Chapter 7: Response of First-Order RL & RC Circuits

EEL 3112c – Circuits-II

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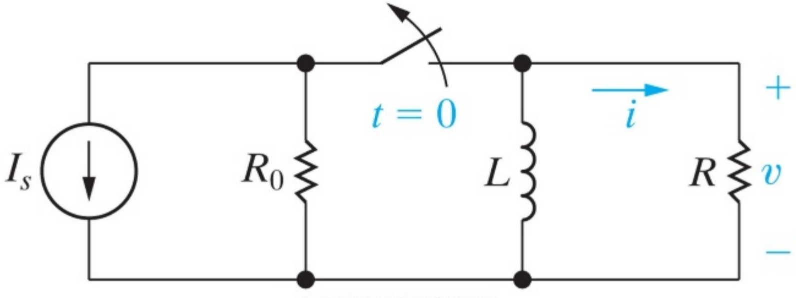
The Natural response of an *RL* circuit

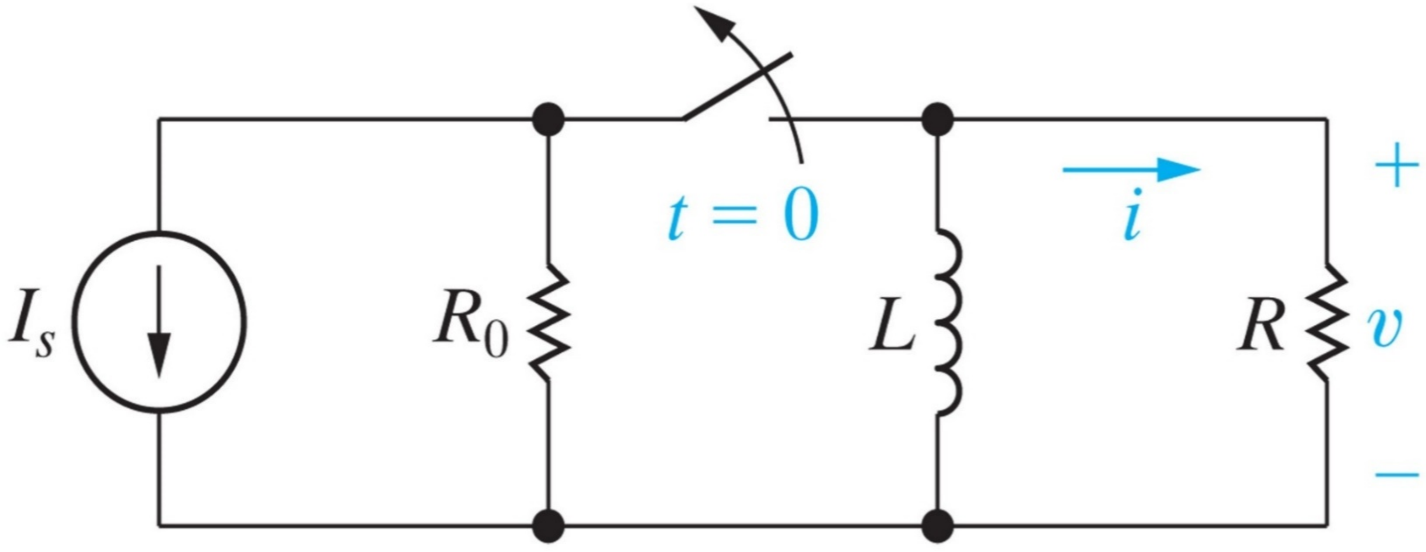
- The natural response of an ***RL*** circuit can best be described in terms of the circuit shown in this Fig.



- We assume that the independent current source generates a constant current of I_s A, and that the switch has been in a closed position for a long time
  - "*a long time*" means that all currents and voltages have reached a constant value
- Thus only constant, or dc, currents exist in the circuit just prior to the switch's being opened, and therefore the inductor appears as a short circuit ($L\frac{di}{dt} = 0$) prior to the release of the stored energy

The Natural response of an *RL* circuit – cont.

