Mathematical Physics Exercises

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1 Topology

Szekeres 10.1
Give an example in $\mathbb{R}^2$ of each of the following:
(a) A family of open sets whose intersection is a closed set that is not open.
Answer Take the family $\mathcal{U} = \{\mathcal{U}_n\}_{n>0}$ where $\mathcal{U}_n = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \frac{1}{n}\}$ which are open disks in the plane of radius $\frac{1}{n}$. The intersection is the point $(0,0)$ since

$$\bigcap_{n=1}^{\infty} \mathcal{U}_n = \lim_{n\to\infty} \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \frac{1}{n}\} = \{(0,0)\}$$

and singleton sets are closed with the standard topology in the plane.
(b) A family of closed sets whose union is an open set that is not closed.
Answer Notice that the interval $(0,1)$ is closed in the standard topology in the plane, since its complement is clearly open. Take the family, therefore, $\{\mathcal{U}_\alpha\} = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, y = \alpha\}_\alpha$ where $\alpha \in (0,1)$. The union is

$$\bigcup_{\alpha\in(0,1)} \mathcal{U}_\alpha = \{(x,y) \in \mathbb{R}^2 : 0 < |x| < 1, 0 < |y| < 1\}$$

which is the open unit square (notice that this is an uncountable union since the index set is the unit interval).
(c) A set that is neither open nor closed.
Answer Take the set $A = \{(x,y) \in \mathbb{R}^2 : |x| < 1, |y| \leq 1\}$. This is a box with closed top and bottom but open sides.
It is not open because we can take a point on the top boundary, for example $(0,1)$. If the set were open, then there would be an $\epsilon > 0$ stuch that $B \equiv B_\epsilon((0,1)) \subset A$, but clearly if $y = (0, 1 + \epsilon/2)$, then $y \in B$ but $y \notin A$.
(d) A countable dense set.
Answer Take $A = \{(x,y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}\} = \mathbb{Q}^2$, points with rational coordinates. Recall that the Cartesian product of countable sets is countable. Slightly less obvious is the fact that it is still a dense set. Consider an arbitrary point $p = (a,b) \in \mathbb{R}^2$ and a neigborhood for some $\epsilon > 0$ the open ball $B_\epsilon(p)$. This then reduces to the 1-dimensional case, since in the interval $(a - \epsilon, a + \epsilon)$, there is a rational number r (likewise for b)
(e) A sequence of continuous functions $f_n : \mathbb{R}^2 \longrightarrow \mathbb{R}$ whose limit is a discontinuous function.
Answer Define the functions as follows (trivially the same for all values of y)

$$f_n(x,y) = \begin{cases} 0 & x \leq 1 - \frac{1}{n} \\ nx - (n-1) & 1 - \frac{1}{n} \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

As $n \to \infty$ at $x = 1$, we have that $n(x-1)+1 \to 1$ so that $f_n(x,y) \to 0$ for $x < 1$, which is clearly discontinuous at the points of the form $(1, y)$.

Szekeres 10.2 If $\mathcal{U}$ generates the topology on X show that $\{A \cap U : U \in \mathcal{U}\}$ generates the relative topology on A

Proof Recall that if $\mathcal{U}$ generates a topology (X, ξ), then by definition, $\xi = \{\bigcup \mathcal{V} : \mathcal{V} \subseteq \mathcal{U}\}$, the union of all subsets of the generating set. Furthermore, the relative topology α on A is defined by all intersections of the topology ξ with the set A: $\alpha = \{A \cap U : U \in \xi\}$. Define the generating set given in the problem (intersections of the original generating set with A) as $\mathcal{B} = \{A \cap U : U \in \mathcal{U}\}$.

The proof is evident directly from the definitions above, for $\alpha = \{\bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B}\} = \{\bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{U} \cap A\} = \{A \cap U : U \in \xi\}$ by definition. $\square$

Szekeres 10.3 Let X be a topological space and $A \subset B \subset X$. If B is given the relative topology, show that the relative topology induced on A by B is identical to the relative topology induced on it by X.

Proof Define $\mathcal{X}$ to be the topology of X, $\xi = \{A \cap U : U \in \mathcal{X}\}$ to be the topology induced by X on A, and $\mathcal{B} = \{B \cap U : U \in \mathcal{X}\}$ to be the topology induced by X on B. Furthermore, let $\beta = \{A \cap U : U \in \mathcal{B}\}$ be the relative topology induced on A by B. We want to show that $\xi = \beta$

$\xi \subseteq \beta$: Let $Y \in \xi$, then $Y = A \cap U = A \cap B \cap U = A \cap V$ for some $U \in \mathcal{X}, V \in \mathcal{B}$. Therefore, $Y \in \beta$.

