Exploring the effects of spatial structure in EGT-games

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Abstract

The evolutionary game theory is an important tool to investigate biological evolution. The two most classic games to simulate the behavior of animals are the Hawk-Dove game and the Prisoner‘s Dilemma. In the games the players contest over a sharable resource. The both models have different payoffs, for which reason it is important to know how a simulation reacts for diverse settings. That is required so simulate biological processes proper. We mainly explored the effect of spatial structure with different cost-to-benefit ratios. Furthermore we introduced variable neighborhood-sizes and a mutation-rate. The simulations were programmed with NetLogo and plotted in R. In the program the probabilities to be a cooperator or a defector are calculated according to the payoff matrix of the games. We compared the simulations using the frequency of the cooperators and the cost-to-benefit ratio.

1 Introduction

1.1 Evolutionary Game Theory

In 1944 the economic scientists Neumann and Morgenstern (Von Neumann and Morgenstern 1944) developed the theory of games as an economic model. Much later, in the 1970s, the evolutionary game theory originated. Instead of playing the game once, like in the game theory, the game in evolutionary game theory is played over and over again by biologically or socially conditioned players. In most cases a player is randomly drawn from a large population and has a specific behavior (Weibull 1997) Our study compares these non-spatial games in well-mixed populations with spatial games and varying variables.

Two of the most popular and most intensively studied games are the “Prisoner’s Dilemma” (Axelrod and Hamilton 1981) (abbreviated “PD”) and the “Hawk-Dove Game” (Sugden 1986) (“HD”). In general the Prisoner’s Dilemma and the Hawk-Dove game have the following notation for a two player game with the two strategies “cooperation” or “defection”. R, T, S and P are the payoffs of the game.

	C	D
C	R	S
C	T	P

(R: Reward; T: Temptation; S: Sucker’s reward; P: Punishment; C: Cooperator; D: Defector)

1.2 The Prisoner‘s Dilemma:

	C	D
C	b-c	-c
D	b	o

(b: benefit, c: cost)

In a PD defection is the evolutionarily stable strategy. The defector gets the benefit b when he plays with a cooperator, who gets a punishment: The cost c. The relationship between payoffs is $T > R > P > S$. When cooperation is mutual, both have the benefit $R = b - c$, but pay a cost for that. Mutual defection results in Payoff $P = 0$ for both players. Because of the punishment of the cooperator, when the other player defects, it is best to defect, regardless of the co-players’ decision (Hauert and Doebeli 2004). Though, in nature there are examples which prove, that not only defection can be evolutionarily stable. Alarm calls warn other animals from predators. It seems to be that cooperation only works when the cost-to-benefit ratio is not too high and spatial structure exists (Clutton-Brock et al. 1999).

1.3 The Hawk-Dove game:

	C	D
C	b-c/2	b-c
D	b	o

Field and experimental studies had problems with the PD as the only model to discuss behavior of ”players”. Its difficult to estimate proper fitness payoffs. That caused various problems between theory and field - and experimental studies. Scientists needed another model for the payoffs of cooperative behavior (Milinski et al. 1997; Nowak and May 1992). In the HD mutual cooperators are better off, because they share the cost of cooperation and receiving the whole benefit. Although cooperation gets rewarded, not punished by playing with a defector. The payoffs P and S have a reverse order in the HD which differs from the PD. P becomes the value $b - c$, so P and S have the reverse order. The new payoff matrix ($T > R > S > P$) leads to persistence of cooperative players beside defectors except for very high costs ($2 * b > c > b > o$) which would recover the PD. When $b > c > 0$ the best action in the Hawk-Dove game depends on the co-player. In case that players always play the opposite strategy of their game partner, you have stable coexistence of cooperators and defectors in well-mixed populations - thus, an equillibrium of cooperators and defectors (Hauert and Doebeli 2004).

