Chapter 14: Introduction to Frequency Selective Circuits

EEL 3112c – Circuits-II

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Topics to be Covered in this Chapter

- In this chapter we will discuss:
    - Introduction to Frequency Selective Circuits
    - Low-Pass Filters
    - High-Pass Filters
    - Bandpass Filters
    - Bandreject Filters
- We will cover sections 14.1 – 14.5

Introduction

- **Up to this point** in our analysis of circuits with sinusoidal sources, the source frequency was held constant
- In this chapter, we analyze the effect of **varying source frequency** on circuit voltages and currents
    - The result of this analysis is the **frequency response** of a circuit
- We've seen in previous chapters that a circuit's response depends on the types of elements in the circuit, the way the elements are connected, and the impedance of the elements
- Although varying the frequency of a sinusoidal source does not change the element types or their connections, it does alter the impedance of capacitors and inductors
    - The impedance of these elements is a function of frequency

Introduction – cont.

- As we will see, the careful choice of circuit elements, their values, and their connections to other elements enables us to construct circuits that pass to the output only those input signals that reside in a desired range of frequencies
  - Such circuits are called **frequency-selective circuits**
  - Frequency-selective circuits are also called **filters** because of their ability to filter out certain input signals on the basis of frequency
- Note that no practical frequency-selective circuit can completely filter out selected frequencies
  - Filters **attenuate** the effect of any input signals with frequencies outside a particular frequency band of interest
- We begin this chapter by analyzing circuits from each of the four major categories of filters: low pass, high pass, band pass, and band reject filters

Some Preliminaries

- Recall from Ch13 that the transfer function of a circuit provides an easy way to compute the steady-state response to a sinusoidal input
  - There, we considered only fixed-frequency sources
- To study the frequency response of a circuit, we replace a fixed-frequency sinusoidal source with a varying frequency sinusoidal source
- The steady-state response due to a sinusoidal input $Acos(\omega t)$ is determined by studying the transfer function $H(s)$ along the imaginary axis, i.e. $H(j\omega)$
  - $H(j\omega)$ is called the frequency response of a system or a circuit
- Since $H(j\omega) \in C$ (i.e. is a complex number), a frequency response plot consists of two parts: (1) magnitude plot $|H(j\omega)|$, (2) phase angle plot $\theta(j\omega)$

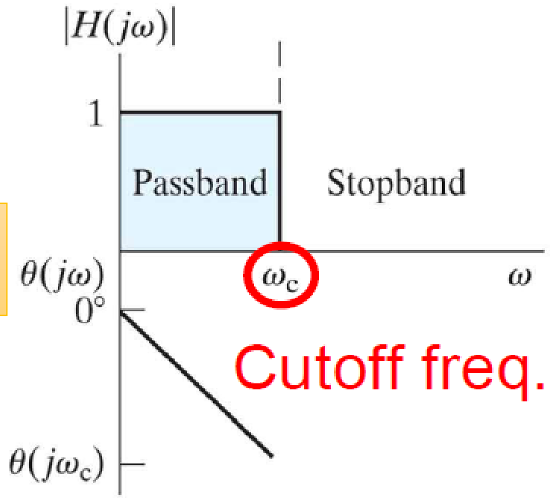
Some Preliminaries – cont.

- The signals passed from the input to the output fall within a band of frequencies called the **passband**
  - Input voltages outside this band have their magnitudes attenuated by the circuit and are effectively prevented from reaching the output terminals of the circuit
- Frequencies not in a circuit's passband are in its **stopband**
- Frequency-selective circuits are categorized by the location of the passband
- One way of identifying the type of frequency-selective circuit is to examine a **frequency response plot**
  - A frequency response plot shows how a circuit's transfer function (both amplitude and phase) changes as the source frequency changes
- A frequency response plot has two parts
  - One is a graph of $|H(j\omega)|$ versus frequency ω called the **magnitude plot**
  - Second graph is a graph of $\theta(j\omega)$ versus frequency ω called the **phase angle plot**

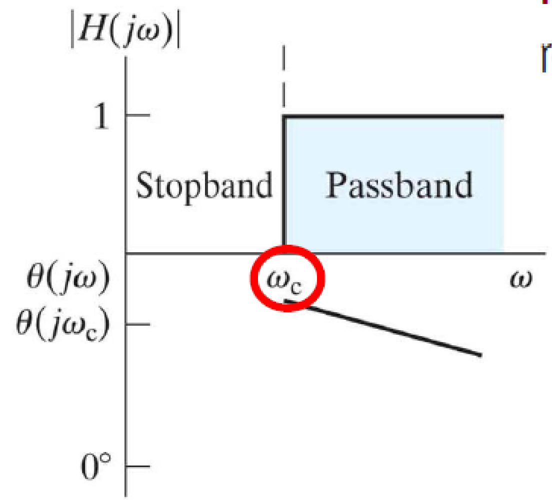
Four Types of **Ideal Filters**

Figure 14.1 ▲ The action of a filter on an input signal results in an output signal.

Low pass filter

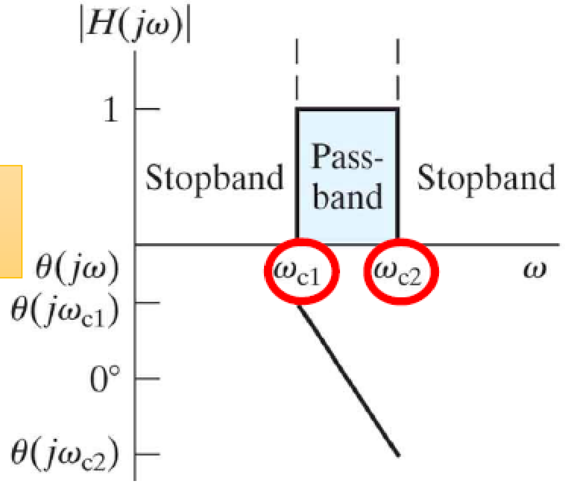


High pass filter

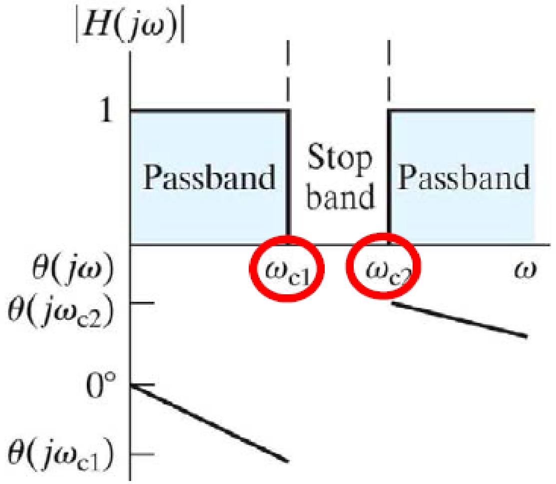


All of the filters we will consider in this chapter are **passive filters**, because their filtering capabilities depend only on the passive elements: resistors, capacitors, and inductors

Band pass filter

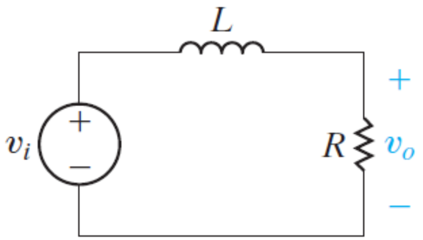


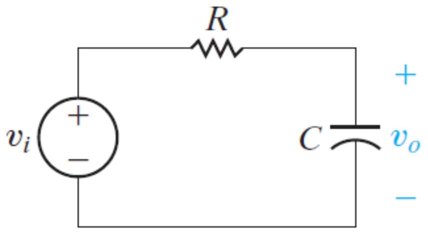
Band reject filter



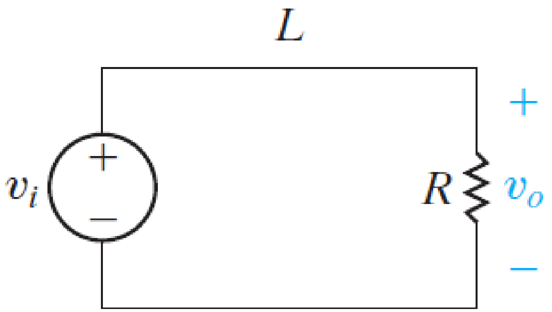
Low-Pass Filters: The series *RL* circuit

- Two different circuit connections can behave as a low-pass filter
  - The series *RL* circuit and the series *RC* circuit

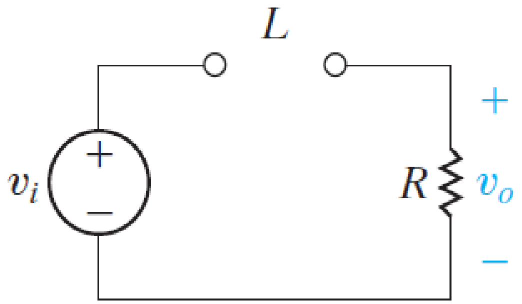




- **The series *RL* circuit**
  - Recall that the impedance of the inductor is $Z_L = j\omega L$, it is a function of frequency



- At low freqs, Z_L is very small compared to R, and the inductor operates as a short circuit
- The term ***low freqs*** refers to frequencies for which $\omega L \ll R$
- In this equivalent circuit for $\omega = 0$, the o/p and the i/p voltages are equal in magnitude & phase



- At high freqs, Z_L is very large compared to R, and inductor functions as an open circuit
- The term ***high freqs*** refers to frequencies for which $\omega L \gg R$
- In this equivalent circuit for $\omega = \infty$, the inductor is an open circuit

- Based on the behavior of the output voltage magnitude, this series *RL* circuit selectively passes low-freq i/p to the o/p, and it blocks high-freq i/p from reaching the o/p

Low-Pass Filters: The series RL circuit – cont.

