ propagating in a background medium without any scatterer and scattered field $P_S(x_m,\omega)$,

$$P(x_m,\omega) = P_B(x_m,\omega) + P_S(x_m,\omega) \quad .$$

Since $P_B(x_m,\omega)$ is the free space field, the Kirchhoff formula for it reduces to

$$P_B(x_m,\omega) = A(\omega)G(x_m,x_m^{(0)},\omega) \quad .$$

This implies that for any closed surface Σ containing $x_m^{(0)}$ and x_m, the following identity must hold:

$$\iint\limits_{\Sigma} [P_{B,i}(x'_m,\omega)n_i G(x_m,x'_m,\omega) - P_B(x'_m,\omega)G_{,i}(x_m,x'_m,\omega)n_i]dS = 0 .$$

Using the above relations and expanding the surface S_r to infinity (due to which the integral over the surface S_r vanishes), we get for $P_S(x_m,\omega)$ from the Kirchhoff formula

$$P_S(x_m,\omega) = \iint\limits_{\Sigma_s} [P_{S,i}(x'_m,\omega)n_i G(x_m,x'_m,\omega) - P_S(x'_m,\omega)G_{,i}(x_m,x'_m,\omega)n_i]dS(x'_m) .$$

This is an exact equation relating the scattered field to its values on the scattering surface Σ_s. To solve it as it stands would not be an easy task. Most frequently used approximation for its solution is the so-called *Kirchhoff approximation*. It relates to the approximation made by Kirchhoff when he tried to evaluate the diffracted light field behind the slit in a screen. The Kirchhoff approximation consists in substitution of the values $P_S(x_m,\omega)$ on Σ_S by $RP_B(x_m,\omega)$. Here R is an appropriately specified reflection coefficient. In this way, the above integral equation can be used for solving forward problems of surface scattering by specifying R and seeking $P_S(x_m,\omega)$, or for solving inverse problems by specifying $P_S(x_m,\omega)$ and seeking R.

6.4.2 Voluminal scatterer - Born approximation

Let us now expand the volume V to infinity. Let us further consider that the velocity distribution in it can be expressed as a sum of a simple background velocity distribution

$c_o(x_m)$ (e.g., constant velocity), for which the solution of the Helmholtz equation is known, and some perturbation $\alpha(x_m)$ representing the voluminal scatterer. We assume that

$$c^{-2}(x_m) = c_o^{-2}(x_m)(1 + \alpha(x_m)) \quad .$$

The velocity distribution $c_o(x_m)$ is usually called the *background* velocity distribution, the function $\alpha(x_m)$ represents its *perturbation*. If we insert this expression for $c(x_m)$ into the Helmholtz equation, we get

$$P_{,ii}(x_m,\omega) + \omega^2 c_o^{-2}(x_m) P(x_m,\omega) = -\bar{Q}(x_m,\omega) - \frac{\omega^2\alpha(x_m)}{c_o^2(x_m)} P(x_m,\omega) \quad .$$

This is an inhomogeneous Helmholtz equation for the background velocity distribution with the source term composed of two parts. The first term represents volume velocity source, the second term is responsible for the generation of the scattered wavefield. To distinguish between both sources, we call the former one the *active source* and the latter one the *passive source*.

To solve the above equation, we can use the Kirchhoff inner formula. We must remember that we are considering the problem in the whole space so that the surface integral in the Kirchhoff's formula vanishes. We consider again the Green function to be reciprocal. Then we can write

$$\begin{aligned} P(x_m,\omega) &= \iiint_{-\infty}^{+\infty} \bar{Q}(x'_m,\omega) G(x_m,x'_m,\omega)\, dV(x'_m) \\ &+ \omega^2 \iiint_{-\infty}^{+\infty} \frac{\alpha(x'_m)}{c_o^2(x'_m)} P(x'_m,\omega) G(x_m,x'_m,\omega)\, dV(x'_m) \quad . \end{aligned}$$

The Green function corresponds to the *unperturbed background* medium. For the pressure wave field generated by the volume velocity point source

$$\bar{Q}(x_m,\omega) = A(\omega)\delta(x_m - x_m^{(s)})$$

we get

$$\begin{aligned} P(x_m,\omega) &= A(\omega) G(x_m, x_m^{(s)},\omega) \\ &+ \omega^2 \iiint_{-\infty}^{+\infty} \frac{\alpha(x'_m)}{c_o^2(x'_m)} G(x_m,x'_m,\omega) P(x'_m,\omega)\, dV(x'_m) \quad . \end{aligned}$$

The pressure wave field at x_m due to the pressure source at $x_m^{(s)}$ is expressed above as the sum of the wavefield in the background medium and the integral representing scattered wavefield due to the perturbations of the background medium. The above integral equation is *exact.* It holds for any perturbation $\alpha(x_m)$. It can be used for solving both forward and inverse problems. This kind of integral equation is known as the *Lippman-Schwinger equation.* It can be rewritten in an alternative form if we express the pressure wavefield $P(x_m,\omega)$ as a sum of the wavefield in the background medium, $P_B(x_m,\omega)$, and the scattered wavefield $P_S(x_m,\omega)$

$$P(x_m,\omega) = P_B(x_m,\omega) + P_S(x_m,\omega) \quad .$$

Then, taking into account that

$$P_B(x_m,\omega) = A(\omega)G(x_m, x_m^{(s)},\omega) \, ,$$

the Lippman-Schwinger equation can be rewritten as

$$P_S(x_m,\omega) = \omega^2 \iiint\limits_{-\infty}^{+\infty} \frac{\alpha(x'_m)}{c_o^2(x'_m)} G(x_m, x'_m,\omega)[P_B(x'_m,\omega) + P_S(x'_m,\omega)]dV(x'_m) \quad .$$

To solve the Lippman-Schwinger equation is not an easy task. If $\alpha(x_m)$ is small, i.e., if the deviations of the actual velocity distribution from the background one are sufficiently small, the problem can be linearized. If $\alpha(x_m)$ is small, then the scattered wavefield $P_s(x_m,\omega)$ will be weak and the term

$$\alpha(x'_m)P_S(x'_m,\omega)$$

under the integral can be neglected since it is a product of two small quantities. The term, which will remain under the integral is

$$\frac{\alpha(x'_m)}{c_o^2(x'_m)} G(x_m, x'_m,\omega)P_B(x'_m,\omega) \quad .$$

The Lippman-Schwinger equation thus attains a form

$$\begin{aligned} P(x_m,\omega) \sim\ & A(\omega)[G(x_m, x_m^{(s)},\omega) \\ & + \ \omega^2 \iiint\limits_{-\infty}^{+\infty} \frac{\alpha(x'_m)}{c_o^2(x'_m)} G(x_m, x'_m,\omega)G(x'_m, x_m^{(s)},\omega)dV(x'_m)] \end{aligned}$$

or alternatively

$$P_S(x_m,\omega) \sim A(\omega)\omega^2 \iiint\limits_{-\infty}^{+\infty} \frac{\alpha(x'_m)}{c_o^2(x'_m)} G(x_m, x'_m,\omega)G(x'_m, x_m^{(s)},\omega)dV(x'_m) \, .$$

This is the first order approximation of the solution of the Lippman-Schwinger equation. This approximate solution is known as the *Born approximation* (it is similar to the approximation used under the same name in the theoretical physics for potential scattering). It has found recently many applications in seismology, both in forward and inverse modeling.

If used for *forward modeling*, there are no problems with the evaluation of the expressions on the RHS of the above equations since all the quantities are assumed to be known. The first order values of $P(x_m, \omega)$ (or $P_S(x_m, \omega)$) can be used again under the integral in the Lippman-Schwinger equation to yield the second-order Born approximation. Described procedure is very convenient for solving problems in which the standard ray method can be used only with modifications, e.g. diffraction problems.

The Born approximation is also convenient for solving *inverse* problems. The above equations relate linearly the unknown perturbation $\alpha(x_m)$ with values of $P(x_m, \omega)$ (or $P_S(x_m, \omega)$), which are assumed to be known at some points from measurements. The above integral equations in the Born approximation are the Fredholm integral equations of the first kind for $\alpha(x_m)$. They can be solved in a closed form if the background velocity $c_o(x_m)$ is assumed to be constant. Then the formulae for the Green function in a homogeneous space derived above can be used and the resulting integral equations can be solved by the Fourier method.

The above described procedures can be also applied to elastic solids including both isotropic and anisotropic media.

CHAPTER 7

RAY METHOD

The ray method is a powerful tool for solving wave propagation problems in rather general laterally varying layered inhomogeneous isotropic and/or anisotropic structures. Competing methods which can be applied to similar types of complex structures are methods of direct numerical solution of equations of motion (acoustic equation or elastodynamic equation) such as finite difference (FD) or finite element (FE) methods. The latter approaches are, however, limited to models, which do not exceed several wavelengths in the extent. With increasing extent of the model, they become extremely time consuming.

The ray method has its advantages and disadvantages. The basic advantage is the applicability of the ray method to inhomogeneous media and its effectivity in such computations. In contrast to the FD and FE methods, the ray method can be applied without difficulties to the models whose extent is much greater than several wavelengths. Another great virtue of the ray method is the physical insight, which it yields into the wave propagation process. It allows to separate individual wave phases in the wavefield and even to follow the way by which energy of these waves propagates through the medium. The ray method also represents an important basis for other related, more sophisticated methods, such as the Gaussian beam summation method, the paraxial ray method, the Maslov method, etc.

The ray method has also some limitations. First, it is only approximate. It is applicable only to smooth media, in which the characteristic dimensions of inhomogeneities are considerably larger than the prevailing wavelength of the considered waves. The ray method can yield distorted results or no results at all in some special regions called singular regions (caustic regions, critical regions, transition zones between the illuminated and shadow regions).

The ray method is known under various names such as the geometrical seismics, the asymptotic ray method, the ray series method. The name the geometrical seismics (derived from "geometrical optics") refers more to the method of construction of ray paths along which seismic energy propagates. The other names specify more general approach, which yields approximate description of the whole wave process.

The ray method was not developed originally for the applications in seismology. The seismic ray method owes a lot to optics and radiophysics. Although the techniques used in different branches of physics are very similar, there are some substantial differences in applications. The ray method is usually applied to much more complicated structures than in optics or radiophysics. There are also different numbers and types of waves considered in different branches of physics.

First seismological applications date to the end of the last century. Then, only *kinematics*, specifically travel times were used. The assumed structure was simple radially symmetric Earth. First attempts to use also *dynamics* (amplitudes and waveforms) were made probably by Sir H. Jeffreys. The ray series solutions of the elastodynamic equations were first suggested by Babič and his colleagues in the USSR (1956) and Karal & Keller in the US (1959). Since then it has been further developed by many scientists all over the world. Substantial contribution has been made by Červený in Czech Republic. The development of computer technique had also strong impact on the development and the applicability of the ray method. In the exploration seismology, the ray method belongs to the basic tools.

7.1 Basics of the ray method

7.1.1 Basic assumptions

Let us consider a homogeneous acoustic equation for pressure $p(x_m, t)$ in a medium with a variable velocity $c(x_m)$ and constant density,

$$p_{,ii}(x_m, t) - c^{-2}(x_m)p_{,tt}(x_m, t) = 0 \quad .$$

In the previous sections we were able to find solutions of the above equation but only under rather strong assumptions about the velocity distribution. For $c(x_m) = \text{const}$, we found, for example, the solution of the acoustic equation in the form of the plane wave

$$p(x_m, t) = PF(t - T(x_m)) \quad .$$

Here P is a constant amplitude of the plane wave, F is an analytic signal and $T(x_m)$ is a linear function of the coordinates x_m,

$$T(x_m) = N_i x_i / c \quad .$$

The symbol N_i denotes the unit vector of the phase normal, constant in the whole medium.

Let us now consider a slight variation of the velocity c in dependence on x_m. Then, we can also expect slight variations of the above plane wave solution, specifically variations of

P and N_i, as the wave proceeds through the inhomogeneous medium. Locally, however, we can still consider the wave to be nearly plane wave. It has thus the form

$$p(x_m, t) = P(x_m)F(t - \tau(x_m)) \quad .$$

Let us now consider a homogeneous medium and in it a wave generated by a point source. In the vicinity of the source, we can again approximate the wave by the above formula, in which $P(x_m)$ and $N_i(x_m)$ vary due to the curvature of the phase front of the wave.

In both mentioned cases the amplitude $P(x_m)$ is no more constant and the eikonal $\tau(x_m)$ is no more linear function of x_m. Spatial variation of all involved quantities is considered to be small. This may be expressed more specifically in the following way. The change of any of the involved quantities on the distance of a prevailing wavelength of the studied wave is required to be negligible in comparison with the value of the considered quantity. For velocity c, amplitude P and slowness vector $\tau_{,i}$ this reads

$$\lambda|\nabla c| \ll c\,, \quad \lambda|\nabla P| \ll |P|\,, \quad \lambda|\nabla \tau_{,i}| \ll |\tau_{,i}| \quad .$$

This can be rewritten

$$\lambda \ll \frac{c}{|\nabla c|}\,, \quad \lambda \ll \frac{|P|}{|\nabla P|}\,, \quad \lambda \ll \frac{|\tau_{,i}|}{|\nabla \tau_{,i}|} \quad .$$

The quantities on the RHS of the above inequalities have dimensions of length. The one of them, which is minimum, can be denoted by L and then all the three relations can be written as

$$\lambda/L \ll 1 \quad .$$

L can be understood as the length, on which the considered parameter can change so that its change is comparable with the value of the parameter itself. The quantity λ/L can be chosen as a small parameter of the problem and the formal solution of the acoustic equation can be expanded in terms of it. Thus the solution will have a structure of a series in powers of λ/L. Instead of λ, we can consider circular frequency ω, $\lambda = (2\pi c)/\omega$. Then the above inequality reads

$$\frac{1}{\omega L} \ll 1 \quad .$$

Then the small parameter is inversely proportional to the frequency ω. The solution can be now expressed in the form of a series in inverse powers of frequency. The series has an asymptotic character. From the above reasons, the ray method is sometimes called *the high-frequency asymptotic method*. Note, however, that in the condition $(\omega L)^{-1} \ll 1$, the frequency ω does not stand alone. If L is sufficiently large (i.e. when the variations of all the considered parameters are very small), then the condition can be satisfied even for a relatively low frequencies.

The above local formula is designed for a study of a separately propagating wave. It cannot be used for a study of interference waves. It fails, therefore, in cases, in which, for example, two waves propagate with nearly the same phase velocity. This usually does not occur in isotropic media (if the spatial variation of elastic parameters is modest) since the difference between the P and S wave phase velocity is rather large there. It can, however, occur in anisotropic media for qS waves, which may, in some directions, propagate with the same phase velocity. To exclude such situations, we must consider the following condition

$$\omega^{-1}\bar{c}|c_{,i}| \ll \Delta c \quad .$$

Here $\bar{c}$ denotes an average of the phase velocities of both qS waves propagating in the same direction, $|c_{,i}|$ denotes the magnitude of the larger of the gradients of the phase velocities, Δc is the difference between the phase velocities and ω is the prevailing frequency of the waves. The inequality follows from the requirement that the distance on which the two qS waves separate by a prevailing wavelength be considerably smaller than the distance on which the phase velocities change by an amount equal to $\bar{c}$.

7.1.2 High-frequency solutions of equation of motion

As was suggested above, the solutions of equations of motion can be sought in the form of a series with the first term of the following form:

$$w(x_m, t) = W(x_m)F(t - \tau(x_m)) \quad .$$

Here $w(x_m, t)$ can stand either for the scalar solution of the acoustic equation (e.g. pressure $p(x_m, t)$) or for a component of a vectorial solution of the elastodynamic equation. Similarly, generally complex-valued quantity $W(x_m)$ can be either scalar or vector. It is called the scalar (vectorial) *ray amplitude*. The scalar real-valued quantity $\tau(x_m)$ is called the *eikonal* or the *phase function*. To satisfy the conditions of Sec.7.1.1, the analytic signal $F(\theta)$ is assumed to be the *high-frequency signal*. It means that the spectrum of the signal between 0 and ω^*, where ω^* is large, is considered to be effectively zero.

The above solution represents the first term in the solution written in the form of a series - the *ray series solution*. The above term is called the *leading term of the ray series* or the *zero-order ray approximation*. The ray series reads

$$w(x_m, t) = \sum_{n=0}^{\infty} W^{(n)}(x_m)F^{(n)}(t - \tau(x_m)) \quad .$$

Here $W^{(n)}$ is the ray amplitude (or coefficient) of the n-th order term of the ray series. If $W^{(n)}$ is a component of a vector, then the component of this vector into the vector $W^{(0)}$ is called the *principal component*. The remaining part of $W^{(n)}$ is called the *additional component*. Functions $F^{(n)}(\theta)$ are again high-frequency analytic signals which satisfy the following relation

$$dF^{(n)}(\theta)/d\theta = F^{(n-1)}(\theta) \quad .$$

For high-frequency signals, this implies that

$$|F^{(n-1)}(\theta)| \gg |F^{(n)}(\theta)| \quad ,$$

i.e., the higher-order terms of the ray series can be neglected. Let us emphasize that the upper-right index in brackets *does not* denote differentiation.

The above representation of the ray solution also includes solutions, which are nonanalytic along wavefronts. The solutions can be expressed in terms of distributions such as the δ-function, the Heaviside function, etc. The phase front then represents a discontinuity and we speak about *propagation of discontinuities*. The discontinuities also satisfy the above relation between $F^{(n)}$ and $F^{(n-1)}$. Let us consider $F^{(0)}$ as the Heaviside function, i.e.

$$F^{(0)}(\theta) = H(\theta) \quad .$$

The Heaviside function is defined as

$$H(\theta) = \begin{cases} 0 & \text{for } \theta \leq 0 \quad , \\ 1 & \text{for } \theta > 0 \quad . \end{cases}$$

For $F^{(n)}(\theta)$ we then have

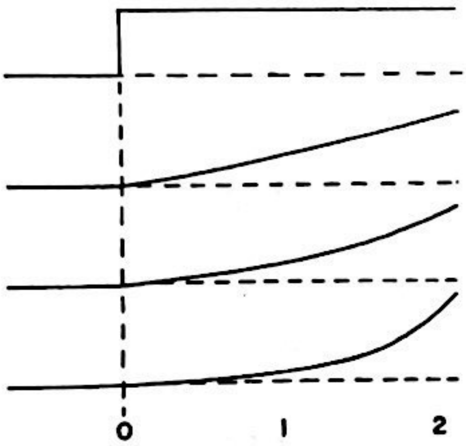
$$F^{(n)}(\theta) = \begin{cases} 0 & \text{for } \theta \leq 0 \quad , \\ \theta^n/n! & \text{for } \theta > 0 \quad . \end{cases}$$

For $F^{(0)}(\theta) - F^{(3)}(\theta)$, see the following picture. We can see that the strongest discontinuity exists for the lowest term. With increasing number n, the function $F^{(n)}$ becomes smoother and smoother.

The ray solutions are often written in terms of harmonic waves. We get such a ray series solution if we put $F^{(0)}(\theta) = e^{-i\omega t}$ and take into account that $dF^{(n)}(\theta)/d\theta = F^{(n-1)}(\theta)$. Then we get

$$w(x_m, t) = \sum_{n=0}^{\infty} \frac{W^{(n)}(x_m)}{(-i\omega)^n} exp[-i\omega(t - \tau(x_m))] \quad .$$

The solution is now expressed in terms of a series in inverse powers of frequency.



