Chapter 12: Introduction to the Laplace Transform

EEL 3112c – Circuits-II

Dr. Suleiman Alsweiss

ECE Department

Florida Polytechnic University

Topics to be Covered in this Chapter

- In this chapter we will discuss:
  - Definition of the Laplace transform
  - The step & the impulse function
  - Functional & operational transforms
  - Applying Laplace Transform
  - Inverse Transforms
  - Poles and Zeros of $F(s)$
  - Initial and final value theorem
- We will cover sections 12.1 – 12.9

Introduction to the Laplace Transform

- We now introduce a powerful analytical technique that is widely used to study the behavior of linear circuits
- The method is based on the Laplace transform, which we define mathematically in the coming slides
- But why do we need another analytical technique?
  - Among the many advantages that we will notice as we learn about the Laplace transform, it will provide a great tool to relate the behavior of a circuit in the time domain to its behavior in the frequency domain
  - Also, we can use the Laplace transform to introduce the concept of transfer functions as a tool to analyze the steady state sinusoidal response of a circuit

Definition of the Laplace Transform

- The **Laplace transform** of a function is given by the expression:

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,dt,$$

  - The symbol $\mathscr{L}\{f(t)\}$ is read "the Laplace transform of $f(t)$"
- The Laplace transform of $f(t)$ is also donated as $F(s)$
  - This means that the resulting expression is a function of s and not t anymore
  - Thus it transforms the signal from **time domain** to **frequency domain**
- After obtaining the frequency-domain expression for the unknown, we inverse-transform it back to the time domain

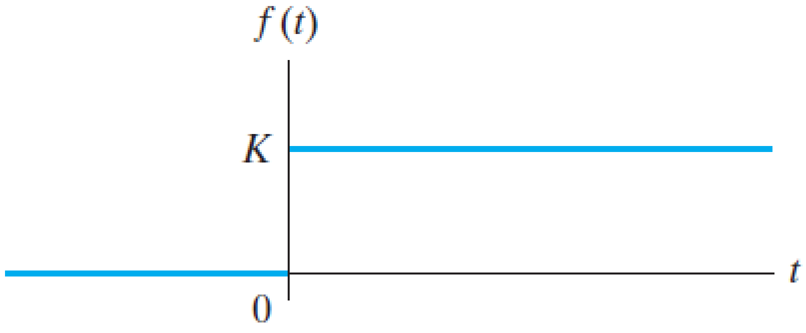
Definition of the Laplace Transform

- Note that the integral in the equation for the Laplace transform is improper
  - The upper limit of the integral is infinite
- Thus we are confronted immediately with the question of whether the integral converges. In other words, does a given $f(t)$ have a Laplace transform?
  - Obviously, the functions of primary interest in engineering analysis have Laplace transforms; otherwise we would not be interested in the transform
- Also, note that the lower limit of the integral in the Laplace transform equation is zero
  - This is a special case of the Laplace transform called the one sided or unilateral transform
  - For those who took/are taking systems & signals class, you will notice that we will define the limits of the integral from $-\infty$ to ∞ which is the general case
    - The unilateral Laplace transform is suitable for causal systems which is the case with circuits analysis
    - Causal systems depends on past and present value of the input, but not the future
      - Non-anticipative systems

Definition of the Laplace Transform

- The one-sided (unilateral) Laplace transform ignores $f(t)$ for $t < 0^-$
- What happens prior to 0^- is accounted for by the initial conditions
  - Thus we use the Laplace transform to predict the response to a disturbance that occurs after initial conditions have been established
- In the discussion that follows, we divide the Laplace transforms into two types: functional transforms and operational transforms
- A **functional transform** is the Laplace transform of a specific function, such as $\sin(\omega t)$
- An **operational transform** defines a general mathematical property of the Laplace transform,
  - Such as finding the transform of the derivative of $f(t)$
- Before considering functional and operational transforms, however, we need to introduce the step and impulse functions

