Math 342W Lecture 11

$V = [\vec{v}_1 \mid \vec{v}_2] \Rightarrow \text{proj}_V(\vec{a}) = \text{proj}_{\vec{v}_{\cdot 1}}(\vec{a}) + \text{proj}_{\vec{v}_{\cdot 2}}(\vec{a})$

only if $\vec{v}_{\cdot 1} \perp \vec{v}_{\cdot 2}$

Let $V = [\vec{v}_1 \mid \vec{v}_2 \mid \dots \mid \vec{v}_d]$, where $\vec{v}_i \perp \vec{v}_j \ \forall i \neq j$

$$\Rightarrow \text{proj}_V(\vec{a}) = \text{proj}_{\vec{v}_{\cdot 1}}(\vec{a}) + \text{proj}_{\vec{v}_{\cdot 2}}(\vec{a}) + \dots + \text{proj}_{\vec{v}_{\cdot d}}(\vec{a})$$

$$= \frac{\vec{v}_1 \vec{v}_1^T}{\|\vec{v}_1\|^2}\vec{a} + \dots + \frac{\vec{v}_d \vec{v}_d^T}{\|\vec{v}_d\|^2}\vec{a}$$

$$= \left(\text{"} \qquad\qquad \text{"} \right)\vec{a}$$

Let $Q = [\vec{v}_i \mid \dots \mid \vec{v}_j]$, where $\vec{v}_i \perp \vec{v}_j$, $\forall i \neq j$ & $\|\vec{v}_i\| = 1 \ \forall i$

We normalized the vectors to length 1

$\text{proj}_V(\vec{a}) = (\vec{v}_1\vec{v}^T + \dots + \vec{v}_d\vec{v}_d^T)\vec{a} \Leftrightarrow$ orthonormal Matrix

orthogonal Projection $= QQ^T\vec{a} = V(V^TV)^{-1}V^T\vec{a} = H = X(X^TX)^{-1}X^T$

Gram-Schmidt Algorithm

invertible change of basis matrix

$\rightarrow$ Takes X and returns Q, R s.t. $X = QR$

Let $X = [\vec{x}_1 \mid \vec{x}_2 \mid \dots \vec{x}_d]$

1a) Find an orthogonal basis $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_d$ for col $[X]$

$\vec{v}_1 = \vec{x}_1$

$\vec{v}_2 = \vec{x}_2 - \text{proj}_{\vec{v}_2}(\vec{x}_2)$

$\vec{v}_3 = \vec{x}_3 - \text{proj}_{[\vec{v}_1, \vec{v}_2]}(\vec{x}_3)$

....

$$\Rightarrow \vec{V}_n = \vec{x}_n - \sum_{j=1}^{n-1} \text{proj}_{\vec{v}_j}(\vec{x}_n)$$

1b) $$\vec{q}_j = \frac{\vec{v}_j}{\|v_j\|^2}, \forall j$$

$X = Q[\quad] \rightsquigarrow$ Our R matrix $\Rightarrow$ Changing our basis

$$[\vec{x}_1 \mid \dots \mid \vec{x}_d] = [\vec{q}_1 \mid \dots \mid \vec{q}_d] \begin{bmatrix} a & b & & \\ 0 & c & & \\ 0 & 0 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 \end{bmatrix} \rightsquigarrow \text{upper Triangular Matrix}$$

$a = \|\vec{x}_1\|$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots \\ & r_{22} & r_{23} & \cdots \\ & & r_{33} & \cdots \\ & & & \ddots \\ & & & & r_{dd} \end{bmatrix} \Rightarrow \overset{\text{Formula}}{\boxed{r_{i,j} = \vec{q}_i^{\,T} \vec{x}_j}} \rightsquigarrow \text{QR decomposition}$$

$$\vec{b} = (X^TX)^{-1}X^T\vec{y}$$

$\downarrow$

$$X^TX\vec{b} = \vec{y} \Rightarrow (QR)^T(QR)\vec{b} = (QR)^T\vec{y}$$

$$\Rightarrow R^T\overset{I}{\cancel{Q^TQ}}R\vec{b} = R^TQ^T\vec{y}$$

$$\Rightarrow R^TR\vec{b} = R^T\underbrace{Q^T\vec{y}}_{\vec{z}}$$

$$\Rightarrow R^TR\vec{b} = R^T(\vec{z})$$

$$\Rightarrow (R^T)^{-1}R^TR\vec{b} = (R^T)^{-1}R^T(\vec{z})$$

$$\Rightarrow R\vec{b} = \vec{z}$$

More Computationally efficient

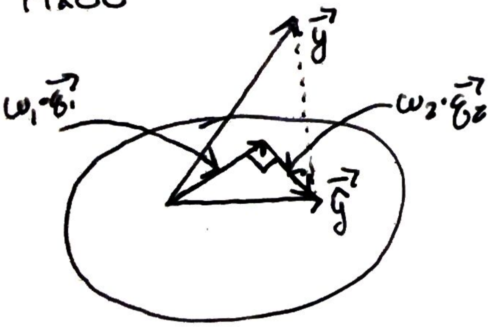
$$SST = SSR + SSE$$

$\downarrow$

$$\sum(y_i - \bar{y})^2 \Rightarrow \text{SSR}\uparrow \Rightarrow \text{SSE}\downarrow \Rightarrow R^2\uparrow \Rightarrow \text{RMSE}\downarrow$$

$\downarrow$

Fixed



$$\hat{\vec{y}} = proj_{\vec{q_0}}(\vec{y}) + \dots + proj_{\vec{q_p}}(\vec{y})$$

$$\vec{q_0} = \frac{\vec{1}_n}{\sqrt{n}}$$

P-Dimensional Pythagorean Theorem

$$||\hat{\vec{y}}||^2 = ||proj_{\vec{q_0}}(\vec{y})||^2 + \dots + ||proj_{\vec{q_p}}(\vec{y})||^2$$