Linear Algebra and Matrix Theory
MATH 3100

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Google's Page Rank Algorithm

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The Google PageRank algorithm is based on the following premise: a website is important if other important websites link to it. To create this thesis Sergey Brin and Larry Page, who later founded Google, made two assumptions. First, a hyperlink is a recommendation of a website. Second, a recommendation is more valuable if the recommender is selective. It follows that the more important a page P_i, the higher its PageRank, r(P_i), which is defined as:

$$r(P_i) = \sum_{P_j \in B_{P_i}} \frac{r(P_j)}{|P_j|} \tag{1}$$

where P_i represents page i = 1, 2, ..., n, B_{P_i} is the set of pages that link to P_i, and $|P_j|$ is the number of sites that P_j points into. These ranks fill the columns of the PageRank vector, defined as $\boldsymbol{\pi}^T$ = (r(P_1),r(P_2),...,r(P_n)), where n is the number of pages being ranked. To find the successive rank of a page, the calculations begin with an estimate of each page, $r_o(P_i)$, and the PageRank is updated iteratively with the following:

$$r_{k+1}(P_i) = \sum_{P_j \in B_{P_i}} \frac{r_k(P_j)}{|P_j|} \tag{2}$$

The hyperlink matrix **H** has elements $h_{ij} = \frac{1}{|P_i|}$ for each page P_j which is linked to by page P_i and $h_{ij} = 0$ otherwise. If each row in **H** sums to either 1 or 0, that matrix is called *substochastic*. This case results in page sinks, if a page has no link to other pages (when a row sums to 0). To avoid PageRank sink due to a row of zeros, replace each element in that row with $1/n$ where n is the number of pages in the web.

With what is defined so far, there still exists a possibility for cycling between states. To avoid this, Google founders introduced the idea of the *random surfer*. According to this concept, there is some probability α ($0 \leq \alpha \leq 1$) that the web surfer moves following a hyperlink. There is also the probability that a web surfer moves by directly linking to one of the other n pages directly, represented by 1-α. With this concept and the hyperlink matrix **H**, the Google PageRank update matrix was formulated.

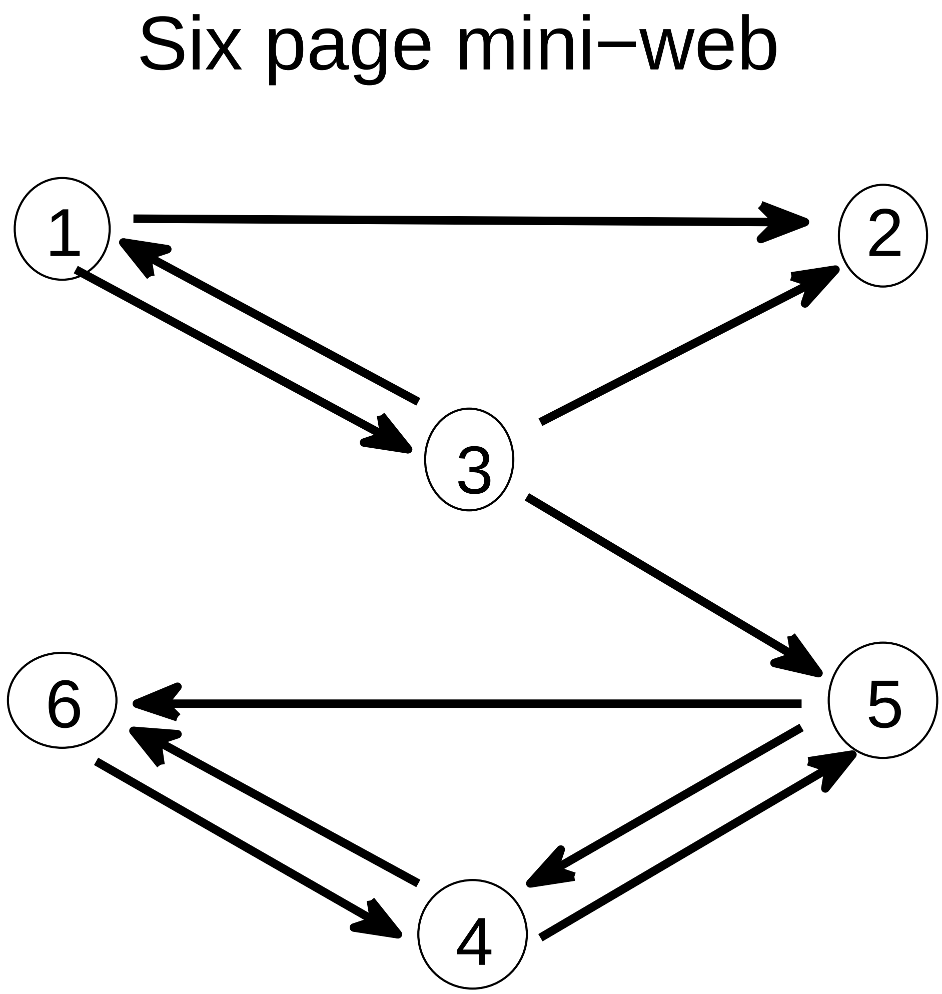
$$\mathbf{G} = \alpha\mathbf{H} + \alpha\mathbf{a}\mathbf{1}^T + \frac{1-\alpha}{n}\mathbf{1}\mathbf{1}^T \tag{3}$$