$\beta \subseteq \xi$: Let $Y \in \beta$, then $Y = A \cap U$ for some $U \in \mathcal{B}$. But $U \in \mathcal{B}$ implies that $U = B \cap V$ for some $V \in \mathcal{X}$. By the definition of a topology, finite intersections are still within the topology, hence $U \in \mathcal{X}$, so that $Y \in \xi$ as desired. $\square$

Szekeres 10.4 Show that for any subsets U, V of a topological space $\overline{U \cup V} = \overline{U} \cup \overline{V}$.

Proof $\overline{U \cup V} \subseteq \overline{U} \cup \overline{V}$: Assume $x \in \overline{U \cup V}$. In particular, if $x \in U, V$ individually, then the proof is trivial, so it remains to show that the accumulation points are properly contained. Thus, assume x is an accumulation point of $U \cup V$: there are other points of $U \cup V$ which are contained in a neighborhood of x, for which one such point is either in U, V, so it is an accumulation point of either set.

$\overline{U} \cup \overline{V} \subseteq \overline{U \cup V}$: Without loss of generality, assume $x \in \overline{U}$. Moreover, assume it is an accumulation point boundary, for the interior is trivial. Some neighborhood, therefore, non-trivially intersects with U, but then this neighborhood intersects with the union of the two sets in question $\square$

Is it true that $\overline{U \cap V} = \overline{U} \cap \overline{V}$?

Answer No, it is not true. Consider $\mathbb{R}$ with the usual topology and take $U = \mathbb{Q}$ and $V = \mathbb{R}\ \mathbb{Q}$. These two sets are disjoint, so $\overline{U \cap V} = \emptyset$, but $\overline{U} = \overline{V} = \mathbb{R}$, so that $\overline{U} \cap \overline{V} = \mathbb{R}$.

What corresponding statements hold for the interior and boundaries of unions and intersections of sets?

Answer *Interiors* The interiors are open sets, so they are closed under arbitrary unions and

finite intersections, so all analogous statements about intersections and unions will hold.
Boundaries $\partial(U \cup V) \subseteq \partial U \cup \partial V$
$\partial(U \cap V) = \partial U \cap \partial V$

2 Measure Theory

Szekeres *Exercise* Show that any countable intersection of measurable sets is measurable
Proof Apply de Morgan's law to rewrite the intersection as a complement of unions

$$\bigcap_{k\in\mathbb{N}} U_k = \bigcup_{k\in\mathbb{N}} U_k^C :$$

This is measurable because if the U_k are measurable, then so are any countable unions and the complement of countable unions by axioms of measurable spaces (sigma algebras). □

Szekeres *Exercise* Prove that the σ-algebra of Borel sets on $\mathbb{R}$ is generated by (a) the infinite left-open intervals (a, ∞), (b) the closed intervals $[a, b]$.
Proof
(a) Take $a < b$ and intersect $(a, \infty) \cap (-\infty, b]$, the latter of which is the complement of (b, ∞). This gives the set $(a, b]$. Similarly, $((b, \infty) \cup (-\infty, a])^C = [a, b)$ is measurable. Together, these sets give us that $[a, b]$ is measurable. Moreover, we can generate any interval of the form (a, b), since we can form $[c, d]$, $[c, a]$, and $[b, d]$ with $c < a < b < d$ and take the appropriate complements and unions. All sets in the measurable space are just unions of such intervals.
(b) This simply repeats the previous logic: we can generate any interval of the form (a, b), since we can form $[c, d]$, $[c, a]$, and $[b, d]$ with $c < a < b < d$ and take the appropriate complements and unions. All sets in the measurable space are just unions of such intervals. □

Szekeres *Exercise* Prove that all singletons $\{a\}$ are Borel sets on $\mathbb{R}$.
Proof Consider sets of the form $[a, b]$ as generators of the Borel sets of $\mathbb{R}$ as per the previous problem. The singleton is measureable, for if $c < a < b$, take $[c, a] \cap [a, b] = \{a\}$. □

Szekeres *Exercise* If $f : X \to Y$ and $g : Y \to Z$ are measurable functions between measure spaces, show that the composition $g \circ f : X \to Z$ is a measurable function.
Proof Name the corresponding measurable spaces as L, M, N for base spaces X, Y, Z, respectively. The assumptions give us that $f^{-1}(M) \subseteq L$ and $g^{-1}(N) \subseteq M$. Note that $(g \circ f)^{-1}(N) = f^{-1} \circ g^{-1}(N) \subseteq f^{-1}(M) \subseteq L$ □

Szekeres *Exercise* Show that for any $a \in \mathbb{R}$, the set $\{x \in X : f(x) = a\}$ is a measurable set of X
Proof There are a number of ways to show this. We already know that $A \equiv \{x \in X : f(x) > a\}$ is measurable since it is the preimage of the interval (a, ∞). Similarly,$A' \equiv \{x \in X : f(x) < a\}$ is also measurable, so that $(A \cup A')^C = \{a\}$ is also measurable. Also, more simply, $\{a\} = f^{-1}(a)$ and the singleton is measurable since f is assumed to be a measurable

function. Lastly(and redundantly), consider the characteristic function:

$$\{x \in X : \chi(x) = a\} = \begin{cases} A & x \in A \\ A^C & x \notin A \end{cases}$$

□

Szekeres *Exercise* Show that for measurable functions $f, g : X \to \mathbb{R}$, the function fg is measurable.

