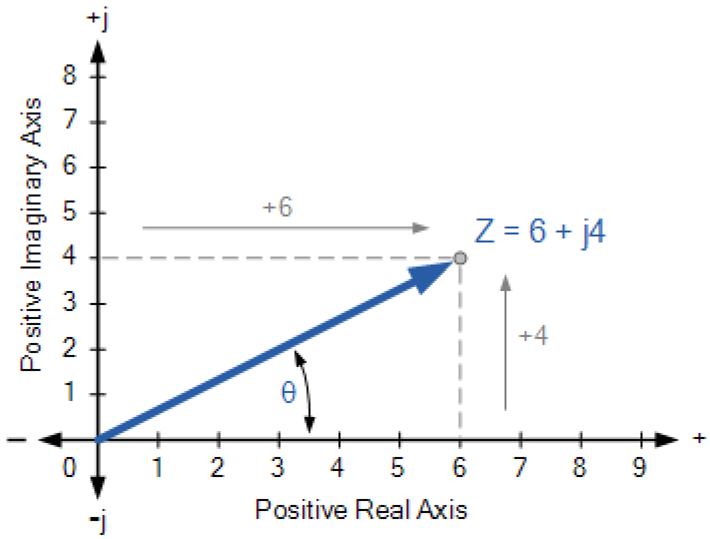


Complex Numbers and Phasors

The mathematics used in Electrical Engineering to add together resistances, currents or DC voltages use what are called "real numbers" used as either integers or as fractions.

But real numbers are not the only kind of numbers we need to use especially when dealing with frequency dependent sinusoidal sources and vectors. As well as using normal or real numbers, **Complex Numbers** were introduced to allow complex equations to be solved with numbers that are the square roots of negative numbers, $\sqrt{-1}$.

In electrical engineering this type of number is called an "imaginary number" and to distinguish an imaginary number from a real number the letter " j " known commonly in electrical engineering as the **j-operator**, is used. Thus the letter "j" is placed in front of a real number to signify its imaginary number operation.

Examples of imaginary numbers are: j3, j12, j100 etc. Then a **complex number** consists of two distinct but very much related parts, a " Real Number " plus an " Imaginary Number ".

Complex Numbers represent points in a two dimensional complex or s-plane that are referenced to two distinct axes. The horizontal axis is called the "real axis" while the vertical axis is called the "imaginary axis". The real and imaginary parts of a complex number are abbreviated as Re(z) and Im(z), respectively.

Complex numbers that are made up of real (the active component) and imaginary (the reactive component) numbers can be added, subtracted and used in exactly the same way as elementary algebra is used to analyse DC Circuits.

The rules and laws used in mathematics for the addition or subtraction of imaginary numbers are the same as for real numbers, j2 + j4 = j6 etc. The only difference is in multiplication because two imaginary numbers multiplied together becomes a negative real number. Real numbers can also be thought of as a complex number but with a zero imaginary part labelled j0.

The **j-operator** has a value exactly equal to $\sqrt{-1}$, so successive multiplication of " j ", (j x j) will result in j having the following values of, -1, -j and +1. As the j-operator is commonly used to indicate the anticlockwise rotation of a vector, each successive multiplication or power of " j ", j^2, j^3 etc, will force the vector to rotate through a fixed angle of 90° in an anticlockwise direction as shown below. Likewise, if the multiplication of the vector results in a -j operator then the phase shift will be -90°, i.e. a clockwise rotation.