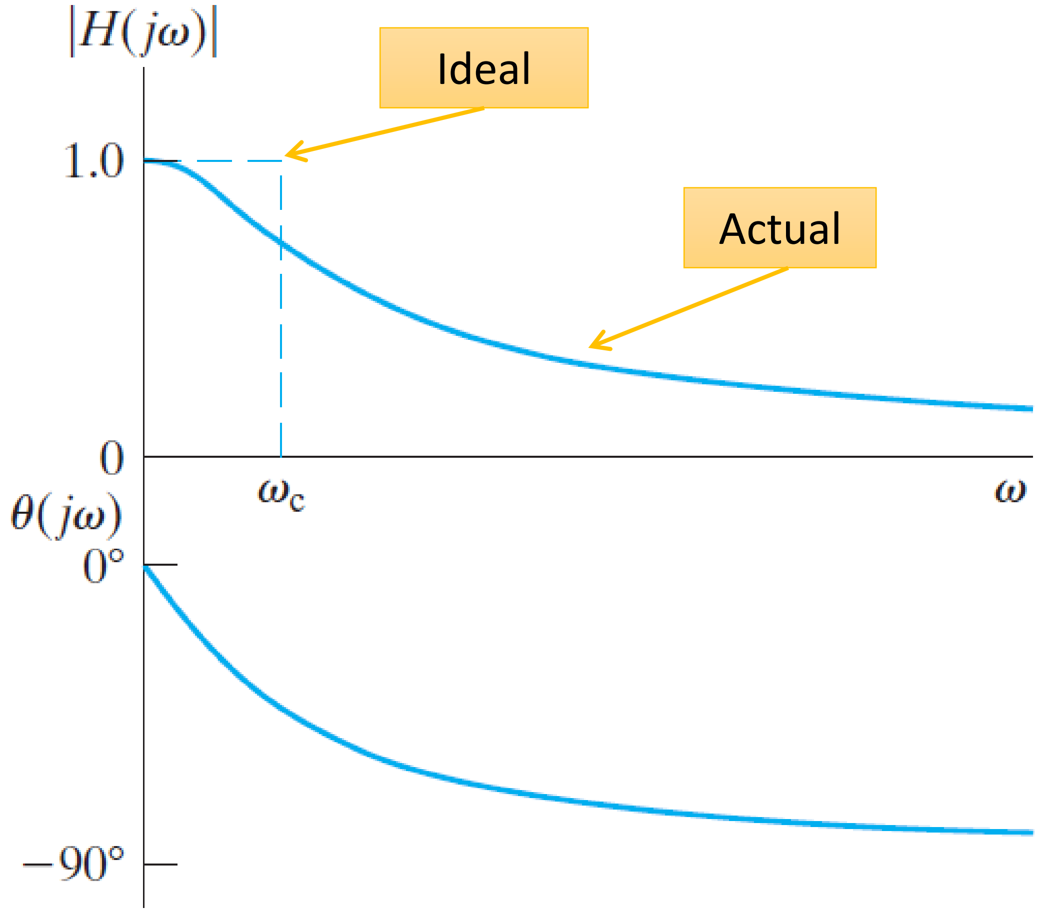


Figure 14.5 ▲ The frequency response plot for the series *RL* circuit in Fig. 14.4(a).

- These two plots comprise the frequency response plots of the series *RL* circuit
  - The upper plot shows how $|H(j\omega)|$ varies with frequency
  - The lower plot shows how $\theta(\omega)$ varies with frequency
- Actual circuits acting as low-pass filters have a magnitude response that changes gradually from the passband to the stopband
  - The magnitude plot of a real circuit requires us to define what we mean by the cutoff frequency ω_c
- For the phase plot, we notice that the phase difference between the input and the output voltage starts with being zero (when the inductor is a short circuit at $\omega = 0$) and continues to gradually increase to reach $-90°$ when $\omega \to \infty$

Defining Cutoff Frequency

- For realistic filter circuits, when the magnitude plot does not allow us to identify a single frequency that divides the passband and the stopband, we need to define the cutoff frequency ω_c
- The definition for cutoff frequency widely used by electrical engineers is the frequency for which the transfer function magnitude is decreased by the factor $1/\sqrt{2}$ from its maximum value:

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{\max}, \qquad (14.1)$$

◀ **Cutoff frequency definition**

  - Where H_{max} is the maximum magnitude of the transfer function
- It follows from this eqn. that the passband of a realizable filter is defined as the range of frequencies in which the amplitude of the output voltage is at least 70.7% of the maximum possible amplitude

Defining Cutoff Frequency – cont.

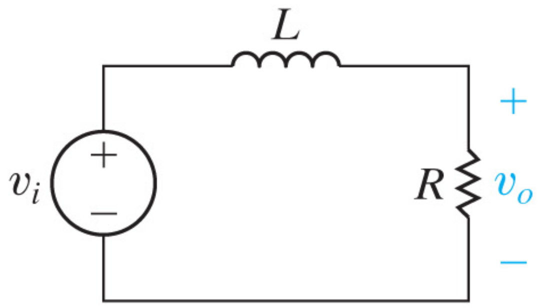
- The constant $1/\sqrt{2}$ used in defining ω_c results in an average power delivered by the circuit is one half the maximum average power
- Thus, ω_c is also called the half power frequency
  - Therefore, in the passband, the average power delivered to the load is at least 50% of the maximum average power

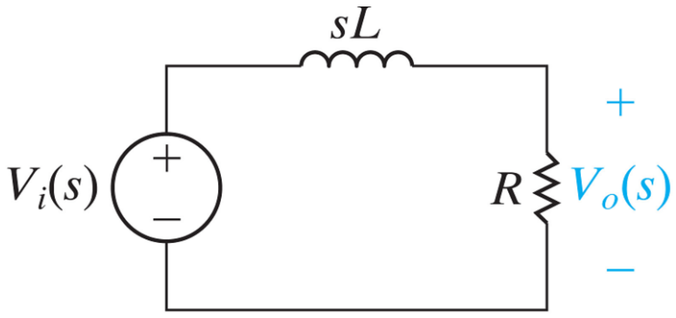
$$
\begin{aligned}
P(j\omega_c) &= \frac{|V_L^2(j\omega_c)|}{R} \\
&= \frac{\left(\frac{1}{\sqrt{2}} V_{L\max}\right)^2}{R} \\
&= \frac{V_{L\max}^2/2}{R} \\
&= \frac{P_{\max}}{2}.
\end{aligned}
$$

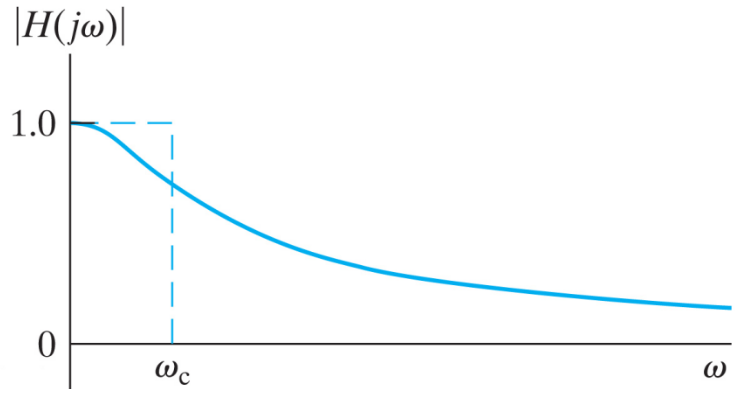
V_L is the voltage drop across the load

The Series *RL* Circuit - Quantitative Analysis

- How is the cutoff frequency, ω_c, related to the component values in the circuit?







$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{R+sL} = \frac{R/L}{s+R/L}$$

$$H(j\omega) = \frac{R/L}{j\omega + R/L}; \qquad |H(j\omega)| = \frac{R/L}{\sqrt{\omega^2 + (R/L)^2}} \; ; \; \theta(j\omega) = -\tan^{-1}\left(\frac{\omega L}{R}\right)$$

For low-pass filters, $H_{max} = |H(j0)|$, at $\omega = 0$ (dc):

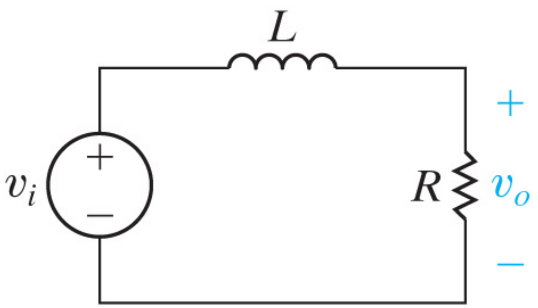
$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}|1| = \frac{R/L}{\sqrt{\omega_c^2 + (R/L)^2}}.$$

Remember that ω_c is defined as the frequency at which $|H(j\omega_c)| = \left(\frac{1}{\sqrt{2}}\right) H_{max}$

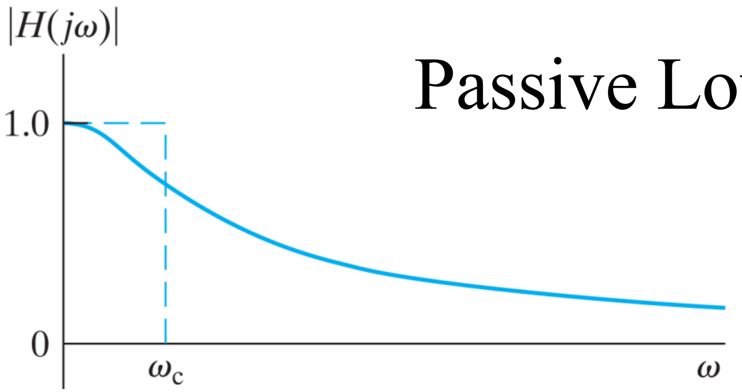
$$\Rightarrow \quad \frac{1}{2} = \frac{(R/L)^2}{\omega_c^2 + (R/L)^2} \quad \Rightarrow \quad 2(R/L)^2 = \omega_c^2 + (R/L)^2 \quad \Rightarrow \quad \boxed{\omega_c = \frac{R}{L}.}$$