- Before we move the switch, the inductor appears as a short circuit, thus the voltage across the inductive branch is zero, and there can be no current in either R_0 or R
- Therefore, all the source current I_s appears in the inductive branch



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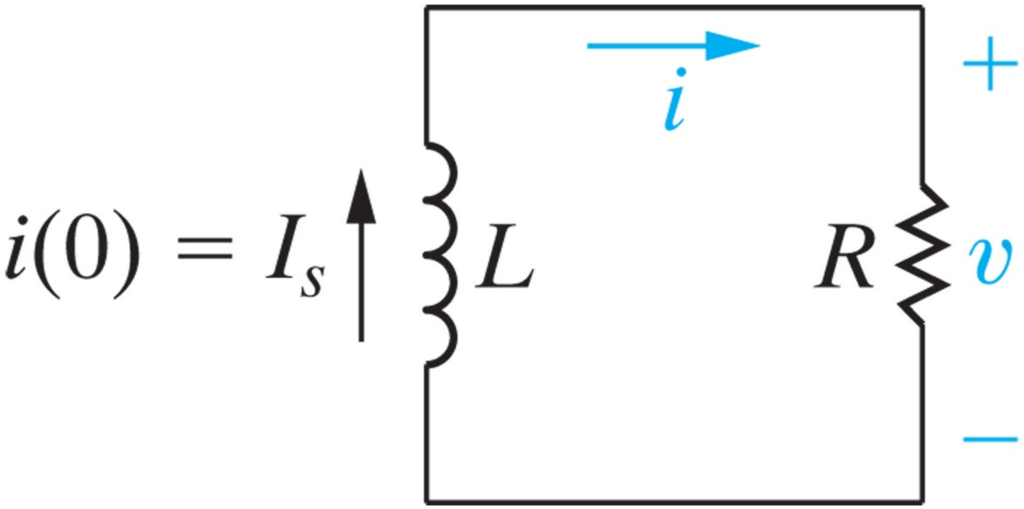
The Natural response of an *RL* circuit – cont.

- Finding the natural response requires finding the voltage and current at the terminals of the resistor after the switch has been opened, that is, after the source has been disconnected and the inductor begins releasing energy
  - If we let $t = 0$ denote the instant when the switch is opened, the problem becomes finding $v(t)$ and $i(t)$ for $t \geq 0$
- For $t \geq 0$, the previous *RL* circuit reduces to the one shown below

Currents and voltages that arise in this configuration are referred to as the **natural response** of the circuit, to emphasize that the nature of the circuit itself , not external sources of excitation, determines its behavior



Using KVL:

$$\boxed{L\frac{di}{dt} + Ri = 0}$$

The Natural response of an *RL* circuit – cont.

- Natural response expression for Current in RL circuits

$$i(t) = I_0 e^{-(R/L)t}, \qquad t \geq 0, \qquad (7.7)$$

◀ **Natural response of an *RL* circuit**

  - where I_0 denotes the initial current in the inductor oriented in the same direction as the reference direction of ***i***
  - The current $i(t)$ starts from an initial value I_0 and decreases exponentially toward zero as t increases
- Recall from Chapter 6 that instantaneous change of current cannot occur in the inductor
  - Therefore, in the first instant after the switch has been opened, the current in the inductor remains uncharged
  - If we use $\mathbf{0^-}$ to denote the time just prior to switching, and $\mathbf{0^+}$ for the time immediately following switching, then

$$i(0^-) = i(0^+) = I_0,$$

◀ **Initial inductor current**

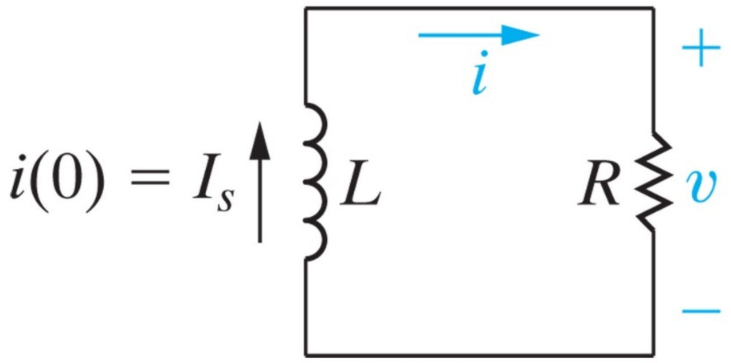
The Natural response of an *RL* circuit – cont.