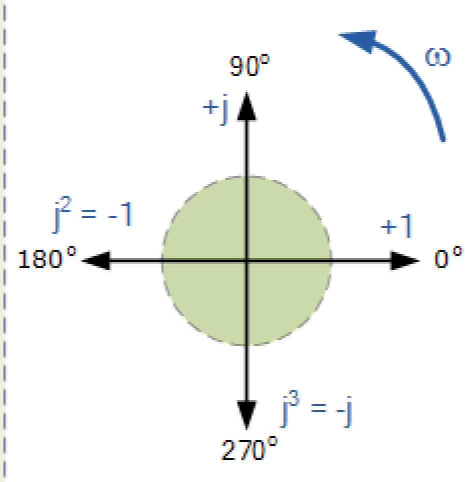
Vector Rotation of the j-operator

$$90^{\circ}\ \text{rotation:}\quad j^{1} = \sqrt{-1} = +j$$

$$180^{\circ}\ \text{rotation:}\quad j^{2} = \left(\sqrt{-1}\right)^{2} = -1$$

$$270^{\circ}\ \text{rotation:}\quad j^{3} = \left(\sqrt{-1}\right)^{3} = -j$$

$$360^{\circ}\ \text{rotation:}\quad j^{4} = \left(\sqrt{-1}\right)^{4} = +1$$



So by multiplying an imaginary number by j^2 will rotate the vector by 180° anticlockwise, multiplying by j^3 rotates it 270° and by j^4 rotates it 360° or back to its original position. Multiplication by j^{10} or by j^{30} will cause the vector to rotate anticlockwise by the appropriate amount. In each successive rotation, the magnitude of the vector always remains the same.

In Electrical Engineering there are different ways to represent a complex number either graphically or mathematically. One such way that uses the cosine and sine rule is called the **Cartesian** or **Rectangular Form**.

Complex Numbers using the Rectangular Form

In the last tutorial about **Phasors**, we saw that a complex number is represented by a real part and an imaginary part that takes the generalised form of:

$$Z = x + jy$$

Where:

Z - is the Complex Number representing the Vector

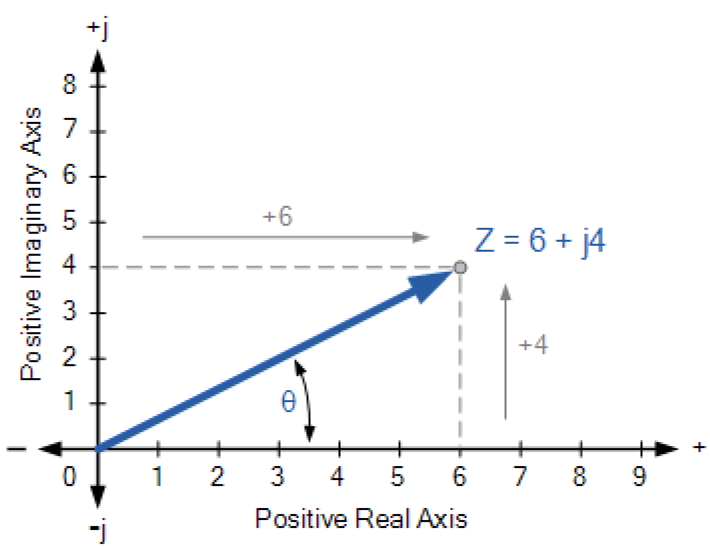
x - is the Real part or the Active component