1.4 Problem and Gap of Knowledge

A lot of researchers of social, economic and biological science worked on the evolutionary game theory. It has become a powerful tool to investigate the emergence of cooperation in groups (Hauert and Doebeli 2004). For the PD it is widely accepted, that spatial structure supports cooperation (Nowak and May 1992). To simulate biological processes there is an increasing discomfort with the PD being the only model to discuss cooperative behavior. The HD is an interesting alternative for describing behavior patterns of field studies(Milinski 1987). The processes were discussed theoretically for several times (Nowak and May 1992; Milinski 1987), but have never been simulated; maybe due to a lack of computing power. To find out the different performance of the PD and the HD, we compare the results of various simulations. Choosing the right model to describe natural processes is a big challenge for scientists. For describing them with the right model, there has to be done further investigation on the Hawk-Dove game, especially to simulate mutations of the behavior, here called Mixed-strategies.

1.5 Our approach and specific questions

In our study we test the following **hypotheses**:

- Spatial structure benefits cooperators especially in the PD.
- Neighborhood-size has an influence to the effect of spatial structure.
- Mixed-strategies change the effect of spatial structure.

For testing the Hypotheses we simulated different HD and PD games with NetLogo 5.1.0 (Wilensky 1999) and performed them with varying variables. The results of the experiments were fitted and visualized in R (R Core Team 2014) to make them comparable. The focus of the Experiments Data was set on the frequency of cooperators with different cost to benefit ratios.

Our experiments examined following questions:

- Which influence has spatial structure in the HD and PD game?
- Which effect have different neighborhood sizes to the games?
- Which influence has a mixed strategy of the players in a spatial and a non-spatial HD?

2 Methods

2.1 Non-spatial PD and HD

The first step for our model was to calculate the average pay-off for cooperators and defectors in the non-spatial PD and HD. We used p for the probability of players being defectors (probability of cooperators is $1 - p$). Due to the payoff matrix of the HD game we used $P_c = 0.5 * (1 - p) * c + b - c$ and $P_d = p * b$. For the PD average pay-off for cooperators changes: $P_c = (1 - p) * b - c$. These average Payoffs reduce or increase the fitness of the single players. In an iterated game fitter players reproduce and get a higher percentage of the population - in our case cooperators or defectors. For the next iteration (reproduction) of the game p is now calculated by comparing the fitness of defectors with the fitness of all players.

2.2 Spatial structure and neighborhood size

In our spatial games we have a 50 X 50 square lattice. Every square represents one player. The whole lattice is updated synchronically. We introduced different neighborhood-sizes from four to 24 neighbors in our model. Therefore we worked with five different sizes of radius (1, $\sqrt[2]{2}$, 2, $2 * \sqrt[2]{2}$, 3, caliber of players as unit) around the players for five neighborhood-sizes (4, 8, 12, 20, 24). Instead of using p, which was the probability to be a defector in the non-spatial game, we inserted a local probability pl in the average payoff terms. These were calculated with the neighbors' probabilities to be a cooperator or defector. Herewith fitness of the players is calculated again facing the neighborhood. As opposed to the non-spatial game the change of the strategy in the next round of the game is not calculated for the whole population, it is calculated for the single player (in our simulation patch). The players randomly choose neighbors for the competition. Then the transition probability (probability to change strategy) is calculated with $p_c = Z/\alpha$. Z is the difference between the fitness of the competitor F_c and the own fitness F. alpha is the maximum difference between the payoffs, which is $\alpha = T - P = b$ in the HD game and $\alpha = T - S = b + c$ in the PD. This correction term ensures p_c values between 0 and 1. Is $Z > 0$ the player changes the strategy with the probability p_c, which illustrates a reproduction of the fitter players.