where **a** is a column vector with elements $a_i = \frac{1}{n}$ if P_i links to no pages or 0 otherwise and **1** is a column vector of ones. Utilizing the PageRank update matrix, the PageRank vector can be iteratively updated as follows.

$$\boldsymbol{\pi}_{k+1}^T = \boldsymbol{\pi}_k^T G \tag{4}$$

Based on a given mini-web, different values for α, and different values for k, we intend to determine what α value Google should use. Using MatLab we will be able to easily compute the matrices and vectors neccessary to determine an appropriate α value.

Dr. Spiller has provided a mini-web consisting of six pages with links as shown below.



Given this web, we start by determining the B_{P_i} and $|P_j|$, as follows.

Page	1	2	3	4	5	6
B_{P_i}	3	1, 3	1	5, 6	3, 4	4, 5
$\lvert P_j \rvert$	2	0	3	2	2	1

From our values for B_{P_i} and $|P_j|$, we are able to calculate $\mathbf{H}$, $\mathbf{a}$, $\mathbf{1}$, and $\mathbf{1}^T$, where $n = 6$.

$$\mathbf{H} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{a} = \begin{bmatrix} 0 \\ 1/6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \mathbf{1}^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Plugging the matrix and vectors determined above into equation 3, the transition matrix **G** can be symbolically determined in terms of α.

$$\mathbf{G} = \begin{bmatrix} 1/6-\alpha/6 & \alpha/3+1/6 & \alpha/3+1/6 & 1/6-\alpha/6 & 1/6-\alpha/6 & 1/6-\alpha/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ \alpha/6+1/6 & \alpha/6+1/6 & 1/6-\alpha/6 & 1/6-\alpha/6 & \alpha/6+1/6 & 1/6-\alpha/6 \\ 1/6-\alpha/6 & 1/6-\alpha/6 & 1/6-\alpha/6 & 1/6-\alpha/6 & \alpha/3+1/6 & \alpha/3+1/6 \\ 1/6-\alpha/6 & 1/6-\alpha/6 & 1/6-\alpha/6 & \alpha/3+1/6 & 1/6-\alpha/6 & \alpha/3+1/6 \\ 1/6-\alpha/6 & 1/6-\alpha/6 & 1/6-\alpha/6 & 5\alpha/6+1/6 & 1/6-\alpha/6 & 1/6-\alpha/6 \end{bmatrix}$$