Proof This is trivial because fg can be written as the composition of two measurable functions, $F : x \mapsto (f(x), g(x)) \in \mathbb{R}^2$ and $\varrho : (a, b) \to ab$. Recall that F is measurable since $F^{-1}(I \times J) = f^{-1}(I) \cap g^{-1}(J)$ for any two intervals I, J. □

Szekeres *Exercise* Show that $f = f^+ + f$ and $|f| = f^+ + f$.

Proof Recall that $f^+ = \sup(f, 0)$ and $f^- = -\inf(f, 0)$. Decompose f as follows:

$$f = \begin{cases} f^+ & f \geq 0 \\ -f^- & f \leq 0 \end{cases}$$

Therefore, when $f > 0$, we have $f = \sup(f, 0) - (-\inf(f, 0)) = f^+ - f^-$, and similarly for the negative case.

For the case of $|f|$, we similarly write

$$|f| = \begin{cases} f^+ = \sup(f, 0) + (-\inf(f, 0)) = \sup(f, 0) = f^+ + f^- & f \geq 0 \\ f^- = \sup(f, 0) + (-\inf(f, 0)) = -\inf(f, 0) = f^+ + f^- & f \leq 0 \end{cases}$$

□

Szekeres 11.1 If (X, M) and (Y, N) are measurable spaces, show that the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ are measurable functions.

Proof We need that $\pi_1^{-1}(M) \subseteq M \times N$. The proof for the other projection map is analogous.

Let $U \in \pi_1^{-1}(M)$. Without loss of generality, take $U \in X$ itself, for if not then we can just carry the logic over to arbitrary intersections and countable unions as necessary. The question reduces to whether $\pi_1^{-1}(X) \subseteq X \times Y$ as sets. Note that the projection maps are surjective, so $X \subseteq \pi_1(X \times Y)$, and by applying the inverse we arrive at the result needed (inverse does not need to be unique for this logic to follow). □

Szekeres 11.2 Find a step function $s(x)$ that approximates $f(x) = x^2$ uniformly to within $\varepsilon > 0$ on $[0, 1]$, in the sense that $|f(x)s(x)| < \varepsilon$ everywhere in $[0, 1]$.

Answer Divide the interval $[0, 1]$ into uniform slices of length $1/n$. Define

$$s(x) = \sum_{k=1}^{n} f(\frac{k}{n})\chi_{[k-1/n,k/n]}(x)^2 = \sum_{k=1}^{n} \frac{k^2}{n^2}\chi_{[k-1/n,k/n]}(x)^2$$

On each interval, the maximal deviation will occur at the leftmost point of the interval. Let $\varepsilon > 0$ be arbitrary and consider the interval $[(j-1)/n, j/n]$ for some j, then

$$|f(x) - s(x)| = |x^2 - \frac{j^2}{n^2}| < |\frac{2j}{n^2}| + |\frac{1}{n^2}| < \varepsilon$$

if we choose $n^2 > \frac{2j+1}{\varepsilon}$.Hence, the difference can be made arbitrarily small for sufficiently large n. In particular, the maximal difference will come from the interval $[1-1/n, 1]$, so choose accordingly.

Szekeres 11.3 Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be measurable functions and $E \subset X$ a measurable set. Show that

$$h(x) = \begin{cases} f(x) & x \in E \\ g(x) & x \notin E \end{cases}$$

is a measurable function on X.

Proof f, g are measurable functions on measurable space M, N; therefore, $f^{-1}(A) \in M$ for each $A \in M$ (and similarly for g). We need $h^{-1}(A) \subseteq M$ for each A. Fix $A = B \cup C$, two disjoint parts such that $B \subseteq E$ and $C \subseteq E^C$. That is, $B = E \cap A$ and $C = E^C \cap A$, and finite intersections are measurable. Then since f, g are measurable, $f^{-1}(B) \cup g^{-1}(C)$ is measurable since the union is of two measurable sets. $\square$

Szekeres 11.4 If $f, g : \mathbb{R} \to \mathbb{R}$ are Borel measurable real functions show that $h(x, y) = f(x)g(y)$ is a measurable function $h : \mathbb{R}^2 \to \mathbb{R}$ with respect to the product measure on $\mathbb{R}^2$.