2.3 Effect of mixed strategies in the spatial and nonspatial Hawk-Dove game

In our simulation for the HD with mixed strategies every player is characterized by the probability to play p to show dove-like behavior with an introduced mutation rate to do further exploration of the game. The initial heterogeneity of the players is randomly chosen from a normal distribution. The mean of that is the equilibrium strategy of well-mixed populations $p_w = (1 - c/(2 * b - v))$, calculated from the cost-to-benefit ratio. The standard deviation is set to 0.02, the boundaries of the deviation are 0 and 1 to get a fitting value for the probability. The following procedure is the same as for the models with spatial structure, but with different mathematics to introduce the mutation. The average payoff $P_{mix} = p_w * p_n * (b - (0.5 * c)) + p_w * (1 - p_n) * (b - c) + (1 - p_w) * p_n * b$ is P with p_n as the mean strategy of all interacting neighbors. More generally the term is $P_{mix} = p_w * p_n * R + p_w * (1 - p_n) * S + (1 - p_w) * p_n * T + (1 - p_w) * (1 - p_n) * P$. Now the payoff differences between neighboring individuals is very small. The update rule for pure strategies has a very small probability of change, which makes the simulation very slow. For that reason a non-linear term is used for the change-probability: $p_{cmix} = [1 + exp(-z/k)]^{-1}$.

2.4 Simulations with NetLogo, plots with R

The simulations were programmed agent-based with NetLogo (Wilensky 1999). The modeling of the non-spatial and the spatial HD and PD with different neighborhood-sizes were ran with the Behavior Space of the program. For having robust results we ran the simulation 10 times for 5000 time-steps with varying costs and benefit set to 1. The costs we calculated according to the replicator dynamics, the equilibrium frequency of cooperators in the HD with $r = c/(2 * b - c)$. Therefore $c = (2 * r/(1 + r))$ with a sequence for r from 0 to 1 with an 0.05 step. The population had the size of 50 x 50 patches, that means 2500 players. The mixed strategy games ran for 10.000 time-steps to make sure that we have the equilibrium level. To compare the models we plotted the results in R (R Core Team 2014) with the frequency of cooperation on the y-axis and the cost-to-benefit ration on the x-axis. The results were compared with frequency of hawk-like behavior in well-mixed populations ($f(wm) = 1 - r$) visualized with a dotted line.

3 Results

??? screenshots of spatial patterns ??? !!! Figure captions !!! !!! Confidence intervals !!! !!! "Code is in the appendix" !!!

3.1 Effect of spatial structure in PD and HD games

In our first simulation experiment, we compared the effect of spatial structure on the persistence of cooperators in the PD and HD games. For the **PD game** we were able to reproduce the theoretical prediction that spatial structure enables cooperators to persist, even if cooperation is not an evolutionarily stable strategy in well-mixed populations. Our simulated spatial PD population with neighborhood size = 8 could maintain an average of 66.5% cooperators (±1.12%) at a cost-benefit ratio of $r = 0.05$. For higher cost-benefit ratios, however, cooperation was not evolutionarily stable at this neighborhood size and ceased within the 5000 time steps. If cooperation did was cost-free, the proportion of cooperators remained close to its initial value. See figure 1 for a comparison of the frequency of cooperation in spatial and nonspatial PD games.