Let us note that it does not matter in which form we are seeking the ray solutions; whether in the form of a series of high-frequency analytic signals, in the form of discontinuities or in the form of high-frequency harmonic waves. In all cases, we shall arrive to the same equations for the amplitude coefficients $W^{(n)}(x_m)$ and eikonal $\tau(x_m)$. Always, the most important term of the above ray series is the leading term, called the zero-order ray approximation.

In the following, we shall concentrate on the determination of the parameters of the leading term. We shall, therefore, consider the ray solution consisting of the zero order ray approximation only.

7.2 Basic system of the equations of the ray method

7.2.1 Acoustic case

Let us consider a homogeneous acoustic equation with the varying velocity $c(x_m)$ and density $\rho(x_m)$,

$$(\rho^{-1}p_{,i})_{,i} - \kappa p_{,tt} = 0 \quad .$$

We shall seek its solution in the form

$$p(x_m, t) = P(x_m)F(t - \tau(x_m)) \quad .$$

Our task is reduced to the determination of the amplitude coefficient $P(x_m)$ and the eikonal $\tau(x_m)$. Let us insert the trial solution into the acoustic equation. We get

$$-\rho^{-2}\rho_{,i}(P_{,i}F - PF'\tau_{,i}) + \rho^{-1}(P_{,ii}F - 2P_{,i}F'\tau_{,i} + PF''\tau_{,i}\tau_{,i} - PF'\tau_{,ii}) - \kappa PF'' = 0 \, .$$

Here F', F'' denote the derivatives of the analytic signal F with respect to its argument. The above equation can be rewritten

$$F''(\tau_{,i}\tau_{,i} - c^{-2})P + F'(\rho^{-1}\rho_{,i}\tau_{,i}P - 2P_{,i}\tau_{,i} - P\tau_{,ii}) + F(P_{,ii} - \rho^{-1}\rho_{,i}P_{,i}) = 0 \,.$$

Because of the relation $|F''| \gg |F'| \gg |F|$, the first term in the above equation has the largest weight. To satisfy the whole equation, we require that the coefficients with F'', F' and F are zero. The coefficient with F'' then yields

$$\tau_{,i}\tau_{,i} = c^{-2}(x_m) \quad .$$

This is the *eikonal equation.* A substantial difference from the eikonal equation which we studied in case of plane wave propagation is that the velocity $c(x_m)$ can vary. Thus the equation holds locally. From the mathematical point of view, the eikonal equation is a nonlinear, first-order partial differential equation for $\tau = \tau(x_m)$. We shall solve it later using rays.

The coefficient with F' yields

$$2P_{,i}\tau_{,i} + P\tau_{,ii} - \rho^{-1}\rho_{,i}\tau_{,i}P = 0 \quad .$$

This is the *transport equation.* It is a linear partial differential equation of the first order for the amplitude coefficient $P(x_m)$. Later, we shall solve it along rays. In that case, it reduces to an ordinary differential equation, which can be solved analytically. The only possibly complex-valued quantity in the transport equation is P. If we multiply the transport equation by $\rho^{-1}P^*$ where P^* denotes complex conjugate of P, if we further multiply the complex conjugate transport equation by $\rho^{-1}P$ and add both resulting equations, we get

$$\rho^{-1}\tau_{,i}(PP^*_{,i} + P_{,i}P^*) + \rho^{-1}\tau_{,ii}PP^* - \rho^{-2}\rho_{,i}\tau_{,i}PP^* = 0 \quad .$$

This yields immediately

$$(PP^*\rho^{-1}\tau_{,i})_{,i} = 0 \quad .$$

The eikonal and transport equations determine the sought quantities $\tau(x_m)$ and $P(x_m)$. Thus, the condition that the coefficient with F also vanishes, is superfluous. The value of the term

$$P_{,ii} - \rho^{-1}\rho_{,i}P_{,i}$$

can be used for checking the precision of the approximate solution. If we used the ray series instead of leading term only in the acoustic equation, then the above term would be a part of the transport equation for the determination of the amplitude of the first order term.

7.2.2 Elastodynamic case - isotropic media

The derivation of basic equations of the ray method for elastodynamic theory for isotropic media is similar to the acoustic case. There are, however, two important differences. First, the ray solution is written for the displacement, which is a vectorial quantity. Second, as we know from plane and spherical wave propagation in isotropic media, we must expect two types of solutions of the elastodynamic equation, the first one corresponding to P waves, the other to S waves. Let us recall the form of the elastodynamic equation for an inhomogeneous isotropic medium,

$$(\lambda+\mu)u_{j,ij} + \mu u_{i,jj} + \lambda_{,i}u_{j,j} + \mu_{,j}(u_{i,j}+u_{j,i}) = \rho u_{i,tt} \quad .$$

The formal solution has now the vectorial form,

$$u_i(x_m,t) = U_i(x_m)F(t-\tau(x_m)) \quad .$$

The quantities $U_i(x_m)$ and $\tau(x_m)$, the vectorial ray amplitude and the eikonal are to be determined. We insert the formal solution into the elastodynamic equation and, similarly as in the acoustic case, we obtain

$$F''N_i(U_m) - F'M_i(U_m) + FL_i(U_m) = 0 \quad .$$

The coefficients N_i, M_i and L_i have the following form

$$\begin{aligned}
N_i(U_m) &= (\lambda+\mu)U_j\tau_{,i}\tau_{,j} + \mu U_i\tau_{,j}\tau_{,j} - \rho U_i \quad , \\
M_i(U_m) &= (\lambda+\mu)[U_{j,i}\tau_{,j} + U_{j,j}\tau_{,i} + U_j\tau_{,ij}] + \mu[2U_{i,j}\tau_{,j} \\
&+ U_i\tau_{,jj}] + \lambda_{,i}U_j\tau_{,j} + \mu_{,j}(U_i\tau_{,j} + U_j\tau_{,i}) \, , \\
L_i(U_m) &= (\lambda+\mu)U_{j,ij} + \mu U_{i,jj} + \lambda_{,i}U_{j,j} + \mu_{,j}(U_{i,j}+U_{j,i}) \, .
\end{aligned}$$

From the same reasons as in the acoustic case, we put the coefficients with F'', F' and F equal zero.

The equation

$$N_i(U_m) = 0$$

can be rewritten into the form, which we used when we studied plane wave propagation

$$\left(\frac{\lambda+\mu}{\rho}\tau_{,i}\tau_{,j} + \frac{\mu}{\rho}\tau_{,k}\tau_{,k}\delta_{ij} - \delta_{ij}\right)U_j = 0 \quad .$$

Substantial difference with respect to the plane wave case is that this equation holds locally. In other words, all the involved quantities vary from point to point and at each point satisfy the above equation. Comparing the equation with a similar one, which we derived in

Sec.3.3.2, for plane waves, we can identify it as the *local Christoffel equation for isotropic media*

$$(\bar{\Gamma}_{ij} - \delta_{ij})U_j = 0$$

with

$$\bar{\Gamma}_{ij} = \frac{\lambda + \mu}{\rho}\tau_{,i}\tau_{,j} + \frac{\mu}{\rho}\tau_{,k}\tau_{,k}\delta_{ij} \quad .$$

The Christoffel equation can be solved if we know the solution of the eigenvalue problem for the matrix $\bar{\Gamma}_{ij}$

$$(\bar{\Gamma}_{ij} - G\delta_{ij})g_j = 0$$

with the eigenvalues G subjected to the condition $G = 1$. Here g_j is the eigenvector corresponding to G. The matrix $\bar{\Gamma}_{ij}$ has three real and positive eigenvalues, the corresponding real eigenvectors are mutually perpendicular. The eigenvalues can be determined from the condition of solvability of the eigenvalue problem, namely

$$det(\bar{\Gamma}_{ij} - G\delta_{ij}) = 0 \quad .$$

A strict determination of all the three eigenvalues was presented in Sec.3.3.2. Here we present an alternative approach.

From the Christoffel equation with U_j substituted by g_j, $(\bar{\Gamma}_{ij} - \delta_{ij})g_j = 0$, we have

$$g_i = \bar{\Gamma}_{ij}g_j = \frac{\lambda + \mu}{\rho}\tau_{,i}(\tau_{,j}g_j) + \frac{\mu}{\rho}\tau_{,k}\tau_{,k}g_i \quad .$$

If $(\tau_{,j}g_j)$ is nonzero, then the above equation can be satisfied only if $g_i \parallel \tau_{,i}$. The other two eigenvectors must then be perpendicular to $\tau_{,i}$. If we take into account that $\tau_{,j} = N_j/c$, where N_j is the phase normal and c is the phase velocity, then we have two choices:

$$g_i = N_i \text{ and } g_i \perp N_i \quad .$$

For the first choice, i.e., $g_i = N_i$, the equation $G = \bar{\Gamma}_{ij}g_ig_j = 1$ yields

$$\alpha^2\tau_{,k}\tau_{,k} = 1 \quad .$$

This is the P wave eikonal equation, $\alpha(x_m) = \sqrt{\frac{\lambda+2\mu}{\rho}}$ is the local P wave velocity.

The P wave polarization is along the local normal to the phase front

$$U_i(x_m) = A(x_m)N_i(x_m) \quad ,$$

where $A(x_m)$ is the scalar amplitude coefficient to be determined. The latter choice, i.e. $g_i \perp N_i$, yields the eikonal equation

$$\beta^2\tau_{,k}\tau_{,k} = 1 \quad ,$$

which describes propagation of S waves, which propagate with the local velocity $\beta(x_m)$

$$\beta(x_m) = \sqrt{\frac{\mu}{\rho}} \quad .$$

The S wave polarization is specified by two arbitrary, mutually perpendicular unit vectors $g_i^{(1)}, g_i^{(2)}$ situated in the plane perpendicular to the local phase front normal $N_i(x_m)$,

$$U_i(x_m) = B(x_m)g_i^{(1)}(x_m) + C(x_m)g_i^{(2)}(x_m) \quad .$$

Here again, $B(x_m)$ and $C(x_m)$ are scalar amplitude coefficients to be determined.

We can see that under the assumption of high-frequency character of the wavefield, we are able to separate it into two independent wave processes even in slightly inhomogeneous media.

We shall now try to determine the unknown amplitude coefficients $A(x_m)$, $B(x_m)$, $C(x_m)$. We shall use for this purpose the equation

$$M_i(U_m) = 0 \quad .$$

Let us start with P waves. We can write

$$U_j = AN_j = A\alpha\tau_{,j}$$

and thus

$$U_{j,i} = A_{,i}\alpha\tau_{,j} + A(\alpha_{,i}\tau_{,j} + \alpha\tau_{,ij}) \quad .$$

Let us insert this into the equation $\alpha\tau_{,i}M_i(U_m) = 0$. We get

$$(\lambda + 2\mu)(2A_{,i}\tau_{,i} + 2\alpha^{-1}\alpha_{,i}\tau_{,i}A + 2A\alpha^2\tau_{,ji}\tau_{,j}\tau_{,i} + A\tau_{,jj}) + (\lambda + 2\mu)_{,i}\tau_{,i}A = 0 \quad .$$

The term $\tau_{,ji}\tau_{,i}\tau_{,j}$ can be rewritten as follows

$$\tau_{,ji}\tau_{,j}\tau_{,i} = \frac{1}{2}(\tau_{,j}\tau_{,j})_{,i}\tau_{,i} = \frac{1}{2}(\alpha^{-2})_{,i}\tau_{,i} = -\alpha^{-3}\alpha_{,i}\tau_{,i} \quad .$$

The above equation then reduces to

$$2\rho\alpha^2 A_{,i}\tau_{,i} + \rho\alpha^2 A\tau_{,jj} + (\rho\alpha^2)_{,i}\tau_{,i}A = 0 \quad .$$

This is the P wave *transport equation for an isotropic medium.* The only possibly complex-valued quantity in this equation is A. If we multiply this equation by A^*, which denotes complex conjugate value of A, if we further multiply the complex conjugate transport equation by A and add both resulting equations, we get

$$\rho\alpha^2\tau_{,i}(AA^*_{,i} + A_{,i}A^*) + \rho\alpha^2\tau_{,ii}AA^* + (\rho\alpha^2)_{,i}\tau_{,i}AA^* = 0 \quad .$$

This yields immediately

$$(AA^*\rho\alpha^2\tau_{,i})_{,i} = 0 \quad .$$

In case of shear waves, we shall proceed in a similar way. Let us first multiply the equation $M_i(U_m) = 0$ sucessively by $g_i^{(1)}$, $g_i^{(2)}$. Since the procedure is the same for both eigenvectors, we consider here only the equation $M_i(U_m)g_i^{(1)} = 0$. We get

$$(\lambda+\mu)(U_{j,i}\tau_{,j}g_i^{(1)}+U_j\tau_{,ij}g_i^{(1)})+\mu(2U_{i,j}\tau_{,j}g_i^{(1)}+U_ig_i^{(1)}\tau_{,jj})+\lambda_{,i}g_i^{(1)}(U_j\tau_{,j})+\mu_{,j}(U_ig_i^{(1)})\tau_{,j}=0,$$

where we took into account that

$$\tau_{,j}g_j^{(1)} = 0 \quad .$$

We use now the expression for the displacement vector of the high-frequency shear wave derived above

$$U_j = Bg_j^{(1)} + Cg_j^{(2)} \quad .$$

Its spatial derivative is

$$U_{j,i} = B_{,i}g_j^{(1)} + C_{,i}g_j^{(2)} + Bg_{j,i}^{(1)} + Cg_{j,i}^{(2)} \quad .$$

Let us insert this into the above equation $M_i(U_m)g_i^{(1)} = 0$ and let us take into account that

$$g_{i,j}^{(1)}g_i^{(1)} = 0 \quad , \quad \tau_{,ij}g_i^{(1)} = -\tau_{,i}g_{i,j}^{(1)} \quad .$$

We get

$$\mu(2\tau_{,j}g_i^{(1)}g_{i,j}^{(2)}C + B\tau_{,jj}) + \mu_{,j}B\tau_{,j} + 2\mu B_{,j}\tau_{,j} = 0$$

and similarly

$$\mu(2\tau_{,j}g_i^{(2)}g_{i,j}^{(1)}B + C\tau_{,jj}) + \mu_{,j}C\tau_{,j} + \mu C_{,j}\tau_{,j} = 0 \quad .$$

These two equations are the *S wave transport equations for an isotropic medium.* We can see that they are mutually coupled through the terms $2\mu\tau_{,j}g_i^{(1)}g_{i,j}^{(2)}C$ and $2\mu\tau_{,j}g_i^{(2)}g_{i,j}^{(1)}B$. Since the specification of vectors $g_i^{(1)}$, $g_i^{(2)}$ in the plane perpendicular to N_i is arbitrary (they must be only mutually orthogonal), we can choose them in such a way that

$$2\mu\tau_{,j}g_i^{(1)}g_{i,j}^{(2)} = 0 \quad \Leftrightarrow \quad 2\mu\tau_{,j}g_i^{(2)}g_{i,j}^{(1)} = 0 \quad .$$

Then the two transport equations decouple. Let us note that along an arbitrary curve, the vector $g_i^{(1)}$ varies as follows (similarly the vector $g_i^{(2)}$)

$$\frac{dg_i^{(1)}}{ds} = g_{i,j}^{(1)}\frac{dx_j}{ds} \quad .$$

If the curve is specified so that it is always perpendicular to the phase front, then

$$\frac{dx_j}{ds} = \beta\tau_{,j}$$

and we have

$$\frac{dg_i^{(1)}}{ds} = \beta g_{i,j}^{(1)}\tau_{,j} \quad .$$

If we recall the above conditions for the decoupling the transport equations for $B(x_m)$ and $C(x_m)$, we can see that they yield

$$\frac{dg_i^{(1)}}{ds} g_i^{(2)} = 0 \quad .$$

If we take into account an obvious fact that

$$\frac{d}{ds}(g_i^{(1)} g_i^{(1)}) = 2g_i^{(1)} \frac{dg_i^{(1)}}{ds} = 0 \quad ,$$

we can see that the conditions for the decoupling of the transport equations yield

$$\frac{dg_i^{(1)}}{ds} = a\tau_{,i}$$

where a is a coefficient of proportionality. We can determine it in the following way

$$a = \frac{dg_i^{(1)}}{ds}\tau_{,i}\beta^2 = \tau_{,i}\tau_{,j}g_{i,j}^{(1)}\beta^3 = -\beta^3\tau_{,ij}\tau_{,j}g_i^{(1)} = -\frac{1}{2}\beta^3(\tau_{,j}\tau_{,j})_{,i}g_i^{(1)} = \beta_{,i}g_i^{(1)} \, .$$

We thus have

$$\frac{dg_i^{(1)}}{ds} = (\beta_{,k}g_k^{(1)})\tau_{,i} \quad , \quad \frac{dg_i^{(2)}}{ds} = (\beta_{,k}g_k^{(2)})\tau_{,i} \quad .$$

Thus, the vectors $g_i^{(1)}$, $g_i^{(2)}$ vary along the curve perpendicular to the wavefront in such a way that the variation has only non-zero projection into the direction of the slowness vector (i.e. into the normal to the phase front). Later, we shall see that the vectors chosen in the above way represent, together with vector N_i, vectorial basis of the so-called ray-centered coordinate system. With the above choice of vectors $g_i^{(1)}$, $g_i^{(2)}$, the originally coupled transport equations decouple. If we substitute μ by $\rho\beta^2$, they read

$$2\rho\beta^2 B_{,i}\tau_{,i} + \rho\beta^2 B\tau_{,jj} + (\rho\beta^2)_{,i}\tau_{,i}B = 0 \quad ,$$
$$2\rho\beta^2 C_{,i}\tau_{,i} + \rho\beta^2 C\tau_{,jj} + (\rho\beta^2)_{,i}\tau_{,i}C = 0 \quad .$$

We can see that they have exactly the same form as the transport equation for P waves, only the velocity α is substituted by β. The equations can be transformed into the more compact form in the same way as in the case of P waves.

In conclusion, let us note that the ray amplitudes $A(x_m)$, $B(x_m)$, $C(x_m)$ may become complex-valued. P waves remain even in this case *linearly polarized*. In the case of S waves, however, the polarization can become *quasielliptical*.

Let us also add that the role of the term $L_i(U_m)$ is the same as of the similar coefficient standing with F in the acoustic case.