y - is the Imaginary part or the Reactive component

j - is defined by $\sqrt{-1}$

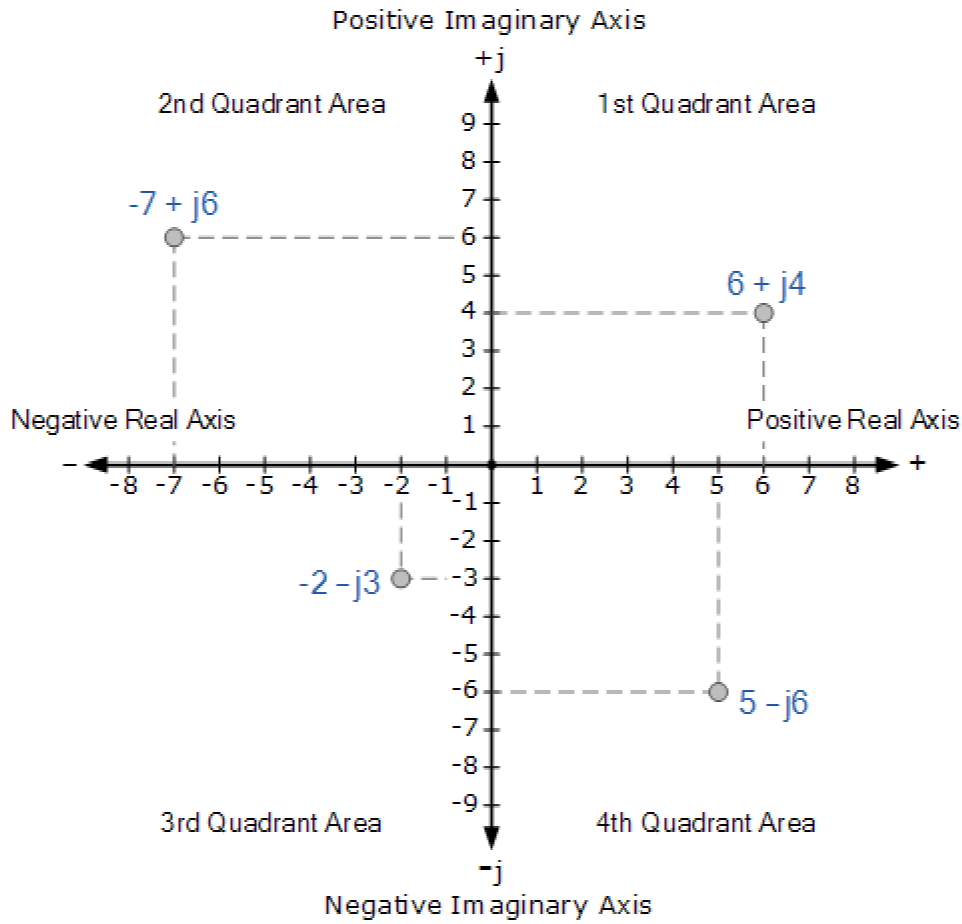
In the rectangular form, a complex number can be represented as a point on a two dimensional plane called the **complex** or **s-plane**. So for example, $Z = 6 + j4$ represents a single point whose coordinates represent 6 on the horizontal real axis and 4 on the vertical imaginary axis as shown.

Complex Numbers using the Complex or s-plane



But as both the real and imaginary parts of a complex number in the rectangular form can be either a positive number or a negative number, then both the real and imaginary axis must also extend in both the positive and negative directions. This then produces a complex plane with four quadrants called an **Argand Diagram** as shown below.

Four Quadrant Argand Diagram



On the Argand diagram, the horizontal axis represents all positive real numbers to the right of the vertical imaginary axis and all negative real numbers to the left of the vertical imaginary axis. All positive imaginary numbers are represented above the horizontal axis while all the negative imaginary numbers are below the horizontal real axis. This then produces a two dimensional complex plane with four distinct quadrants labelled, QI, QII, QIII, and QIV.

The Argand diagram above can also be used to represent a rotating phasor as a point in the complex plane whose radius is given by the magnitude of the phasor will draw a full circle around it for every $2\pi/\omega$ seconds.

Then we can extend this idea further to show the definition of a complex number in both the polar and rectangular form for rotations of 90°.

$$0^\circ = \pm 360^\circ = +1 = 1\angle 0^\circ = 1 + j0$$

$$+90^\circ = +\sqrt{-1} = +j = 1\angle +90^\circ = 0 + j1$$

$$-90^\circ = -\sqrt{-1} = -j = 1\angle -90^\circ = 0 - j1$$

$$\pm 180^\circ = \left(\sqrt{-1}\right)^2 = -1 = 1\angle \pm 180^\circ = -1 + j0$$

Complex Numbers can also have "zero" real or imaginary parts such as: Z = 6 + j0 or Z = 0 + j4. In this case the points are plotted directly onto the real or imaginary axis. Also, the angle of a complex number can be calculated using simple trigonometry to calculate the angles of right-angled triangles, or measured anti-clockwise around the Argand diagram starting from the positive real axis.