◀ **Cutoff frequency for *RL* filters**

The Series *RL* Circuit: Quantitative Analysis – cont.



$$\boxed{\omega_c = \frac{R}{L}}$$



Passive Low Pass Filters

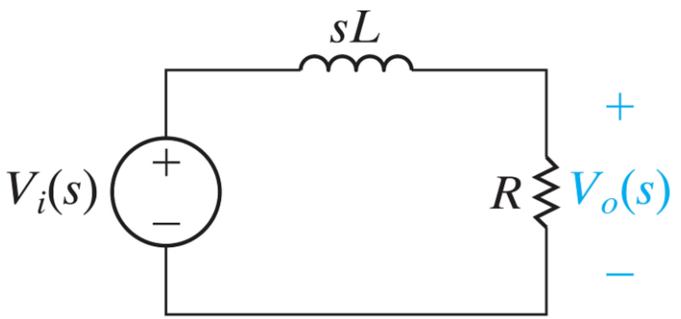
Design an *RL* low-pass filter with a cutoff frequency (f_c) of 2 kHz. Use a 5 kΩ resistor.

$$\omega_c = 2\pi f_c = 2\pi(2000) = \frac{R}{L}$$

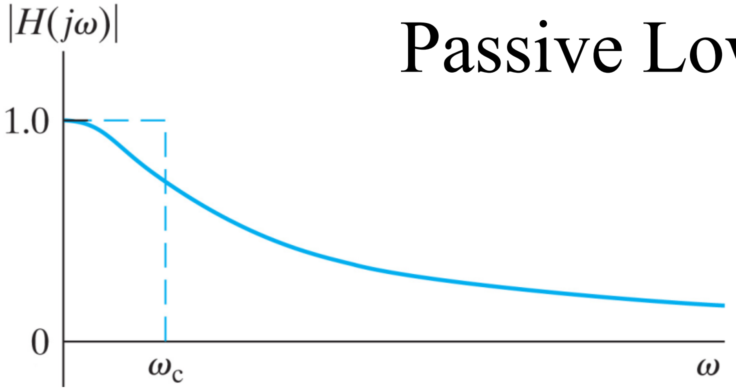
$$\Rightarrow \qquad L = \frac{R}{2\pi(2000)} = \frac{5000}{2\pi(2000)}$$

$$\approx 0.4 \text{ H}$$

The Series *RL* Circuit: Quantitative Analysis – cont.



$$\boxed{\omega_c = \frac{R}{L}}$$



Passive Low Pass Filters

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{R+sL} = \frac{R/L}{s+R/L} = \boxed{\frac{\omega_c}{s+\omega_c} = H_{LPF}(s)}$$

◀ Transfer function for a low-pass filter

Find the value of $|H(j\omega)|$ for:

$\omega << \omega_c$

$$\left|\frac{\omega_c}{j\omega+\omega_c}\right| \approx \left|\frac{\omega_c}{\omega_c}\right| = 1$$

$\omega = \omega_c$

$$\left|\frac{\omega_c}{j\omega+\omega_c}\right| \approx \left|\frac{1}{j+1}\right| = \frac{1}{\sqrt{2}}$$

$\omega >> \omega_c$

$$\left|\frac{\omega_c}{j\omega+\omega_c}\right| \approx \left|\frac{\omega_c}{j\omega}\right| \to 0$$

Low-Pass Filters: The series *RC* circuit

- The series *RC* circuit shown here also behaves as a low-pass filter
  - We can verify this via the same qualitative analysis we used previously
  - Note that the circuit o/p is the voltage across the capacitor

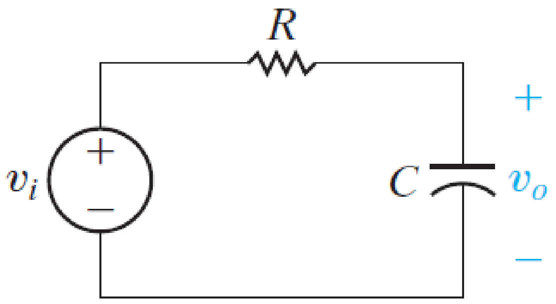


Figure 14.7 ▲ A series *RC* low-pass filter.

- As we did in the previous qualitative analysis, we use three frequency regions to develop the behavior of the series *RC* circuit

1. *Zero frequency* ($\omega = 0$): The impedance of the capacitor is infinite, and the capacitor acts as an open circuit. The input and output voltages are thus the same.
2. *Frequencies increasing from zero*: The impedance of the capacitor decreases relative to the impedance of the resistor, and the source voltage divides between the resistive impedance and the capacitive impedance. The output voltage is thus smaller than the source voltage.
3. *Infinite frequency* ($\omega = \infty$): The impedance of the capacitor is zero, and the capacitor acts as a short circuit. The output voltage is thus zero.

Impedance of the capacitor is:

$$Z_C = \frac{1}{j\omega C}$$

Low-Pass Filters: The series *RC* circuit - cont.

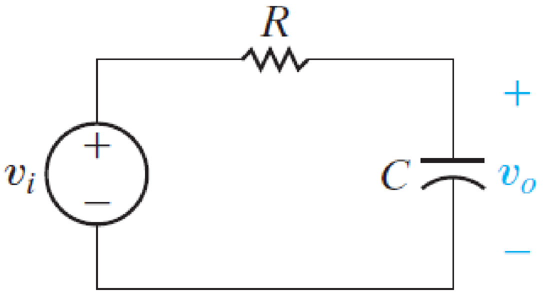
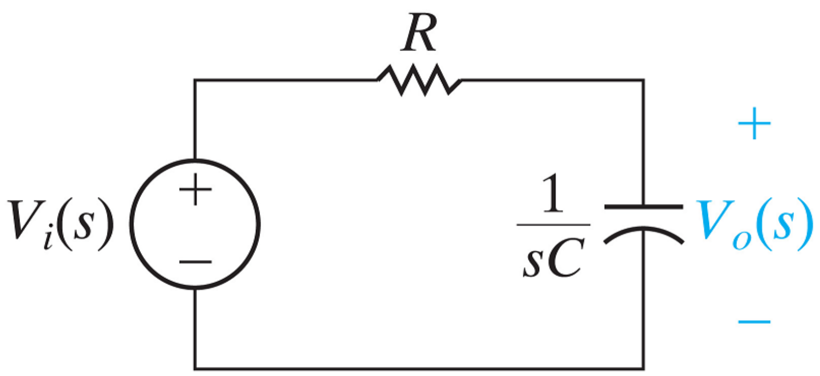
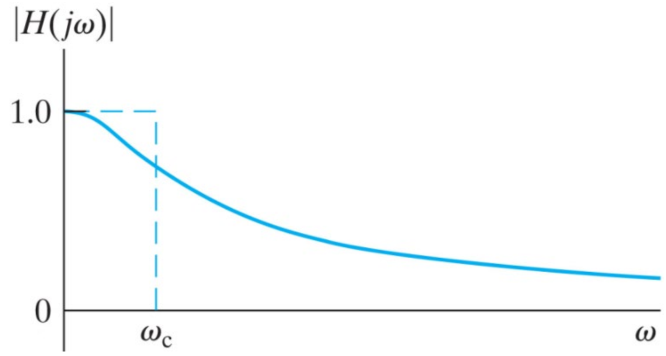


Figure 14.7 ▲ A series *RC* low-pass filter.





$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1/sC}{R+1/sC} = \frac{1}{RsC+1} = \frac{1/RC}{s+1/RC} = \frac{\omega_c}{s+\omega_c} = H_{LPF}(s) \Rightarrow \boxed{\omega_c = \frac{1}{RC}}$$

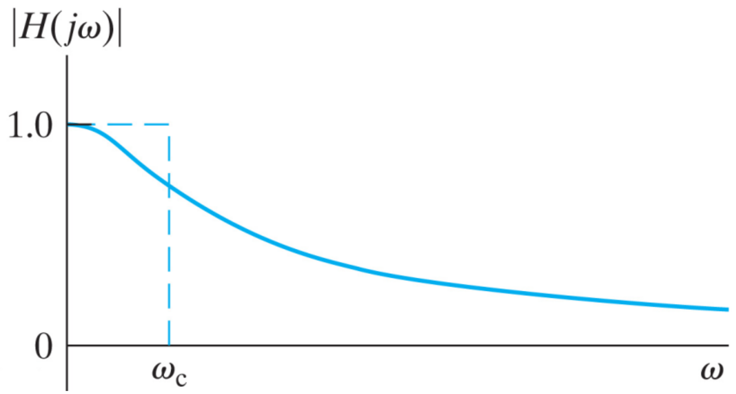
Design an RC low pass filter with a cutoff frequency (f_c) of 8 kHz. Use a 10 kΩ resistor.