As **G** has been determined symbolically in terms of α, we can plug in values for α. With each increasing iteration for $\boldsymbol{\pi}_k$, the values in the vector $\boldsymbol{\pi}_k$ converge to $\boldsymbol{\pi}$. For $\alpha = 0.9$, a minimum of 15 iterations are required to obtain the steady state. The resulting PageRank vector after 15 iterations is

$$\boldsymbol{\pi}_{15} = \boldsymbol{\pi} = \begin{bmatrix} 0.0372 \\ 0.0540 \\ 0.0415 \\ 0.3750 \\ 0.2060 \\ 0.2862 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6101 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.4500 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.4500 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.3705 & 0 \\ 0 & 0 & 0 & 1 & 0 & -0.0896 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 0.0715 & 0.2931 & 0.0000 & 0.0000 & -0.0647 & -0.2123 \\ 0.1036 & 0.5094 & 0.0000 & 0.0000 & 0.0139 & 0.8541 \\ 0.0797 & 0.3415 & 0.0000 & 0.0000 & 0.0730 & -0.3636 \\ 0.7204 & -0.5891 & -0.7071 & 0.7071 & -0.6606 & 0.0184 \\ 0.3957 & -0.1414 & 0.7071 & -0.7071 & 0.7376 & -0.3047 \\ 0.5498 & -0.4135 & 0.0000 & 0.0000 & -0.0992 & 0.0082 \end{bmatrix}$$

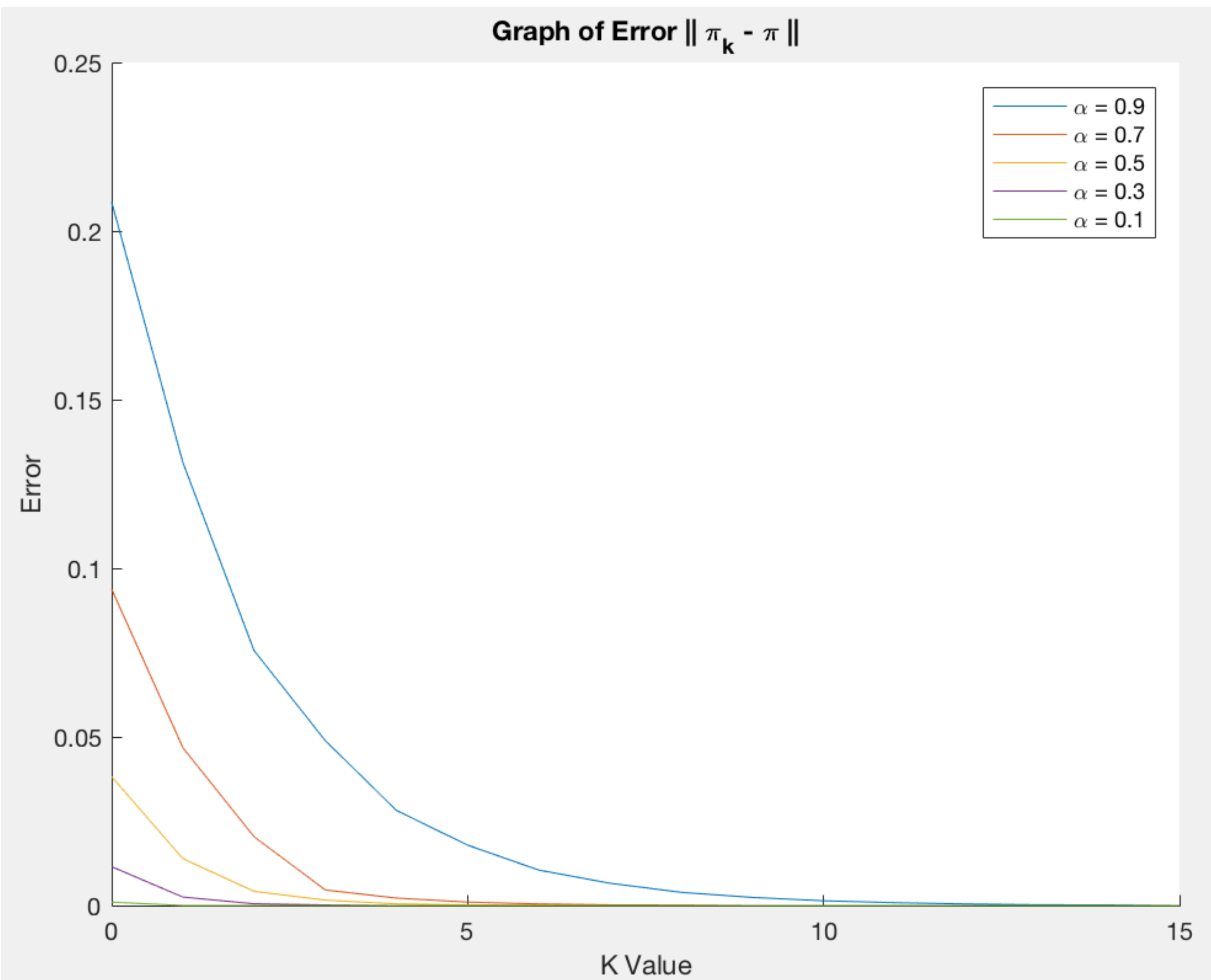
D shows the eigenvalues and **V** shows the corresponding eigenvectors. The PageRank vector is updated iteratively until it converges. The largest eigenvalue and corresponding eigenvector, λ and $\mathbf{v}_{max}$, are important for the update relationship of the Google Matrix, **G**. Because this is designed as a stochastic matrix, as K goes to infinity, the steady state is obtained with the $\boldsymbol{\pi}$ vector. The λ values are raised to the power of K and multiplied by the corresponding coefficient and eigenvector, so that the page vector does not become infinitely large, as would happen if this were a *substochastic* matrix design. The eigenvalue and eigenvector λ and $\mathbf{v}_{max}$ are the last iteration values that contribute to the Page Rank vector, and are the last changes that cause the matrix to converge. The initial vector $\boldsymbol{\pi}_0$ is an arbitrary vector in $\mathbf{R}^6$, and that can be written as a linear combination of eigenvectors of **G**. This is due to the fact that the eigenvectors of **G** form a basis of $\mathbf{R}^6$. While iterating through the $\boldsymbol{\pi}_{k+1}$