7.2.3 Elastodynamic case - anisotropic medium

Let us recall the homogeneous elastodynamic equation for an inhomogeneous anisotropic medium

$$(c_{ijkl}u_{k,l})_{,j} - \rho u_{i,tt} = 0 \quad .$$

As in the isotropic case, we assume the formal solution in the form of the leading term of the ray series,

$$u_i(x_m, t) = U_i(x_m)F(t - \tau(x_m)) \quad .$$

In order to determine the unknown functions $U_i(x_m)$ and $\tau(x_m)$, we shall insert the trial solution into the elastodynamic equation. This will yield

$$F''N_i(U_m) - F'M_i(U_m) + FL_i(U_m) = 0 \quad ,$$

where now

$$\begin{aligned}
N_i(U_m) &= c_{ijkl}U_k\tau_{,l}\tau_{,j} - \rho U_i \quad , \quad \checkmark \\
M_i(U_m) &= (c_{ijkl}U_k\tau_{,l})_{,j} + c_{ijkl}U_{k,l}\tau_{,j} \quad , \\
L_i(U_m) &= c_{ijkl}U_{k,lj} + c_{ijkl,j}U_{k,l} \Rightarrow (c_{ijkl}\,U_{k,l}(x_m))_{,j}
\end{aligned}$$

The equation

$$N_i(U_m) = 0$$

can be rewritten in the form of the *Christoffel equation for inhomogeneous anisotropic media*

$$(\bar{\Gamma}_{ik} - \delta_{ik})U_k = 0 \quad ,$$

where

$$\bar{\Gamma}_{ik} = \rho^{-1}c_{ijkl}\tau_{,l}\tau_{,j} \quad .$$

The Christoffel matrix $\bar{\Gamma}_{ik}$ has three real and positive eigenvalues $G(x_m, \tau_{,m})$ (now functions of spatial variables), to each corresponds one eigenvector $g_i(x_m)$. From the Christoffel equation we get

$$G(x_m, \tau_{,m}) = 1 \quad .$$

This is the *eikonal equation for an inhomogeneous anisotropic medium*. There are thus three eikonal equations corresponding to three different waves. The displacement vector of each of them can be written as

$$U_i(x_m) = A(x_m)g_i(x_m) \quad ,$$

which indicates that the waves are linearly polarized.

We shall now derive the transport equation for the amplitude $A(x_m)$ in an inhomogeneous anisotropic medium. We again insert the above expression for $U_i(x_m)$ into the equation ?? $M_i(U_m)g_i = 0$. We get

$$A_{,j}c_{ijkl}g_k g_i \tau_{,l} + A(c_{ijkl}g_k\tau_{,l})_{,j}g_i + c_{ijkl}A_{,l}g_k g_i\tau_{,j} + Ac_{ijkl}g_{k,l}\tau_{,j}g_i \;=0 \;(4)$$
$$= 2A_{,j}c_{ijkl}g_k g_i\tau_{,l} + A(c_{ijkl}g_k g_i\tau_{,l})_{,j} = 0\,. \;(5) \qquad (4)$$

We used the symmetry of the elastic tensor $c_{ijkl} = c_{klij}$ to obtain the above result. Using the results derived in Sec.3.4.2, we can write

$$v_i^{(g)} = \rho^{-1}c_{ijkl}\tau_{,l}g_j g_k \quad .$$

The above equation represents the *group velocity vector*, which is now function of coordinates. Using it, we can simplify the above equation to the form

$$2\rho A_{,j}v_j^{(g)} + A(\rho v_j^{(g)})_{,j} = 0 \quad .$$

This is the transport equation for an inhomogeneous anisotropic medium. It can be further transformed into the form

$$(\rho AA^* v_j^{(g)})_{,j} = 0 \quad .$$

Again, the term $L_i(U_m)$ can be used to check the accuracy of the determination of U_m.

7.3 Energy considerations

In Secs.2.4.1 and 2.4.2, we introduced several important energy functions: strain energy W, kinetic energy K and energy flux S_i. For the acoustic case they read

$$W = \frac{1}{2}\kappa p^2\,, \quad K = \frac{1}{2}\rho v_i v_i\,, \quad S_i = pv_i\,.$$

We shall now evaluate these expressions in the ray approximation. If we express pressure p and the particle velocity v_i as a ray series, we shall also get the energy functions in the form of series. As before, we are interested only in the leading terms of the ray series for p and v_i and for the energy functions. For p and v_i we have again

$$p(x_m, t) = P(x_m)F(t - \tau(x_m))\,, \quad v_i(x_m, t) = V_i(x_m)F(t - \tau(x_m)) \quad .$$

We must also take into account the first order acoustic equation relating pressure p with the particle velocity v_i,

$$p_{,i}(x_m, t) + \rho v_{i,t}(x_m, t) = 0 \quad .$$

If we insert into it the above approximations of p and v_i, we get

$$P_{,i}F - PF'\tau_{,i} + \rho V_i F' = 0 \quad .$$

As before, the equation is satisfied when the coefficients with F and F' are zero. The coefficient with F' yields

$$V_i = \rho^{-1} P \tau_{,i} \quad .$$

We can see that in the zero order approximation, the relation between $V_i(x_m)$ and $P(x_m)$ is the same as in the case of plane waves.

Since the energy functions are nonlinear in p and/or v_i, we must, as in the case of plane waves, work with the real parts of p and v_i, which are physically meaningful,

$$p = \frac{1}{2}(PF + P^*F^*), \quad v_i = \frac{1}{2}(V_iF + V_i^*F^*) \quad .$$

Using the above expressions, we get

$$W = K = \frac{\kappa}{8}(PF + P^*F^*)^2, \quad E = W + K = \frac{\kappa}{4}(PF + P^*F^*)^2, \quad S_i = \frac{\tau_{,i}}{4\rho}(PF + P^*F^*)^2$$

These results correspond fully to the results derived for instantaneous energy functions in the case of plane waves. We must keep in mind, however, that the above results represent a leading term in the ray approximation of the energy functions, and that the energy functions are generally space dependent. As in the case of plane waves, the above instantaneous values can be time averaged or time integrated. In the latter case, we get formulae analogous to those for plane waves but space dependent,

$$\hat{W} = \hat{K} = \frac{\kappa}{2}PP^*f_a, \quad \hat{E} = \kappa PP^*f_a, \quad \hat{S}_i = \frac{\tau_{,i}}{\rho}PP^*f_a,$$

with

$$f_a = \int_{-\infty}^{+\infty} g^2(t)dt \quad ,$$

where $g(t)$ is the considered real signal.

For the group velocity we get

$$v_i^{(g)} = S_i/E = c^2\tau_{,i} = cN_i \quad .$$

The derivation of energy functions for the elastodynamic case is the same as above and we shall, therefore, not present it here.

7.4 Effects of an interface

Let us consider a slightly curved interface Σ separating two inhomogeneous acoustic halfspaces. Under the slightly curved interface we understand that the variation of the normal to the interface n_i on the distance of a prevailing wavelength λ is substantially smaller than the size of the normal itself, i.e. smaller than unit,

$$\lambda|\nabla n_i| \ll 1 \quad .$$

Since the gradient of the normal is proportional to the curvature of the interface, and thus inversely proportional to the radii of curvature of the interface, we can rewrite the above condition as

$$R_1 \gg \lambda \quad , \quad R_2 \gg \lambda \quad .$$

where R_1, R_2 are the main radii of curvature of the interface Σ. Let us denote the parameters of one halfspace by $c_1(x_m)$, $\rho_1(x_m)$ and of the other by $c_2(x_m)$, $\rho_2(x_m)$. Let us represent the incident wave in the form of the leading term of the ray series

$$p(x_m, t) = P(x_m)F(t - \tau(x_m)) \quad .$$

Similarly, reflected and transmitted waves are

$$p^r(x_m, t) = P^r(x_m)F^r(t - \tau^r(x_m)) \quad ,$$
$$p^t(x_m, t) = P^t(x_m)F^t(t - \tau^t(x_m)) \quad .$$

Similar formulae can be written for the particle velocity. We assume the incident wave to be known, the quantities to be determined are P^r, P^t, F^r, F^t, τ^r, τ^t.

The above waves must guarantee continuity of the pressure and the normal component of the particle velocity across the interface Σ. This yields

$$PF(t-\tau) + P^r F^r(t-\tau^r) = P^t F^t(t-\tau^t) \quad ,$$
$$n_i V_i F(t-\tau) + n_i V_i^r F^r(t-\tau^r) = n_i V_i^t F^t(t-\tau^t) \quad .$$

As in the case of plane waves, we shall try to use the same unknown quantity in both equations, amplitude of the pressure. For this purpose we need again the relation between the particle velocity and the pressure. In Sec.7.3 we found that in the high-frequency approximation

$$V_i = \rho^{-1} P \tau_{,i}$$

so that the second of the continuity equations can be rewritten as

$$\rho_1^{-1} n_i \tau_{,i} PF(t-\tau) + \rho_1^{-1} n_i \tau^r_{,i} P^r F^r(t-\tau^r) = \rho_2^{-1} n_i \tau^t_{,i} P^t F^t(t-\tau^t) \, .$$

If we compare the above equations with the similar system derived in Sec.3.8.1 for plane waves, we can see that both systems are formally equivalent. There are, however, two substantial differences. First, the above system is only approximate. Second, the above system is satisfied locally, all quantities appearing in it may vary from point to point.

Since the relation between the incident and generated waves should be independent of the form of the signal, we get, similarly as in the case of plane waves,

$$F(\theta) = F^r(\theta) = F^t(\theta)$$

at any point of the interface. Along the interface Σ, we also get a natural relation

$$\tau(x_m) = \tau^r(x_m) = \tau^t(x_m) \quad .$$

This relation, called sometimes the *phase matching relation* implies equality of projections of the first and higher derivatives of phase functions into the plane tangent to Σ. For the first derivatives, we can make the following consideration. Let l denotes an arclength along a curve on Σ. Then

$$\frac{d}{dl}(\tau - \tau^r) = \left(\frac{\partial \tau}{\partial x_i} - \frac{\partial \tau^r}{\partial x_i}\right)\frac{dx_i}{dl} = 0 \quad ,$$

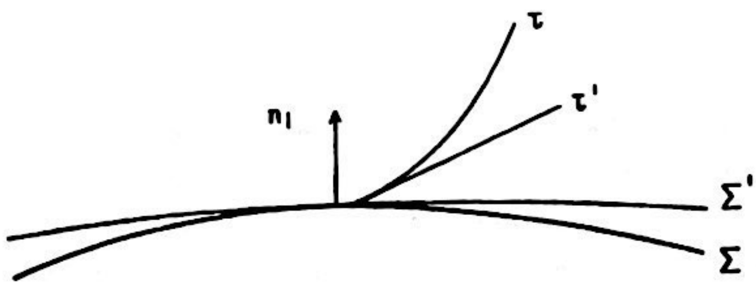
which implies

$$\frac{\partial \tau}{\partial x_i}\frac{dx_i}{dl} = \frac{\partial \tau^r}{\partial x_i}\frac{dx_i}{dl} \quad .$$

Since the equation is satisfied for any curve arbitrarily chosen on Σ, the equation says that the tangential projections of the slowness vector into the Σ are the same for the incident and generated waves. We got the same result also in case of plane waves. From there we remember that it leads to the Snell law. We also remember that if the tangential components of the slowness vector are known, the normal ones can be determined from the corresponding eikonal equation, see Sec.3.8.1.1.

With τ^r, τ^t, F^r, F^t and $\tau^r_{,i}$, $\tau^t_{,i}$ determined, the above system of two continuity equations can be solved to obtain P^r and P^t. Since the system is formally equivalent to the system derived for the plane waves, the result will be the plane wave reflection and transmission coefficients. The coefficients may now, however, vary along Σ.

Thus, we found that in the zero-order ray approximation, reflection and transmission of a wave with the curved phase front τ at the curved interface Σ reduces to the reflection/transmission of the plane wave τ' at the plane interface Σ', see the picture. The phase front τ' of the plane wave is perpendicular to the local slowness vector $\tau_{,i}$, i.e. τ' is tangent to the phase front τ at the point of incidence. Similarly, the plane interface Σ' is perpendicular to the local normal to the interface Σ at the point of incidence, i.e. Σ' is tangent to Σ at



the point of incidence. The amplitude of the reflected or transmitted wave at the point of incidence (in the zero-order ray approximation) depends only on the local values of velocity and density on both sides of the interface, on the angle of incidence and on the amplitude of the incident wave. It does not depend on the curvature of the interface, of the incident wave front or on gradients of velocity and density at the point of incidence. Only higher order terms in the ray series become dependent on the latter mentioned parameters.

The conclusions concerning the waves in the elastodynamic case are very similar, and we shall not, therefore, present them here.

7.5 Solutions of the eikonal equation - rays

We shall separately introduce rays for acoustic, isotropic and anisotropic inhomogeneous media.

7.5.1 Rays in acoustic and isotropic inhomogeneous media

There are several ways, in which it is possible to introduce rays. Perhaps most appropriate it is to consider them as trajectories along which energy of waves propagates. We shall show that we arrive to the same equations if we define rays as characteristics of the eikonal equation, trajectories perpendicular to the phase front or extremals of the Fermat's functional. In the following, we are going to discuss briefly each case.

7.5.1.1 Rays - characteristics of the eikonal equation

Eikonal equation for an acoustic or isotropic medium reads as follows

$$c^2 p_i p_i = 1 \quad .$$

Here c is the phase velocity of the considered wave and

$$p_i = \tau_{,i}$$

are components of the slowness vector. The eikonal equation is nonlinear first order partial differential equation for $\tau = \tau(x_m)$. It can be solved by the *method of characteristics* - the method for solving partial differential equations. The *characteristics* are 3-D curves, along which the solution of the partial differential equation can be determined. The characteristics themselves are determined by a system of *ordinary differential equations*. To solve these equations is much easier task than to solve the original partial differential equation.

For the equation

$$H(x_m, p_m, \tau) = 0,$$

the equations of characteristics are

$$\frac{dx_i}{du} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{du} = \frac{\partial H}{\partial x_i}, \quad \frac{d\tau}{du} = p_i \frac{\partial H}{\partial p_i} \quad .$$

The first six equations are the equations for the determination of characteristics. The last equation specifies how to determine the sought function $\tau(x_m)$ on the characteristics.

In case of the eikonal equation, the function H does not depend explicitely on the sought quantity τ. Equations of the type

$$H(x_m, p_m) = 0$$

are known as *Hamilton-Jacobi equations*. They are broadely used in classical mechanics, where the function H is known as the *Hamiltonian* and usually represents total energy of a mechanical system. The components of the slowness vector p_i are called the *generalized moments* and coordinates x_i the *generalized coordinates*. The system of equations of characteristics is called the *canonical system*. Let us note that originally, the canonical system was proposed by Hamilton for solving problems of light propagation. Let us note that Hamiltonian does not have its meaning of total energy in seismological applications.

In case of the above eikonal equation, the function H has the following form

$$H = G - 1 = c^2 p_i p_i - 1 \quad .$$

Equations of characteristics of this equation read

$$\frac{dx_i}{d\tau} = c^2 p_i \quad , \quad \frac{dp_i}{d\tau} = -c^{-1} c_{,i} \quad .$$

We shall show in Sec.7.5.1.2 that the derived equations are equivalent to the equations describing the trajectories, along which energy propagates, i.e., rays. Thus the above system of ordinary differential equations represents the *ray tracing system*.

In the above equations, we used $u = \frac{1}{2}\tau$ since

$$\frac{d\tau}{du} = p_i \frac{\partial G}{\partial p_i} = p_i 2c^2 p_i = 2c^2 p_i p_i = 2 \quad .$$

We could expect this result if we realized that the function G is a homogeneous function of the second degree, see Sec.2.4.1, for which

$$\frac{\partial G}{\partial p_i} p_i = 2G \quad .$$

Other quantities can be used instead of τ as parameters along the ray. If we use the arclength s, we get

$$\frac{dx_i}{ds} = cp_i \quad , \quad \frac{dp_i}{ds} = -c^{-2} c_{,i} \quad , \quad \frac{d\tau}{ds} = c^{-1} \quad .$$

A convenient system is obtained for the parameter σ related to τ as $d\tau/d\sigma = c^{-2}$. Then

$$\frac{dx_i}{d\sigma} = p_i \quad , \quad \frac{dp_i}{d\sigma} = \frac{1}{2}(c^{-2})_{,i} \quad , \quad \frac{d\tau}{d\sigma} = c^{-2} \quad .$$

The ray tracing systems can be solved analytically when the velocity distribution is specified by a simple analytic function. The ray tracing system with parameter σ along the ray is especially convenient for this purpose. If, for example, the inverse squared velocity (squared slowness) c^{-2} is approximated by a linear function of coordinates, a simple analytic solution for ray and slowness vector and phase function along it can be found. Most common way of solving the ray tracing equations is, however, to solve them numerically. Various standard routines based e.g. on the Runge-Kutta method or Hamming predictor-corrector method can be used for this purpose.

The velocity c in the ray tracing equations is the phase velocity of P waves in acoustic media or the phase velocity of P or S waves in isotropic media. The ray tracing systems for P and S waves are thus different.

The above ray tracing systems can be solved when the initial values of x_i and p_i are given. For the ray tracing system with τ as a parameter along a ray, the initial conditions read

$$x_i(\tau_o) = x_{oi} \quad , \quad p_i(\tau_o) = p_{oi} \quad .$$

We must remember that the slowness vector at any point of the ray, including the one in which the initial conditions are specified, *must satisfy the eikonal equation*. This implies that only 5 of the 6 ray tracing equations are independent. At any point of the ray, we can express p_i in terms of the angles φ and δ

$$p_1 = c^{-1} \cos\varphi \sin\delta \quad , \quad p_2 = c^{-1} \sin\varphi \sin\delta \quad , \quad p_3 = c^{-1} \cos\delta \quad .$$

With this choice, the eikonal equation is automatically satisfied. The angles φ and δ can be used instead of p_i in the ray tracing system, and we thus obtain a system consisting of only 5 independent equations, which satisfy eikonal equation automatically:

$$\begin{aligned}
\frac{dx_1}{d\tau} &= c\cos\varphi\sin\delta \quad , \quad \frac{dx_2}{d\tau} = c\sin\varphi\sin\delta \quad , \quad \frac{dx_3}{d\tau} = c\cos\delta \quad , \\
\frac{d\varphi}{d\tau} &= (c_{,1}\sin\varphi - c_{,2}\cos\varphi)/sin\delta \quad , \\
\frac{d\delta}{d\tau} &= -(c_{,1}\cos\varphi + c_{,2}\sin\varphi)\cos\delta + c_{,3}\sin\delta \quad .
\end{aligned}$$

The initial conditions for this system are

$$x_i(\tau_o) = x_{oi} \quad , \quad \delta(\tau_o) = \delta_o \quad , \quad \varphi(\tau_o) = \varphi_o \quad .$$

A disadvantage of this system for numerical applications is the fact that it contains trigonometric functions, the evaluation of which is rather time consuming. Another disadvantage is its failure when vertical rays are to be traced ($\sin\delta$ in denominator becomes zero).

The above described procedure, in which rays are specified by a point and a direction of the ray in it is called the *initial-value ray tracing.* Often, instead of ray specified by its point and an initial direction, we are interested in tracing rays connecting two prescribed points (e.g. source and receiver). Then we speak about the *two-point ray tracing.*

To find two-point rays is not an easy task. There are two basic approaches how to solve this problem. The first one, called the *bending method* starts from an initial simple curve connecting the two considered points, which is iteratively perturbed so as to satisfy the ray tracing equations. The procedure may fail in case of the *multipathing*, i.e., in case that the considered points can be connected by more than one ray. The procedure tends to iterate to the same ray. The second procedure is called the *shooting method.* It uses initial-value ray tracing to calculate rays from one given point to the vicinity of the second point. The distance of the termination point of the ray from the second point is used to modify the initial direction of the next shot so that the next ray arrives more closely to the second point. Various iterative procedures may be used such as the method of halving of interval, the regula falsi method or the paraxial ray approximation.