$$\omega_c = 2\pi f_c = 2\pi(8000) = \frac{1}{RC}$$

$$\Rightarrow \qquad C = \frac{1}{(10{,}000)(2\pi)(8000)}$$

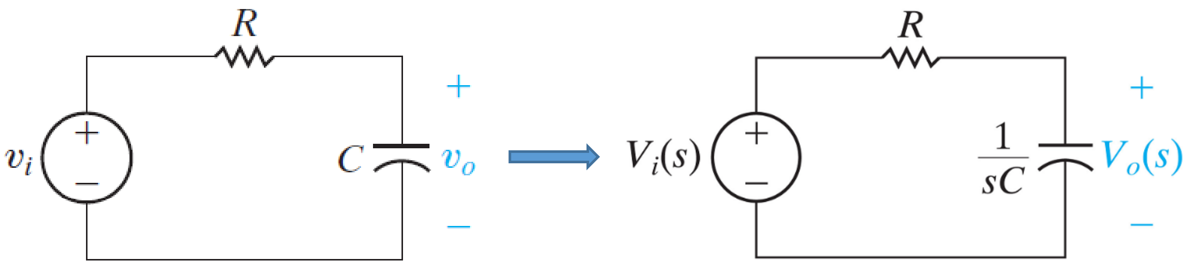
$$\approx 2 \text{ nF}$$

Low-Pass Filters: Summary



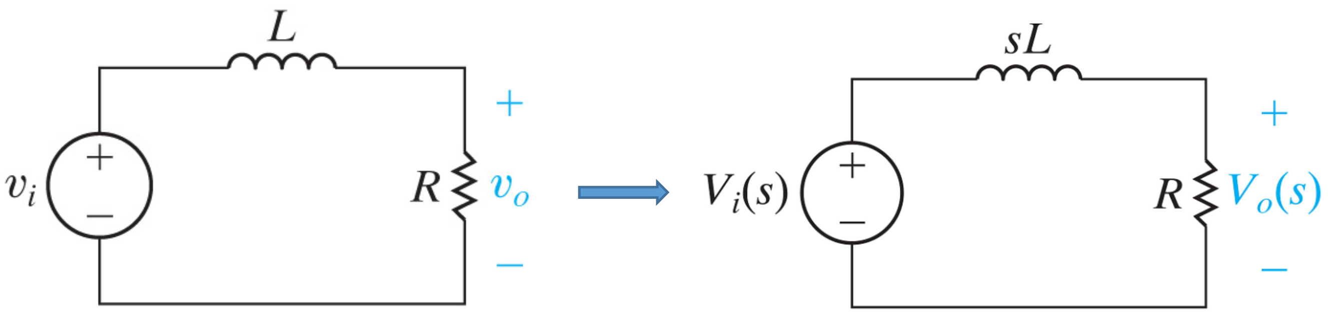
$$\boxed{H_{LPF}(s) = \frac{\omega_c}{s+\omega_c}}$$

◀ **Transfer function for a low-pass filter**



Note that the output voltage is the voltage drop across C

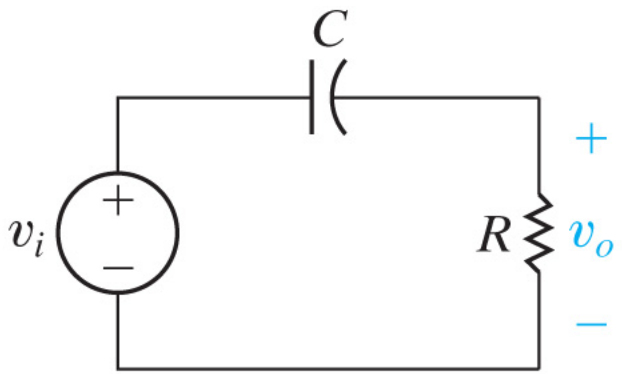
$$\boxed{\omega_c = \frac{1}{RC}}$$



Note that the output voltage is the voltage drop across R

$$\boxed{\omega_c = \frac{R}{L}}$$

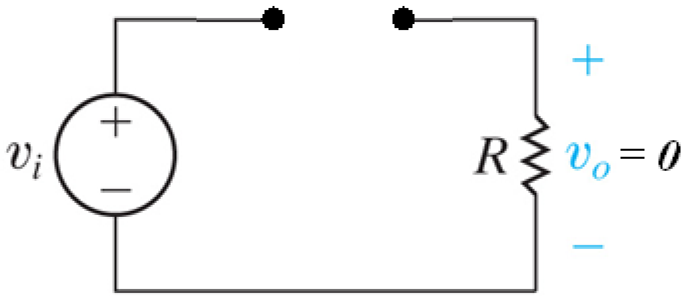
High-Pass Filters: The series *RC* circuit



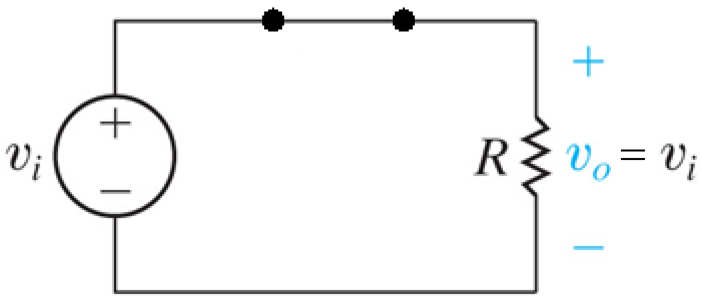
Note that the output voltage is the voltage drop across the resistor *R*

- Think about how the capacitor behaves at ω = 0 and as ω → ∞:

For $\omega = 0$: $Z_c = \dfrac{1}{j(0)C} \rightarrow \infty$

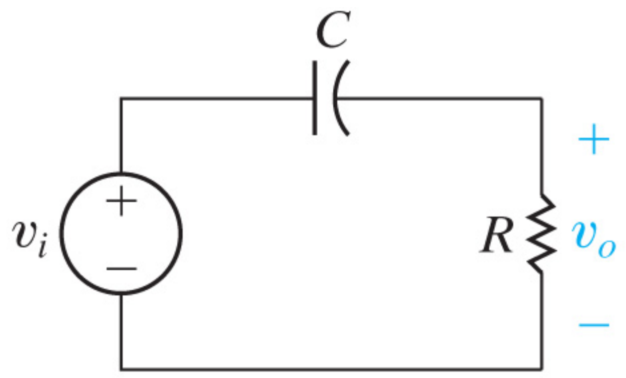


For $\omega \rightarrow \infty$: $Z_c = \dfrac{1}{j(\infty)C} = 0$

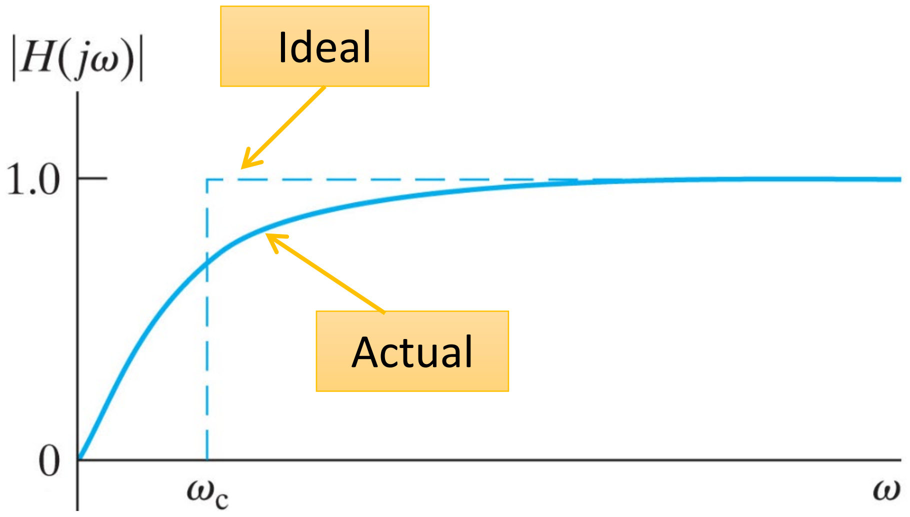


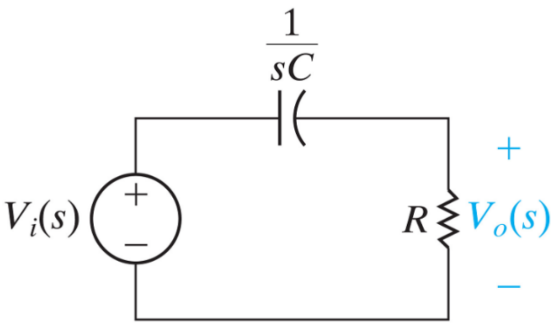
High-Pass Filters: The series *RC* circuit – cont.

- Plot |H(jω)| versus ω, where $H(s) = Vo/Vi$



Note that the output voltage is the voltage drop across the resistor *R*





- We need to relate the values of the circuit components *R* & *C* to the cutoff frequency
- We know that at ω_c the magnitude of $H(j\omega)$ is $\frac{1}{\sqrt{2}}\ H_{max}$
- Let us check if this is true when $\omega_c = \frac{1}{RC}$?

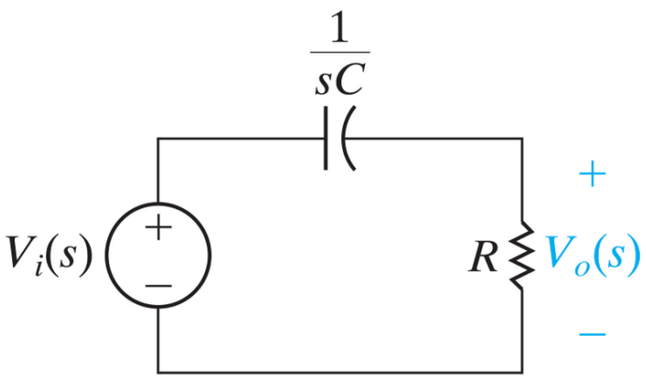
$$H(s) = \frac{R}{R + 1/sC} = \frac{RsC}{RsC + 1} = \frac{s}{s + 1/RC}$$

$$\omega_c \overset{?}{=} 1/RC :$$

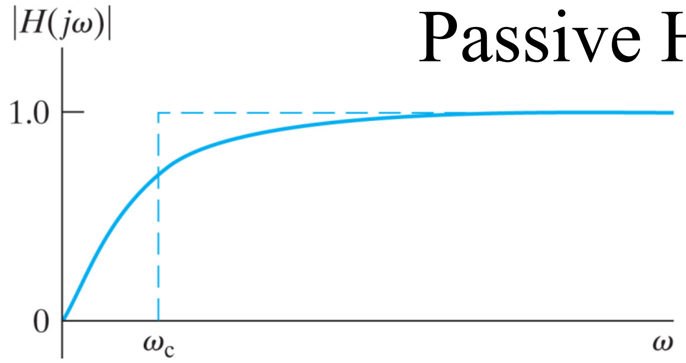
$$\left|H(j\,{}^{1}\!/_{RC})\right| = \left|\frac{j\,{}^{1}\!/_{RC}}{j\,{}^{1}\!/_{RC} + 1/RC}\right| = \frac{1/RC}{\sqrt{(1/RC)^2 + (1/RC)^2}}$$

$$= \frac{1}{\sqrt{2}} \qquad \text{so} \qquad \boxed{\omega_c = 1/RC}$$

High-Pass Filters: The series *RC* circuit – cont.