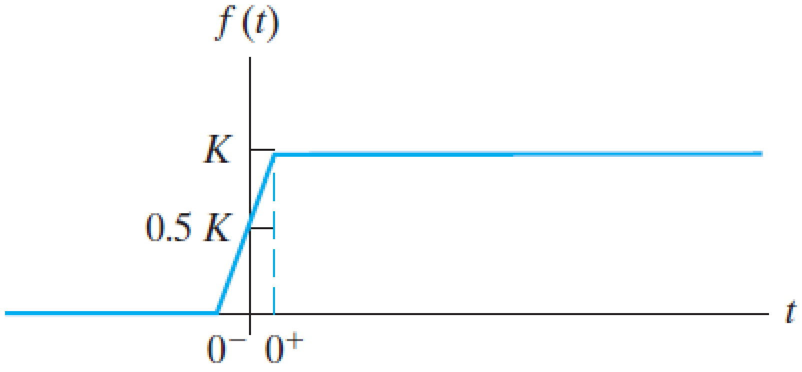
The Step Function



$$Ku(t) = 0, \quad t < 0,$$

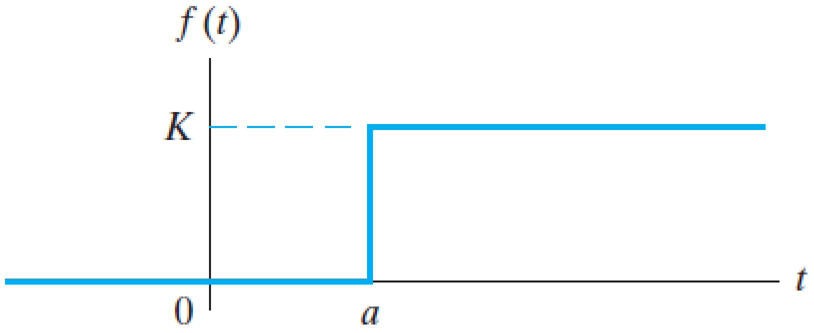
$$Ku(t) = K, \quad t > 0.$$

If K is 1, the function defined by is the **unit step**.



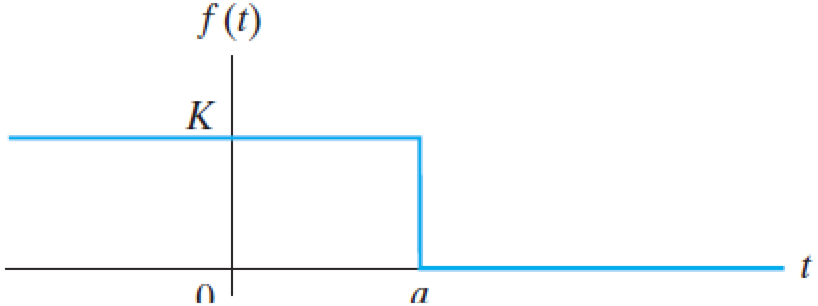
The step function is not defined at $t = 0$. In situations where we need to define the transition between 0^- and 0^+, we assume that it is linear and that

$$Ku(0) = 0.5K.$$



$$Ku(t - a) = 0, \quad t < a,$$

$$Ku(t - a) = K, \quad t > a.$$



$$Ku(a - t) = K, \quad t < a,$$

$$Ku(a - t) = 0, \quad t > a.$$

Example 12.1: Using Step Functions to Represent a Function of Finite Duration

Use step functions to write an expression for the function illustrated in Fig. 12.6.