Proof Again, we need that for every Borel set $A \in \sigma(\mathbb{R})$, $h^{-1}(A) \in \sigma(\mathbb{R}^2)$. We have already that $f^{-1}(A) \in \sigma(\mathbb{R})$ and a similar statement for g. First, we show that for any two sets $h^{-1}(A \times B) \subseteq f^{-1}(A) \times g^{-1}(B)$. If $x \in \{(a, b) \in A \times B : h(a, b) = f(a)g(b) \in \mathbb{R}\}$, $y \in \{a \in A : f(a) \in \mathbb{R}\}$, and $z \in \{b \in B : g(b) \in \mathbb{R}\}$, then by closure if $f(a)g(b) \in \mathbb{R}$, then $f(a) \in \mathbb{R}$ (and similarly for g), so we have the proper set inclusions.
We also have that $(A) \times \sigma(B) \subset \sigma(A \times B)$ trivially. $\square$

Szekeres *Exercise* Show that if $B \subset A$ are measurable sets, then $\mu(B) \leq \mu(A)$.

Proof Note that because $B \subset A$, $\mu(A) = \mu(B) + \mu(A/B)$, and A/B is obviously also measurable and nonnegative. Hence, $\mu(B) \leq \mu(A)$. $\square$

Szekeres *Exercise* Show that if $A \subset \mathbb{R}$ is a countable set, then $\mu(A) = 0$.

Proof Fix $\varepsilon > 0$. For each $x_k \in A$ and every $k \in \mathbb{N}$, cover it with an interval $(x_k - \frac{\varepsilon}{2^{k+1}}, x_k + \frac{\varepsilon}{2^{k+1}})$ which has length $\frac{\varepsilon}{2^k}$. The countable set's measure can be made arbitrarily small:

$$\mu(A) = \mu(\bigcup_{k\in\mathbb{N}} x_k) = \sum_{k\in\mathbb{N}} \mu(x_k) < \sum_{k\in\mathbb{N}} \mu((x_k - \frac{\varepsilon}{2^{k+1}}, x_k + \frac{\varepsilon}{2^{k+1}})) = \sum_{k\in\mathbb{N}} \frac{\varepsilon}{2^k} = \varepsilon$$

$\square$

Szekeres *Exercise* Show that outer measure satisfies $\mu^*(\emptyset) = 0$.

Proof Fix an arbitrary measure μ, and consider the set $\{\mu(U) : U \text{ is open in } \mathbb{R} \text{ and } \emptyset \subset U\}$.

However, all measures are nonnegative and $\mu(U)$ can be made arbitrarily small since all sets contain the empty set. The infimum is, therefore, 0. $\square$

Szekeres *Exercise* For an open interval $I = (a, b)$ show that $\mu^*(I) = \mu(I) = b - a$.
Proof The definition $\mu^*(I) = \inf\{\mu(U) : U \text{ is open in } \mathbb{R} \text{ and } I \subset U\}$ implies that there exists a set U such that for any $\varepsilon > 0$

$$\mu^*(I) \leq \mu(U) + \varepsilon \leq b - a + \varepsilon$$

However, we also have that there is some U' for which

$$\mu(U') < \mu^*(I) + \varepsilon = b - a + \varepsilon$$

By definition, we also have that $\mu^*(I) \leq \mu(I)$. We need to show also that $\mu(I) \leq \mu^*(I)$. This is true since it is an open cove of itself, s the length is clearly bounded below, as necessary. $\square$

Szekeres 11.5 Show that every countable subset of $\mathbb{R}$ is measurable and has Lebesgue measure zero.
Proof We already showed that $\mu(A) = 0$ for such a set A. Now we show that if it is countable, it must be measurable. Let $I = (a, b)$ be an arbitrary open interval. We want to show that $\mu(I) = \mu^*(I \cap A) + \mu^*(I \cap A^C)$. Enumerate the points of A:

$$A = \bigcup_{k \in \mathbb{N}} a_k$$

and cover each point with an open interval $(a_k - \varepsilon/2^{k+1}, a_k + \varepsilon/2^{k+1})$, which can be made arbitrarily small. We know that $b - a = \mu(I)$. Moreover, $I \cap A$ may only interset some points of A; call this subset A' and enumerate that a_j accordingly with $[j][k]$. Then

$$\mu^*(I \in A) = \mu^*\left(\bigcup((a_j - \varepsilon/2^{j+1}, a_j + \varepsilon/2^{j+1}))\right) \leq \sum_j \frac{\varepsilon}{2^j} \leq \varepsilon$$

Similarly, then if l indexes the remaining subsets

$$b - a = \mu(I) \geq \mu^*(I \in A^C) = b - a - \sum_l \frac{\varepsilon}{2^l} \geq b - a - \varepsilon$$

Thus, A satisfies the condition.
Moreover, A has measure zero as we showed in one of the previous exercises. $\square$