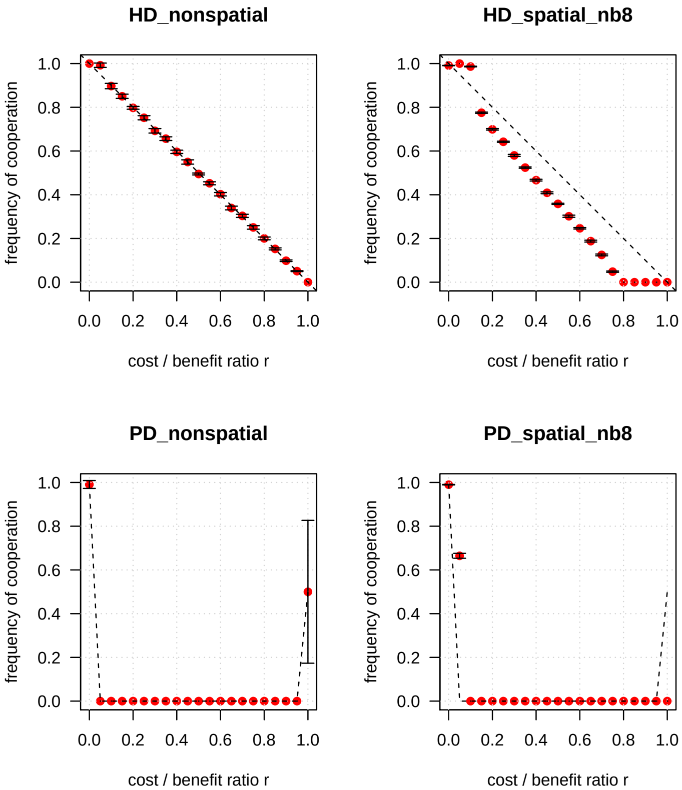

Figure 1: Comparison of HD and PD game simulations, both with and without spatial structure. [**t = 5000, i = 10**]

In the spatial **HD game** we found a quite contrary effect of spatial structure. When simulating a well-mixed population we could reproduce the theoretical prediction that the equilibrium frequency of cooperators is $1-r$. However, adding spatial structure to the model reduced the frequency of cooperators as compared to the non-spatial version for most r values (figure 1). The higher the r value, the bigger the relative disadvantage of cooperators was. For $r \geq 0.8$ cooperation completely vanished from the population. Only at very low cost-benefit ratios, cooperators in the HD game profited from spatial structure. The threshold r value up to which cooperators profited was 0.1 for a neighborhood size of 8.

3.2 Effect of neighborhood size

In the second simulation experiment we varied neighborhood size in the spatial PD and HD games and investigated whether changes in neighborhood size would change the effects of spatiality. Except for neighborhood size, the HD simulations were run with the exact same parameter set as before. In the first PD game simulation spatial structure only came into effect at very small r-values. To achieve a reasonably high x-axis resolution in the most relevant section, we lowered the r-stepwidth from 0.05 to 0.01 for r-values < 0.1 in the second PD experiment.

Spatial Hawk-Dove games

Varying neighborhood size in the HD game yielded three main observations:
In the spatial HD game with eight neighbors, cooperators profited from spatial structure at low cost-benefit ratios and suffered at higher r-values. This also goes for different neighborhood sizes. However, the threshold r-value at which benefit turns into disadvantage varies with neighborhood size: It is higher for small neighborhood sizes and decreases with increasing neighborhood size.
In addition to the increased benefit from spatial structure at small r-values, smaller neighborhood sizes (4 neighbors) had another effect in the second HD experiment: The detriment of spatial structure in comparison to well-mixed populations increased faster for r-values above the threshold r-value. In contrast, bigger neighborhoods (> 8 neighbors) were less detrimental to cooperators at high r-values.
At the upper end of the range of r-values, another effect of neighborhood size became apparent: The r-value above which total extinction of cooperators becomes likely changed with neighborhood size. The smaller the neighborhood, the more likely cooperators are to die out completely. With N being the average number of neighbors, cooperators were likely to die out completely if $1/N > 1 - r$. The effects of varying neighborhood sizes in the HD game are illustrated in figure 2.