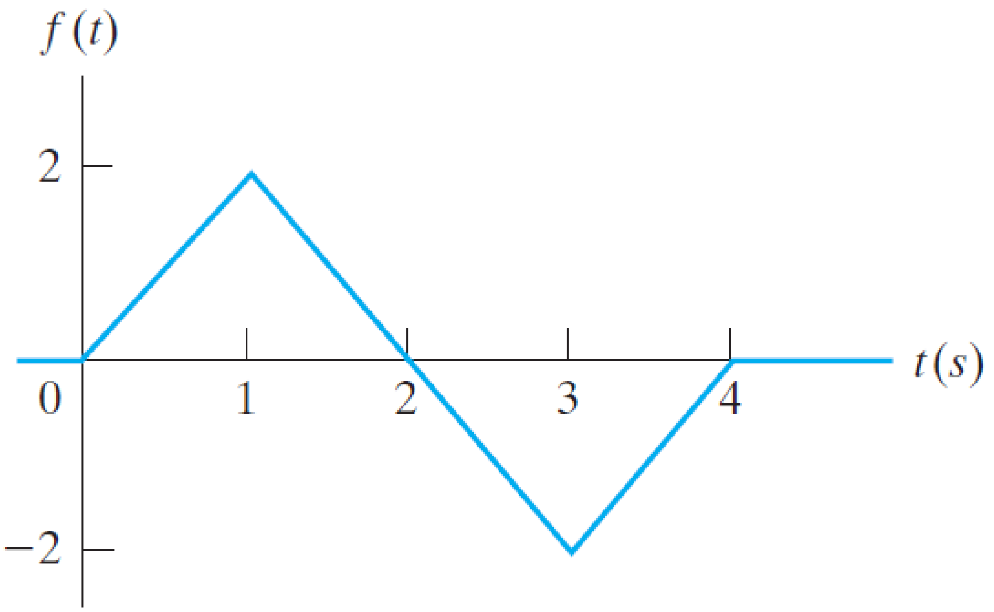
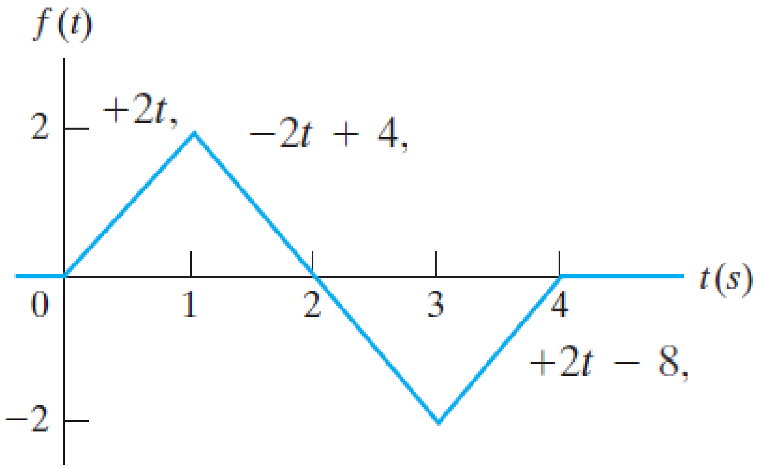


Figure 12.6 ▲ The function for Example 12.1.

Solution



$$f(t) = 2t[u(t) - u(t-1)] + (-2t+4)[u(t-1) - u(t-3)] + (2t-8)[u(t-3) - u(t-4)].$$

The impulse Function

- An **impulse** is a signal of infinite amplitude and zero duration
- Such **impulse** signals don't exist in nature, but some circuit signals come very close to approximate this function, so we need to find a mathematical model of an impulse
- Impulse function allows us to define the derivative at a discontinuity function, and thus to define the Laplace transform of that derivative

Mathematically, the **impulse function** is defined as:

$$\int_{-\infty}^{\infty} K\delta(t)dt = K; \quad \delta(t) = 0, \quad t \neq 0.$$

An important property of the impulse function is the **sifting property**, which is expressed as:

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a).$$

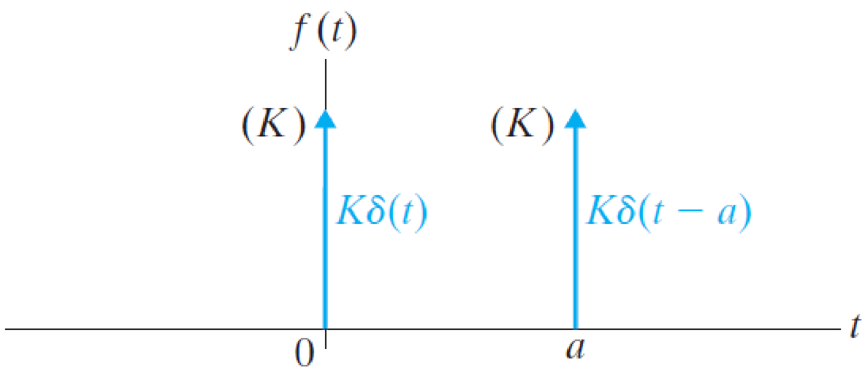
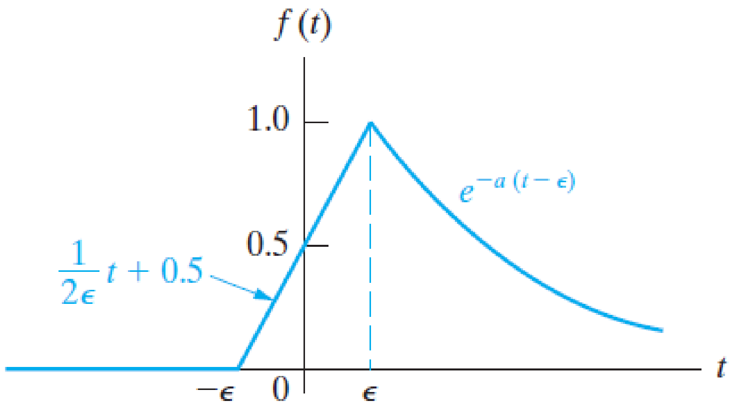
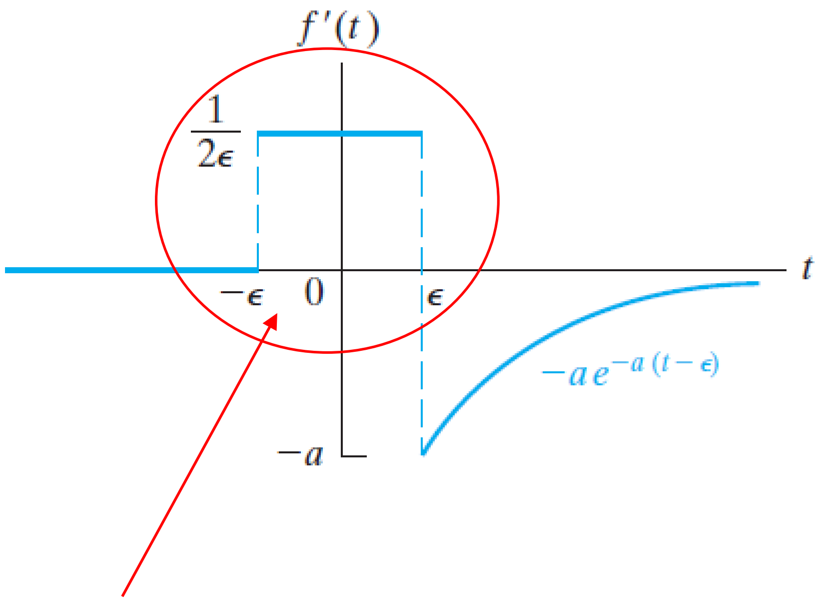


Figure 12.11 ▲ A graphic representation of the impulse $K\delta(t)$ and $K\delta(t-a)$.

The impulse Function – cont.

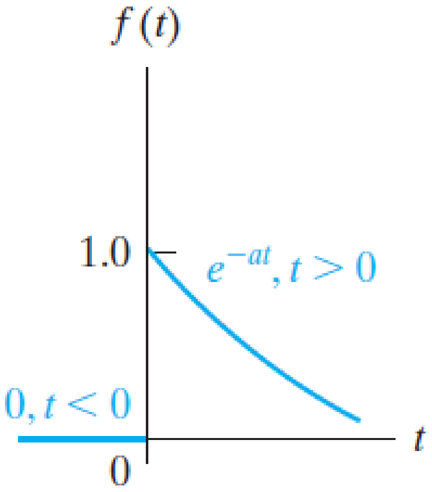


derivative



As $\epsilon \to 0$, then $f(t)$ will look like

The area of this region is 1.
As $\epsilon \to 0$, then the amplitude will reach ∞ and the base will be zero



$$f'(0) \to \delta(t) \quad \text{as } \epsilon \to 0.$$

The impulse Function – cont.

- We use the sifting property of the impulse function to find its Laplace transform

$$\mathscr{L}\{\delta(t)\} = \int_{0^-}^{\infty} \delta(t)e^{-st}\,dt = \int_{0^-}^{\infty} \delta(t)dt = 1$$