Then angles between 0 and 90° will be in the first quadrant (I), angles (θ) between 90 and 180° in the second quadrant (II). The third quadrant (III) includes angles between 180 and 270° while the fourth and final quadrant (IV) which completes the full circle, includes the angles between 270 and 360° and so on. In all the four quadrants the relevant angles can be found from:

$$\tan^{-1}(\text{imaginary component} \div \text{real component})$$

Addition and Subtraction of Complex Numbers

The addition or subtraction of complex numbers can be done either mathematically or graphically in rectangular form. For addition, the real parts are firstly added together to form the real part of the sum, and then the imaginary parts to form the imaginary part of the sum and this process is as follows using two complex numbers A and B as examples.

Complex Addition and Subtraction

$$A = x + jy \qquad B = w + jz$$

$$A + B = (x + w) + j(y + z)$$

$$A - B = (x - w) + j(y - z)$$

Complex Numbers Example No1

Two vectors are defined as, A = 4 + j1 and B = 2 + j3 respectively. Determine the sum and difference of the two vectors in both rectangular (a + jb) form and graphically as an Argand Diagram.

Mathematical Addition and Subtraction

Addition

$$A + B = \left(4 + j1\right) + \left(2 + j3\right)$$

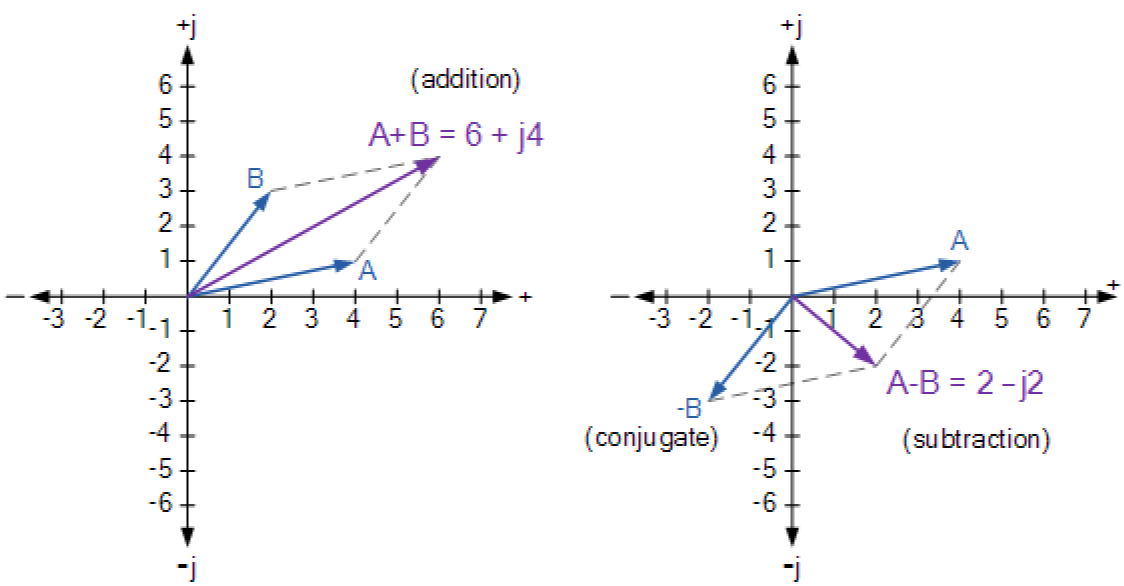
$$A + B = \left(4 + 2\right) + j\left(1 + 3\right) = 6 + j4$$