The above ray tracing systems can be used for tracing rays in 3-D inhomogeneous structures. Often, simpler types of structures are considered, in which the parameters of the medium do not vary in one direction. Such structures are called 2-D. If, for example, the medium does not vary along the coordinate x_2, and if we specify the initial conditions for rays in the following way

$$x_1(\tau_o) = x_{01} \, , \, x_2(\tau_o) = 0 \, , \, x_3(\tau_o) = x_{03} \, , \, p_1(\tau_o) = p_{01} \, , \, p_2(\tau_o) = 0 \, , \, p_3(\tau_o) = p_{03} \quad ,$$

the above 3-D ray tracing system reduces to

$$\frac{dx_1}{d\tau} = c^2 p_1 \, , \quad \frac{dx_3}{d\tau} = c^2 p_3 \, , \quad \frac{dp_1}{d\tau} = -c^{-1} c_{,1} \, , \quad \frac{dp_3}{d\tau} = -c^{-1} c_{,3} \quad .$$

Along the ray the condition

$$p_1^2 + p_3^2 = c^{-2}$$

must be satisfied. We can see that if we choose the initial conditions so that initial slowness vector is situated in the plane (x_1, x_3) then the whole ray is a plane curve situated in the plane (x_1, x_3). The ray tracing in terms of the angles φ and δ with the initial conditions

$$x_1(\tau_o) = x_{01} \, , \; x_2(\tau_o) = 0 \, , \; x_3(\tau_o) = x_{03} \, , \; \varphi(\tau_o) = 0 \, , \; \delta(\tau_o) = \delta_o$$

in the above 2-D structure reduces to the following system

$$\frac{dx_1}{d\tau} = c^2 \sin\delta \, , \quad \frac{dx_3}{d\tau} = c^2 \cos\delta \, , \quad \frac{d\delta}{d\tau} = -c_{,1} \cos\delta + c_{,3} \sin\delta \quad .$$

If we did not specify $p_{02} = 0$, the ray tracing for 2-D structure would also simplify but the ray would remain a general 3-D curve. In such a case

$$\begin{aligned} \frac{dx_1}{d\tau} &= c^2 p_1 \, , \quad \frac{dx_3}{d\tau} = c^2 p_3 \, , \quad \frac{dp_1}{d\tau} = -c^{-1} c_{,1} \, , \quad \frac{dp_3}{d\tau} = -c^{-1} c_{,3} \quad , \\ x_2 &= x_2(\tau_o) + \int_{\tau_o}^{\tau} c^2 p_{02} d\xi \, , \quad p_2(\tau_o) = p_{02} = \text{const.} \end{aligned}$$

In the first line, we have the same ray tracing system as for the 2-D case. The slowness vector must now, however, satisfy a different condition,

$$p_1^2 + p_{02}^2 + p_3^2 = c^{-2} \quad .$$

Here p_{02} is a constant related to the deviation of the ray from the plane (x_1, x_3). For $p_{02} = 0$, the above ray tracing system reduces to the 2-D one. We can see that for $p_{02} \neq 0$, we must solve 4 differential equations as in the standard 2-D case. In addition, we must integrate along the obtained curve in the (x_1, x_3) plane to get x_2 coordinate of the ray. Because we are using 2-D procedure and compute 3-D rays, we speak often about $2\,\frac{1}{2}-$ D *modeling*.

In case that the parameters of the medium depend on one coordinate only (1-D models), the ray tracing equations can be further simplified. If we are not interested in tracing rays but only in travel times, the equations for the range x_1 and travel time τ can be written in the closed form (in form of integrals over the coordinate, along which parameters of the medium vary, see Červený et al (1977)).

7.5.1.2 Rays - energy flux trajectories

In Sec.7.3, we have shown that in acoustic case, the group velocity equals the phase velocity both in the size and in the direction,

$$v_i^{(g)} = cN_i \quad .$$

The same result could be derived also for isotropic case, see Sec.3.4.3. This means that the trajectory along which energy propagates must be at each point tangent to N_i,

$$dx_i/ds = aN_i \quad .$$

Since dx_i/ds and N_i are unit vectors, $a = 1$ and the above equation can be rewritten as

$$dx_i/ds = cp_i \quad ,$$

which represents the first set of the ray tracing equations. The second set for p_i can be derived in the following way

$$dp_i/ds = \frac{\partial p_i}{\partial x_k}\frac{dx_k}{ds} = \frac{\partial p_k}{\partial x_i}\frac{dx_k}{ds} = \frac{\partial p_k}{\partial x_i}p_k c = \frac{1}{2}\frac{\partial c^{-2}}{\partial x_i}c = -c^{-2}c_{,i} \, .$$

7.5.1.3 Rays - trajectories perpendicular to the phase front

Let us consider a phase front $\tau(x_m) = \text{const}$. The slowness vector $p_i = \tau_{,i}$ is thus perpendicular to the phase front. The trajectory perpendicular at each point to a phase front must have, therefore, slowness vector as its tangent vector. This is, however, trajectory, which we studied in the previous section. We found that it represents a ray. This definition of rays is independent of the choice of the coordinates, in which the ray is to be determined. It is, therefore, convenient for the derivation of ray tracing equations in curvilinear coordinates.

7.5.1.4 Rays - extremals of Fermat's functional

Let us consider the velocity $c(x_m)$ and its first derivatives to be continuous functions of spatial coordinates in the considered medium. Let us choose two arbitrary fixed points A and B in the medium and let us consider the following integral

$$I = \int_A^B dt = \int_A^B \frac{ds}{c} \quad .$$

The integration can be performed along an arbitrary curve connecting the points A and B. The above integral is known as the *Fermat's functional* and it is generally considered in the form

$$I = \int_A^B F(x_i, \dot{x}_i) du$$

with $x_i = x_i(u)$, $\dot{x}_i = dx_i/du$ and u is an arbitrary parameter along the curve.

The *Fermat's principle* specifies the extremal connecting the points A and B as that curve, which makes the Fermat's functional stationary, i.e. for which the variation δI of the integral I is zero

$$\delta I = \delta \int_A^B \frac{ds}{c} = \delta \int_A^B F(x_i, \dot{x}_i) du = 0 \quad .$$

The curve, which satifies this condition, is called the *extremal of the Fermat's functional.* It is known from the calculus of variation that the curve, which satisfies the condition $\delta I = 0$ can be obtained as a solution of a system of ordinary differential equations known as the *Euler's equations.* For the Fermat's functional

$$I = \int_A^B F(x_i, \dot{x}_i) du$$

the Euler equations read

$$\frac{d}{du}\left(\frac{\partial F}{\partial \dot{x}_i}\right) - \frac{\partial F}{\partial x_i} = 0 \quad .$$

In our case, the function $F(x_i, \dot{x}_i)$ has the form

$$F(x_i, \dot{x}_i) = c^{-1}\frac{ds}{du} = c^{-1}(\dot{x}_i \dot{x}_i)^{1/2} \quad .$$

We have, therefore,

$$\frac{\partial F}{\partial \dot{x}_i} = c^{-1}\frac{\dot{x}_i}{(\dot{x}_k \dot{x}_k)^{1/2}} \quad , \quad \frac{\partial F}{\partial x_i} = -c^{-2}c_{,i}(\dot{x}_i \dot{x}_i)^{1/2} \quad .$$

The Euler equations have thus the form

$$\frac{d}{du}\left(c^{-1}\frac{\dot{x}_i}{(\dot{x}_k \dot{x}_k)^{1/2}}\right) + c^{-2}c_{,i}(\dot{x}_i \dot{x}_i)^{1/2} = 0 \quad .$$

For $u = s$ we get

$$\frac{d}{ds}\left(c^{-1}\frac{dx_i}{ds}\right) = -c^{-2}c_{,i} \quad .$$

This is second order ordinary differential equation. It can be formally rewritten into the system of first order ordinary differential equations. If we denote by p_i the expression in the brackets, we get

$$\frac{dx_i}{ds} = cp_i \quad , \quad \frac{dp_i}{ds} = -c^{-2}c_{,i} \quad .$$

From the first set of equations, we immediately see that the symbol p_i represents the slowness vector. The whole system of equations is again identical to the ray tracing system, which we have obtained in Sec.7.5.1.2. We can thus see that the rays are extremals of the Fermat's functional.

Let us make an interesting comment. In the original version of the Fermat's principle, the rays were sought as the curves which made the Fermat's functional *minimum*. This specification excludes very frequent case of multipathing between the points A and B, which correponds to loops in travel time curves. For this reason, the Fermat's principle was generalized for the stationary values of the Fermat's functional.

7.5.2 Rays in anisotropic inhomogeneous media

In Sec.7.5.1, we have shown that rays in inhomogeneous isotropic media can be introduced in several different ways. Here we shall introduce rays only as characteristics of the eikonal equation and as trajectories, along which energy propagates.

7.5.2.1 Rays - characteristics of the eikonal equation

Eikonal equation for an inhomogeneous anisotropic medium reads

$$G(x_m, p_m) = 1 \quad ,$$

where G is one of the three eigenvalues of the Christoffel matrix $\bar{\Gamma}_{ik}$

$$\bar{\Gamma}_{ik} = c_{ijkl} p_j p_l \quad .$$

We shall first assume that all the three eigenvalues of the matrix $\bar{\Gamma}_{ik}$ are mutually different, which means that the corresponding eigenvectors, which specify the polarization of the considered waves can be uniquely determined. We shall start from the same system of ordinary differential equations of characteristics as in Sec.7.5.1.1. Since $H = G - 1$, we get:

$$\frac{dx_i}{du} = \frac{\partial G}{\partial p_i} \quad , \quad \frac{dp_i}{du} = -\frac{\partial G}{\partial x_i} \quad , \quad \frac{d\tau}{du} = p_i \frac{\partial G}{\partial p_i} \quad .$$

This system represents ray tracing system for anisotropic media. The eigenvalue $G(x_m, p_m)$ can be expressed as

$$G(x_m, p_m) = a_{ijkl} p_j p_l g_i g_k \quad .$$

We can immediately see that G is a homogeneous quadratic function of p_i, which implies, using the Euler theorem on homogeneous functions

$$\frac{d\tau}{du} = 2G = 2 \quad .$$

We can thus rewrite the ray tracing system as

$$\frac{dx_i}{d\tau} = \frac{1}{2}\frac{\partial G}{\partial p_i} \quad , \quad \frac{dp_i}{d\tau} = -\frac{1}{2}\frac{\partial G}{\partial x_i}$$

To evaluate $\partial G/\partial p_i$, $\partial G/\partial x_i$, we can proceed in several ways. The most straightforward but not very effective would be to find G as the solution of the equation

$$\det(\bar{\Gamma}_{ik} - G\delta_{ik}) = 0$$

and then to differentiate it. The expressions for G can be, however, quite complicated and we can succeed to derive them only in case of higher symmetry anisotropic media. If we realize that we are not interested in the function G itself but only in its derivatives, we can use the theorem on implicit functions and differentiate the above equation using it. In such a way, we get

$$\frac{\partial G}{\partial p_i} = \frac{\partial \bar{\Gamma}_{jk}}{\partial p_i} D_{jk}/D \quad , \quad \frac{\partial G}{\partial x_i} = \frac{\partial \bar{\Gamma}_{jk}}{\partial x_i} D_{jk}/D \quad ,$$

where

$$D_{ij} = \frac{1}{2}\epsilon_{ikl}\epsilon_{jmn}(\bar{\Gamma}_{km} - \delta_{km})(\bar{\Gamma}_{ln} - \delta_{ln}) \quad ,$$
$$D = D_{11} + D_{22} + D_{33} \quad .$$

We can thus write the ray tracing equations as follows

$$\frac{dx_i}{d\tau} = a_{ijkl}p_l D_{jk}/D \quad , \quad \frac{dp_i}{d\tau} = -\frac{1}{2}a_{njkl,i}p_n p_l D_{jk}/D \quad .$$

The above ray tracing system can be used as long as all the three eigenvalues of the Christoffel matrix differ. If two eigenvalues coincide, then $D = 0$ and the ray tracing cannot be used.

We can use an alternative form of the ray tracing system, in which the above problem can be avoided. We can determine the derivatives $\partial G/\partial p_i$, $\partial G/\partial x_i$ from the expression for G

$$G(x_m, p_m) = a_{ijkl}p_i p_l g_j g_k \quad .$$

For $\partial G/\partial p_i$, for example, we get

$$\frac{\partial G}{\partial p_i} = 2a_{ijkl}p_l g_j g_k + 2a_{njkl}p_n p_l \frac{\partial g_j}{\partial p_i} g_k \quad .$$

Let us investigate the second term on the RHS of the above equation and let us take into account the relation

$$\bar{\Gamma}_{jk}g_k = a_{njkl}p_n p_l g_k = G g_j = g_j \quad .$$

We get

$$2a_{njkl}p_np_l\frac{\partial g_j}{\partial p_i}g_k = 2g_j\frac{\partial g_j}{\partial p_i} = \frac{\partial}{\partial p_i}(g_jg_j) = 0 \quad .$$

For $\partial G/\partial x_i$ we would proceed in the same way. The derivatives $\partial G/\partial p_i$, $\partial G/\partial x_i$ thus read

$$\frac{\partial G}{\partial p_i} = 2a_{ijkl}p_lg_jg_k \quad , \quad \frac{\partial G}{\partial x_i} = a_{njkl,i}p_np_lg_jg_k \quad .$$

Inserted in the ray tracing equations, this yields the alternative ray tracing system

$$\frac{dx_i}{d\tau} = a_{ijkl}p_jg_kg_l \quad , \quad \frac{dp_i}{d\tau} = -\frac{1}{2}a_{njkl,i}p_np_lg_jg_k \quad .$$

This system will be useful as long as we are able to specify the polarization vector at each point of the ray. The polarization vector can be determined in several ways (from the solution of the eigenvalue problem, using the perturbation approach). We shall show here the most straightforward way following from the eigenvalue equation

$$(\bar{\Gamma}_{jk} - G\delta_{jk})g_k = 0 \quad .$$

This equation says that the polarization vector g_k is perpendicular to each row of the matrix $(\bar{\Gamma}_{jk} - G\delta_{jk})$. By definition, g_k must be unit vector, therefore, it must satisfy the normalization condition

$$g_kg_k = 1 \quad .$$

We can thus get g_k by normalizing the following vectorial product

$$\epsilon_{klm}(\bar{\Gamma}_{il} - G\delta_{il})(\bar{\Gamma}_{mn} - G\delta_{mn}) \quad , \quad i \neq n\,.$$

Equation $(\bar{\Gamma}_{jk} - G\delta_{jk})g_j = 0$ implies that, at least, two rows of the matrix $(\bar{\Gamma}_{jk} - G\delta_{jk})$ must be dependent and thus at least one of the three possible vectorial products will be zero. It may be, therefore, reasonable to form a sum of all three vectorial products of rows of the matrix $(\bar{\Gamma}_{jk} - \delta_{jk})$ since that must be nonzero (unless two eigenvalues of $\bar{\Gamma}_{jl}$ do coincide, see below). The resulting vector must be then normalized as described above.

The above procedure fails in case of waves, whose eigenvalues coincide (shear waves in isotropic media, shear waves along singular directions in anisotropic media). The only polarization vector, which can be uniquely determined in such a situation is the polarization vector, which would correspond to qP waves. The polarization vectors of qS waves are then any two mutually perpendicular vectors situated in the plane perpendicular to the uniquely determined polarization vector.

The ray tracing equations are not independent. They must satisfy the eikonal equation $G = 1$ at each point of the ray.

The ray tracing system can be solved if the initial values of the computed quantities are specified at some point

$$x_i(\tau_o) = x_{0i} \quad , \quad p_i(\tau_o) = p_{0i}$$

To determine p_{0i} is slightly more complicated than in the isotropic case. The slowness vector p_{0i} can be expressed as

$$p_{0i} = N_i^0 / c(N_k^0) \quad .$$

For its determination, we need to find $c(N_k^0)$. It can be found from the equation of solvability of the Christoffel equation

$$\det(\Gamma_{ik} - c^2\delta_{ik}) = 0 \quad .$$

This cubic equation yields three values of the phase velocities $c(N_k^0)$, corresponding to the three waves, qS_1, qS_2 and qP, which can propagate in an anisotropic medium.

Let us add that the initial direction of the ray is specified by

$$dx_i/d\tau|_0 = a^0_{ijkl} p_{0l} g_{0j} g_{0k}$$

and is thus, generally, different from the initial direction of the slowness vector. Similarly as in isotropic media, we are not free in choosing independently all the three components of the slowness vector. They must be chosen in such a way that they satisfy the eikonal equation

$$G(x_{0m}, p_{0m}) = 1 \quad .$$

As was shown before, the eikonal equation must be satisfied not only at the initial point of the ray but at any of its points. Since the ray tracing equations have the same form for all the waves, which can propagate in an anisotropic medium, the type of the selected wave is specified only by the choice of the initial conditions.

The above ray tracing system, in which rays are parameterized by the phase function τ, can be rewritten so that the parameter along rays is the arclength s. In Sec.7.2.3, we noted that in the high frequency approximation the group velocity is given by the expression

$$v_i^{(g)} = a_{ijkl} p_l g_j g_k \quad .$$

We can immediately see that the first set of the ray tracing equations yields

$$\frac{dx_i}{d\tau} = v_i^{(g)} \quad .$$

This implies for the arclength s along the ray

$$\frac{ds}{d\tau} = v^{(g)} \quad , \quad v^{(g)} = (v_i^{(g)} v_i^{(g)})^{1/2} \quad .$$

We can thus easily transform the ray tracing system into the new form

$$\frac{dx_i}{ds} = (v^{(g)})^{-1} a_{ijkl} p_j g_k g_l, \quad \frac{dp_i}{ds} = -\frac{1}{2}(v^{(g)})^{-1} a_{njkl,i} p_n p_l g_j g_k, \quad \frac{d\tau}{ds} = (v^{(g)})^{-1} \quad .$$

The solution of the ray tracing system yields a series of interesting quantities along the ray. It yields the slowness vector p_i, from which phase velocity vector c_i can be determined,

$$c_i = (p_k p_k)^{-1} p_i \quad .$$

The first set of the ray tracing equations parameterized by τ yields the group velocity vector $v_i^{(g)}$,

$$v_i^{(g)} = a_{ijkl} p_l g_j g_k \quad .$$

Since the polarization vector g_i must also be known during the ray tracing, we know all the three important directions and can determine their mutual orientation: direction of the slowness vector (phase normal, phase velocity vector), direction of the group velocity vector (energy flux direction), direction of the displacement vector (polarization vector).