$$\boxed{\omega_c = \frac{1}{RC}}$$



Passive High Pass Filter

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{R + 1/sC} = \frac{RsC}{RsC+1} = \frac{s}{s+1/RC} = \boxed{\frac{s}{s+\omega_c} = H_{HPF}(s)}$$

◀ Transfer function for a **High-pass filter**

Find the value of |H(jω)| for:

$\omega \ll \omega_c$	$\omega = \omega_c$	$\omega \gg \omega_c$
$\left\lvert\frac{j\omega}{j\omega+\omega_c}\right\rvert \approx \left\lvert\frac{\omega}{\omega_c}\right\rvert \to 0$	$\left\lvert\frac{j\omega}{j\omega+\omega_c}\right\rvert \approx \left\lvert\frac{j}{j+1}\right\rvert = \frac{1}{\sqrt{2}}$	$\left\lvert\frac{j\omega}{j\omega+\omega_c}\right\rvert \approx \left\lvert\frac{j\omega}{j\omega}\right\rvert = 1$

High-Pass Filters: The series *RL* circuit

- Following the same analysis we did with the series *RC* circuit, we can construct a high-pass filter using a series *RL* circuit

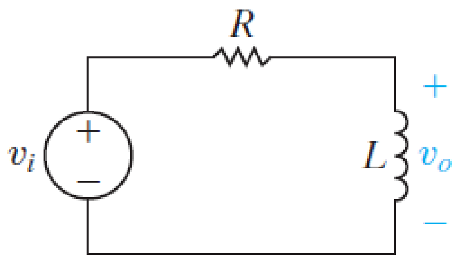
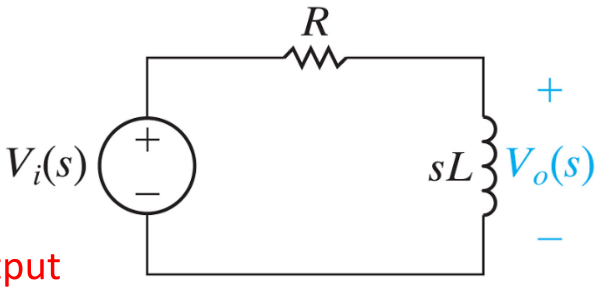
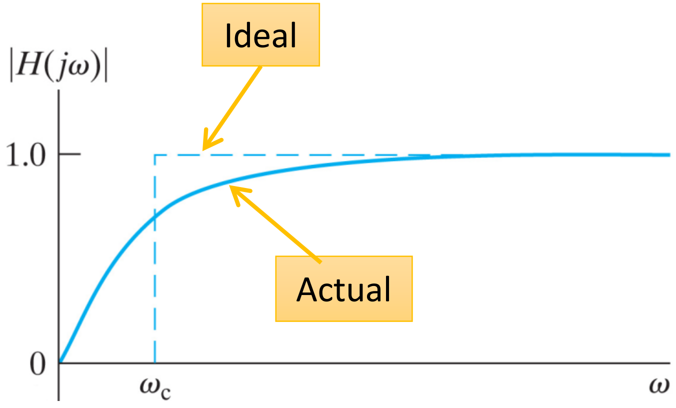


Figure 14.13 ▲ The circuit for Example 14.3.



Note that the output voltage is the voltage drop across L



$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{sL}{R + sL} = \frac{s}{s + R/L} = \boxed{\frac{s}{s + \omega_c} = H_{HPF}(s)}$$

◀ **Transfer function for a High-pass filter**

$$\boxed{\omega_c = \frac{R}{L}}$$

High-Pass Filters: Loading the Series *RL* HPF

- We want to examine the effect of placing a load resistor in parallel with the inductor in the *RL* high-pass filter circuit

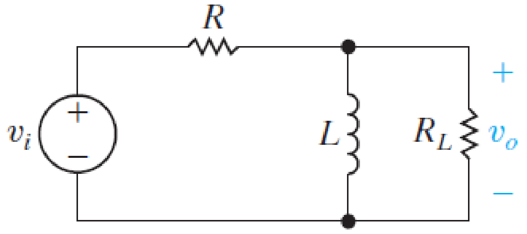


Figure 14.15 ▲ The circuit for Example 14.4.

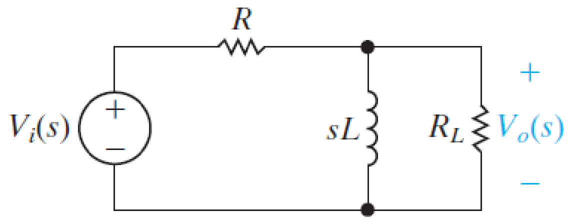
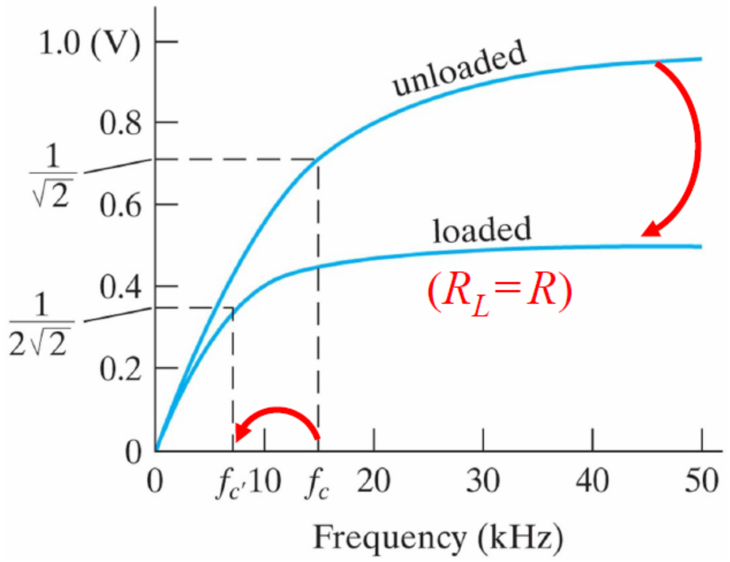


Figure 14.16 ▲ The *s*-domain equivalent of the circuit in Fig. 14.15.



Begin by transforming the circuit in Fig. 14.15 to the *s*-domain, as shown in Fig. 14.16. Use voltage division across the parallel combination of inductor and load resistor to compute the transfer function:

$$H(s) = \frac{\frac{R_L sL}{R_L + sL}}{R + \frac{R_L sL}{R_L + sL}} = \frac{\left(\frac{R_L}{R + R_L}\right)s}{s + \left(\frac{R_L}{R + R_L}\right)\frac{R}{L}} = \frac{Ks}{s + K\left(\frac{R}{L}\right)} = \frac{Ks}{s + K\omega_c}$$

Where $K = \frac{R}{R+R_L}$ and is < 1

$\omega_{c(loaded)} = K\omega_{c(unloaded)}$, which yields to the fact that $\omega_{c(loaded)} < \omega_{c(unloaded)}$

Effects of Loading on Filter Performance

- Lower the cutoff frequency
- Reduce the gain in the passband
- The response varies with the load

Solution: Use active filters (Ch15)

High-Pass Filters: Superposition of Sinusoids

- Let $x(t) = x_1(t) + x_2(t) = \cos(0.1\omega_c t) + \sin(3\omega_c t)$
  - The above signal is a superposition of two sinusoids one with frequency $\omega_1 = 0.1\omega_c$, and the second one is with frequency $\omega_2 = 3\omega_c$
- The transfer function for a high-pass filter is:

$$H_{HPF}(s) = \frac{s}{s + \omega_c} \qquad \& \qquad H_{HPF}(j\omega) = \frac{j\omega}{j\omega + \omega_c}$$

- Thus,

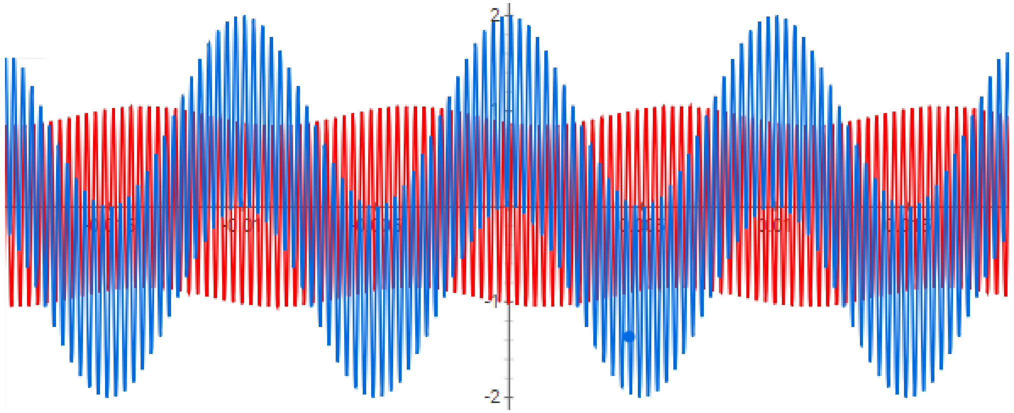
$$H(j\omega_1) = \frac{j\omega_1}{j\omega_1 + \omega_c} = \frac{j(0.1\omega_c)}{j(0.1\omega_c) + \omega_c} = \frac{0.1j}{0.1j + 1} \approx \frac{0.1\angle 90°}{1.005\angle 5.7°} \approx 0.1\angle 84°$$

$$H(j\omega_2) = \frac{j\omega_2}{j\omega_2 + \omega_c} = \frac{j(3\omega_c)}{j(3\omega_c) + \omega_c} = \frac{3j}{3j + 1} \approx \frac{3\angle 90°}{3.16\angle 71.6°} \approx 0.95\angle 18°$$

High-Pass Filters: Superposition of Sinusoids – cont.