Subtraction

$$A - B = \left(4 + j1\right) - \left(2 + j3\right)$$

$$A + B = \left(4 - 2\right) + j\left(1 - 3\right) = 2 - j2$$

Graphical Addition and Subtraction



Multiplication and Division of Complex Numbers

The multiplication of complex numbers in the rectangular form follows more or less the same rules as for normal algebra along with some additional rules for the successive multiplication of the j-operator where: $j^2 = -1$. So for example, multiplying together our two vectors from above of A = 4 + j1 and B = 2 + j3 will give us the following result.

$$A \times B = (4 + j1)(2 + j3)$$

$$= 8 + j12 + j2 + j^2 3$$

$$\text{but } j^2 = -1,$$

$$= 8 + j14 - 3$$

$$A \times B = 5 + j14$$

Mathematically, the division of complex numbers in rectangular form is a little more difficult to perform as it requires the use of the denominators conjugate function to convert the denominator of the equation into a real number. This is called "rationalising". Then the division of complex numbers is best carried out using "Polar Form", which we will look at later. However, as an example in rectangular form lets find the value of vector A divided by vector B.

$$\frac{A}{B} = \frac{4 + j1}{2 + j3}$$

Multiply top & bottom by Conjugate of 2 + j3

$$\frac{4 + j1}{2 + j3} \times \frac{2 - j3}{2 - j3} = \frac{8 - j12 + j2 - j^2 3}{4 - j6 + j6 - j^2 9}$$

$$= \frac{8 - j10 + 3}{4 + 9} = \frac{11 - j10}{13}$$

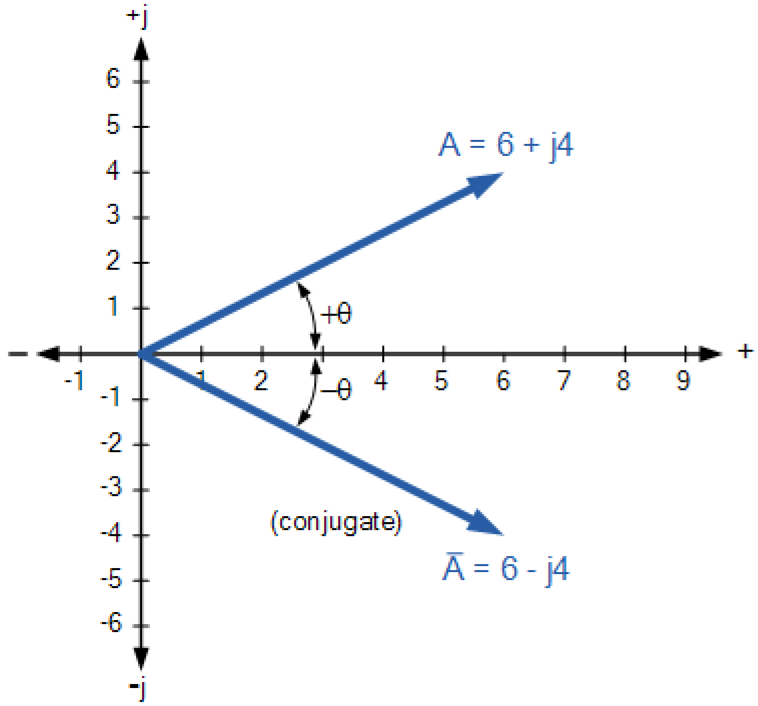
$$= \frac{11}{13} + \frac{-j10}{13} = 0.85 - j0.77$$

The Complex Conjugate

The **Complex Conjugate**, or simply **Conjugate** of a complex number is found by reversing the algebraic sign of the complex numbers imaginary number only while keeping the algebraic sign of the real number the same and to identify the complex conjugate of z the symbol $\bar{z}$ is used. For example, the conjugate of z = 6 + j4 is $\bar{z}$ = 6 – j4, likewise the conjugate of z = 6 – j4 is $\bar{z}$ = 6 + j4.