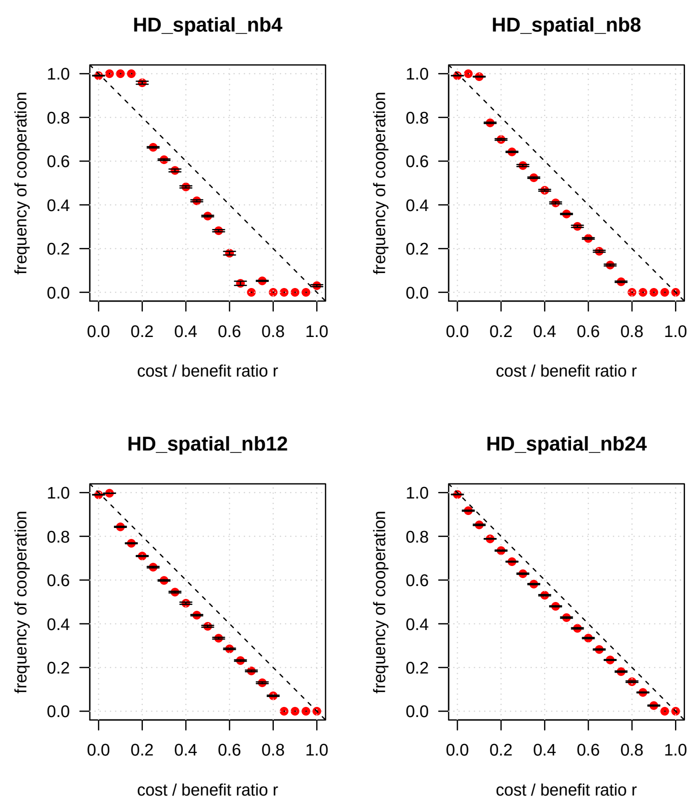

Figure 2: Effect of varying neighborhood size in the HD game. [**t = 5000, i = 10**]

Spatial Prisoner's Dilemma games

For all simulated neighborhood sizes, we could reproduce our finding from the first experiment: Spatial structure allows for the evolution of cooperation in PD games. Nonetheless, our second experiment revealed some constraints to this finding: Firstly, cooperation is only evolutionarily stable for r-values ≤ 0.09, for bigger r-values it disappears from the population. Secondly, neighborhood size has a big influence on a population's ability to maintain cooperation.
In very small neighborhoods (4 neighbors) , cooperators are generally much less likely to persist and therefore make up for smaller ratios of the population than in bigger neighborhoods (see figure 3). The extinction threshold for cooperators is at $r = 0.04$ and thereby significantly lower than in bigger neighborhoods.

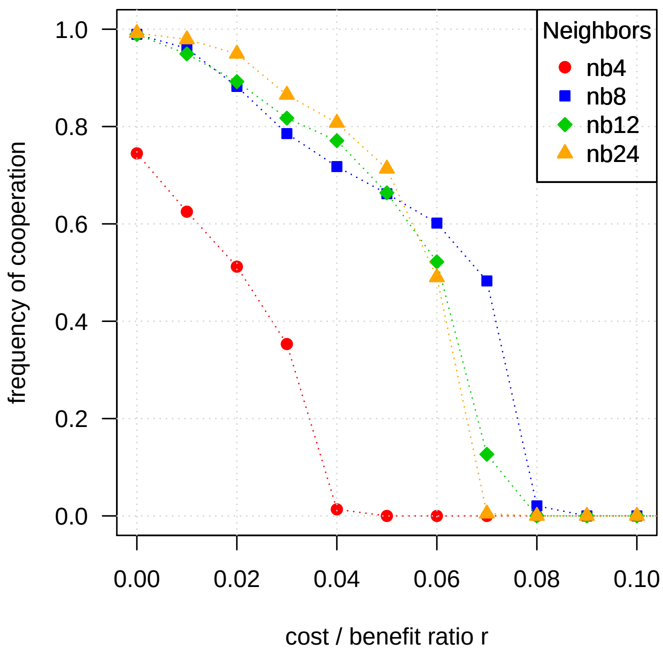

Figure 3: Spatial PD game simulations with different neighborhood sizes. [$\mathbf{t = 5000, i = 10}$]

For neighborhood sizes of 8 or more neighbors, however, the results are ambiguous: Up to a threshold of $r = 0.05$, bigger neighborhoods result in slightly higher numbers of cooperators. At r-values between 0.05 and 0.08 however, the order inverts so that populations with neighborhood size 8 can maintain more cooperation than those with neighborhood size 12 or 24.
Figure 4 better illustrates this anomality: We ran the spatial PD game for two fixed r-values below (0.03) and above (0.065) the tipping point and at the same time varied neighborhood size. When looking at the results, it becomes apparent that at r-values close to the extinction threshold, the population with neighborhood size = 8 constitutes an optimum for cooperators whereas populations with bigger neighborhood size can not maintain the high ratio of cooperators they supported at lower r-values.