Szekeres 11.6 Show that the union of a sequence of sets of measure zero is a set of Lebesgue measure zero.
Proof Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of measure zero sets. For any $\varepsilon > 0$, $\mu(A_n) < \varepsilon/2^n$. So

$$\bigcup A_n < \sum_n \frac{\varepsilon}{2^n} = \varepsilon$$

which can be made arbitarily small (and must be nonnegative). $\square$

Szekeres 11.7 If $\mu^*(N) = 0$ show that for any set E, $\mu^*(E \cup N) = \mu^*(E - N) = \mu^*(E)$. Hence show that $E \cup N$ and EN are Lebesgue measurable if and only if E is measurable.
Proof We have that definition $\mu^*(I) = \inf\{\mu(U) : U \text{ is open in } \mathbb{R} \text{ and } I \subset U\}$ for any open interval, I. Indeed, then for any open $U \supset E \cup N$, consider the measure μ. By properties of the measure, we have

$$\mu(U) > \mu(E \cup N)$$

but then there exists an $\varepsilon > 0$ such that

$$\mu(U) + \varepsilon \geq \mu(E) + \mu(N) \geq \mu(E \cup N) \geq \mu(E)$$

.Assuming that E is measurable, over all U the upper bound converges to $\mu(E)$. Therefore, $\mu(E \cup N) = \mu(E)$ as desired. The proof for case is similar. □

Szekeres 11.8 A measure is said to be complete if every subset of a set of measure zero is measurable. Show that if $A \subset \mathbb{R}$ is a set of outer measure zero, $\mu^*(A) = 0$, then A is Lebesgue measurable and has measure zero. Hence show that Lebesgue measure is complete. **Proof** To be Lebesgue measureable, we need that $\mu(I) = \mu^*(I \cap A) + \mu^*(I \cap A^C)$.In particular, if $I \subset A$ where A has measure zero, then it is, indeed, the case that the Lebesgue measure is complete. Note first that

$$0 = \mu^*(A) \geq \mu^*(I \cap A) \geq 0$$

so that A has (outer) measure zero. Moreover, $\mu^*(I) = \mu^*(I \cap A)$, and we are only considering measurable intervals in the first place. □

Szerkes 11.9 Show that a subset E of $\mathbb{R}$ is measurable if for all $\varepsilon > 0$ there exists an open set $U \supset E$ such that $\mu^*(U - E) < \varepsilon$.
Proof Fix $\varepsilon > 0$. This simply follows from the definition of $\mu^*(E) == \inf\{\mu(U) : U \text{ is open in } \mathbb{R} \text{ and } I \subset U\}$.That is, it follows directly from the definition that $\exists U \supset E$ with $\mu^*(U) < \mu^*(E) + \varepsilon$. This implies the conclusion, since

$$\mu^*(U - E) < \mu^*(U) - \mu^*(E) < \varepsilon$$

□

Szerkes 11.12 Show that if f and g are Lebesgue integrable on $E \subset \mathbb{R}$ and $f \geq g$ a.e., then

$$\int_E f d\mu \geq \int_E g d\mu$$

Proof Since the sets in question are measurable, $E = \bigsqcup A_k$. We rewrite the definitons in terms of sums for the case of simple functions ($\{a_k\}$ are the simple-function coefficients for f):

$$\sum_{k=0}^{n} a_k \mu(A_k) \geq \sum_{k=0}^{n} b_k \mu(A_k)$$

. Next we need to take suprema of both sides over the sets of all step functions approximating our given functions $f, g : X \to \mathbb{R}$. We unravel the definitions of the Lebesgue integral (i.e.

that it only takes on a non-zero value on the support E):

$$\int_E f d\mu = \int f \chi_E d\mu$$

The result then follows trivially from the fact that the supremum of the dominated function cannot exceed the dominating function (almost everywhere, so except on an arbitrarily small set). $\Box$

Szerkes 11.13 If f and g are Lebesgue integrable real functions, then for any $a, b \in \mathbb{R}$ the function $af + bg$ is Lebesgue integrable and for any measurable set E

$$\int_E (af + bg) d\mu = a \int_E f d\mu + b \int_E g d\mu$$

Proof We unravel the definitions. First, we show that $af + bg$ is Lebesgue measurable on $\mathbb{R}$. Let $\mathcal{B}$ denote the Borel sigma algebra on $\mathbb{R}$. If $B\mathcal{B}$, then for non-zero a, b (otherwise trivial), we have that the inverse function $h^{-1} \equiv (af+bg)^{-1}$ exists and $h(B) \in \mathcal{B}$ since linear combinations of measurable functions are measurable. Without loss of generality, assume that the function is positive. Since we are dealing with a linear combination, this extension to both positive and negative parts is trivial. The quality follows because sums are linear, and we are taking suprema, which behave nicely:

$$\int_E (af + bg) d\mu = \int (af + bg) \chi_E d\mu = \sup \sum_{k=0}^{n} h_k \mu(A_k) =$$

$$a \sup \sum_{k=0}^{n} f_k \mu(A_k) + b \sup \sum_{k=0}^{n} g_k \mu(A_k) = a \int_E f d\mu + b \int_E g d\mu$$