The points on the Argand diagram for a complex conjugate have the same horizontal position on the real axis as the original complex number, but opposite vertical positions. Thus, complex conjugates can be thought of as a reflection of a complex number. The following example shows a complex number, 6 + j4 and its conjugate in the complex plane.

Conjugate Complex Numbers

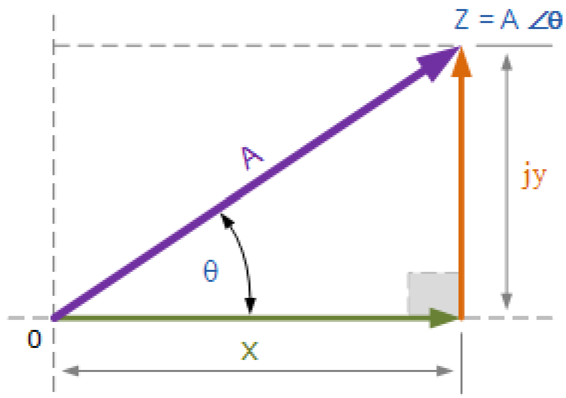


The sum of a complex number and its complex conjugate will always be a real number as we have seen above. Then the addition of a complex number and its conjugate gives the result as a real number or active component only, while their subtraction gives an imaginary number or reactive component only. The conjugate of a complex number is an important element used in Electrical Engineering to determine the apparent power of an AC circuit using rectangular form.

Complex Numbers using Polar Form

Unlike rectangular form which plots points in the complex plane, the **Polar Form** of a complex number is written in terms of its magnitude and angle. Thus, a polar form vector is presented as: $Z = A \angle \pm \theta$, where: Z is the complex number in polar form, A is the magnitude or modulo of the vector and θ is its angle or argument of A which can be either positive or negative. The magnitude and angle of the point still remains the same as for the rectangular form above, this time in polar form the location of the point is represented in a "triangular form" as shown below.

Polar Form Representation of a Complex Number



As the polar representation of a point is based around the triangular form, we can use simple geometry of the triangle and especially trigonometry and Pythagoras's Theorem on triangles to find both the magnitude and the angle of the complex number. As we remember from school, trigonometry deals with the relationship between the sides and the angles of triangles so we can describe the relationships between the sides as:

$$A^2 = x^2 + y^2$$

$$A = \sqrt{x^2 + y^2}$$

$$\text{Also,} \quad x = A.\cos\theta, \quad y = A.\sin\theta$$

Using trigonometry again, the angle θ of A is given as follows.

$$\theta = \tan^{-1}\frac{y}{x}$$

Then in Polar form the length of A and its angle represents the complex number instead of a point. Also in polar form, the conjugate of the complex number has the same magnitude or modulus it is the sign of the angle that changes, so for example the conjugate of $6 \angle 30^o$ would be $6 \angle - 30^o$.

Converting between Rectangular Form and Polar Form

In the rectangular form we can express a vector in terms of its rectangular coordinates, with the horizontal axis being its real axis and the vertical axis being its imaginary axis or j-component. In polar form these real and imaginary axes are simply represented by "$A \angle\theta$". Then using our example above, the relationship between rectangular form and polar form can be defined as.

Converting Polar Form into Rectangular Form, (P→R)

$$6\angle 30^{o} = x + jy$$

However,

$$x = A.\cos\theta \qquad y = A.\sin\theta$$

Therefore,

$$\begin{aligned} 6\angle 30^{0} &= (6\cos\theta) + j(6\sin\theta) \\ &= (6\cos 30^{o}) + j(6\sin 30^{o}) \\ &= (6\times 0.866) + j(6\times 0.5) \\ &= 5.2 + j3 \end{aligned}$$

We can also convert back from rectangular form to polar form as follows.