Based on the vector, $\boldsymbol{\pi}_{15}$, the ranks of the websites would be ordered P_4, P_6, P_5, P_2, P_3, P_1.

In addition, we can calculate $\boldsymbol{\pi}_k$ for $\alpha = 0.7$, 0.5, 0.3, and 0.1. For each value of α, there is a different number of iterations needed in order for the values of $\boldsymbol{\pi}_k$ to converge to $\boldsymbol{\pi}$. The table below shows the number of iterations necessary and the value of $\boldsymbol{\pi}$ for each value of α.

α	0.9	0.7	0.5	0.3	0.1
k	15	10	7	4	1
$\boldsymbol{\pi}$	$\begin{bmatrix}0.0372\\0.0540\\0.0415\\0.3750\\0.2060\\0.2862\end{bmatrix}$	$\begin{bmatrix}0.0852\\0.1150\\0.0932\\0.2899\\0.1866\\0.2302\end{bmatrix}$	$\begin{bmatrix}0.1162\\0.1452\\0.1245\\0.2390\\0.1759\\0.1992\end{bmatrix}$	$\begin{bmatrix}0.1392\\0.1601\\0.1456\\0.2044\\0.1699\\0.1808\end{bmatrix}$	$\begin{bmatrix}0.1581\\0.1661\\0.1607\\0.1781\\0.1670\\0.1700\end{bmatrix}$

Using this information, we obtain the error that occurs as the iteration increases for each value of α shown in the following graph.



As shown, while the value of α decreases, the number of iterations needed for $\boldsymbol{\pi}_k$ to converge also decreases. Based on the gathered information the most appropriate value that Google should use is 0.1 because it requires only 1 iteration for it to be the most accurate.