with the coefficients defined intuitively. $\Box$

Szerkes 11.14 If f is a Lebesgue integrable function on $E \subset \mathbb{R}$ then show that the function ψ defined by $\psi(a) = \mu(x \in E : |f(x)| > a) = O(a^1)$ as $a \to \infty$ is Lebesgue measurable.
Proof If f is Lebesgue integrable on E, then the integral of the function is the supremum over all simple functions

$$\int_E f d\mu = \sup \int h \chi_E d\mu$$

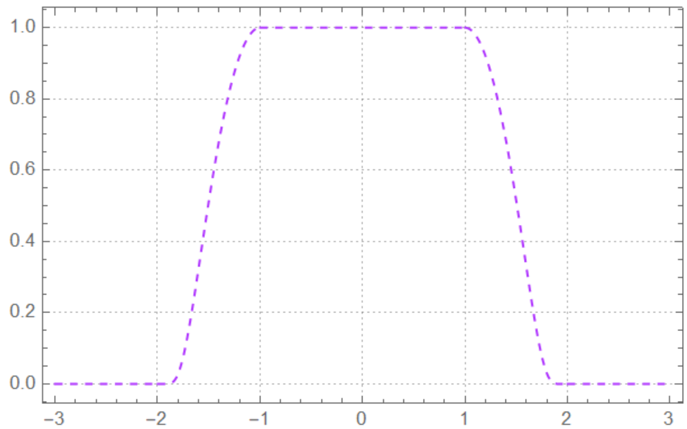
. Note further that ψ is always positive. Let A denote the set on which $f(x) > a$. Thus, we look at the set $E \cap A$, which is further a measurable set. Observe that $(f - a)^{-1}(0, \infty)$ is indeed a measurable set for all a so that ψ is a measurable function. $\Box$

3 Distributions

Szerkes 12.1 Construct a test function such that $\phi(x) = 1$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \leq 2$. **Answer** We take the lead from the book, where it uses the function $e^{-1/x}$ to construct a continuous test function. Our is a test function whose support is the closed disk of radius 2. Indeed, it is a C^∞ function as verified trivially:

$$\phi(x) = \begin{cases} Nf(x+1) \text{ for } -2 \le x \le -1 \\ 1 \text{ for } -1 < x < 1 \\ Nf(x-1) \text{ for } 1 \le x \le 2 \\ 0 \text{ otherwise} \end{cases}$$

where we define $f(x) = e^{\frac{2}{x-1}}$ and $N = 1/f(0)$ is the normalization. Furthermore, we include a plot:



Szerkes 12.2 For every compact set $K \subset \mathbb{R}^n$ let $\mathcal{D}(K)$ be the space of C^∞ functions of compact support within K. Show that if all integer vectors k are set out in a sequence where $N(k)$ denotes the position of k in the sequence, then

$$||f||_K = \sup_{x\in K} \sum_k \frac{1}{2^{N(k)}} \frac{|D_k f(x)|}{1+|D_k f(x)|}$$

is a norm on $\mathcal{D}(K)$. Let a set U be defined as open in $\mathcal{D}(\mathbb{R}^n)$ if it is a union of open balls $\{g \in K : ||g-f||_K < a\}$. Show that this is a topology and sequence convergence with respect to this topology is identical with convergence of sequences of functions of compact support to all orders.

Proof First, we show that $||\cdot||$ is a norm on the space of test functions.

1. Triangle inequality/subadidivity:

$$||f+g||_K = \sup_{x\in K} \sum_k \frac{1}{2^{N(k)}} \frac{|D_k[f(x)+g(x)]|}{1+|D_k[f(x)+g(x)]|} \le \sup_{x\in K} \sum_k \frac{1}{2^{N(k)}} \frac{|D_k f(x)|+|D_k g(x)]|}{1+|D_k[f(x)+g(x)]|}$$
$$\le \sup_{x\in K} \sum_k \frac{1}{2^{N(k)}} \frac{|D_k f(x)|+|D_k g(x)]|}{1+|D_k f(x)|} + \sup_{x\in K} \sum_k \frac{1}{2^{N(k)}} \frac{|D_k f(x)|+|D_k g(x)]|}{1+|D_k g(x)|}$$
$$= ||f||_K + ||g||_K$$

2. Absolute homogeneity: First, we have the following naive bound:

$$||\lambda f||_K = \sup_{x\in K}\sum_k \frac{1}{2^{N(k)}}\frac{|D_k\lambda f(x)|}{1+|D_k\lambda f(x)|} \le |\lambda|||f||_K$$

   Secondly, we must show the opposite inequality. As $x \to \partial K$, $f(x) \to 0$ since it is a function of compact support. Indeed, the derivative also goes to zero because it is the unique extension of the function to the boundary. In particular, there exists an $\varepsilon > 0$, $x \in K$ such that

$$||\lambda f||_K \ge \sum_k \frac{1}{2^{N(k)}}\frac{|D_k\lambda f(x)|}{1+|D_k\lambda f(x)|} - \varepsilon = |\lambda|\sum_k \frac{1}{2^{N(k)}}\frac{|D_k f(x)|}{1+|D_k\lambda f(x)|} - \varepsilon$$

   If the denominator goes to 1, this implies when we take the supremum that $||\lambda f||_K \ge |\lambda|||f||_K$

3. Positive definiteness: Supppose $||f||_K = 0$. The only way this is possive is if $\sum_k |D_k f(x)| = 0$. Since all the values are nonnegative, this requires that for each k, $D_k f(x) = 0$. However, if all derivatives of all orders are zero (and since it is a test function is it everywhere continuous, including through its extenion on the boundary), it must be that $f(x) = 0$ for all $x \in K$.