Converting Rectangular Form into Polar Form, (R→P)

$$\left(5.2 + j3\right) = A\angle\theta$$

$$\text{where:} \quad A = \sqrt{5.2^{2} + 3^{2}} = 6$$

$$\text{and} \quad \theta = \tan^{-1}\frac{3}{5.2} = 30^{o}$$

$$\text{Hence,} \quad \left(5.2 + j3\right) = 6\angle 30^{0}$$

Polar Form Multiplication and Division

Rectangular form is best for adding and subtracting complex numbers as we saw above, but polar form is often better for multiplying and dividing. To multiply together two vectors in polar form, we must first multiply together the two modulus or magnitudes and then add together their angles.

Multiplication in Polar Form

$$Z_1 \times Z_2 = A_1 \times A_2 \angle \theta_1 + \theta_2$$

Multiplying together $6\angle 30^o$ and $8\angle -45^o$ in polar form gives us.

$$Z_1 \times Z_2 = 6 \times 8 \angle 30^o + (-45^o) = 48 \angle -15^o$$

Division in Polar Form

Likewise, to divide together two vectors in polar form, we must divide the two modulus and then subtract their angles as shown.

$$\frac{Z_1}{Z_2} = \left(\frac{A_1}{A_2}\right) \angle \theta_1 - \theta_2$$

$$\frac{Z_1}{Z_2} = \left(\frac{6}{8}\right) \angle 30^o - (-45^o) = 0.75 \angle 75^o$$

Fortunately today's modern scientific calculators have built in mathematical functions (check your book) that allows for the easy conversion of rectangular to polar form, ($R \rightarrow P$) and back from polar to rectangular form, ($R \rightarrow P$).

Complex Numbers using Exponential Form

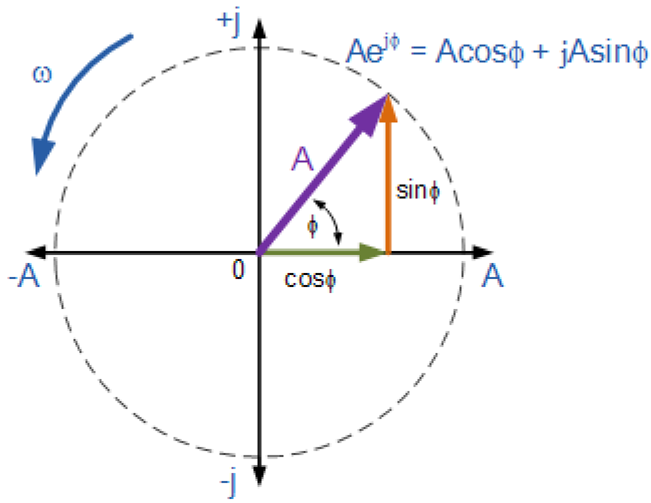
So far we have considered complex numbers in the **Rectangular Form**, ($a + jb$) and the **Polar Form**, ($A \angle \pm\theta$). But there is also a third method for representing a complex number which is similar to the polar form that corresponds to the length (magnitude) and phase angle of the sinusoid but uses the base of the natural logarithm, e = 2.718 281.. to find the value of the complex number. This third method is called the **Exponential Form**.

The **Exponential Form** uses the trigonometric functions of both the sine ($\sin$) and the cosine ($\cos$) values of a right angled triangle to define the complex exponential as a rotating point in the complex plane. The exponential form for finding the position of the point is based around **Euler's Identity**, named after Swiss mathematician, Leonhard Euler and is given as:

$$Z = Ae^{j\phi}$$

$$Z = A\left(\cos\phi + j\sin\phi\right)$$

Then Euler's identity can be represented by the following rotating phasor diagram in the complex plane.



We can see that Euler's identity is very similar to the polar form above and that it shows us that a number such as $Ae^{j\theta}$ which has a magnitude of 1 is also a complex number. Not only can we convert complex numbers that are in exponential form easily into polar form such as: $2e^{j30} = 2\angle 30$, $10e^{j120} = 10\angle 120$ or $-6e^{j90} = -6\angle 90$, but Euler's identity also gives us a way of converting a complex number from its exponential form into its rectangular form. Then the relationship between, Exponential, Polar and Rectangular form in defining a complex number is given as.