7.5.2.2 Rays - trajectories along which energy propagates

Direction of the energy flux is specified by the group velocity vector

$$v_i^{(g)} = a_{ijkl} p_l g_j g_k \quad .$$

The trajectory tangent at each point to $v_i^{(g)}$ will have the form

$$\frac{dx_i}{d\tau} = v_i^{(g)} = a_{ijkl} p_l g_j g_k \quad .$$

The ray tracing will be complete if we have a possibility to determine components of the slowness vector. We can proceed as in Sec.7.5.1.2:

$$\begin{aligned} \frac{dp_i}{d\tau} &= \frac{\partial p_i}{\partial x_m}\frac{dx_m}{d\tau} = \frac{\partial p_m}{\partial x_i} a_{mjkl} p_l g_j g_k = \frac{1}{2} a_{mjkl} \frac{\partial p_m p_l}{\partial x_i} g_j g_k \\ &= \frac{1}{2}\frac{\partial}{\partial x_i}(\underbrace{a_{mjkl} p_m p_l g_j g_k}_{1}) - \frac{1}{2} a_{mjkl,i} p_m p_l g_j g_k \\ &- a_{mjkl} p_m p_l g_j \frac{\partial g_k}{\partial x_i} = -\frac{1}{2} a_{mjkl,i} p_m p_l g_j g_k - \underbrace{g_k \frac{\partial g_k}{\partial x_i}}_{0} \quad . \end{aligned}$$

The resulting ray tracing system has thus the same form as the ray tracing derived in the preceding section.

Let us make one important comment. From all the approaches used to derive equations, which govern the propagation of high-frequency waves in inhomogeneous media only the approach based on solving elastodynamic equations and approach based on energy considerations give a complete solution of the problem without necessity of any additional assumptions. Specifically, only both named approaches yield the separation of high frequency seismic waves into three types in anisotropic media and two types in isotropic media. We have shown this with the former approach. We show in what follows how separation follows from energy considerations.

Without invoking elastodynamic equations, we can write the high-frequency expressions for the displacement vector in the following form,

$$u_i = \frac{1}{2}(AF + A^*F^*)g_i \quad .$$

As before, F is a high-frequency analytic signal, A is the amplitude term and g_i is a unit vector specifying the polarization of the wave. After inserting this into the expressions for strain energy W, kinetic energy K and energy flux S_i, we get

$$\begin{aligned} W &= \frac{1}{8}\rho a_{ijkl}p_i p_l g_j g_k (AF' + A^*F^{*\prime})^2 \quad , \\ K &= \frac{1}{8}\rho g_j g_j (AF' + A^*F^{*\prime})^2 \quad , \\ S_i &= \frac{1}{4}\rho a_{ijkl}p_l g_j g_k (AF' + A^*F^{*\prime})^2 \quad . \end{aligned}$$

For the group velocity vector we can write

$$v_i^{(g)} = \frac{S_i}{W + K} \quad .$$

We shall now use the identity, which we have used before,

$$v_i^{(g)}p_i = 1 \quad ,$$

which follows from the simple consideration

$$p_i v_i^{(g)} = \frac{\partial \tau}{\partial x_i}\frac{dx_i}{d\tau} = \frac{d\tau}{d\tau} = 1 \quad .$$

From the above expression for $v_i^{(g)}$, we get

$$v_i^{(g)}p_i = \frac{S_i p_i}{W + K} = \frac{(1/4)\rho a_{ijkl}p_i p_l g_j g_k}{(1/8)\rho a_{ijkl}p_i p_l g_j g_k + (1/8)\rho g_j g_j} = 1$$

This equation yields

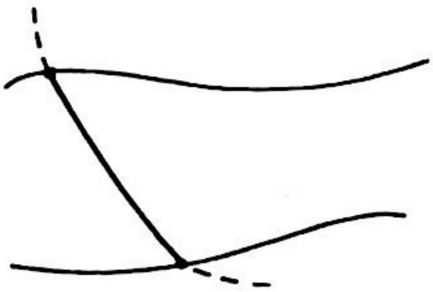
$$a_{ijkl}p_i p_l g_j g_k = g_j g_j = 1 \quad .$$

The expression on the LHS is the eigenvalue $G(x_m, p_m)$ of the matrix $\bar{\Gamma}_{jk} = a_{ijkl}p_ip_l$. Since the matrix $\bar{\Gamma}_{jk}$ has generally, three different eigenvalues, the equation $v_i^{(g)}p_i = 1$ can be satisfied in three different cases, for each of them $G = 1$. The three different cases correspond to the three independent waves, which can propagate in an inhomogeneous anisotropic medium.

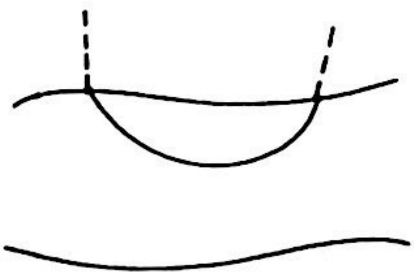
7.5.3 Rays across interfaces

The above derived ray tracing systems can be used in media with smoothly varying parameters of the medium. In the case of the existence of an interface at which the parameters of the medium change abruptly, the point of intersection of the ray with the interface - the *point of incidence* - must be first found. At this point, we must first specify what will be the type of the generated wave (reflected, transmitted, P, S, qP, qS_1, qS_2) and for this chosen wave, we must determine the initial values of x_i and p_i for tracing the ray of the generated wave. The initial values of the x_i are the coordinates of the point of incidence. The determination of the values of p_i is as described in Secs.7.4 and 3.8. Specification of the type of the wave is done by the *wave code*. The wave code consists of a chain of numbers (isotropic case) or pairs of numbers (anisotropic case), each specifying one *single element of the ray*. By the single element of the ray we understand a part of the ray situated in one layer, with its endpoints situated on *both* interfaces bounding the layer, see Fig.A. If both endpoints are situated on

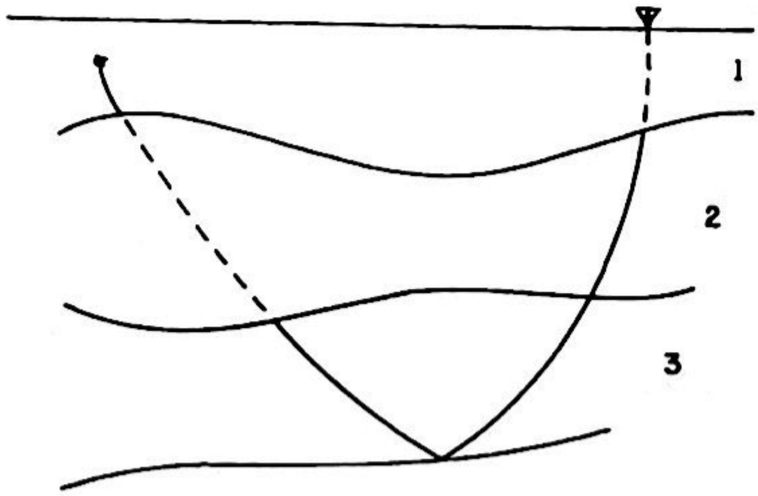
A



B



the same interface, the element is formally understood as composed of two single elements (of course of the same wave type), see Fig.B. Such an element is described in the code by two numbers (two pairs of numbers). The code thus consists of numbers corresponding to the elements of the ray from the source towards the receiver. In the isotropic case, the absolute value of each number specifies the number of the layer, in which the element is situated; the negative sign corresponds to an S wave, the positive sign corresponds to a P wave. In anisotropic media, the first number from the pair specifies the number of the layer, in which the corresponding element of the ray is situated. The second number specifies the type of the wave along this element: $1 - qS_1$, $2 - qS_2$, $3 - qP$.



If the wave, whose ray is shown in the picture propagates in an isotropic medium and the full line denotes the elements along which the wave propagates as a P wave and the dashed line denotes the elements along which the wave propagates as an S wave, the wave code would read:

$$6; 1\text{-}2\ 3\ 3\ 2\text{-}1 \quad .$$

The first number specifies the number of elements of the ray. If the wave propagates in an anisotropic medium and the full and dashed lines denote qP and qS_2 waves respectively, the wave code would read:

$$6; (13)(22)(33)(33)(23)(12) \quad .$$

7.6 Solution of transport equations

Before we solve the transport equations, we introduce some important concepts, which will make the solution easier. While for the determination of the eikonal τ it was sufficient to study individual rays, for the determination of ray amplitudes information about closest neighbourhood of the rays is also necessary. For this purpose, we need to work with systems of rays. We shall, therefore, introduce the *ray fields* and characterize their properties. We shall also introduce the *ray parameters* and the *ray coordinates*. When the ray coordinates are defined, we can specify the transformation laws from the ray to Cartesian coordinates. In this way, we can define the *transformation matrix from the ray to Cartesian coordinates* which plays a very important role in the ray method. Its *Jacobian* is related to the *geometrical spreading* and the *cross-section of the ray tube*.

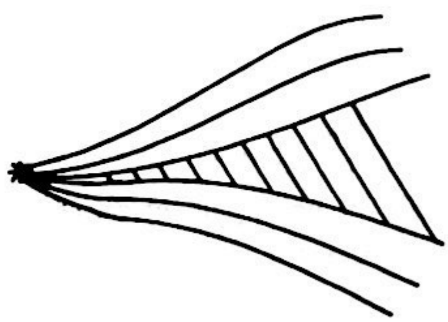
7.6.1 Ray fields - important definitions

Each ray can be specified by two parameters γ_1, γ_2, which we call the *ray parameters*. In various situations, they can be introduced in a different way. In case of a point source, we can take two take-off angles (or radiation angles) φ_o, δ_o as the ray parameters. We can also take any two slowness vector components (the third one is automatically determined from the eikonal equation). In case of an initial surface, along which the distribution of initial time is given, we can choose as ray parameters any curvilinear coordinates on the surface. Thus the rays are specified by their intersections with the initial surface. Let us note that for non constant distribution of the initial time along the surface, the rays are not perpendicular to the surface. Let us also note that all the above specifications hold for both isotropic and anisotropic media. We must only remember that in anisotropic media the ray parameters φ_o, δ_o do not specify directly the ray direction. They specify the direction of the slowness vector which must be used for the specification of the ray direction.

Let us now introduce the *ray coordinates* γ_1, γ_2, γ_3, where γ_1, γ_2 are the above introduced ray parameters and γ_3 is a parameter along the ray. It may be the arclength s, the eikonal τ or any other monotonic parameter along the ray. The ray coordinates are curvilinear coordinates, which can be used for the specification of a point in a space. In regions, where the ray coordinates are defined and are single-valued functions of Cartesian coordinates, we can write,

$$x_i = x_i(\gamma_j) \quad .$$

In the left picture, the shaded area denotes a *shadow region*, region to which no rays



penetrate. In such a region, the above transformation cannot be applied because the ray coordinates are not defined there. The right picture shows the situation, in which two neighbouring rays are intersecting each other. Such a point is described by two different sets of ray coordinates and thus, the ray coordinates are not single valued at it.

If γ_1, γ_2 are constant, the above equation describes a ray. If γ_3 is constant, the above equation describes a surface of constant γ_3. In case of $\gamma_3 = \tau$, this surface is a phase front.

If the Jacobian

$$\mathcal{J} = \partial(x_1, x_2, x_3)/\partial(\gamma_1, \gamma_2, \gamma_3)$$

exists and is non-zero, the ray field is called *regular*. In the opposite case, the ray field is called *singular*. The Jacobian $\mathcal{J}$ is connected with very important transformation matrix Q_{ij} from the ray to Cartesian coordinates,

$$Q_{ij} = \partial x_i/\partial \gamma_j \quad .$$

We can thus write

$$\mathcal{J} = \det Q_{ij} \quad .$$

The matrix Q_{ij} has many important applications. We shall show that it is related to *the geometrical spreading*, one of the most important factors controlling the ray amplitudes. It can also be used for the determination of *paraxial rays*, the rays whose parameters only slightly differ from the ray parameters of the considered ray. The relation of the matrix Q_{ij} to the ray and Cartesian coordinates can also be used for an effective solution of the two-point ray tracing problem. The matrix can be determined in several ways, one of them being the integration of a system of linear ordinary differential equations called the *dynamic ray tracing system*, see Sec. 7.6.2.2.

Let us note that for the evaluation of the transformation matrix Q_{ij} or the Jacobian $\mathcal{J}$, we can also use the results of the ray tracing. The third column of Q_{ij} contains derivatives $\partial x_i/\partial \gamma_3$, which can be determined from the ray tracing equations. Both the matrix Q_{ij} and the Jacobian $\mathcal{J}$ depend, of course, on the choice of the parameter γ_3 along the ray. We shall therefore in the following denote the Jacobian $\mathcal{J}_u$, where u is the parameter used along the ray.

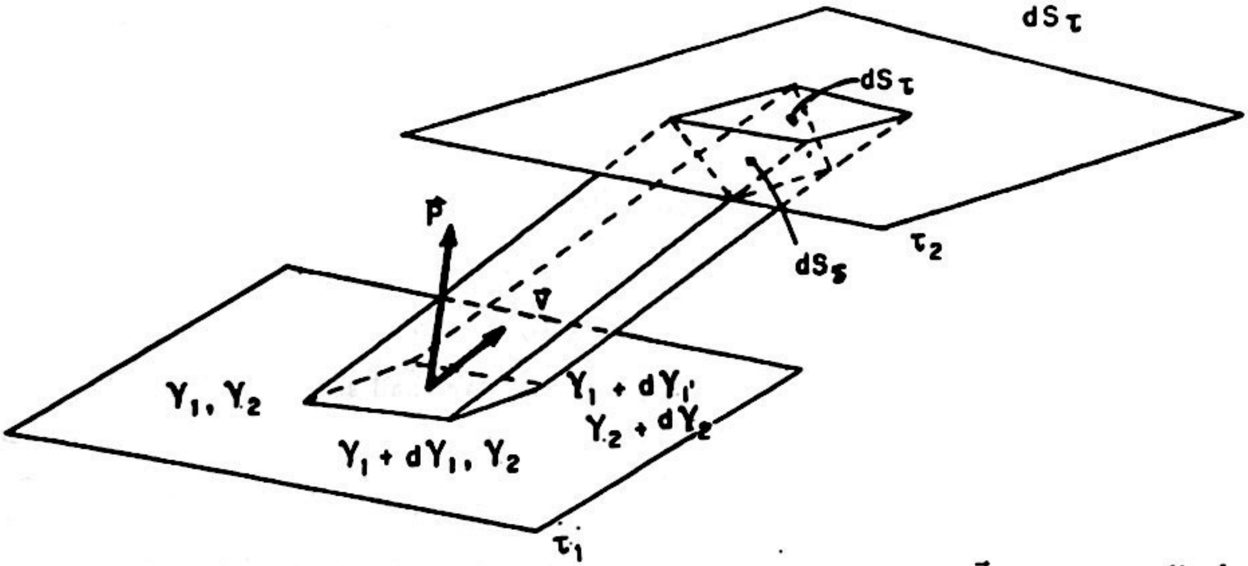
For $\mathcal{J}_s$ we have

$$\mathcal{J}_s = \begin{vmatrix} \frac{\partial x_1}{\partial \gamma_1}\Big|_s & \frac{\partial x_1}{\partial \gamma_2}\Big|_s & \frac{\partial x_1}{\partial s} \\ \frac{\partial x_2}{\partial \gamma_1}\Big|_s & \frac{\partial x_2}{\partial \gamma_2}\Big|_s & \frac{\partial x_2}{\partial s} \\ \frac{\partial x_3}{\partial \gamma_1}\Big|_s & \frac{\partial x_3}{\partial \gamma_2}\Big|_s & \frac{\partial x_3}{\partial s} \end{vmatrix} = \left(\frac{\partial \vec{r}}{\partial \gamma_1} \times \frac{\partial \vec{r}}{\partial \gamma_2}\right)_s \frac{d\vec{r}}{ds} \quad .$$

An important characteristics of the ray field is its density. The denser is the ray field, the higher are the amplitudes. To characterize the density of the rays, we introduce the terms the *ray tube* and the *cross-sectional area of the ray tube*. The narrower is the ray tube (and the denser is the ray field), the higher are the ray amplitudes, and vice versa. We shall thus define the ray tube as a system of rays with ray parameters specified in the interval

$$(\gamma_1, \gamma_1 + d\gamma_1) \times (\gamma_2, \gamma_2 + d\gamma_2)$$

By the cross-sectional area of the ray tube we understand the part of the surface $\gamma_3 =$ const cut out by the ray tube. In the following picture, which represents a ray tube in an



anisotropic medium, we show two such cross-sections. The first one, $d\vec{S}_s$ is a perpendicular cross-section of the ray tube and can be expressed in the vectorial form as

$$d\vec{S}_s = \left(\frac{\partial \vec{r}}{\partial \gamma_1} \times \frac{\partial \vec{r}}{\partial \gamma_2}\right)_s d\gamma_1 d\gamma_2 \quad .$$

The other one is the cross-section $d\vec{S}_\tau$ of the ray tube with the phase front and has again the vectorial form

$$d\vec{S}_\tau = \left(\frac{\partial \vec{r}}{\partial \gamma_1} \times \frac{\partial \vec{r}}{\partial \gamma_2}\right)_\tau d\gamma_1 d\gamma_2 \quad .$$

Let us now express the Jacobians $\mathcal{J}_s$ and $\mathcal{J}_\tau$ in terms of the cross-sectional areas. We get

$$\mathcal{J}_s = (d\vec{S}_s \frac{d\vec{r}}{ds})/d\gamma_1 d\gamma_2 = dS_s/d\gamma_1 d\gamma_2, \mathcal{J}_\tau = (d\vec{S}_\tau v^{(g)} \frac{d\vec{r}}{ds})/d\gamma_1 d\gamma_2 = dS_s v^{(g)}/d\gamma_1 d\gamma_2 = v^{(g)} \mathcal{J}_s \quad .$$

We can express J_τ also in an alternative way,

$$J_\tau = \left(d\vec{S}_\tau v^{(g)} \frac{d\vec{r}}{ds}\right)/d\gamma_1 d\gamma_2 = \left(dS_\tau c\, \vec{p}\, \vec{v}^{(g)}\right)/d\gamma_1 d\gamma_2 = \Omega c \quad .$$

Here we used the identity $p_i v_i^{(g)} = 1$ derived in Sec.7.5.2.2 and introduced a new quantity Ω,

$$\Omega = \left|\frac{\partial \vec{r}}{\partial \gamma_1} \times \frac{\partial \vec{r}}{\partial \gamma_2}\right|_\tau \quad .$$

Using the above equations, we can relate $\mathcal{J}_s, \mathcal{J}_\tau$ and Ω,

$$\boxed{J_\tau = v^{(g)} J_s = c\,\Omega} \quad .$$

Let us note that all the three quantities may change their sign along a ray. This means that the quantities may pass through zero. Such a situation occurs at points where the ray tube shrinks to a point. Such points are known as the *caustic points*.

The above introduced quantity Ω plays an important role in the following considerations. We shall call its absolute value $|\Omega|$ *geometrical spreading*. It can be determined at an arbitrary point where J_τ or J_s are known. Note that in *isotropic* media, where $v^{(g)} = c$,

$$J_s = \Omega \quad .$$

7.6.2 Determination of the matrix Q_{ij} and the Jacobian $\mathcal{J}$

The Jacobian $\mathcal{J}$ and the matrix Q_{ij} can be evaluated along rays in several ways. We describe here two principal approaches. The first one is based on an approximate measure of the cross-sectional area of the ray tube by substituting the derivatives by finite differences. The other one is based on solving the dynamic ray tracing equations along the ray.