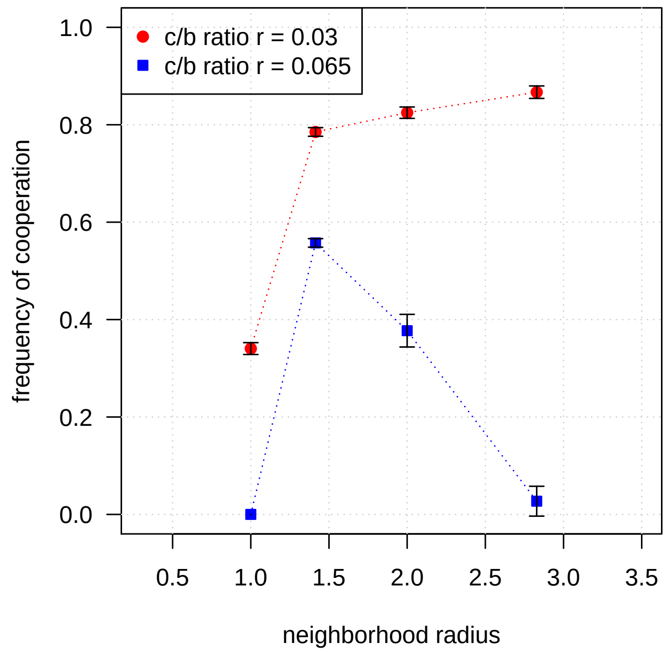

Figure 4: Spatial PD game simulations with fixed cost-benefit-ratio and different neighborhood sizes. Radius 1 is adequate to 4 neighbors, radius 1.4 = 8 neighbors, radius 2 = 12 neighbors and radius 2.8 = 24 neighbors. [$\mathbf{t = 10000, i = 10}$]

3.3 Effect of mixed strategies

In our third experiment, we implemented mixed strategies in the HD game in order to compare the effect of spatial structure in the mixed-strategy HD game with that in the pure-strategy game. In the pure-strategy HD game, spatial structure lowers the ration of cooperators, except for very small r-values where it poses a benefit to them. In mixed-strategy HD games however, this ambivalence does not exist. For any r-values the probability to play cooperator is reduced in comparison to the non-spatial version of the game. On the other hand, the detriment of spatial structure on cooperators is not as severe as in the pure-strategy HD game, especially at high r-values (figure 5).
The general effect of varying neighborhood size is similar to the pure-strategy HD game: Bigger neighborhoods reduce the detriment of spatial structure to cooperators. As the frequency of cooperators in the mixed-strategy HD game is very close to their $1 - r$ frequency in well-mixed populations anyway, the change induced by varying neighborhood size becomes negligibly small.

