Chapter 7: Superposition

- Introduction
  - The wave equation is linear, that is if $\psi_1(x,t)$ and $\psi_2(x,t)$ satisfy the wave equation, then so does $\psi(x,t) = \psi_1(x,t) + \psi_2(x,t)$.
  - This suggests the "Principle of Superposition": the resultant disturbance at any point in a medium is the algebraic sum of the separate constituent waves.
- The Addition of Waves of the Same Frequency
  - A solution of the wave equation can be written as

$$E(x,t) = E_0 sin[\omega t - (kx + \epsilon)],$$

    in which E_0 is the amplitude of the harmonic disturbance propogating along the x axis. This is

$$E(x,t) = E_0 sin[\omega t + \alpha(x, \epsilon].$$

  - Suppose there are two such waves,

$$E_1 = E_{01} sin(\omega t + \alpha_1),$$

$$E_2 = E_{02} sin(\omega t + \alpha_2),$$

    each with the same frequency and speed, coexisting in space.
  - Then $E = E_1 + E_2$ is such that

$$E = E_0 sin(\omega t + \alpha),$$

    where

$$E_0^2 = E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} cos(\alpha_2 - \alpha_1),$$

    and

$$tan\alpha = \frac{E_{01} sin\alpha_1 + E_{02} sin\alpha 2}{E_{01} cos\alpha_1 + E_{02} cos\alpha_2}.$$

  - That is the composite wave is harmonic and of the same frequency as the constituents, although its amplitude and phase are different.
  - This phase difference between these two waves may arise from a difference in path lengths and is δ, where

$$\delta = \frac{2\pi}{\lambda}(x_1 - x_2) + (\epsilon_1 - \epsilon_2).$$

- If the wave emitters are initially in phase, then $\epsilon_1 = \epsilon_2$, and because $n = c/v = \lambda_0/\lambda$
$$\delta = \frac{2\pi}{\lambda_0} n(x_1 - x_2).$$
- The quantity $n(x_1 - x_2)$ is known as the optical path difference or $OPD = \Lambda$. Thus $\delta = k_0 \Lambda$.
- Waves for which $\epsilon_1 - epsilon_2$ is constant are said to be coherent.
- When the OPD is equal to $0, \pm 2\pi, \pm 4\pi, ...$, the resultant amplitude is a maximum because of constructive interference. When the OPD is equal to $\pm\pi, \pm 3\pi, ...$, the resultant amplitude is a minimum because of destructive interference.

- Superposition of Many Waves
  - Consider the addition of N waves such that
$$E = \sum_{i=1}^{i=N} E_{0i} cos(\alpha_i \pm \omega t).$$
  - Then
$$E = E_0 cos(\alpha \pm \omega t),$$
where
$$E_0^2 = \sum_{i=1}^{i=N} E_{0i}^2 + 2 \sum_{j \geq i}^{N} \sum_{i=1}^{i=N} cos(\alpha_i - \alpha_j),$$
and
$$tan\alpha = \frac{\sum_{i=1}^{i=N} E_{0i} sin\alpha_i}{\sum_{i=1}^{i=N} E_{0i} cos\alpha_i}.$$
  - Consider N atomic emitters compising an ordinary light source. Each photon emitted is associated with a short duration oscillatory wave pulse. That is each atom is an independent source of wave trains with each wave train lasting about 1 to 10 ns. After each wave train, the atom will emit a new wavetrain with a totally random phase completely independent of the phase of the previous wavetrains and of the phase of those emitters around it.
  - The resultant flux density asrising from N independent sources having random, rapidly varying phases is given by N times the flux density of any one sourca $E_0^2 = N E_{01}^2$.
  - If sources are coherent, then $E_0^2 = N^2 E_{01}^2$.
  - Coherent sources generally alter the spatial distribution of energy but not the total energy.

- Standing Waves
  - Previousy considered two harmonic waves propogating in the same direction - now consider two waves propogating in te same direction.
  - COnsider $E_l = E_{0l} sin(kx + \omega t + \epsilon_l)$ and $E_r = E_0 r sin(kx - \omega t + \epsilon_r)$.
  - The composite disturbance is

$$E = E_l + E_r,$$

    and with some trignometric manipulation,

$$E(x,t) = 2E_{0l} sinkxcos\omega t,$$

    where without too much loss of generality, we have assumed $E_{0l} = E_{0r}$.
  - This is the equation for a standing or stationary wave. Its profile does not move through space.
  - At certain points, nodes, namely $x = 0, \lambda/2, \lambda, 3\lambda/2, ...$the disturbance is zero at all times.
  - Halfway between the nodes, that is, at $x = \lambda/4, 3\lambda/4, ...$, the amplitude has a maximum value of $\pm 2E_{ol}$: these are the anitnodes.
- The addition of Waves of Different Frequency
  - Consider the addition of two waves travelling in the same direction but with different frequency.
  - Let $E_1 = E_{01} cos(k_1 x - \omega_1 t)$, and $E_2 = E_{02} cos(k_2 x - \omega_2 t)$, where $k_1 > k_2$ and $\omega_1 > \omega_2$.
  - The net composite wave can be formulated as

$$E = 2E_{01} cos(\frac{1}{2}[(k_1 + k_2)x - (\omega_1 + \omega_2)t]) \times cos\frac{1}{2}[(k_1 - k_2) - (\omega_1 - \omega_2)t].$$

