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Squared skewness minus kurtosis bounded by 186/125 for unimodal distributions

Chris A.J. Klaassen *, Philip J. Mokveld, Bert van Es

Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, Netherlands

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Abstract

The sharp inequality for squared skewness minus kurtosis is derived for the class of unimodal distributions. © 2000 Elsevier Science B.V. All rights reserved

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1. Notation, the inequality and its history

Let F be a nondegenerate distribution with finite fourth moment. The coefficient of kurtosis of F is defined as

$$E_F(X - E_F X)^4 \sigma_F^{-4} - 3$$

and denoted by κ_F with σ_F the standard deviation. We will denote the coefficient of skewness

$$E_F(X - E_F X)^3 \sigma_F^{-3}$$

by τ_F.

By Cauchy–Schwarz we have

$$(E_F(X - E_F X)^3)^2 \leqslant E_F(X - E_F X)^4 E_F(X - E_F X)^2 \tag{1}$$

and hence we obtain the inequality

$$\tau_F^2 \leqslant \kappa_F + 3. \tag{2}$$

Pearson (1916) has improved (2) to the sharp inequality

$$\tau_F^2 \leqslant \kappa_F + 2. \tag{3}$$

* Corresponding author.

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Equality holds in (3) iff X is Bernoulli under F in the broad sense; i.e. X takes on two distinct values a.s. For unimodal distributions Pearson's inequality (3) can be improved still further.

A distribution function F is unimodal if it is convex–concave. Such an F has at most one atom, which may occur only at the mode. If F is linear on its support with an atom at one of its boundary points, we will call F a one-sided boundary-inflated uniform distribution. These distributions are extremal in the following sense.

Theorem 1.1. *If F has a nondegenerate unimodal distribution with finite fourth moment, then*

$$\tau_F^2 \leqslant \kappa_F + \tfrac{186}{125} \tag{4}$$

holds with equality iff F is a one-sided boundary-inflated uniform distribution with mass $\frac{1}{2}$ at the atom.

This theorem is our main result. It will be proved in Section 2. In this proof we will use Pearson's inequality (3) itself. Therefore, a proof of (3) will be given in Section 2 too.

For symmetric unimodal distributions inequality (4) can be improved to

$$\kappa_F \geqslant -\tfrac{6}{5} \tag{5}$$

as may be seen from our proof of (4); see Remark 2.1. This inequality (5) has been proved in a somewhat different way in Boon et al. (1999). It has been used there to show that in confidence intervals for the location parameter of symmetric unimodal distributions based on the t-statistic the quantiles should be used for the distribution of the t-statistic under the assumption of uniformity of the underlying distribution.

An obvious generalization of (5) to unimodal, but not necessarily symmetric, distributions would be

$$\tau_F^2 \leqslant \kappa_F + \tfrac{6}{5}, \tag{6}$$

which has been conjectured in Rohatgi and Székely (1989). However, for the one-sided boundary-inflated uniform distribution with mass $\frac{1}{2}$ at the atom this inequality does not hold. Nevertheless, it does hold for all unimodal F for which the mean and mode coincide, as we will show by Remark 2.2 in Section 2.

Finally, we note that for distributions with bounded support a sharp *upper* bound on the kurtosis κ_F has been derived by Teuscher and Guiard (1995) in terms of the endpoints of the support.

2. Proofs and remarks

Straightforward computation shows that the trivial inequality

$$0 \leqslant E_F\{(X - E_F X - \tfrac{1}{2}\sigma_F a)^2 (X - E_F X - \tfrac{1}{2}\sigma_F b)^2\} \tag{7}$$

with

$$a = \sqrt{\kappa_F + 2} - \sqrt{\kappa_F + 6}, \qquad b = \sqrt{\kappa_F + 2} + \sqrt{\kappa_F + 6} \tag{8}$$

is equivalent to

$$0 \leqslant \kappa_F + 2 - \sqrt{\kappa_F + 2}\ \tau_F, \tag{9}$$

which proves Pearson's inequality (3). Essentially, this is the proof of Rohatgi and Székely (1989) and Wilkins (1944), which is related to Watson's proof from Pearson (1916, pp. 432).

To prove Theorem 1.1, we also note that a random variable X with unimodal distribution F may be represented as

$$X = UZ \tag{10}$$

with U uniform on $(0,1)$ and U and Z independent; cf. Theorem V.9, p. 158, of Feller (1971). Here we have assumed, without loss of generality, that the mode is at 0.

Let G denote the distribution of Z and write μ_G, σ_G, τ_G and κ_G for mean, standard deviation, skewness, and kurtosis respectively of G. Tedious computation shows

$$\begin{aligned}\tau_F^2 - \kappa_F = {} & 3 + 108(\tau_G + \mu_G\sigma_G^{-1})^2(4 + \mu_G^2\sigma_G^{-2})^{-3} \\ & -\tfrac{9}{5}(16\kappa_G + 48 + 24\mu_G\sigma_G^{-1}\tau_G + 16\mu_G^2\sigma_G^{-2} + \mu_G^4\sigma_G^{-4})(4 + \mu_G^2\sigma_G^{-2})^{-2}. \qquad (11)\end{aligned}$$