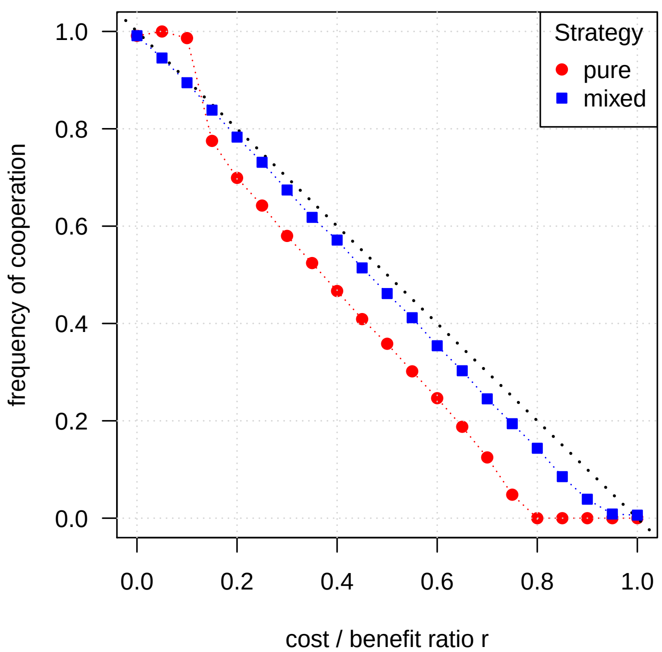

Figure 5: Spatial HD game simulations with neighborhood size = 8 and different strategies. The dotted black line depicts the frequency of cooperation in non-spatial HD games which equals $1 - r$ for both pure- and mixed-strategy games. [**t = 10000, i = 10**]

4 Discussion and Conclusions

4.1 Main findings

Spatial structure

If the **Prisoner's Dilemma** is played in well-mixed populations, cooperation is not an evolutionarily stable strategy. However, previous research has shown that associative interactions, such as spatial structure can allow for the evolution of cooperation (Nowak and May 1992; Doebeli and Knowlton 1998; Killingback et al. 1999). In our simulations, we could reproduce this finding: In spatial PD games, cooperators form spatial clusters that reduce the exploitation by defectors. However, cooperation could only be maintained at very low cost-benefit ratios ($r < 0.1$). At higher r-values, defectors "eat up" the clusters from their borders so that the cooperators vanish from the population. Our finding matches well with the results of Ohtsuki et al. (2006) who investigated different ratios of $benefit/cost$ found a threshold value: If b/c is bigger than the average number of neighbors, cooperation can evolve.

In the **Hawk-Dove Game**, the effect of spatial structure is more ambiguous. Due to the payoff structure of the HD game, it is most beneficial to use different strategies than neighboring cells. For this reason, even well-mixed populations maintain a ratio of $1 - r$ cooperators. The same mechanism, however, inhibits the emergence of larger clusters. Instead, clusters in the shape of crosses or filaments are formed. Especially at higher r-values where the natural proportion of defectors is high, cooperators are much more prone to exploitation in the contact zones where they encounter adjacent defectors. As a consequence, the ratio of cooperators is lower than in well-mixed population for most (higher) r-values. Unlike in the well-mixed population, cooperation can vanish completely at high r-values.
At very low r-values, we found that cooperators profit from spatial structure. A possible explanation could be that even if the co-player defects, in the HD game the benefit of cooperation can still outweigh the cost. Furthermore, cooperators profit from cooperating neighbors, which are much more frequent at low r-values (Hauert and Doebeli 2004).

Different neighborhood sizes

Our simulations on the effect of varying neighborhood sizes again produced very different results for PD and HD games. In the **HD game**, both benefits and disadvantages through spatial structure were most pronounced in the small neighborhood (4 neighbors). The bigger the neighborhood, the curves leveled out and converged towards the linear $1 - r$ relationship from the non-spatial HD game. We think that this effect is caused by the fact that in the HD game, spatial structure only works over very small distances because there are no larger clusters of the same strategy. When the co-player for the next round is drawn, in bigger neighborhoods the ratios of the different strategies are closer to the population average. Besides the fact that spatial structure has a larger effect in small neighborhoods, we found that the extinction threshold for cooperators also varies with neighborhood size. This confirms the findings of Hauert and Doebeli (2004).
Regarding the **PD game**, our results leave more room for interpretation.

Ohtsuki et al. (2006)

(Wang et al. 2012)

very small neighborhoods (4) are not sufficiently as capable to reduce exploitation from defectors as bigger ones turning point / other processes

Mixed strategies

compare the effect of spatial structure in the mixed-strategy HD game with that in the pure-strategy game

"What does cooperation mean in die Hawk-Dove game?"