7.6.2.1 Use of finite differences

The above derived relation

$$\mathcal{J}_s = dS_s/d\gamma_1 d\gamma_2$$

can be modified approximately as follows

$$\mathcal{J}_s \sim (\Delta\vec{S}_s \frac{d\vec{r}}{ds})/\Delta\gamma_1\Delta\gamma_2 \quad .$$

It is usually more convenient to use ΔS_τ instead of ΔS_s since the rays are commonly parameterized by τ. Then we have

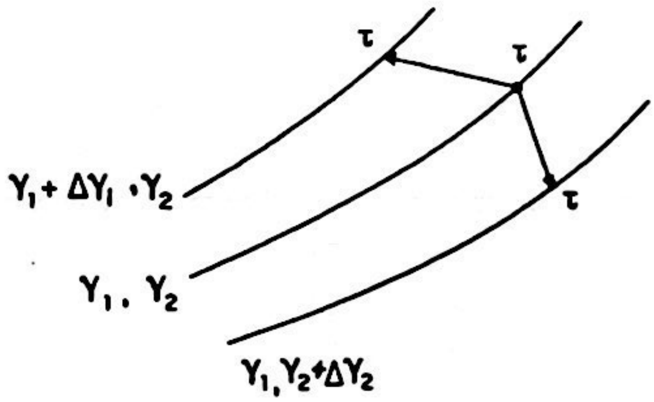
$$\mathcal{J}_\tau = \mathcal{J}_s v^{(g)} = \frac{d\vec{S}_\tau \frac{d\vec{r}}{ds} v^{(g)}}{d\gamma_1 d\gamma_2} = \frac{dS_\tau \vec{N}\vec{v}^{(g)}}{d\gamma_1 d\gamma_2} = \frac{c dS_\tau}{d\gamma_1 d\gamma_2} = \frac{c d\vec{S}_\tau \vec{N}}{d\gamma_1 d\gamma_2}$$

and thus

$$\mathcal{J}_s \sim \frac{c\Delta\vec{S}_\tau \vec{N}}{v^{(g)}\Delta\gamma_1\Delta\gamma_2} \quad ,$$

where $\vec{N}$ is normal to the phase front and c denotes the phase velocity. For the evaluation of $\mathcal{J}_s$ we need to compute three rays with ray parameters (γ_1, γ_2) - the central ray, $(\gamma_1 + \Delta\gamma_1, \gamma_2)$, $(\gamma_1, \gamma_2 + \Delta\gamma_2)$ - two auxiliary rays. To construct $\Delta\vec{S}_\tau$, we use points on the three rays corresponding to the same time τ.

We can then construct two vectors as shown in the picture. The vectorial product of these



vectors yields the sought vectorial surface. This procedure can fail if points corresponding to the same time τ on all the three rays lie in a line. To avoid such a situation, it is better to shoot more rays and to make their selection before using them for calculating $\mathcal{J}_s$.

If we are interested not only in the Jacobian $\mathcal{J}_s$ but also in the elements of the matrix Q_{ij}, i.e. in derivatives $\partial x_i/\partial\gamma_J$ $(J = 1,2)$, we can write for the points corresponding to time τ on the K-th auxiliary ray (now the auxiliary rays may differ from the central ray in both ray parameters γ_1 and γ_2) in the first approximation

$$\begin{aligned} x_i(\gamma_1 + \Delta\gamma_{1K}, \gamma_2 + \Delta\gamma_{2K}) &= x_i(\gamma_1, \gamma_2) + Q_{i1}\Delta\gamma_{1K} + Q_{i2}\Delta\gamma_{2K} \quad , \\ p_i(\gamma_1 + \Delta\gamma_{1K}, \gamma_2 + \Delta\gamma_{2K}) &= p_i(\gamma_1, \gamma_2) + P_{i1}\Delta\gamma_{1K} + P_{i2}\Delta\gamma_{2K} \quad . \end{aligned}$$

Here we denoted

$$Q_{iJ} = \partial x_i/\partial\gamma_J \quad , \quad P_{iJ} = \partial p_i/\partial\gamma_J \quad .$$

The above equations written for both auxiliary rays form six independent systems of two linear algebraic equations for Q_{iJ} and $P_{iJ}(J = 1,2)$. For Q_{iJ} and P_{iJ} we get

$$\begin{aligned} Z_{i1} &= (\Delta z_{i1}\Delta\gamma_{22} - \Delta z_{i2}\Delta\gamma_{21})B^{-1} \quad , \\ Z_{i2} &= (\Delta z_{i2}\Delta\gamma_{11} - \Delta z_{i1}\Delta\gamma_{12})B^{-1} \quad , \end{aligned}$$

where

$$\Delta z_{iJ} = (z_{iJ} - \bar{z}_i) \quad , \quad B = \Delta\gamma_{11}\Delta\gamma_{22} - \Delta\gamma_{12}\Delta\gamma_{21} \quad .$$

Here Z_{iJ} can be either Q_{iJ} or P_{iJ}, $\bar{z}_i$ are x_i or p_i measured on the central ray (γ_1, γ_2), z_{iJ} are x_i or p_i measured on the J-th auxiliary ray, $\Delta\gamma_{IJ}$ denotes the difference between the I-th ray parameter of the J-th auxiliary ray and the central ray. The quantities Q_{iJ} can be supplemented by $dx_i/d\tau$ determined during the ray tracing and the Jacobian can be determined. The Jacobian can be both positive and negative. The changes of the sign define the *index of trajectory*, see later.

7.6.2.2 Dynamic ray tracing

Another and more frequently used way how to determine the matrix Q_{ij} (and P_{ij}) and the Jacobian $\mathcal{J}$ is to use the so-called *dynamic ray tracing equations*. In this approach, all the quantities are evaluated exactly and no additional (auxiliary) rays are required in the vicinity of the ray. The dynamic ray tracing equations can be simply obtained by differentiation of the ray tracing equations with respect to the ray parameters γ_1, γ_2. If we write the ray tracing equations in the form including both isotropic and anisotropic media, i.e.

$$\frac{dx_i}{d\tau} = \frac{1}{2}\frac{\partial G}{\partial p_i} \quad , \quad \frac{dp_i}{d\tau} = -\frac{1}{2}\frac{\partial G}{\partial x_i} \quad ,$$

the dynamic ray tracing has the form

$$\begin{aligned} \frac{dQ_{iJ}}{d\tau} &= \frac{1}{2}\left[\frac{\partial^2 G}{\partial p_i \partial x_k} Q_{kJ} + \frac{\partial^2 G}{\partial p_i \partial p_k} P_{kJ}\right] , \\ \frac{dP_{iJ}}{d\tau} &= \frac{1}{2}\left[\frac{\partial^2 G}{\partial x_i \partial x_k} Q_{kJ} + \frac{\partial^2 G}{\partial x_i \partial p_k} P_{kJ}\right] , \end{aligned}$$

where $Q_{kJ} = \partial x_k/\partial\gamma_J|_\tau$, $P_{kJ} = \partial p_k/\partial\gamma_J|_\tau$. The dynamic ray tracing system consists of 12 linear ordinary differential equations. Similarly as in the case of the ray tracing system, not all the equations are independent. *Two* conditions follow immediately from the derivative of the eikonal equation with respect to γ_1, γ_2

$$\frac{\partial G}{\partial x_k} Q_{kJ} + \frac{\partial G}{\partial p_k} P_{kJ} = 0 \quad .$$

Using the ray tracing equations, this can be rewritten

$$\frac{dp_k}{d\tau} Q_{kJ} - \frac{dx_k}{d\tau} P_{kJ} = 0 \quad .$$

Other *two* conditions follow from the requirement that the slowness vector p_k is perpendicular to the phase front and thus also to any vector tangent to it. Since the vectors $Q_{kJ} = \partial x_k/\partial\gamma_J|_\tau$ for $J = 1, 2$ are tangent to the phase front, we have

$$p_k Q_{kJ} = 0 \quad .$$

We can see that the dynamic ray tracing system consists, in fact, from 8 independent linear differential equations.

In isotropic case when

$$G = c^2 p_n p_n \quad ,$$

we have

$$\frac{\partial G}{\partial p_i} = 2c^2 p_i \quad , \quad \frac{\partial G}{\partial x_i} = 2cc_{,i} p_n p_n \quad ,$$

$$\frac{\partial^2 G}{\partial p_i \partial p_k} = 2c^2 \delta_{ik} \quad , \quad \frac{\partial^2 G}{\partial p_i \partial x_k} = 4cc_{,k} p_i \quad ,$$

$$\frac{\partial^2 G}{\partial x_i \partial p_k} = 4cc_{,i} p_k \, , \frac{\partial^2 G}{\partial x_i \partial x_k} = 2c_{,k} c_{,i} c^2 + 2c^{-1} c_{,ik} \quad .$$

This yields the dynamic ray tracing system

$$\frac{dQ_{iJ}}{d\tau} = 2cc_{,k} p_i Q_{kJ} + c^2 P_{iJ} \quad ,$$

$$\frac{dP_{iJ}}{d\tau} = -c_{,k} c_{,i} c^{-2} Q_{kJ} - c^{-1} c_{,ik} Q_{kJ} - 2cc_{,i} p_k P_{kJ} \quad .$$

The last equation could be further simplified if we take into acount the condition following from the derivative of the eikonal equation

$$c^{-3} c_{,k} Q_{kJ} = -p_k P_{kJ} \quad .$$

For solving the dynamic ray tracing equations, it is necessary to specify appropriate initial and boundary conditions for Q_{iJ} and P_{iJ}. The initial conditions differ according to the specification of the source and ray parameters. For example, for a point source situated in an inhomogeneous anisotropic medium and for the slowness vector specified by the ray parameters $\gamma_1 = \varphi_o$, $\gamma_2 = \delta_o$, where φ_o and δ_o are take-off angles at the source such that the slowness vector at the source has the form

$$p_i(\tau_0) = N_i^0 c_0^{-1}(N_k^0) \equiv c_0^{-1}(\cos\varphi_0 \cos\delta_0, \quad \sin\varphi_0 \cos\delta_0, \quad \sin\delta_0) \quad ,$$

the initial conditions are as follows:

$$Q_{iJ}(\tau_0) = 0, \qquad P_{iJ} = c_0^{-1}\left(\frac{\partial N_i^0}{\partial \gamma_J} - p_i^0 v_k^{(g)}(\tau_0) \frac{\partial N_k^0}{\partial \gamma_J}\right) \quad .$$

Here

$$\frac{\partial N_1^0}{\partial \varphi_0} = -\cos\delta_0 \sin\varphi_0 \, , \quad \frac{\partial N_2^0}{\partial \varphi_0} = \cos\delta_0 \cos\varphi_0 \, , \quad \frac{\partial N_3^0}{\partial \varphi_0} = 0 \, ,$$

$$\frac{\partial N_1^0}{\partial \delta_0} = -\cos\varphi_0 \sin\delta_0 \, , \quad \frac{\partial N_2^0}{\partial \delta_0} = -\sin\varphi_0 \sin\delta_0 \, , \quad \frac{\partial N_3^0}{\partial \delta_0} = \cos\delta_0 \quad .$$

In case of an inhomogeneous isotropic medium, in which $c_o(N_k) = c_o = v^{(g)}(\tau_o)$, the above initial conditions simplify to

$$Q_{iJ}(\tau_o) = 0\,, \quad P_{11}(\tau_o) = -c_o^{-1}\sin\varphi_o\cos\delta_o\,, \quad P_{21}(\tau_o) = c_o^{-1}\cos\varphi_o\cos\delta_o\,, \quad P_{31}(\tau_o) = 0\,,$$

$$P_{12}(\tau_o) = -c_o^{-1}\cos\varphi_o\sin\delta_o\,, \quad P_{22}(\tau_o) = -c_o^{-1}\sin\varphi_o\sin\delta_o\,, \quad P_{32}(\tau_o) = c_o^{-1}\cos\delta_o\,.$$

The boundary conditions specifying the transformation of quantities Q_{iJ}, P_{iJ} across an interface can be derived using the phase matching procedure.

In a homogeneous medium, the dynamic ray tracing system considerably simplifies and can be solved analytically. Let us consider a homogeneous isotropic medium. Then the above dynamic ray tracing equations reduce to

$$\frac{dQ_{iJ}}{dt} = c^2 P_{iJ}\,, \qquad \frac{dP_{iJ}}{dt} = 0 \quad ,$$

from which we get

$$Q_{iJ}(\tau) = c^2 P_{iJ}(\tau - \tau_o)\,, \qquad P_{iJ} = const.$$

Let us remind that the ray tracing equations can also be solved analytically in this case. They yield straight rays specified as

$$Q_{i3} = \partial x_i/\partial s = cp_i\,, \qquad p_i = const.$$

We can thus evaluate the geometrical spreading, which we have defined as $|\Omega| = |\mathcal{J}_s| = |detQ_{ij}|$. If we take into account the initial conditions specified above, we get

$$detQ_{ij} = c^2(\tau-\tau_o)^2 \begin{vmatrix} -\sin\varphi_o\cos\delta_o & -\cos\varphi_0\sin\delta_o & \cos\varphi_o\cos\delta_o \\ \cos\varphi_o\cos\delta_o & -\sin\varphi_o\sin\delta_o & \sin\varphi_o\cos\delta_o \\ 0 & \cos\delta_o & \sin\delta_o \end{vmatrix} = c^2(\tau-\tau_o)^2\cos\delta_o \quad .$$

Thus,

$$|\Omega| = |\mathcal{J}_s| = r^2|\cos\delta_o| \quad ,$$

where r is the distance between the source and the observer.

In the following section, we shall see that the amplitudes are proportional to $(\Omega(\tau_o)/\Omega(\tau))^{1/2}$. In this respect, we can make here three important conclusions.

First, we can see that in the case of a point source, $\Omega(\tau_o) = 0$. This could imply zero ray amplitudes everywhere in the space. We shall discuss this apparent paradox in Sec.7.7.

Second, we can see that the geometrical spreading appears in the amplitude formula in the form of a ratio. Thus, if we multiply the geometrical spreading by a nonzero constant, the resulting amplitude will be unchanged. Let us note that the multiplication of the initial conditions for the dynamic ray tracing by a nonzero constant is equivalent to the multiplication of the corresponding resulting elements Q_{iJ}. This is a consequence of the linearity of the dynamic ray tracing. Thus, let us modify the above initial conditions for the isotropic case in the following way,

$$Q_{iJ}(\tau_o) = 0, \quad P_{11}(\tau_o) = -\sin\varphi_o, \quad P_{21}(\tau_o) = \cos\varphi_o, \quad P_{31}(\tau_o) = 0,$$

$$P_{12}(\tau_o) = -\cos\varphi_o \sin\delta_o, \quad P_{22}(\tau_o) = -\sin\varphi_o \sin\delta_o, \quad P_{32}(\tau_o) = \cos\delta_o \quad .$$

We multiplied the original vector P_{i1} by $c_o/\cos\delta_o$ and the vector P_{i2} by c_o. The resulting modified geometrical spreading $|\Omega_M|$ will not approach zero for δ_o approaching $\pi/2$ as the spreading $|\Omega|$ would. Moreover, it is not difficult to show that the geometrical spreading $|\Omega_M|$ defined in the described way is *reciprocal*. It means that if we choose two arbitrary points A and B on a ray, $|\Omega_M|$ calculated from A to B is equal to $|\Omega_M|$ calculated from B to A. In a homogeneous isotropic medium, the modified geometrical spreading reduces to

$$|\Omega_M| = c^2 r^2 \quad .$$

Third, we can see that the ray amplitudes of a wave generated by a point source are inversely proportional to r as were the amplitudes of the spherical waves. The ray method gives in a homogeneous isotropic medium exact solution, which in our case is a spherical wave.

The dynamic ray tracing equations can be solved simultaneously with the ray tracing or only along selected, already computed, rays. Since the dynamic ray tracing yields quantities important for performing the paraxial two-point ray tracing, the former procedure is usually more convenient.

In isotropic case, it is very convenient to perform the dynamic ray tracing in the so-called *ray-centered coordinate system* instead of in the Cartesian coordinates. The ray-centered coordinates q_i are introduced as follows. In the plane perpendicular to the ray, we specify two mutually perpendicular unit vectors $g_i^{(1)}, g_i^{(2)}$ in the way described in Sec.7.2.2. Along these vectors, we can introduce Cartesian coordinates q_1, q_2. These coordinates supplemented by the third coordinate q_3, which may be value of any of the parameters along the ray, e.g., τ, s, σ at the point of intersection of the perpendicular plane and the ray, form the ray-centered coordinate system. The Cartesian dynamic ray tracing system consisting of 12 linear differential equations (the solutions of which are subject to 4 additional conditions) can be, in the ray-centered coordinates, reduced to the system of eight independent, substantially simplified equations.

7.6.3 Solution of the transport equation

We consider the transport equation for an inhomogeneous anisotropic medium. The results for the isotropic and the acoustic case can be deduced in the same way. We shall use two approaches. The first one is based on the transport equation derived in Sec.7.2.3 in the form

$$(\rho A A^{*} v_j^{(g)})_{,j} = 0 \quad .$$

The volume integral of the above equation and successive use of the Gauss theorem yield

$$\iiint_V (\rho A A^{*} v_j^{(g)})_{,j} dV = \iint_S \rho A A^{*} v_j^{(g)} n_j dS = 0 \quad .$$

Here S is a surface surrounding the volume V, n_j is the outer normal to S.

Let us now consider the volume V to be formed by a part of a ray tube limited by two different phase fronts, with which the ray tube makes cross-sections $dS_\tau(\tau_o)$ and $dS_\tau(\tau)$. On the sides of the ray tube, we have

$$v_j^{(g)} n_j = 0$$

and on the cross-sections with phase fronts, we have

$$v_j^{(g)}(\tau_o) n_j(\tau_o) = -c(\tau_o) \, , \quad v_j^{(g)}(\tau) n_j(\tau) = c(\tau) \quad .$$

Thus the only elements of the surface surrounding the elementary volume contributing to the surface integral are the cross-sectional areas dS_τ of the ray tube. The above integral equation thus reduces to the following form

$$\iint_{S_\tau(\tau)} \rho(\tau) A(\tau) A^{*}(\tau) c(\tau) dS_\tau(\tau) - \iint_{S_\tau(\tau_o)} \rho(\tau_o) A(\tau_o) A^{*}(\tau_o) c(\tau_o) dS_\tau(\tau_o) = 0 \quad .$$

If we use the relation between the cross-sectional area dS_τ and the Jacobian $\mathcal{J}_\tau$,

$$dS_\tau = \mathcal{J}_\tau c^{-1} d\gamma_1 d\gamma_2 \quad ,$$

we can rewrite the above integral equation as follows

$$\iint_{\gamma_1 \gamma_2} d\gamma_1 d\gamma_2 [\rho(\tau) A(\tau) A^{*}(\tau) \mathcal{J}_\tau(\tau) - \rho(\tau_o) A(\tau_o) A^{*}(\tau_o) \mathcal{J}_\tau(\tau_o)] = 0 \quad .$$

Since the ray tube can be chosen arbitrarily and since ρ and J_τ are real-valued quantities, the above integral equation yields

$$A(\tau) = A(\tau_o) \sqrt{\frac{\rho(\tau_o) J_\tau(\tau_o)}{\rho(\tau) J_\tau(\tau)}} = A(\tau_o) \sqrt{\frac{\rho(\tau_o) v^{(g)}(\tau_o) J_s(\tau_o)}{\rho(\tau) v^{(g)}(\tau) J_s(\tau)}} = A(\tau_o) \sqrt{\frac{\rho(\tau_o) c(\tau_o) \Omega(\tau_o)}{\rho(\tau) c(\tau) \Omega(\tau)}} \quad .$$

The above equations are called the *continuation formulae* since they give a possibility to continue evaluation of the amplitudes from a point on the ray corresponding to time τ_o to another point on the ray.