- From the definition of the transfer function, we know that it is the ratio between the output and the input in the frequency domain
  - Looking at $H(j\omega_1) \approx 0.1\angle 84°$, this means that the response of the high-pass circuit to the input $x_1(t)$ will be $x_1(t)$ but attenuated by ~90%
    - Note that $x_1(t)$ is the low frequency component of $x(t)$
  - Looking at $H(j\omega_1) \approx 0.95\angle 18°$, this means that the response of the high-pass circuit to the input $x_2(t)$ will be $x_2(t)$ but attenuated by only ~5%
    - Note that $x_2(t)$ is the high frequency component of $x(t)$
  - You can note that this HPF will actually pass the high frequency components of the input signal and rejects the low frequency components of the input

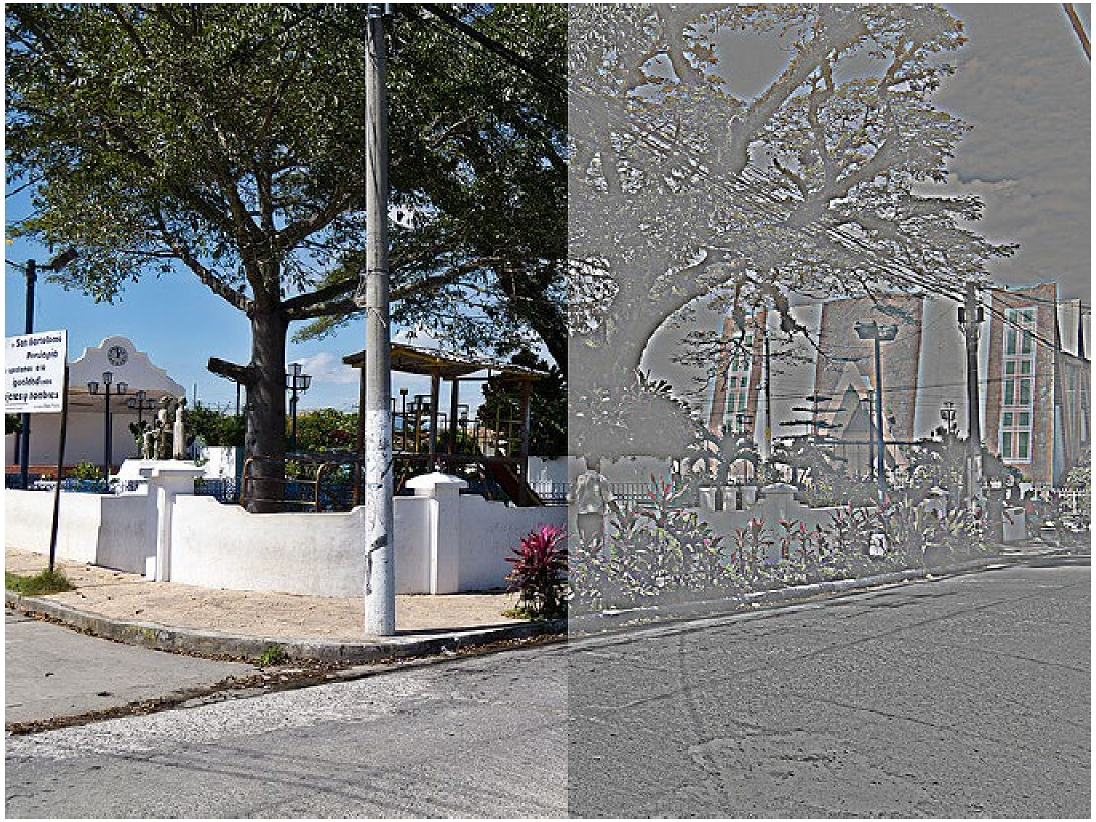
A plot for $\boldsymbol{x(t)}$ & $\boldsymbol{y(t)}$ assuming $\omega_c = 1kHz$



Note that the cos term in $x(t)$ which modulates $x(t)$ to have a cos shape envelop no longer exists in the output signal $y(t)$, because it was rejected by the HPF

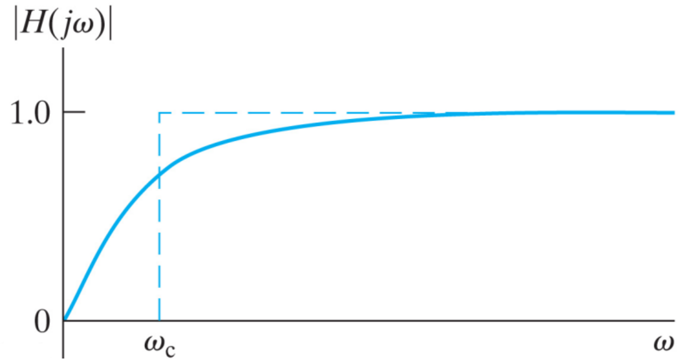
High-Pass Filters: Image Processing

- An image can be looked at as a signal with different frequency components
- Filtering is a very useful technique in image processing to enhance images
  - High pass filters can be used to enhance contrast
  - Low pass filters can be used to remove noise or abrupt changes in the contrast



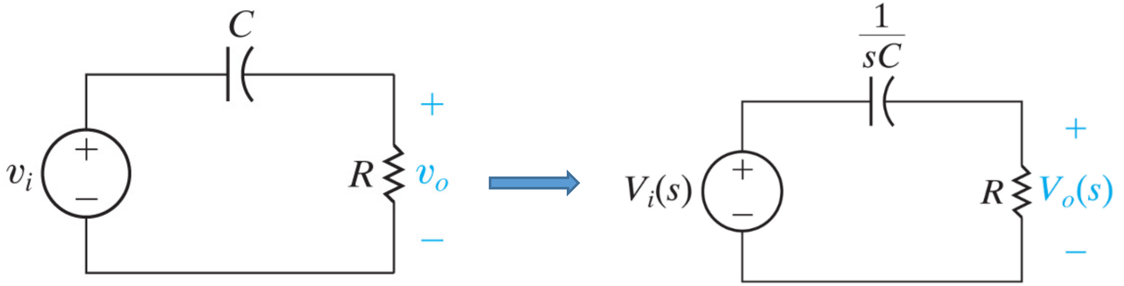
Wikipedia.org

High-Pass Filters: Summary



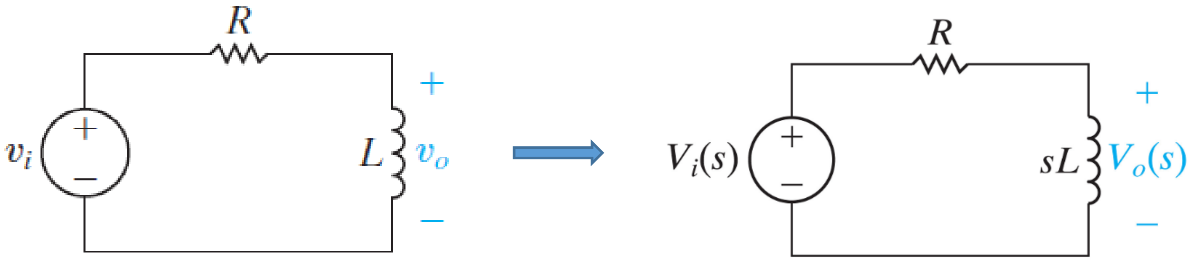
$$H_{HPF}(s) = \frac{s}{s+\omega_c}$$

◀ Transfer function for a **High-pass filter**



Note that the output voltage is the voltage drop across R

$$\omega_c = \frac{1}{RC}$$



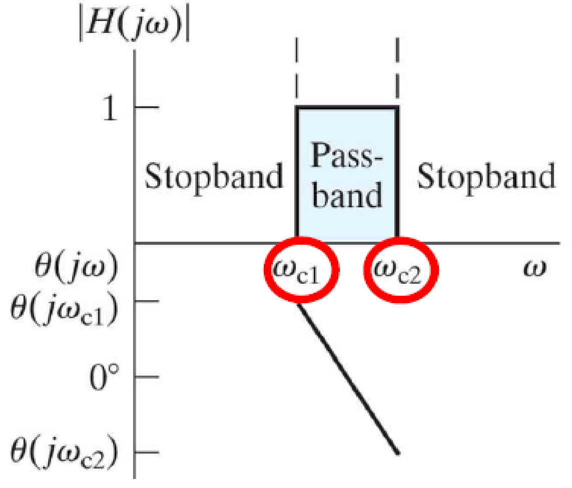
Note that the output voltage is the voltage drop across L

$$\omega_c = \frac{R}{L}$$

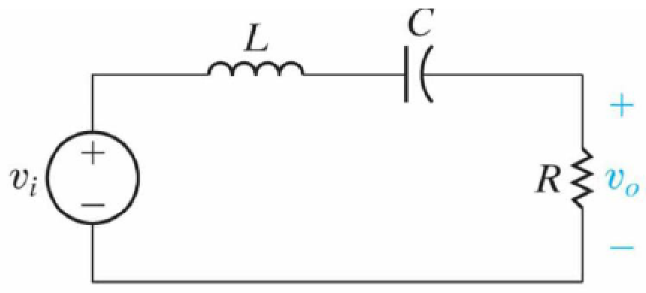
Bandpass Filters

- The next filters we examine are those that pass voltages within a band of frequencies to the output while filtering out voltages at frequencies outside this band
  - These filters are somewhat more complicated than the lowpass and high-pass filters of the previous sections
- As we have already seen, ideal bandpass filters have two cutoff frequencies ω_{c1} and ω_{c2} which defines the pass band
- For realistic bandpass filters, those cutoff frequencies are again defined as the freq. for which $|H(j\omega)| = \frac{1}{\sqrt{2}} H_{max}$