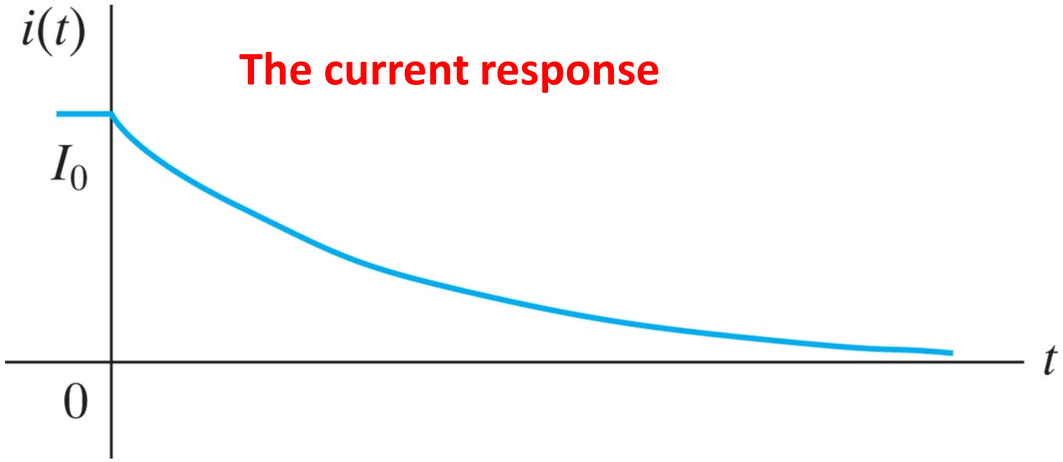
- The current $i(t)$ starts from an initial value I_0 and decreases exponentially toward zero as t increases
- We derive the voltage v across the resistor R from a direct application of Ohm's law:

$$\boxed{v = iR = I_0 R e^{-(R/L)t}, \qquad t \geq 0^+}$$

- Because instantaneous change can occur in inductor voltage, the voltage at 0^+ & 0^-:

$$\boxed{\begin{aligned} v(0^-) &= 0, \\ v(0^+) &= I_0 R, \end{aligned}}$$





The Natural response of an *RL* circuit – cont.

- The expression of $i(t)$ include a term of the form $e^{-\left(\frac{R}{L}\right)t}$

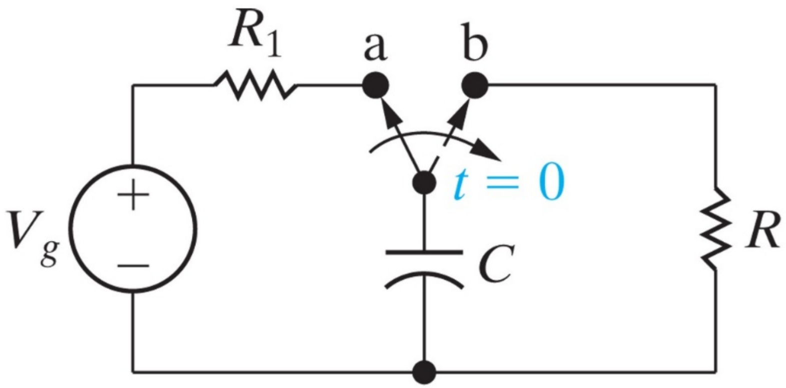
$$i(t) = I_0 e^{-(R/L)t}, \qquad t \geq 0$$

  - The coefficient of t namely: R/L, determines the rate at which the current or voltage approaches zero
  - The reciprocal of this ratio is the **time constant** of the circuit (τ)

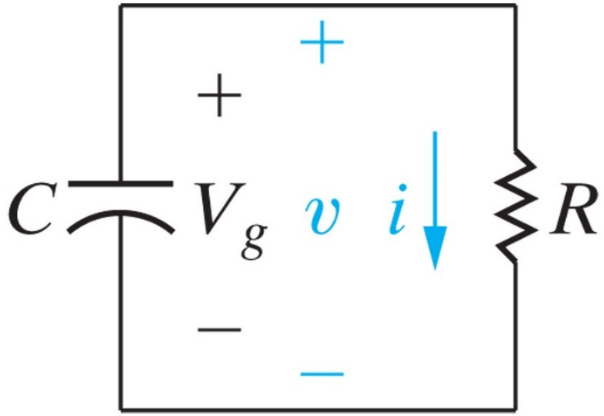
Time constant for *RL* circuit ▶

$$\tau = \text{time constant} = \frac{L}{R}.$$

The Natural response of an *RC* circuit



For $t < 0$, C is open circuit and biased by voltage V_g, while R_1& R carry no current



For $t > 0$, the circuit reduces to this, and the capacitor voltage will decrease (capacitor is discharging) and the energy will dissipate via R