Another way how to solve the transport equation is to use the following important relation

$$v^{(g)}_{i,i} = \mathcal{J}_s^{-1}\frac{d}{ds}(v^{(g)}\mathcal{J}_s) = \mathcal{J}_\tau^{-1}\frac{d\mathcal{J}_\tau}{d\tau} \quad .$$

We shall insert this relation into the above transport equation and we take into account that $(\rho AA^*)_{,j}v_j = d(\rho AA^*)/dt$. We thus get

$$\frac{d(\rho AA^*)}{d\tau} + \rho AA^*\mathcal{J}_\tau^{-1}\frac{d\mathcal{J}_\tau}{d\tau} = 0 \quad ,$$

which yields

$$\frac{d}{d\tau}(\rho AA^*\mathcal{J}_\tau) = 0 \quad .$$

The solution of this equation reads

$$A(\tau) = \frac{\psi(\gamma_1,\gamma_2)}{\sqrt{\rho(\tau)\mathcal{J}_\tau(\tau)}} = \frac{\psi(\gamma_1,\gamma_2)}{\sqrt{\rho(\tau)v^{(g)}(\tau)\mathcal{J}_s(\tau)}} = \frac{\psi(\gamma_1,\gamma_2)}{\sqrt{\rho(\tau)c(\tau)\Omega(\tau)}} \quad ,$$

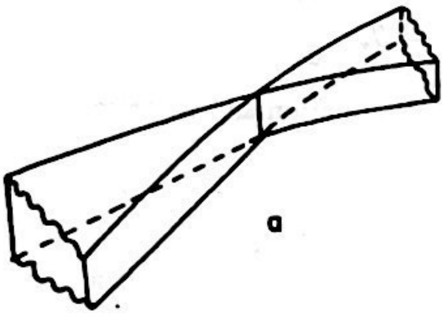
where $\psi(\gamma_1,\gamma_2)$ is a function constant on a considered ray. The above formula can be simply rewritten into the continuation formula fully corresponding to the one derived above.

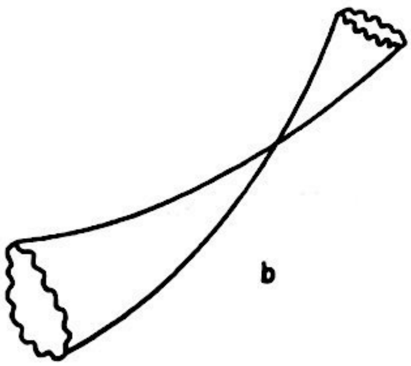
We said above that the quantity Ω can be both positive or negative. Its sign can change along the considered ray even several times. It is, therefore, more practical to rewrite the above equations as follows

$$\begin{aligned} \Omega(\tau) &= |\Omega(\tau)|e^{i\pi k(\tau_o,\tau)} \quad , \\ A(\tau) &= A(\tau_0)\left[\frac{\rho(\tau_o)c(\tau_o)|\Omega(\tau_o)|}{\rho(\tau)c(\tau)|\Omega(\tau)|}\right]^{1/2} e^{-i\frac{\pi}{2}k(\tau_o,\tau)} \quad , \end{aligned}$$

where $|\Omega(\tau)|$ is the above introduced geometrical spreading. The quantity $k(\tau_o,\tau)$ is known as *an index of the ray trajectory*, or briefly the *KMAH index*. It gives a number of points at which the sign of Ω changed. These points are the *caustic points*, thus k gives the number of caustic points encountered along the ray. The caustic points can be of the first and second order. Caustic points of the first order are taken into account once, the caustic points of the second order twice.

At a caustic point of the first order, the ray tube shrinks to an arc perpendicular to the direction of propagation. In case of the caustic point of the second order, the ray tube shrinks to a point, see pictures.





7.7 Ray Green function

We could see in the previous chapters that the calculation of the Green function is of great importance in many applications. Calculation of the complete Green function, however, may be rather complicated task even in homogeneous media, see Chap.5. In slightly inhomogeneous media, it is therefore reasonable not to seek the complete Green function but only its zero-order ray approximation, which we call the *ray Green function*. We shall seek it as a zero-order ray solution of the following inhomogeneous elastodynamic equation

$$(c_{ijkl}G_{kn,l})_{,j} - \rho G_{in,tt} = -\delta_{in}\delta(x_m - x_{om})\delta(t - t_o) \quad .$$

The ray solution of this equation has a form of a sum over the considered elementary waves (direct, multiply reflected, refracted, etc.). We shall call the solution corresponding to a single wave the *elementary ray Green function*. For the sake of brevity, when it does not lead to a confusion, we shall often refer to this solution also as the ray Green function.

Using the expressions for the zero-order ray solution of the elastodynamic equation derived in the previous sections, we can write the elementary ray Green function as follows

$$G^R_{in}(x_m, t; x_{om}, t_o) = A(x_m)g_i(x_m)\delta^{(A)}(t - t_o - \tau(x_m)) \quad ,$$

where

$$A(x_m) = A(x_{om})\left[\frac{\rho(x_{om})c(x_{om})\Omega_M(x_{om})}{\rho(x_m)c(x_m)\Omega_M(x_m)}\right]^{1/2} \quad .$$

Here $g_i(x_m)$ is the polarization vector, $\delta^{(A)}$ denotes the analytic delta function, $\delta^{(A)}(\xi) = \delta(\xi) - \frac{i}{\pi\xi}$, see Sec.1.3, and $\tau(x_m)$ is the eikonal. The quantity Ω_M represents the reciprocal geometrical spreading calculated with the modified initial conditions, see Sec.7.6.2.2, $\Omega_M = c_0^2\Omega/\cos\delta_o$. The index R in the symbol G^R_{in} denotes the *ray* Green function.

As we have shown before, the formula for $A(x_m)$ expresses the preservation of the quantity

$$A(x_m)[\rho(x_m)c(x_m)\Omega_M(x_m)]^{1/2}$$

along the ray. In the previous section, we have found that $\Omega_M(x_m)$ is zero at a point source, which implies that $A(x_m)$ tends there to infinity in order to keep the above expression finite. Since ρ and c are finite, it is obvious that the quantity

$$A(x_m)\Omega_M^{1/2}(x_m)$$

must also be finite everywhere including the source. We shall introduce a quantity $g(x_{om})$ such that

$$g(x_{om}) = \lim_{x_m \to x_{om}} A(x_m)\Omega_M^{1/2}(x_m) \quad ,$$

where the limit is performed along a ray. We shall call $g(x_{om})$ the *radiation pattern of the source*. From this definition, we can see that we can understand $g(x_{om})$ as a spreading-free amplitude at the source. The radiation pattern defines the directivity of the source. If it is constant, we speak about *isotropic* radiation pattern, i.e. the radiation is the same to all directions. Here we are interested in the ray Green function, we shall therefore look for the directivity of the single force.

Let us consider a unit force oriented along the x_n-axis. It has the direction and size of the unit base vector $\vec{i}_n$ of the Cartesian coordinate system. Let us further consider the polarization vector $g_i(x_{om})$ of the studied elementary wave at the source. The contribution of the single force to the direction specified by $g_i(x_{om})$ will thus be proportional to $\vec{g}(x_{om})\vec{i}_n = g_n(x_{om})$. The radiation pattern of the unit single force can thus be written

$$g(x_{om}) = Cg_n(x_{om}) \quad ,$$

where C is a factor of proportionality to be determined. Formal expression for the ray Green function can be finally written as

$$G^R_{in} = C\left[\frac{\rho(x_{om})c(x_{om})}{\rho(x_m)c(x_m)|\Omega_M(x_m)|}\right]^{1/2} e^{-i\frac{\pi}{2}k(x_{om},x_m)}g_n(x_{om})g_i(x_m)\delta^{(A)}(t - t_o - \tau(x_m)) \, .$$

7.7.1 Ray Green function for an isotropic medium

In an isotropic medium, the above expression for the ray Green function can be written separately for P and S waves. If we denote, as before, by $N_i(x_m)$ a unit vector specifying the polarization of P waves and by $g_i^{(1)}(x_m)$ and $g_i^{(2)}(x_m)$ unit vectors specifying the polarization of S waves, we can write for the P wave ray Green function

$$G_{in}^{RP} = C_P \left[\frac{\rho(x_{om})\alpha(x_{om})}{\rho(x_m)\alpha(x_m)|\Omega_M^P(x_m)|} \right]^{1/2} e^{-i\frac{\pi}{2}k_p(x_m,x_{om})} N_n(x_{om}) N_i(x_m) \delta^{(A)}(t - t_o - \tau_P(x_m))$$

and for the S wave ray Green function

$$G_{in}^{RS} = C_S \left[\frac{\rho(x_{om})\beta(x_{om})}{\rho(x_m)\beta(x_m)|\Omega_M^S(x_m)|} \right]^{1/2} e^{-i\frac{\pi}{2}k_s(x_m,x_{om})}$$

$$\times [g_n^{(1)}(x_{om}) g_i^{(1)}(x_m) + g_n^{(2)}(x_{om}) g_i^{(2)}(x_m)] \delta^{(A)}(t - t_o - \tau_S(x_m)) \, .$$

The formal expression for S waves can be further simplified if we take into account that

$$g_n^{(1)} g_i^{(1)} + g_n^{(2)} g_i^{(2)} = \delta_{in} - N_i N_n \quad .$$

This may be easily proved if we realize that $g_n^{(1)}$, $g_n^{(2)}$, N_n are projections of the vector $\vec{i}_n$ into the vector basis $\vec{g}^{(1)}$, $\vec{g}^{(2)}$, $\vec{N}$. Then

$$\vec{i}_i \vec{i}_n = \vec{g}_i^{(1)} \vec{g}_n^{(1)} + \vec{g}_i^{(2)} \vec{g}_n^{(2)} + N_i N_n = \delta_{in} \, .$$

We shall now determine the factors of proportionality C_P, C_S for the ray Green function for *homogeneous* isotropic medium. In other words, we shall seek the zero-order ray solution of the following inhomogeneous elastodynamic equation

$$(\lambda + \mu) G_{jn,ij} + \mu G_{in,jj} - \rho G_{in,tt} = -\delta_{in} \delta(x_m - x_{om}) \delta(t - t_o) \, .$$

The above formal expressions for the ray Green functions reduce in a homogeneous medium to

$$G_{in}^{RP} = C_P \frac{N_i N_n}{\alpha r} \delta \left(t - t_o - \frac{r}{\alpha} \right)$$

and

$$G_{in}^{RS} = C_S \frac{(\delta_{in} - N_i N_n)}{\beta r} \delta \left(t - t_o - \frac{r}{\beta} \right) \quad .$$

Here we used the fact that in a homogeneous isotropic medium

$$|\Omega_M| = c^2 r^2 \quad ,$$

see Sec.7.6.2.2. The symbol r denotes the distance between the source and an observer.

Since we work with the zero-order ray approximation of the Green function, its differentiation will lead to a system of terms of different importance. Asymptotically largest contribution will be of the terms with highest derivative of the analytic signal. In this way, we can write

$$G^{RP}_{jn,ij} \sim C_P \frac{N_j N_n}{\alpha r} \frac{N_i N_j}{\alpha^2} \ddot{\delta} = C_P \frac{N_i N_n}{\alpha^3 r} \ddot{\delta} \,,$$

and thus

$$G^{RP}_{jn,ij} N_i N_n \sim C_P \frac{\ddot{\delta}}{\alpha^3 r} \quad .$$

Similarly, we get

$$(G^{RP}_{in} N_i N_n)_{,jj} \sim C_P \frac{\ddot{\delta}}{\alpha^3 r} \quad .$$

We can see that in our approximation, we can write

$$G^{RP}_{jn,ij} N_i N_n \sim (G^{RP}_{in} N_i N_n)_{,jj} \quad .$$

In the same way, we get

$$G^{RP}_{in,jj} N_i N_n \sim (G^{RP}_{in} N_i N_n)_{,jj} \quad .$$

If we multiply the inhomogeneous elastodynamic equation by $N_i N_n$, we get

$$(\lambda + \mu) G_{jn,ij} N_i N_n + \mu G_{in,jj} N_i N_n - \rho G_{in,tt} N_i N_n = -\delta(x_m - x_{om}) \delta(t - t_o) \,.$$

If we insert into this equation the previous asymptotic results and $\lambda + 2\mu = \rho\alpha^2$, we get

$$\rho\alpha^2 (G^{RP}_{in} N_i N_n)_{,jj} - \rho (G^{RP}_{in} N_i N_n)_{,tt} = -\delta(x_m - x_{om}) \delta(t - t_o)$$

This is, however, the inhomogeneous acoustic equation, which we studied in Sec.5.1. The solution of such an equation is as follows

$$(G^{RP}_{in} N_i N_n) = \frac{1}{4\pi\rho\alpha^2 r} \delta\left(t - t_o - \frac{r}{\alpha}\right) \quad .$$

Using the formal expression for G^{RP}_{in} on the left hand side of the equation, we get

$$\frac{C_P}{\alpha r} \delta\left(t - t_o - \frac{r}{\alpha}\right) = \frac{1}{4\pi\rho\alpha^2 r} \delta\left(t - t_o - \frac{r}{\alpha}\right) \quad ,$$

which yields for the factor of proportionality

$$C_P = \frac{1}{4\pi\rho\alpha} \quad .$$

In a very similar way, we can find that

$$G^{RS}_{in,ij} \sim C_S \frac{\delta_{jn} - N_j N_n}{\beta r} \frac{N_i N_j}{\beta^2} \ddot{\delta} = 0 \quad ,$$

$$G^{RS}_{in,jj}(\delta_{in} - N_i N_n) \sim [G^{RS}_{in}(\delta_{in} - N_i N_n)]_{,jj} \quad .$$

If we multiply the inhomogeneous elastic equation by $(\delta_{in} - N_i N_n)$, we get

$$(\lambda+\mu)G_{jn,ij}(\delta_{in}-N_iN_n)+\mu G_{in,jj}(\delta_{in}-N_iN_n)-\rho G_{in,tt}(\delta_{in}-N_iN_n) = -2\delta(x_m-x_{om})\delta(t-t_o).$$

If we again insert the previous asymptotic results and $\mu = \rho\beta^2$, we get

$$\rho\beta^2[G^{RS}_{in}(\delta_{in} - N_i N_n)]_{,jj} - \rho[G^{RS}_{in}(\delta_{in} - N_i N_n)]_{,tt} = -2\delta(x_m - x_{om})\delta(t - t_o) \, .$$

This is again the inhomogeneous acoustic equation, whose solution yields for C_S

$$C_S = \frac{1}{4\pi\rho\beta} \quad .$$

The ray Green function for a *homogeneous isotropic* medium has thus the following form

$$G^R_{in} = \frac{N_i N_n}{4\pi\rho\alpha^2 r}\delta\left(t - t_o - \frac{r}{\alpha}\right) + \frac{\delta_{in} - N_i N_n}{4\pi\rho\beta^2 r}\delta\left(t - t_o - \frac{r}{\beta}\right) \, .$$

We can see that the above expression corresponds exactly to the far-field part of the complete Green function derived in Sec.5.2.

We can now use the above determined proportionality factors C_P, C_S for writing down the final expression for the ray Green function for *inhomogeneous isotropic* media

$$G^R_{in} = \frac{1}{4\pi}\frac{N_i(x_m)N_n(x_{om})e^{-i\frac{\pi}{2}k_p(x_{om},x_m)}\delta^{(A)}(t - t_o - \tau_P(x_m))}{[\rho(x_{om})\alpha(x_{om})\rho(x_m)\alpha(x_m)|\Omega^P_M(x_m)|]^{1/2}}$$

$$+\frac{1}{4\pi}\frac{[\delta_{in} - N_i(x_m)N_n(x_{om})]e^{-i\frac{\pi}{2}k_s(x_{om},x_m)}\delta^{(A)}(t - t_o - \tau_S(x_m))}{[\rho(x_{om})\beta(x_{om})\rho(x_m)\beta(x_m)|\Omega^S_M(x_m)|]^{1/2}} \quad .$$

As we have mentioned in Sec.7.6.2.2, both quantities Ω^P_M and Ω^S_M are reciprocal and thus also the above *ray Green function for inhomogeneous isotropic media is reciprocal.*

7.7.2 Ray Green function for an anisotropic medium

The determination of the ray Green function for an inhomogeneous anisotropic medium can be performed in a way quite similar to that used in the previous section. Without derivation, we show the expression for the radiation pattern of an arbitrary of the three waves, which can propagate in anisotropic media,

$$g(x_{om}) = \pm \frac{g_m(x_{om})}{4\pi\rho(x_{om})c(x_{om})} \quad .$$

The corresponding ray Green function for inhomogeneous anisotropic media thus reads

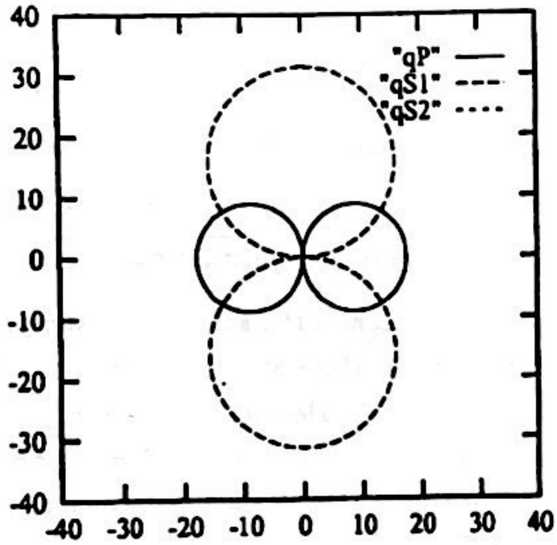
$$G^R_{in} = \pm \frac{g_i(x_m)g_n(x_{om})e^{-i\frac{\pi}{2}k(x_{om},x_m)-i\delta_K}\delta^{(A)}(t-t_o-\tau(x_m))}{4\pi[\rho(x_{om})\rho(x_m)c(x_{om})c(x_m)|\Omega_M(x_m)|]^{1/2}}$$

The selection of the proper sign and the selection of the proper value of δ_K is controlled by the properties of the slowness surface at the source. For the directions, in which the larger of the principal curvatures of the slowness surface is negative (the corresponding section of the surface is convex outward), the sign "+" must be considered. If the larger of the principal curvature is positive, the sign "−" must be considered. As to δ_K, it is zero in the directions, in which the Gaussian curvature of the slowness surface is positive and $\delta_K = \pi/2$ in the directions, in which the Gaussian curvature is negative. The quantity $\Omega_M(x_m)$ is again the reciprocal geometric spreading corresponding to the modified initial conditions, see Sec.7.6.2.2. Thus the above *ray Green function for inhomogeneous anisotropic media is reciprocal.*

As an illustration, we show here examples of behavior of several of the above discussed quantities for a homogeneous anisotropic medium. We consider again the anisotropic model introduced in Sec.3.6.1, i.e. the model of transversely isotropic medium with vertical axis of symmetry. In this model, we study behavior of the radiation pattern, of the modified geometrical spreading on a unit sphere (radius $10^3 m$) around the source and of the *complete radiation pattern.* Under the complete radiation pattern, we understand the combination of the radiation pattern defined above with the geometrical spreading. The complete radiation pattern thus represents a radiation which could be observed on a unit sphere surrounding a source. The radiation pattern is displayed in $[m^3/s]$, the modified spreading in $[10^{12} m^4/s^2]$ and the complete radiation pattern in $[10^{-6} m]$. The considered forces are $10^6 N$, the moment is $10^6 Nm$.