  - Now define $\bar{\omega}$ and $\bar{k}$ as the average angular frequency and average propogation number. Similarly the quantities ω_m and k_m are the modulation frequency and modulation propogation number respectively. That is

$$\bar{\omega} = \frac{1}{2}(\omega_1 + \omega_2), \omega_m = \frac{1}{2}(\omega_1 - \omega_2),$$

    and

$$\bar{k} = \frac{1}{2}(k_1 + k_2), k_m = \frac{1}{2}(k_1 - k_2).$$

  - Then

$$E = 2E_{0l} cos(k_m x - \omega_m t) cos(\bar{k}x - \bar{\omega}t).$$

- The total disturbance may be viewed as a travelling wave of frequency $\bar{\omega}$ having a time-varying or modulated amplitude $E_0(x,t)$ such that,

$$E(x,t) = E_0(x,t)cos(\bar{k} - \bar{\omega}t),$$

where

$$E_0(x,t) = 2E_{0l}cos(k_m x - \omega_m t).$$

- If $\omega_1 \approx \omega_2$, then $\bar{\omega}$ is much greater than ω_m and $E_0(x,t)$ will change slowly.
- It can be shown that

$$E_0^2(x,t) = 2E_{01}^2[1 + cos(2k_m x - 2\omega_m t)].$$

- Thus $E_0^2(x,t)$ oscillates about a value of $2E_{01}^2$ with an angular frequency of $(\omega_1 - \omega_2$: the beat frequency.
- Thus E_0, the amplitude of the resulting wave varies at the modulation frequency but E_0^2 varies at twice that - the beat frequency.

- Group Velocity
  - When a number of different-frequency harmonic waves superimpose to form a composite disturbance, the resulting modulation envelope will travel at a speed different from that of the constituent waves.
  - Group velocity verses phase velocity.
  - The phase velocity in this situation is $v = \bar{\omega}/\bar{k}$, that is the phase velocity of the carrier wave.
  - The group velocity relates to the rate at which the modulation envelope advances i.e. $v_g = \omega_m/k_m$.
  - If $v_1 = v_2$ then $v_g = v$.
  - Phase velocity can be greater than c, but phase velocity doesnt carry information or energy.
  - Group velocity can also be greater than c.
  - In either case, information or energy cannot be transmitted at a speed greater than c.

- Anharmonic Periodic Waves
  - Superposition of harmonic functions having different amplitudes and frequencies produces an anharmonic wave.
  - Fourier Series: A function $f(x)$ can be synthesized by a sum of harmonic functions whose wavelengths are integral submultiples of λ (ie. λ, $\lambda/2$, $\lambda/3$....).

– Any periodic function, $f(x)$,

$$f(x) = \frac{A_0}{2} + \sum_{m=1}^{\infty} A_m cosmkx + \sum_{m=1}^{\infty} B_m sinmkx$$

where, knowing $f(x)$,

$$A_m = \frac{2}{\lambda}\int_0^{\lambda} f(x)cosmkxdx,$$

$$B_m = \frac{2}{\lambda}\int_0^{\lambda} f(x)sinmkxdx.$$

– ω is the fundamental. Subsequent omegas like $2\omega, 3\omega, ...$ are the harmonics.

– Fourier series so represented has the form

$$f(x) = C_0 + C_1 cos(\frac{2\pi}{\lambda}x + \epsilon_1) + C_2 cos(\frac{2\pi}{\lambda}x + \epsilon_2) + ...$$

– Can extend this to non-periodic waves abd write

$$f(x) = \frac{1}{\pi}[\int_0^{\infty} A(k)coskxdk + \int_0^{\infty} B(k)sinkxdk],$$

$$A(k) = \int_{-\infty}^{\infty} f(x)coskxdx,$$

$$B(k) = \int_{-\infty}^{\infty} f(x)sinkxdx.$$

– $A(k), B(k)$ are the contributions to angular spatial frequency between $k, k + dk$.

– The frequency bandwidth is Δk or $\Delta\omega$.