Unlike in the PD game where a cooperator can only benefit if the co-player also cooperates, in the HD game the benefit of cooperation can still outweigh the cost.

–¿ two variables influence whether a system is stable: neighborhood size and cost-benefit ratio

(Ohtsuki et al 2006: "natural selection favours cooperation, if the benefit of the altruistic act, b, divided by the cost, c, exceeds the average number of neighbours, k, which means b/c ¿ k. In this case, cooperation can evolve as a consequence of social viscosity even in the absence of reputation effects or strategic complexity.")

4.2 Limitations

While gender and age do not have a significant influence on attitude, answers differ significantly among places where the interviews were conducted. We therefore assume that the selection of interview sites is relevant for the outcome. Due to the limited time and number of researchers in our study, our results can not be fully generalized. For reproducing our study on a large scale one should aim to select the interviewees as representative as possible.
Besides the selection of interview sites, the scheme of interviews might constitute a certain limitation to our study. As the interviews were conducted orally, the responses might be influenced by interactions between interviewer and interviewees. For example, interviewees might attribute a pro-refugee opinion to the interviewer and therefore not be honest when having a different opinion themselves. This goes especially for interviewers speaking English. Interviewees might assume that they are talking to migrants and might therefore not want to express a negative attitude towards refugees. Several times it occurred that interviewees were in company of other people who would try to influence the answers by making comments. Although we explicitly asked the interviewees for their personal opinion, the results may have been biased in some cases. To increase objectivity and avoid bias by personal interactions, one could switch to anonymized printed questionnaires in future studies.

4.3 Final conclusions, applications and further research

- **spatial structure does not automatically make a system stable**

- **two variables influence whether a system is stable: neighborhood size and cost-benefit ratio**

In conclusion, we found that people who oppose hosting more refugees tend to overestimate the real numbers whereas people who favor hosting more refugees tend to underestimate the real numbers. However, our data do not support a correlation between precision of estimate and attitude. We cannot tell from our data if this means that subtle perception rather than knowledge influences people's attitude towards refugees. We therefore recommend further investigating this question with a more differentiated questionnaire that incorporates a well-defined proxy for the interviewees knowledge on refugees. While gender and age did not significantly influence attitude in our study, answers differed significantly among places where the interviews were conducted. When reproducing this study, places for the interviews should be selected so that they represent an average of the population. Furthermore, we recommend using printed questionnaires to ensure objective and honest answers.

It is widely assumed that spatial structure allows for the evolution of cooperation in PD games. However, this goes only for a small range of cost-benefit ratios. Cooperation is only evolutionarily stable for r-values $\leq$ 0.09, for bigger r-values it disappears from the population.

In spatial HD games: small neighborhoods = bigger profit when r is small, and small neighborhoods = bigger disadvantage when r is big.

In spatial PD games:

Found anomality

r 0.03, nb4 -¿ stable at propC 0.35 after 2500 steps -¿ stable r 0.065, nb4 -¿ cooperators die after 900 steps -¿ unstable r 0.03, nb8 -¿ stable at propC 0.77 after 10000 steps -¿ stable r 0.065, nb8 -¿ stable at propC 0.55 after 2500 steps -¿ stable r 0.03, nb12 -¿ stable at propC 0.825 after 7500 steps -¿ stable r 0.065, nb12 -¿ randomly oscillating around propC 0.33 after 10000 steps -¿ half-stable r 0.03, nb24 -¿ stable at propC 0.85 after 4000 steps -¿ stable r 0.065, nb 24 -¿ cooperators die after 7400 steps -¿ unstable

fixed r: at 0.03 and 0.065 varying neighborhood size 5000 steps, 5 repetitions

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Appendix

All Netlogo models and R scripts we created and used for this report can be accessed on the projects' github page at https://github.com/peterantkowiak/EGT_course.

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