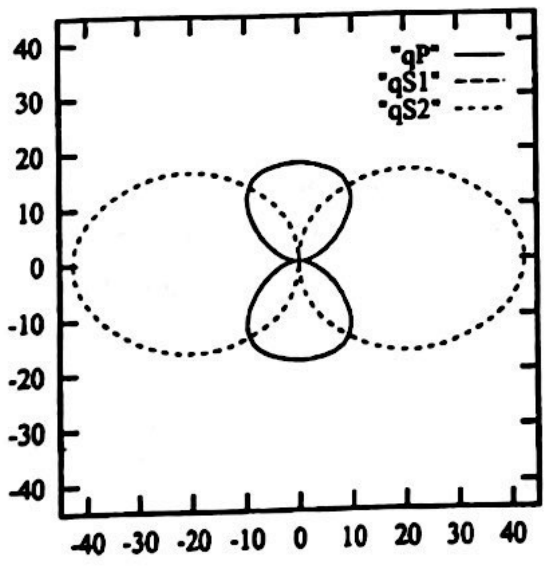
The first picture shows a horizontal section of the radiation pattern of the horizontal force. Because the horizontal plane is the plane of isotropy in the studied model, the radiation pattern has exactly the same form as the radiation pattern of the horizontal force in an isotropic medium. The presented polar diagram is just another form of the diagrams of the elastodynamic far-field Green function shown in Sec.5.2. The qS_1 wave is polarized in the horizontal plane, polarization of the qS_2 wave is perpendicular to the horizontal plane and, therefore, this wave is not generated by the horizontal force.

HORIZONTAL FORCE RAD. PAT., XY PLANE

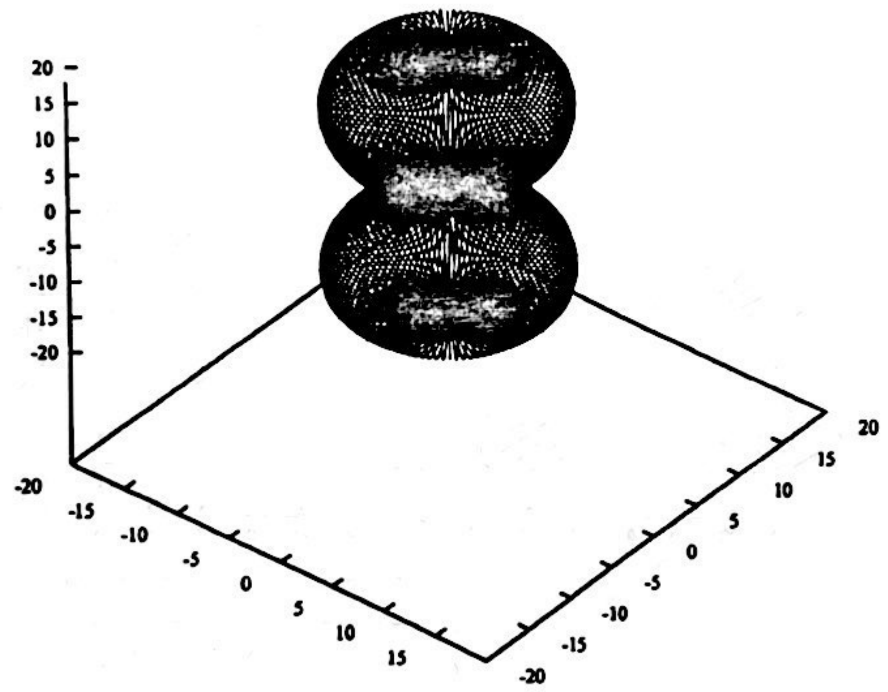


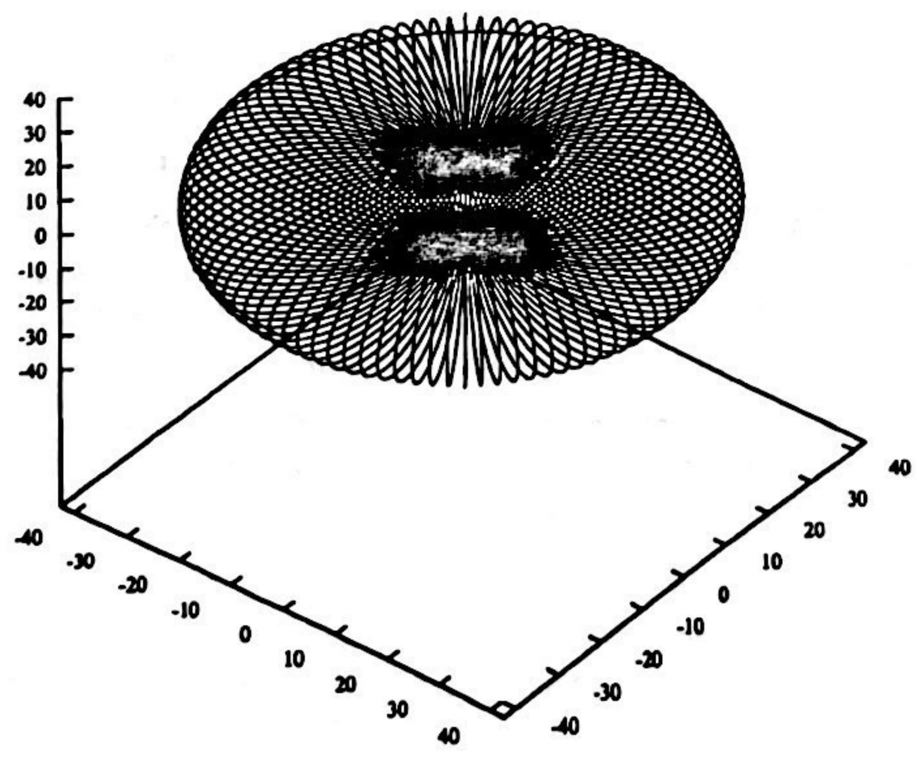
In the next picture, a vertical section of the radiation pattern of the vertical force (the force is now parallel to the axis of symmetry of the medium) is shown. The deformation of the pattern due to anisotropy is obvious. The qS_1 wave is now polarized horizontaly and it is, therefore, not generated. The polarization of the qS_2 wave is in the plane of the picture.

VERTICAL FORCE RAD. PAT., XZ PLANE

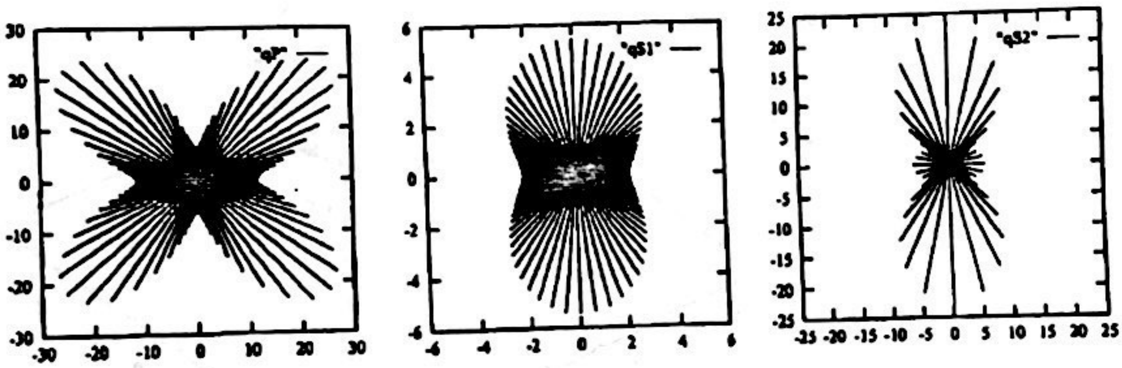


Next two figures show the 3-D radiation patterns of the vertical force for the qP and qS_2 waves. The vertical sections on the previous figure were obtained from these 3-D diagrams.

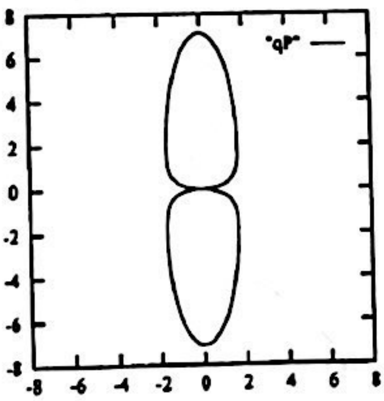

20
15
10
5
0
-5
-10
-15
-20
-20
-15
-10
-5
0
5
10
15
20
15
10
5
0
-5
-10
-15
-20


40
30
20
10
0
-10
-20
-30
-40
-40
-30
-20
-10
0
10
20
30
40
40
30
20
10
0
-10
-20
-30
-40

The three figures below show vertical sections of the geometrical spreading displayed together with the rays shot with equally spaced initial angles of the phase normal. In an isotropic medium, the figures would show circles with equally spaced radial lines. Due to anisotropy this pattern is deformed into the displayed forms. The spreading is higher where the density of rays is higher and vice versa. Notice concentrations of rays in different



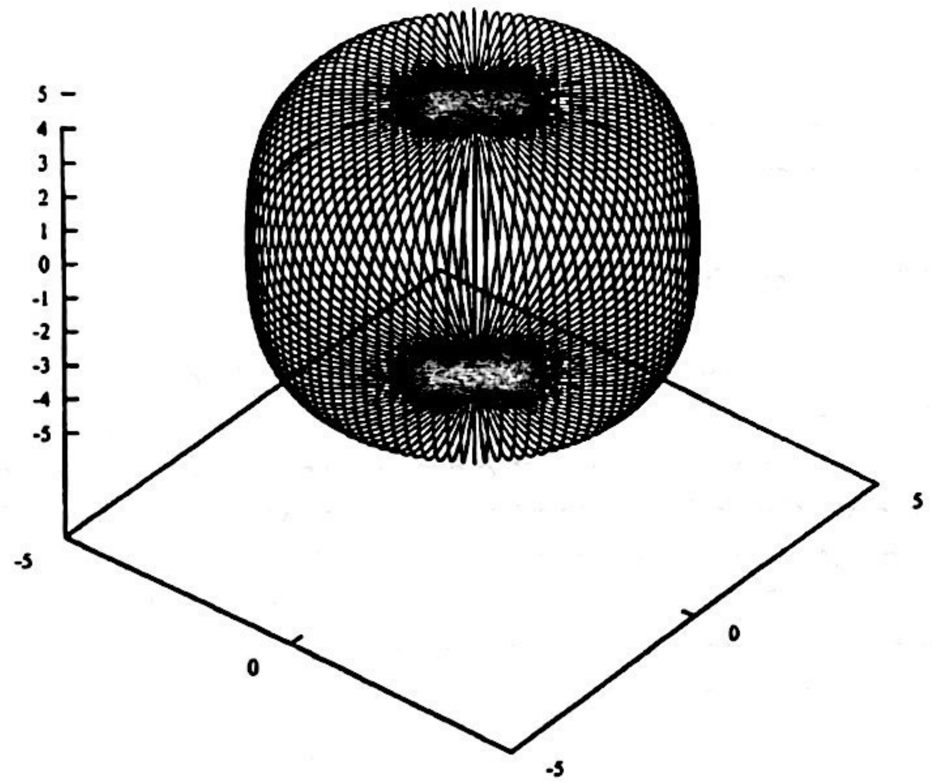
FOCAL SPHERE RAD. PAT.- VERT. FORCE, XZ PLANE

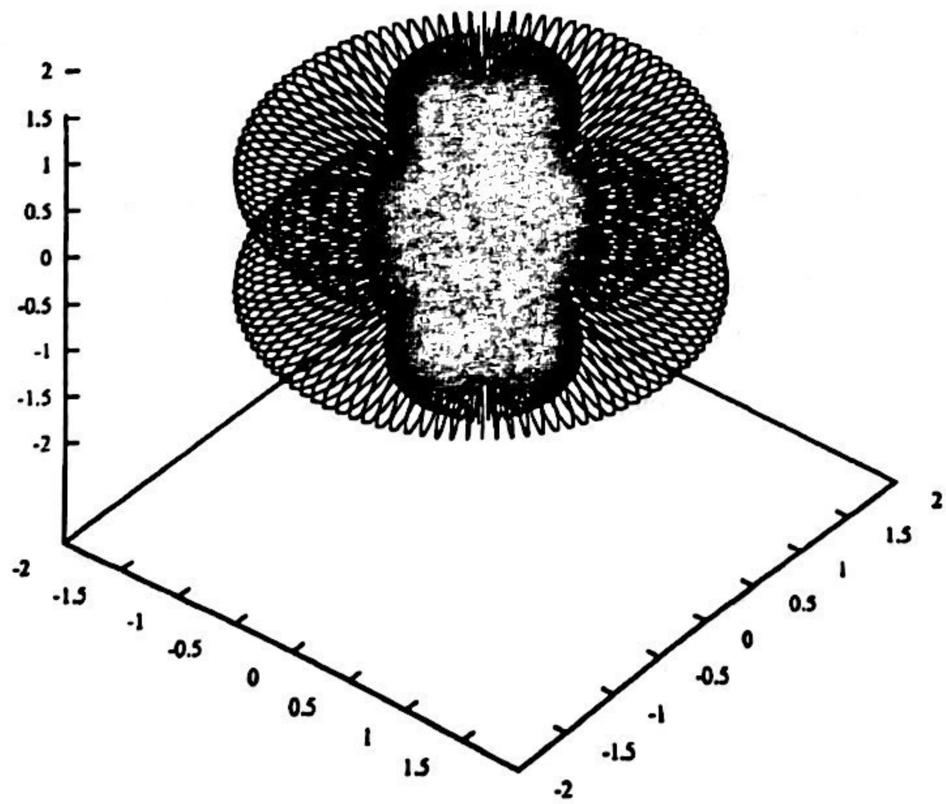


directions for different types of waves. Let us emphasize that in contrast to the radiation pattern, which is a characteristics of a source, the geometrical spreading is indepenedent of the considered type of the source and its orientation. It depends only on the parameters of the medium and the type of the considered wave.

The figure above shows vertical section of the complete radiation pattern of the qP wave due to the vertical force. It shows combined effects of the radiation pattern and the geometrical spreading shown before.

As was mentioned before, the Green function can be used in many seismological applications. One of them is the calculation of the radiation patterns of various types of point sources. For illustration, the following two figures are shown here. The figure below shows 3-D display of the radiation pattern of the qP wave generated by an explosive point source situated in the above described anisotropic structure.


5
4
3
2
1
0
-1
-2
-3
-4
-5
-5
0
-5
0
5


2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

In an isotropic medium, the figure would show a sphere. Anisotropy is responsible for its slight deformation. Anisotropy is also responsible for more striking feature shown in the last figure. It shows qS_2 wave radiation pattern. Let us note that an explosive source in an isotropic medium does not generate shear waves at all.

7.8 Ray synthetic seismograms

The results of the preceding sections can be used to construct *ray synthetic seismograms* for general 3-D inhomogeneous isotropic and/or anisotropic layered media. Under the ray synthetic seismogram at a point R due to the source at a point S, we understand any component of the displacement vector $u_i(x_{Rm}, t)$ as a function of time. The displacement vector $u_i(x_{Rm}, t)$ is a sum of contributions $u_i^{(j)}(x_{Rm}, t)$ of individual *elementary waves* propagating from S to R. The elementary wave can be P or S direct wave or any multiply reflected, possibly converted wave whose ray connects the points R and S. Each elementary wave has its *wave code*, see Sec.7.5.3.

To construct a ray synthetic seismogram, we must thus proceed in the following way:

1. to specify the wave codes of all elementary waves we are interested in;

2. to compute rays connecting the points S and R for the selected wave codes (this will require application of two-point ray tracing procedure). Along the computed rays, travel times τ and generally complex-valued vectorial amplitudes A_i must be determined (the amplitudes include effects of the geometrical spreading in smooth parts of the medium as well as the effects of reflection/transmission at internal interfaces or free surface);

3. for each (j-th) elementary wave to evaluate complex-valued ray elementary seismogram
$$u_i^{(j)}(x_{Rm}, t) = A_i^{(j)} F^{(j)}(t - \tau^{(j)}(x_{Rm})) \quad ;$$

4. to determine the ray synthetic seismogram as a real part of the sum of the complex-valued ray elementary seismograms
$$u_i(x_{Rm}, t) = Re\left[\sum_{(j)} A_i^{(j)} F^{(j)}(t - \tau^{(j)}(x_{Rm}))\right] \quad .$$

Instead of the above described approach, which is performed in the time domain, a frequency-domain approach or an approach based on convolution could be used.

LIST OF REFERENCES

Achenbach, J.D., 1975. Wave propagation in elastic solids, North-Holland Publ. Co., Amsterdam.

Auld, B.A., 1990. Acoustic fields and waves in solids, J.Wiley, New York.

Aki, K. & Richards, P.G., 1980. Quantitative seismology, W.H. Freeman, S. Francisco.

Babič, V.M. & Buldyrev, V.S., 1991. Short-wavelength diffraction theory (Asymptotic methods), Springer, Berlin.

Bleinstein, N., 1984, Mathematical methods for wave phenomena, Academic Press, New York.

Brekhovskih, L.M., 1960. Waves in layered media, Academic Press, New York.

Červený, V., 1989. Lecture notes, PPPG/UFBa, Salvador.

Červený, V., 1994. Seismic ray theory, in preparation.

Červený, V., Molotkov, I.A. & Pšenčík, I., 1977. Ray method in seismology, Universita Karlova, Praha.

Červený, V. & Ravindra, R., 1971. Theory of seismic head waves, University of Toronto Press, Toronto.

Fedorov, F.I., 1968. Theory of elastic waves in crystals, Plenum Press, New York.

Fokkema, J.T., 1988. Seismic methods, lecture notes, PPPG/UFBa, Salvador.

Helbig, K., 1994. Foundations of anisotropy for exploration seismics. Handbook of geophysical exploration, Section I. Seismic exploration, Vol.22, Pergamon.

Kennett, B.L.N., 1983. Seismic wave propagation in stratified media, Cambridge Univ. Press, Cambridge.

Kravtsov, Yu. A. & Orlov Yu. I., 1990. Geometrical optics of inhomogeneous media, Springer Verlag, Berlin.

Pillant, W.L., 1979. Elastic waves in the Earth, Elsevier, Amsterdam.

Tygel, M. & Hubral, P., 1987. Transient waves in layered media, Elsevier, Amsterdam.

Weglein, A.B. & Stolt, R.H., 1992. The wave physics of wavelet estimation, downward continuation and surface and volume scattering, Lecture notes, PPPG/UFBa, Salvador.