Using KCL at V_g:

$$\boxed{C\frac{dv}{dt} + \frac{v}{R} = 0}$$

We can easily find the voltage by thinking in terms of node voltages. Using the lower junction between *R* and *C* as the reference node and summing the currents away from the upper junction between *R* and *C*

The Natural response of an *RC* circuit – cont.

$$v(0^-) = v(0) = v(0^+) = V_g = V_0, \quad (7.23)$$

◀ **Initial capacitor voltage**

Recall from Chapter 6 that instantaneous change of voltage cannot occur in the capacitor

$$\tau = RC. \quad (7.24)$$

◀ **Time constant for *RC* circuit**

The time constant of the *RC* circuit will equal the product of the Thevenin resistance (as seen from the capacitor) and the capacitance

$$v(t) = V_0 e^{-t/\tau}, \quad t \geq 0, \quad (7.25)$$

◀ **Natural response of an *RC* circuit**

This indicates that the natural response of an *RC* circuit is an exponential decay of the initial voltage. The time constant *RC* controls the rate of decay

Summary of Natural Response

1. Find the initial current, I_0, through the inductor.
2. Find the time constant of the circuit, $\tau = L/R$.
3. Use Eq. 7.15, $I_0 e^{-t/\tau}$, to generate $i(t)$ from I_0 and τ.

◀ **Calculating the natural response of *RL* circuit**

Calculating the natural response of an *RC* circuit ▶

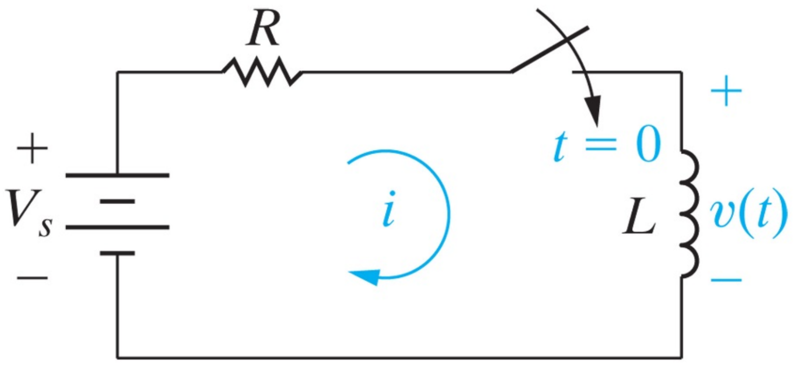
1. Find the initial voltage, V_0, across the capacitor.
2. Find the time constant of the circuit, $\tau = RC$.
3. Use Eq. 7.25, $v(t) = V_0 e^{-t/\tau}$, to generate $v(t)$ from V_0 and τ.

The Step Response of *RL* & *RC* Circuits

- The **natural response is the sudden removal** of the constant voltage or current source in the circuit and then analyze the current and voltage that arise in the discharge path
- The response of a circuit to **the sudden application** of a constant voltage or current source is referred to as the **step response** of the circuit
- We will start with the step response of the *RL* circuit then the *RC* circuit

The Step Response of *RL*

- We will use the circuit shown below to develop the step response of an *RL* circuit
  - The task is to find the expressions for the current in the circuit and for the voltage across the inductor after the switch has been closed
  - We will use circuit analysis to derive the differential equation that describes the circuit in terms of the variable of interest



After switch is closed, KVL requires that:

$$\boxed{V_s = Ri + L\frac{di}{dt},} \Rightarrow \boxed{\frac{di}{dt} = \frac{-Ri + V_s}{L} = \frac{-R}{L}\left(i - \frac{V_s}{R}\right).}$$

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-(R/L)t}. \qquad (7.35)$$

◀ **Step response of *RL* circuit**

After the switch has been closed, the current increases exponentially from zero to a **final value of $\boldsymbol{V_s/R}$**. The time constant of the circuit, L/R, determines the rate of increase

- I_0 is the current at t=0 and $i(t)$ is the current at any $t > 0$
- When the initial energy in the inductor is zero, I_0 is zero