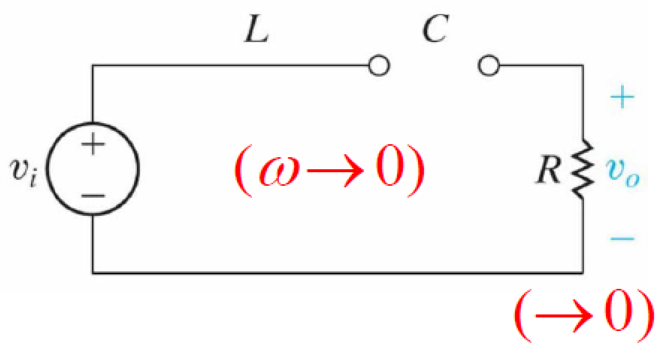
BPF



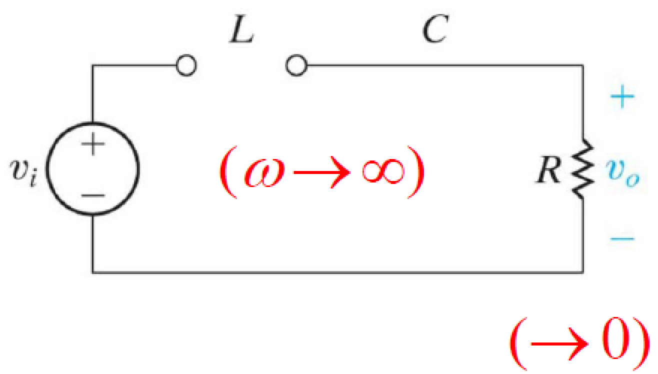
Bandpass Filters: Series *RLC* Circuit



$$Z_c = 1/(j\omega C),\ Z_L = j\omega L$$



As $\boldsymbol{\omega \to 0}, Z_C \to \infty \;\&\; Z_L \to 0 \Longrightarrow\ v_0 = 0$



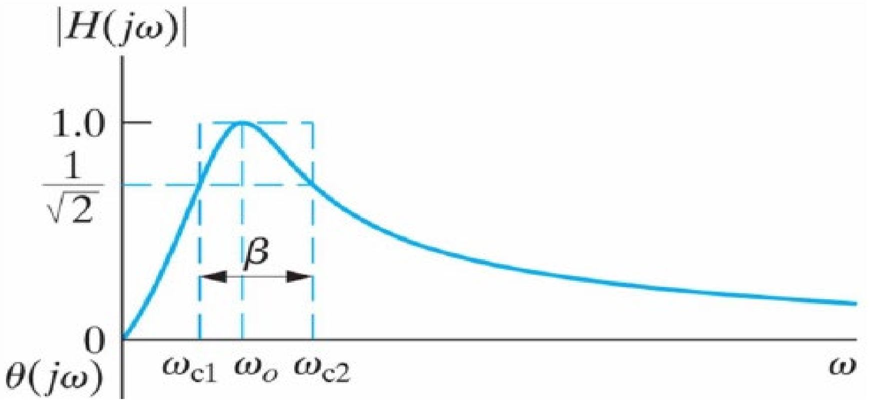
As $\boldsymbol{\omega \to \infty}, Z_C \to 0 \;\&\; Z_L \to \infty \Longrightarrow\ v_0 = 0$

Note that there will be one frequency $\omega_0 \in (0, \infty)$ for which the impedance of the inductor is equal and opposite to the impedance of the capacitor $Z_C + Z_L = 0$ $(short\ circuit) \Longrightarrow\ \boldsymbol{v_0 = v_i}$

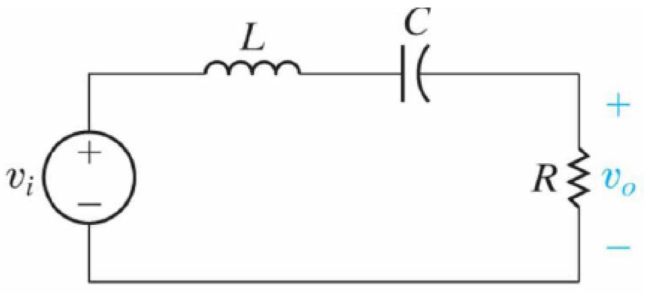
Bandpass Filters: Series *RLC* Circuit – cont.

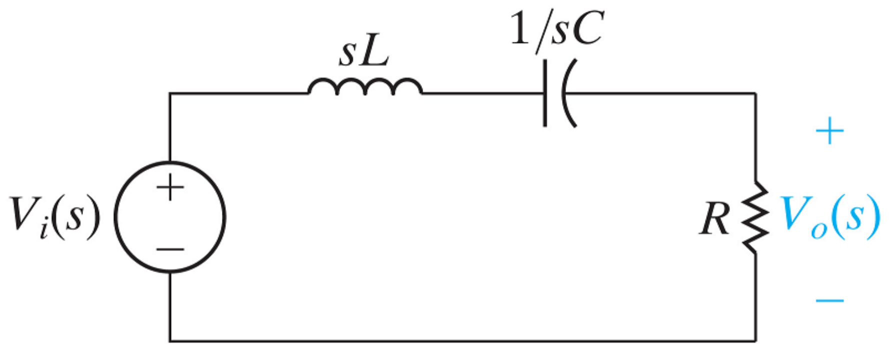
- Five parameters that characterize the frequency response of a BPF:
  - ω_0: the center frequency: the frequency for which the magnitude of the transfer function is maximum
    - The frequency for which the transfer function is purely real
    - $Z_C + Z_L = 0$, and we will have the resistor R in the circuit by itself
  - The cutoff frequencies (ω_{c1} and ω_{c2}): the frequencies at which the magnitude of the transfer function is reduced from its maximum by $1/\sqrt{2}$
  - Bandwidth (β): $\beta = \omega_{c2} - \omega_{c1}$ (units in rad/s or Hz)
  - The quality factor (Q): $Q = \frac{\omega_0}{\beta}$ (unitless)
- The freq. response characteristics are related as:

$$\boxed{\begin{aligned} \omega_o &= \sqrt{\omega_{c1}\omega_{c2}} \\ \beta &= \omega_{c2} - \omega_{c1} \\ Q &= \omega_o/\beta \end{aligned}} \quad \omega_o \neq \frac{\omega_{c1} + \omega_{c2}}{2}$$



Bandpass Filters: Series *RLC* Circuit (Quantitative Analysis)





$$H(s) = \frac{R}{R + sL + 1/sC} = \frac{(R/L)s}{s^2 + (R/L)s + 1/LC}$$

To find the frequency response we substitute $s = j\omega$

$$H(s) = \frac{\left(\frac{R}{L}\right) j\omega}{(j\omega)^2 + \left(\frac{R}{L}\right) j\omega + \frac{1}{LC}} = \frac{\left(\frac{R}{L}\right) j\omega}{-\omega^2 + \left(\frac{R}{L}\right) j\omega + \frac{1}{LC}}$$

Thus, the magnitude and phase equations are:

$$|H(j\omega)| = \frac{\omega(R/L)}{\sqrt{[(1/LC) - \omega^2]^2 + [\omega(R/L)]^2}},$$

$$\theta(j\omega) = 90^\circ - \tan^{-1}\left[\frac{\omega(R/L)}{(1/LC) - \omega^2}\right].$$

- Now, we can move on to calculate the five parameters that characterize *RLC* bandpass filter

Bandpass Filters: Series *RLC* Circuit (Quantitative Analysis) – cont.

- Recall that ω_0 is the freq. for which the circuit's transfer function is purely real
  - The transfer function for the series *RLC* circuit will be real when the frequency of the voltage source makes the sum of the capacitor and inductor impedances zero

$$j\omega_0 L + \frac{1}{j\omega_0 C} = 0 \;\rightarrow j\omega_0 L = \frac{-1}{j\omega_0 C} \;\rightarrow (j\omega_0)^2 = \frac{1}{LC} \;\rightarrow \boldsymbol{\omega_0 = \frac{1}{\sqrt{LC}}}$$

- We know that when $\omega = \omega_{c1}\ or\ \omega_{c2}, |H(j\omega)| = \frac{1}{\sqrt{2}} H_{max}$
  - H_{max} occurs at the center frequency ω_0 and it equals 1 for ideal BPF

$$\omega_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)},$$

$$\omega_{c2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}.$$

- Given that, the bandwidth $\boldsymbol{\beta = \omega_{c2} - \omega_{c1} = R/L}$
- Finally, quality factor $\boldsymbol{Q = \frac{\omega_0}{\beta} = \frac{\sqrt{1/LC}}{R/L} = \sqrt{\frac{L}{R^2 C}} = \frac{\sqrt{L/C}}{R}}$

Bandpass Filters: Series *RLC* Circuit (Quantitative Analysis) – cont.