  - Which is an important Laplace transform pair that we make good use of in circuit analysis
- Finally, an impulse function can be thought of as a derivative of a step function; that is:

$$\delta(t) = \frac{du(t)}{dt}$$

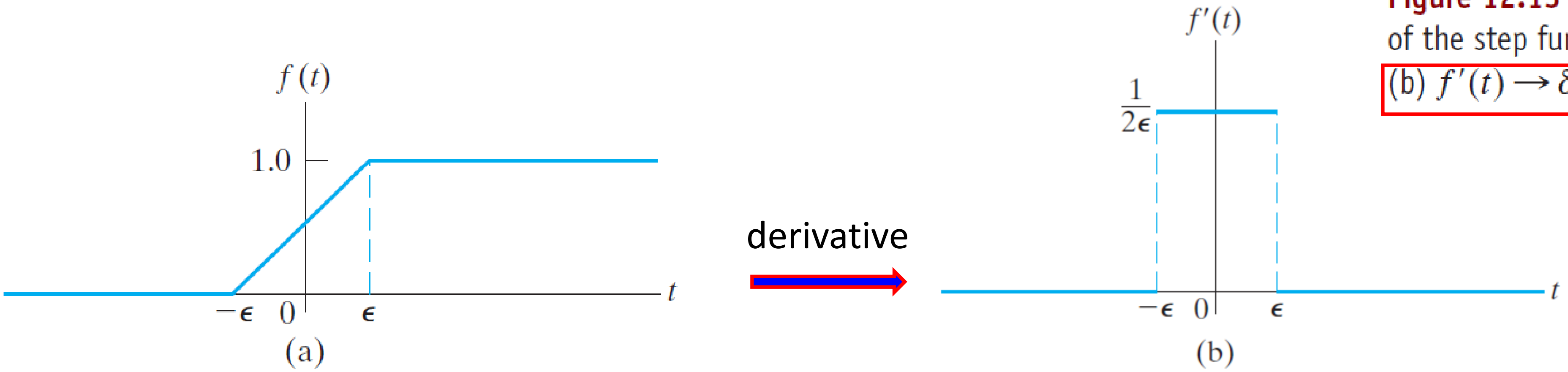


Figure 12.13 ▲ The impulse function as the derivative of the step function: (a) $f(t) \to u(t)$ as $\epsilon \to 0$; and (b) $f'(t) \to \delta(t)$ as $\epsilon \to 0$.

Functional Transforms

- Functional transform is simply the Laplace transform of a specified function of t
  - Remember that we are limiting our introduction to the unilateral, or one-sided, Laplace transform

Laplace transform of the unit step function

$$\mathscr{L}\{u(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}\,dt = \int_{0^+}^{\infty} 1e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_{0^+}^{\infty} = \frac{1}{s}$$

Laplace transform of the decaying exponential function

$$\mathscr{L}\{e^{-at}\} = \int_{0^+}^{\infty} e^{-at}e^{-st}\,dt = \int_{0^+}^{\infty} e^{-(a+s)t}\,dt = \frac{1}{s+a}.$$

Laplace transform of the sin function

$$\begin{aligned}
\mathscr{L}\{\sin \omega t\} &= \int_{0^-}^{\infty} (\sin \omega t)e^{-st}\,dt \\
&= \int_{0^-}^{\infty} \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)e^{-st}\,dt \\
&= \int_{0^-}^{\infty} \frac{e^{-(s-j\omega)t} - e^{-(s+j\omega)t}}{2j}\,dt \\
&= \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) \\
&= \frac{\omega}{s^2+\omega^2}.
\end{aligned}$$

Functional Transforms – cont.

Common Laplace transform pairs

TABLE 12.1 An Abbreviated List of Laplace Transform Pairs

Type	$f(t)$ $(t > 0-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

First note that $\cos x = \frac{e^{jx}+e^{-jx}}{2}$ and $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$.

$$\Rightarrow \mathscr{L}\{\cos \omega t\} = \frac{1}{2}\left(\mathscr{L}\{e^{j\omega t}\} + \mathscr{L}\{e^{-j\omega t}\}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right)$$

$$= \frac{1}{2}\left(\frac{(s+\cancel{j\omega})+(s-\cancel{j\omega})}{s^2-j^2\omega^2}\right) = \frac{\cancel{2}s}{\cancel{2}(s^2+\omega^2)}$$

$$\boxed{\therefore \mathscr{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2}}$$

Operational Transforms

- Operational transforms indicate how mathematical operations performed on either $f(t)$ or $F(s)$ are converted into the opposite domain
  - The properties of the Laplace transform
- The operations of primary interest are (1) multiplication by a constant; (2) addition (subtraction); (3) differentiation; (4) integration; (5) translation in the time domain; (6) translation in the frequency domain; and (7) scale changing

Operational Transforms – cont.