Next, we show that this gives a topology on the space of test functions. Indeed, this must be true because it is a metric on the space, and the metric will induce the corresponding topology generated by open balls of the form $\{g \in \mathcal{D}(K) : ||f - g|| < r\}$ for some radius r.
Lastly, we show that a sequence that coverges in this topology also converge to all orders. Suppose $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of test functions, that converges to f, then for $n > N$ for some fixed N large enough and for arbitary $\varepsilon > 0$

$$||f - f_n||_K = \sup_{x\in K}\sum_k \frac{1}{2^{N(k)}}\frac{|D_k[f(x) - f_n(x)]|}{1+|D_k[f(x) - f_n(x)]| < \varepsilon}$$

The above implies that each term in the sum must be accordingly bounded:

$$0 \le |D_k f(x)| - |D_k f_n(x)||D_k[f(x) - f_n(x)]| < \frac{|D_k[f(x) - f_n(x)]|}{1+|D_k[f(x) - f_n(x)]|} < \frac{\varepsilon}{2^{N(k)}}$$

For each k, the derivatives converge, so the dirstribution converges to all orders. The reverse direction is trivial by substituting into the definition of the norm. $\square$
Szerkes 12.3 Which of the following is a distribution?

1. $T(\phi) = \sum_{n=1}^m \lambda_n \phi^{(n)}(0)$
2. $T(\phi) = \sum_{n=1}^m \lambda_n \phi(x_n)$
3. $T(\phi) = \phi(0)^2$
4. $T(\phi) = \sup \phi$

5. $T(\phi) = \int_{-\infty}^{\infty} |\phi(x)| dx$

Answer We simply need to check whether each is linear on test functions

1. **Yes**

$$T(\alpha\phi + \beta\psi) = \sum_{n=1}^{m} \lambda_n(\alpha\phi^{(n)}(0) + \beta\psi^{(n)}(0))$$

$$= \alpha \sum_{n=1}^{m} \lambda_n \phi^{(n)}(0) + \beta \sum_{n=1}^{m} \psi^{(n)}(0) = \alpha T(\phi) + \beta T(\psi)$$

2. **Yes**

$$T(\alpha\phi + \beta\psi) = \sum_{n=1}^{m} \lambda_n(\alpha\phi(x_n) + \beta\psi(x_n))$$

$$= \alpha \sum_{n=1}^{m} \lambda_n \phi(x_n) + \beta \sum_{n=1}^{m} \psi(x_n) = \alpha T(\phi) + \beta T(\psi)$$

3. **No** $T(\alpha\phi) = (\alpha\phi(0))^2 = \alpha^2\phi(0)^2 \neq \alpha\phi(0)^2 = \alpha T(\phi)$
4. **No** $T(\phi + \psi) = \sup(\phi + \psi) \neq \sup\phi + \sup\psi = T(\phi) + T(\psi)$
5. **No** $T(\phi + \psi) = \int_{-\infty}^{\infty} |\phi(x) + \psi(x)| dx \leq \int_{-\infty}^{\infty} |\phi(x)| + |\psi(x)| dx = T(\phi) + T(\psi)$

Szerkes 12.4 We say a sequence of distributions T_n converges to a distribution T , written $T_n \to T$,if $T_n(\phi) \to T(\phi)$ for all test functions $\phi \in \mathcal{D}$ (this is sometimes called weak convergence). If a sequence of continuous functions f_n converges uniformly to a function $f(x)$ on every compact subset of $\mathbb{R}$, show that the associated regular distributions $T_{fn} \to T_f$. In the distributional sense, show that we have the following convergences: (1) $f_n(x) = \frac{n\pi}{1+n^2x^2} \to \delta(x)$ and (2) $g_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \to \delta(x)$

Proof Fix $K \in \mathbb{R}$ a compact set, and suppose that there exists $N \in \mathbb{N}$large enough so that $\forall n > N$ and every $\varepsilon > 0$, $||f - f_n|| < \varepsilon$. Consider the regular distributions generated by the functions:

$$|T_{fn} - T_f| = |\int \phi(x)[f_n(x) - f(x)]dx| \leq \int \phi(x)|f_n(x) - f(x)|dx < \varepsilon \cdot C$$