- Some alternative forms to the cutoff frequency equations are:

$$\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_o^2},$$

$$\omega_{c2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_o^2}.$$

$$\omega_{c1} = \omega_o \cdot \left[-\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}\right],$$

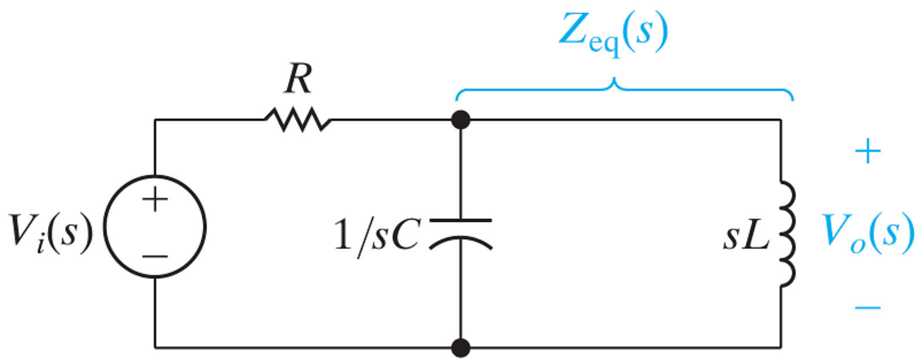
$$\omega_{c2} = \omega_o \cdot \left[\frac{1}{2Q} + \sqrt{1 + \left(\frac{1}{2Q}\right)^2}\right].$$

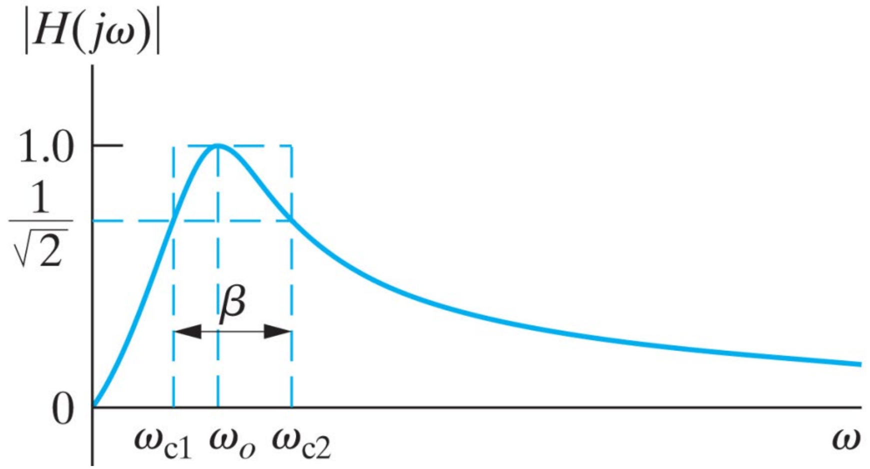
- The transfer function of a BPF can also be written in terms of β & ω_0:

$$H(s) = \frac{R}{R + sL + 1/sC} = \frac{RsC}{RsC + s^2LC + 1}$$

$$= \frac{(R/L)s}{s^2 + (R/L)s + (1/LC)} = \boxed{\frac{\beta s}{s^2 + \beta s + \omega_o^2} = H_{BPF}(s)}$$

Bandpass Filters: Parallel *RLC* Circuit





$$H(s) = \frac{Z_{eq}}{R + Z_{eq}}, \qquad Z_{eq} = sL \,\|\, 1/sC = \frac{L/C}{sL + 1/sC}$$

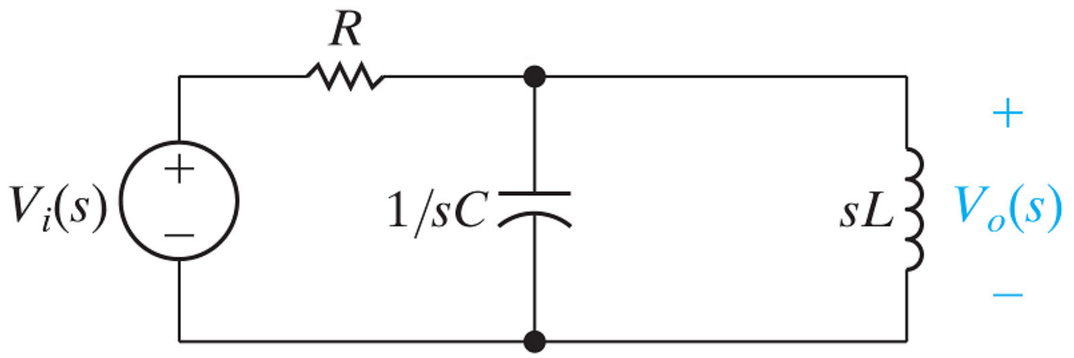
$$H(s) = \frac{\frac{L/C}{sL+1/sC}}{R + \frac{L/C}{sL+1/sC}} = \frac{L/C}{R(sL + 1/sC) + L/C}$$

$$\frac{(1/RC)s}{s^2 + (1/RC)s + (1/LC)} = H_{BPF}(s) \rightarrow \boxed{\frac{\beta s}{s^2 + \beta s + \omega_o^2} = H_{BPF}(s)}$$

$$\boxed{\begin{aligned} \omega_o &= \sqrt{1/LC} \\ \beta &= 1/RC \end{aligned}}$$

Bandpass Filters: Parallel *RLC* Circuit – cont.

- **Example:**
- Design a bandpass filter with a center frequency of 2 kHz and a bandwidth of 500 Hz. Use a 250 Ω resistor
- **Answer:**



$$\beta = \frac{1}{RC}, \qquad \omega_0 = \sqrt{\frac{1}{LC}}$$

$$\Rightarrow \qquad C = \frac{1}{R\beta} = \frac{1}{(250)[2\pi(500)]} \approx 1.27\,\mu\mathrm{F}$$

$$\Rightarrow \qquad L = \frac{1}{C\omega_0^2} = \frac{1}{(1.27\mu)[2\pi(2000)]^2} \approx 5\,\mathrm{mH}$$

Bandpass Filters: Parallel *RLC* Circuit – cont.

- **Example:**
- Find the cutoff frequencies for a bandpass filter with a center frequency of 2 kHz and a bandwidth of 500 Hz.
- **Answer:**
- We can find the cutoff frequencies by simultaneously solving the following two equations:

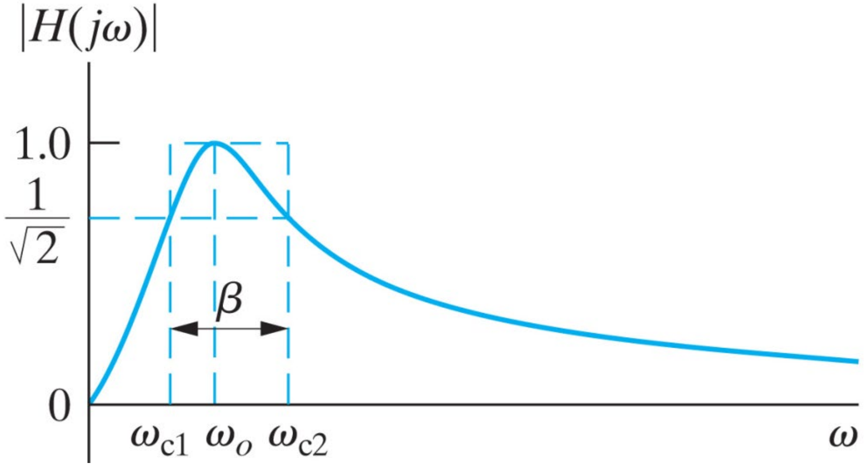
$$\beta = \omega_{c2} - \omega_{c1}, \qquad \omega_0 = \sqrt{\omega_{c1}\omega_{c2}}$$

$$\Rightarrow \qquad \omega_0^2 = \omega_{c1}(\beta + \omega_{c1}) \quad \Rightarrow \quad \omega_{c1}^2 + \beta\omega_{c1} - \omega_0^2 = 0$$

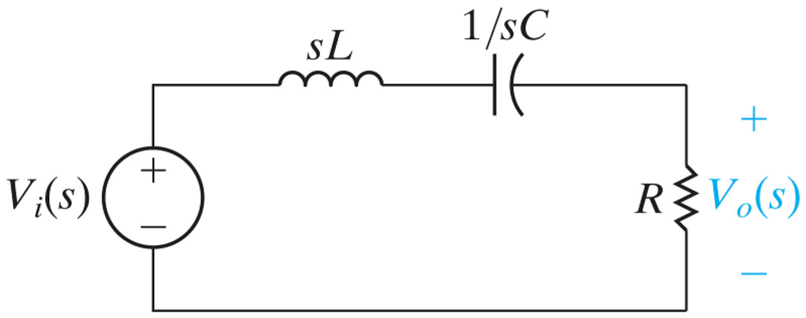
$$\Rightarrow \qquad \omega_{c1} = \frac{-\beta \pm \sqrt{\beta^2 - 4(-\omega_0^2)}}{2} = \frac{-500 + \sqrt{500^2 + 4(2000^2)}}{2} = 1765.56\ \text{Hz}$$

$$\Rightarrow \qquad \omega_{c2} = \omega_{c1} + \beta = 1765.56 + 500 = 2265.56\ \text{Hz}$$

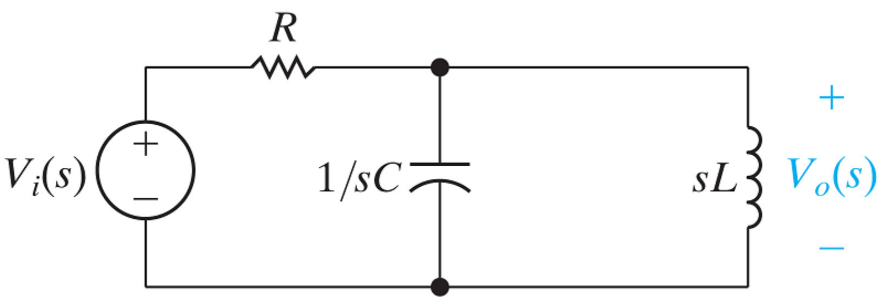
Bandpass Filters: Summary



$$H_{HPF}(s) = \frac{\beta s}{s^2 + \beta s + \omega_o^2}$$



$$\boxed{\omega_o = \sqrt{1/LC} \qquad \beta = R/L}$$



$$\boxed{\omega_o = \sqrt{1/LC} \qquad \beta = 1/RC}$$

Summary of Topics Covered in this Chapter

- In this chapter we discussed:
  - Introduction
  - Low-Pass Filters
  - High-Pass Filters
  - Bandpass Filters
  - Bandreject Filters
- We covered sections 14.1 – 14.5