By Pearson's inequality (3) applied to G this yields

$$\begin{aligned}\tau_F^2 - \kappa_F \leqslant {} & 3 + \frac{1}{5}(4 + \mu_G^2\sigma_G^{-2})^{-3}\left\{ -36(1 + 4\mu_G^2\sigma_G^{-2})\left(\tau_G - 3\frac{\mu_G}{\sigma_G}\frac{1 - \mu_G^2\sigma_G^{-2}}{1 + 4\mu_G^2\sigma_G^{-2}}\right)^2 \right. \\ & \left. +324\frac{\mu_G^2}{\sigma_G^2}\frac{(1 - \mu_G^2\sigma_G^{-2})^2}{1 + 4\mu_G^2\sigma_G^{-2}} - 180\frac{\mu_G^2}{\sigma_G^2} - 576 - 180\frac{\mu_G^4}{\sigma_G^4} - 9\frac{\mu_G^6}{\sigma_G^6} \right\}. \qquad (12)\end{aligned}$$

Bounding the square containing τ_G, by 0 and substituting $y = \mu_G^2\sigma_G^{-2}/4 \geqslant 0$ we obtain

$$\begin{aligned}\tau_F^2 - \kappa_F &\leqslant 3 - \tfrac{9}{5}(1 + y)^{-3}(1 + 16y)^{-1}(1 + 15y + 43y^2 + 45y^3 + 16y^4) \\ &= 3 - \tfrac{9}{5}(1 + y)^{-1}(1 + 16y)^{-1}(1 + 13y + 16y^2) \qquad (13) \\ &= \tfrac{6}{5} + \tfrac{36}{5}y(1 + y)^{-1}(1 + 16y)^{-1}.\end{aligned}$$

The right-hand side of (13) attains its maximum at $y = \frac{1}{4}$ thus implying (4).

From (7)–(9) we see that Pearson's inequality reduces to an equality iff X puts all its mass at two points, i.e. iff X has a Bernoulli distribution. Consequently, the above proof shows that (4) is an equality iff G is Bernoulli, $\mu_G^2 = \sigma_G^2$, and $\tau_G = 0$, i.e. iff $P_G(Z = 0) = P_G(Z = z_0) = \frac{1}{2}$ for some $z_0 \neq 0$. This means that equality holds in (4) iff F is one-sided boundary-inflated uniform with mass $\frac{1}{2}$ at the atom.

Remark 2.1. Let F be symmetric. It follows from (10) that G has to be symmetric too. Consequently, μ_G vanishes and (13) holds with $y = 0$. This proves (5) and it also shows that equality holds iff G is symmetric and Bernoulli, i.e. iff F is symmetric and uniform.

Remark 2.2. Let mean and mode of F coincide, and assume, without loss of generality, that they vanish. Then, μ_G vanishes too and again (13) holds with $y = 0$. This shows the validity of (6) for all unimodal F with mode at the mean. Studying this argument we see that equality holds here iff G is Bernoulli with $\mu_G = \tau_G = 0$, i.e. iff F is (symmetric and) uniform. Since symmetric unimodal distributions have their mode at their mean, we may note that inequality (5) and its sharpness follow from the present inequality (6).

Remark 2.3. The set

$$\{(\tau_F^2, \kappa_F + 3)\colon F \text{ unimodal}\} \qquad (14)$$

is described via a polynomial equation of degree 4 in Johnson and Rogers (1951).

Remark 2.4. Let $X_1, \ldots, X_n$ be independent random variables with variance $\sigma_i^2 = E(X_i - EX_i)^2$, skewness $\tau_i = \sigma_i^{-3/2}E(X_i - EX_i)^3$, and kurtosis $\kappa_i = \sigma_i^{-4}E(X_i - EX_i)^4 - 3$. If F_n is the distribution of $\sum_{i=1}^n X_i$, then

Table 1
Upper bounds for $\tau_F^2 - \kappa_F$

Bound	F	Equality at and only at
2	All	Bernoulli
186/125	Unimodal	One-sided boundary-inflated uniform
6/5	Mean=mode	Uniform
6/5	Symmetric unimodal	Uniform
0	Infinitely divisible	Normal, Poisson

straightforward computation shows

$$\tau_{F_n}^2 - \kappa_{F_n} = \frac{\sum_{i=1}^n \sigma_i^4(\tau_i^2 - \kappa_i)}{(\sum_{i=1}^n \sigma_i^2)^2} - \frac{\sum_{i=1}^n \sum_{j=1}^n \sigma_i^2\sigma_j^2(\sigma_i\tau_i - \sigma_j\tau_j)^2}{2(\sum_{i=1}^n \sigma_i^2)^3}. \tag{15}$$

If X_i has a normal or Poisson distribution, then we have

$$\tau_i^2 = \kappa_i, \quad \sigma_i\tau_i = \begin{cases} 0 & \text{normal,} \\ 1 & \text{Poisson.} \end{cases} \tag{16}$$

Consequently, if F_n is the distribution of the sum of independent normal and Poisson random variables $X_1,\dots,X_n$, we obtain

$$\tau_{F_n}^2 - \kappa_{F_n} \leqslant 0 \tag{17}$$

with equality iff either all X_i have a normal distribution or all X_i are Poisson.

Any infinitely divisible distribution F is a limit of F_n of the above type (see Example XVII.3(b), p. 566, of Feller, 1971). If F_n is chosen in this way and if moreover $E_{F_n}X^4 \to E_F X^4 < \infty$ holds, then (17) implies

$$\tau_F^2 - \kappa_F \leqslant 0, \tag{18}$$

a result formulated in Theorem 2 of Rohatgi and Székely (1989). In fact, equality holds in (18) only if F is either Poisson or normal. This has been shown very elegantly in the proof of Proposition 1 of Gupta et al. (1994).

We summarize the bounds discussed in this paper in Table 1.

Related inequalities for multivariate distributions have been presented by Móri et al. (1993).

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