The Step Response of *RL* – *cont.*

- When I_0 is zero, then one time constant after the switch has been closed, the current will have reached approximately 63% of its final value, or

$$i(\tau) = \frac{V_s}{R} - \frac{V_s}{R}e^{-1} \approx 0.6321\frac{V_s}{R}.$$

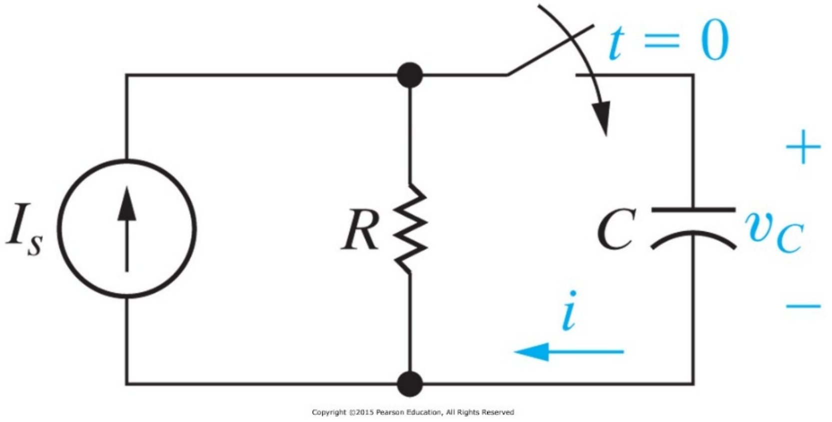
- The voltage across an inductor is $L di/dt$, thus for $t \geq 0^+$

$$\boxed{v = L\left(\frac{-R}{L}\right)\left(I_0 - \frac{V_s}{R}\right)e^{-(R/L)t} = (V_s - I_0 R)e^{-(R/L)t}.}$$

  - This equation indicates that the inductor voltage jumps to $V_s - I_0 R$ at the instant the switch is closed and then decays exponentially to zero

The Step Response of *RC*

- We can find the step response of a first-order *RC* circuit by analyzing the circuit shown below
  - For mathematical convenience, we choose the Norton equivalent of the network connected to the equivalent capacitor



After switch is closed, KCL at the lower node requires that:

$$\boxed{C\frac{dv_C}{dt} + \frac{v_C}{R} = I_s.}$$

$$v_C = I_sR + (V_0 - I_sR)e^{-t/RC}, \quad t \geq 0. \qquad (7.51)$$

◀ **Step response of an *RC* circuit**

When the initial energy in the capacitor is zero, V_0 is zero.

$$i = \left(I_s - \frac{V_0}{R}\right)e^{-t/RC}, \qquad t \geq 0^+,$$

A General Solution

- The general approach to finding either the natural response or the step response of the first-order *RL* and *RC* circuits
- When computing the step and natural responses of circuits, it may help to follow these steps:
    1. Identify the variable of interest for the circuit
        - For RC circuits, it is most convenient to choose the capacitive voltage; for RL circuits, it is best to choose the inductive current
    2. Determine the initial value of the variable, which is the value at t_0
        - Note that if you choose capacitive voltage or inductive current as your variable of interest, it is not necessary to distinguish between $t = 0^-$ and $t = 0^+$. This is because they both are continuous variables. If you choose another variable, you need to remember that its initial value is defined at $t = 0^+$
    3. Calculate the final value of the variable, which is its value as $t \rightarrow \infty$.
    4. Calculate the time constant for the circuit.

A General Solution – cont.

General solution for natural and step responses of *RL* and *RC* circuits ▶

$$x(t) = x_f + [x(t_0) - x_f]e^{-(t-t_0)/\tau}. \quad (7.59)$$

$$\begin{matrix}\text{the unknown} \\ \text{variable as a} \\ \text{function of time}\end{matrix} = \begin{matrix}\text{the final} \\ \text{value of the} \\ \text{variable}\end{matrix} + \left[\begin{matrix}\text{the initial} \\ \text{value of the} \\ \text{variable}\end{matrix} - \begin{matrix}\text{the final} \\ \text{value of the} \\ \text{variable}\end{matrix}\right] \times e^{\frac{-[t-(\text{time of switching})]}{(\text{time constant})}}$$

In many cases, the time of switching—that is t_0—is zero

Sequential Switching

- Whenever switching occurs more than once in a circuit, we have what we call **sequential switching**
  - For example, a single, two-position switch may be switched back and forth, or multiple switches may be opened or closed in sequence
  - The time reference for all switching cannot be $t = 0$
- We determine the voltages and currents generated by a switching sequence by using the techniques described previously in this chapter
- We derive the expressions for $v(t)$and $i(t)$ for a given position of the switch or switches and then use these solutions to determine the initial conditions for the next position of the switch or switches

Sequential Switching Example

The two switches in the circuit shown in Fig. 7.31 have been closed for a long time. At $t = 0$, switch 1 is opened. Then, 35 ms later, switch 2 is opened.