To show that the associated distributions converge, we need only show that the densities converge. Note that $\delta(\phi) = \phi(0)$ for any test function. Let us split up the integral into three parts and show that the distribution acting on the test function will converge to $\phi(0)$:

$$T_{fn}(\phi) = \int_{\mathbb{R}} f_n(x)\phi(x)dx = \int_{-\infty}^{-\frac{1}{\sqrt{n}}} \frac{1}{\pi} \frac{n}{1+n^2x^2}\phi(x)dx + \int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} \frac{1}{\pi} \frac{n}{1+n^2x^2}\phi(x)dx + \int_{\frac{1}{\sqrt{n}}}^{\infty} \frac{1}{\pi} \frac{n}{1+n^2x^2}\phi(x)dx$$

where, for shorthand, we refer to the three integrals in the summand as I, J, K. First, we show $I, K \to 0$. Note that $\phi < C = \sup_{x \in K} \phi(x)$ is the bound for ϕ which exists since it is a test funciton on compact support. We choose $\frac{1}{\sqrt{n}}$ as the bound for the integrals so that the tails converge quickly enough (any exponent less that 1 would do). Therefore,

$$0 \le K \le C \int_{\frac{1}{\sqrt{n}}}^{\infty} \frac{1}{\pi} \frac{n}{1+n^2x^2} dx \le \frac{C}{\pi} \left[\arctan(nx)\right]_{\frac{1}{\sqrt{n}}}^{\infty} = \frac{C}{\pi} (\arctan \sqrt{n} - \frac{\pi}{2})$$

Indeed, as $\lim_{n\to\infty} \arctan \sqrt{n} \to \frac{\pi}{2}$. Therefore, $I, K \to 0$. Lastly, we show that $J \to \phi(0)$. Note that for any $\varepsilon > 0$, there is $N \in \mathbb{N}$, such that for all $n \ge N$ $|\phi(0) - \phi(x)| < \varepsilon$ for every $x \in [-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}]$, in particular for the maximum and minimum on the interval, m and M, respectively . Thus,

$$m \cdot 2 \arctan \sqrt{n} < \int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} \frac{1}{\pi} \frac{n}{1+n^2x^2} \phi(x) dx < M \cdot 2 \arctan \sqrt{n}$$

implying that $|J| \le \phi(0) + \varepsilon$ in the limit of $n \to \infty$. (Trivially, the integral over $\mathbb{R}$ is 1). The argument for g_n is analogous. $\square$

Szerkes 12.6 Evaluate (a) $\int_{-\infty}^{\infty} e^{at} \sin bt \delta^{(n)}(t) dt$ for $n = 0, 1, 2$ and (b) $\int_{-\infty}^{\infty} (\cos t + \sin t) \delta^{(n)}(t^3 + t^2 + t) dt$ for $n = 0, 1$

Answer Recall first the key identities: $T'(\phi) = -T(\phi')$ and $\delta \circ f = \frac{1}{|f'(a)|} \delta_a$

(a)

$$\int_{-\infty}^{\infty} e^{at} \sin bt \delta(t) dt = \left[e^{at} \sin(bt)\right]_{t=0} = 0$$

$$\int_{-\infty}^{\infty} e^{at} \sin bt \delta'(t) dt = -\left[e^{at} \sin bt\right]'_{t=0} = -\left[be^{at} \cos bt + ae^{at} \sin bt\right]_{t=0} = b$$

$$\int_{-\infty}^{\infty} e^{at} \sin bt \delta''(t) dt = \left[e^{at} \cos bt + ae^{at} \sin bt\right]'_{t=0}$$

$$= \left[-b^2 e^{at} \sin bt + abe^{at} \cos bt + abe^{at} \cos bt + a^2 e^{at} \sin bt\right]'_{t=0} = 2ab$$

(b)

$$\int_{-\infty}^{\infty} (\cos t + \sin t) \delta(t^3 + t^2 + t) dt = 1$$

$$\int_{-\infty}^{\infty} (\cos t + \sin t) \delta'(t^3 + t^2 + t) dt = -\frac{1}{2} \left[(-\sin t + \cos t)\right]_{t=0} = -\frac{1}{2}$$

Szerkes 12.7 Show the following identities: (a) $\delta((x-a)(x-b)) = \frac{1}{b-a}(\delta(x-a) + \delta(x-b))$, (b) $\frac{d}{dx}\theta(x^2 - 1) = \delta(x-1) - \delta(x+1) = 2x\delta(x^2 - 1)$

Answer (a)

$$\delta((x-a)(x-b)) = \frac{1}{|2a-a-b|} \delta(x-a) + \frac{1}{|2b-a-b|} \delta(x-b) = \frac{1}{b-a} (\delta(x-a) + \delta(x-b))$$

(b)

$$\frac{d}{dx}\theta(x^2-1) = \delta(x-1) - \delta(x+1) = 2x\delta(x^2-1)$$