Complex Number Forms

$$Z = x + jy = A\angle\theta = A\left(\cos\phi + j\sin\phi\right)$$

Phasor Notation

So far we have look at different ways to represent either a rotating vector or a stationary vector using complex numbers to define a point on the complex plane. Phasor notation is the process of constructing a single complex number that has the amplitude and the phase angle of the given sinusoidal waveform.

Then phasor notation or phasor transform as it is sometimes called, transfers the real part of the sinusoidal function: $A_{(t)} = A_m \cos(\omega t \pm \Phi)$ from the time domain into the complex number domain which is also called the frequency domain. For example:

$$V_{(t)} = V_m \cos(\omega t + \theta) \quad \Leftrightarrow \quad \text{Euler's identity: } e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$V_{(t)} = 20\cos\left(\omega t + 30^\circ\right) \text{Volts} \Rightarrow \left(\frac{20}{\sqrt{2}}\right)\angle 30^\circ \Rightarrow V_{RMS} = 14.14\angle 30^\circ$$

or

$$V_{(t)} = 35\cos\left(\omega t + 45^\circ\right) \text{Volts} \Rightarrow \left(\frac{35}{\sqrt{2}}\right)\angle 45^\circ \Rightarrow V_{RMS} = 24.75\angle 45^\circ$$

or

$$V_{(t)} = 10\cos\left(\omega t - 30^\circ\right) \text{Volts} \Rightarrow \left(\frac{10}{\sqrt{2}}\right)\angle\text{-}30^\circ \Rightarrow V_{RMS} = 7.07\angle\text{-}30^\circ$$

or

$$V_{(t)} = 100\cos(\omega t) \text{ Volts} \Rightarrow \left(\frac{100}{\sqrt{2}}\right)\angle 0^\circ \Rightarrow V_{RMS} = 70.72\angle 0^\circ$$

Please note that the $\sqrt{2}$ converts the maximum amplitude into an *effective* or RMS value with the phase angle given in radians, (ω).

Summary of Complex Numbers

Then to summarize this tutorial about **Complex Numbers** and the use of complex numbers in electrical engineering.

- Complex Numbers consist of two distinct numbers, a real number plus an imaginary number.
- Imaginary numbers are distinguish from a real number by the use of the j-operator.
- A number with the letter " j " in front of it identifies it as an imaginary number in the complex plane.
- By definition, the j-operator $j \equiv \sqrt{-1}$
- Imaginary numbers can be added, subtracted, multiplied and divided the same as real numbers.
- The multiplication of " j " by " j " gives $j^2 = -1$
- In Rectangular Form a complex number is represented by a point in space on the complex plane.
- In Polar Form a complex number is represented by a line whose length is the amplitude and by the phase angle.

- In Exponential Form a complex number is represented by a line and corresponding angle that uses the base of the natural logarithm.
- A complex number can be represented in one of three ways:

$Z = x + jy$ » Rectangular Form

$Z = A\angle\Phi$ » Polar Form

$Z = A\,e^{j\Phi}$ » Exponential Form

- Euler's identity can be used to convert Complex Numbers from exponential form into rectangular form.

In the previous tutorials including this one we have seen that we can use phasors to represent sinusoidal waveforms and that their amplitude and phase angle can be written in the form of a complex number. We have also seen that **Complex Numbers** can be presented in rectangular, polar or exponential form with the conversion between each complex number algebra form including addition, subtracting, multiplication and division.

In the next few tutorials relating to the phasor relationship in AC series circuits, we will look at the impedance of some common passive circuit components and draw the phasor diagrams for both the current flowing through the component and the voltage applied across it starting with the AC Resistance.

53 Comments