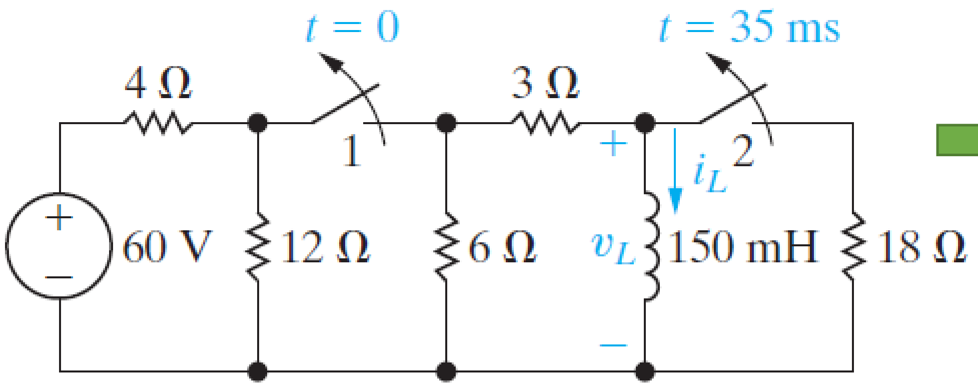


Figure 7.31 ▲ The circuit for Example 7.11.

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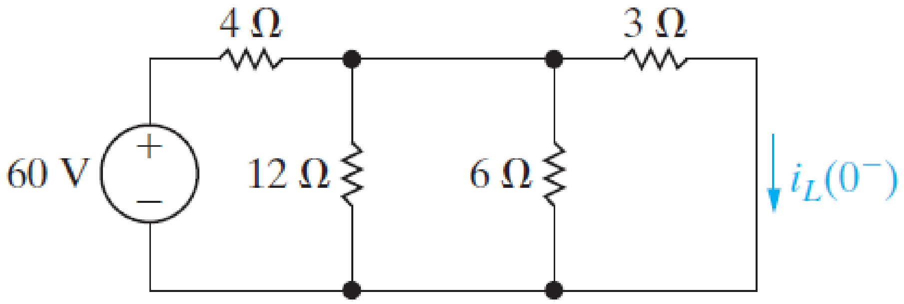

Figure 7.32 ▲ The circuit shown in Fig. 7.31, for $t < 0$.

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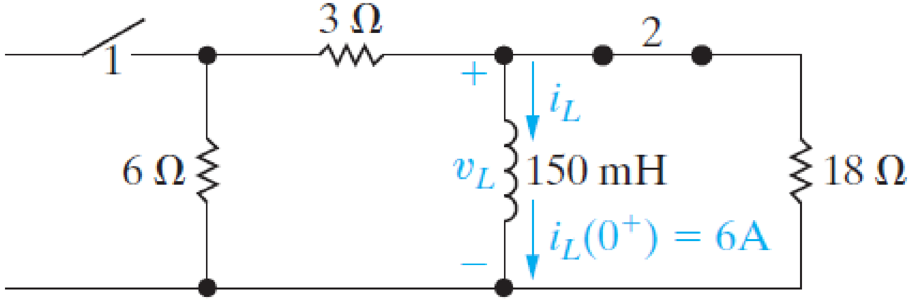

Figure 7.33 ▲ The circuit shown in Fig. 7.31, for $0 \leq t \leq 35$ ms.

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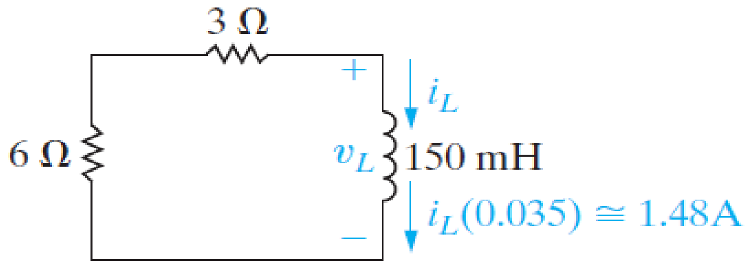

Figure 7.34 ▲ The circuit shown in Fig. 7.31, for $t \geq 35$ ms.

Next Class

- Today we reviewed Ch7
- Next class we will start our discussion about RLC circuits
  - Chapter 8 in the book