- Multiplication by a constant (Linearity):
  - Multiplication of $f(t)$ by a constant corresponds to multiplying $F(s)$ by the same constant

From the defining integral, if

$$\mathscr{L}\{f(t)\} = F(s),$$

then

$$\mathscr{L}\{Kf(t)\} = KF(s).$$

- Addition/Subtraction (Linearity):

$$\mathscr{L}\{f_1(t) + f_2(t) - f_3(t)\} = F_1(s) + F_2(s) - F_3(s),$$

Operational Transforms – cont.

- Differentiation:
  - Differentiation in the time domain corresponds to multiplying $F(s)$ by s and then subtracting the initial value of $f(t)$, that is $f(0^-)$, from this product

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-),$$

  - To generalize the differentiation property for n^{th} order derivative:

$$\mathcal{L}\left\{\frac{d^nf(t)}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt}$$

$$- s^{n-3}\frac{d^2f(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}.$$

Operational Transforms – cont.

- Integration:
  - Integration in the time domain corresponds to dividing by *s* in the *s* domain

$$\mathscr{L}\left\{\int_{0^-}^{t} f(x)\,dx\right\} = \frac{F(s)}{s}$$

- Scale changing:
  - The scale-change property gives the relationship between $f(t)$ and $F(s)$ when the time variable is multiplied by a positive constant

$$\mathscr{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a > 0,$$

$$\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}, \quad \Longrightarrow \quad \mathscr{L}\{\cos\omega t\} = \frac{1}{\omega}\frac{s/\omega}{(s/\omega)^2+1} = \frac{s}{s^2+\omega^2}.$$

Operational Transforms – cont.

- Translation in time domain (Time shifting)
  - For the function $f(t)u(t) \xrightarrow{L} F(s)$, then

$$\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s), \quad a > 0.$$

  - For example, if $\mathscr{L}\{tu(t)\} = \frac{1}{s^2}$, then the Laplace transform of $(t-a)u(t-a)$ can be found directly as:

$$\mathscr{L}\{(t-a)u(t-a)\} = \frac{e^{-as}}{s^2}.$$

Operational Transforms – cont.

- Translation in the frequency domain (Frequency Shifting)
  - Translation in the frequency domain corresponds to multiplication by an exponential in the time domain

$$\mathscr{L}\{e^{-at} f(t)\} = F(s + a),$$

  - For example:

$$\mathscr{L}\{\cos \omega t\} = \frac{s}{s^2 + \omega^2}, \Longrightarrow \mathscr{L}\{e^{-at} \cos\omega t\} = \frac{s + a}{(s + a)^2 + \omega^2}$$

Operational Transforms – cont.

Properties of the Laplace transform

TABLE 12.2 An Abbreviated List of Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\dfrac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\dfrac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \dfrac{df(0^-)}{dt}$
nth derivative (time)	$\dfrac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dfrac{df(0^-)}{dt} - s^{n-3}\dfrac{df^2(0^-)}{dt^2} - \cdots - \dfrac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\dfrac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\dfrac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\dfrac{d^nF(s)}{ds^n}$
s integral	$\dfrac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

Assessment Problem 12.2

- Use the appropriate operational transform from Table 12.2 to find the Laplace transform of t^2e^{-at}

TABLE 12.2 An Abbreviated List of Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\dfrac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\dfrac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \dfrac{df(0^-)}{dt}$
nth derivative (time)	$\dfrac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\dfrac{df(0^-)}{dt} - s^{n-3}\dfrac{df^2(0^-)}{dt^2} - \cdots - \dfrac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\dfrac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\dfrac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\dfrac{d^nF(s)}{ds^n}$
s integral	$\dfrac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

Assessment Problem 12.2

- Use the appropriate operational transform from Table 12.2 to find the Laplace transform of t^2e^{-at}
- Answer:
- From table 12.2, we know that:

nth derivative (s)	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$

- In this case, $f(t) = e^{-at}, and\ F(s) = \frac{1}{s+a}$, from table 12.1
- Then to find the answer:

$$n = 2 \ \rightarrow (-1)^2 \left(\frac{d^2F(s)}{ds^2}\right) = \left(\frac{d^2}{ds^2}\left[\frac{1}{s+a}\right]\right) = \left(\frac{d^2}{ds^2}[s+a]^{-1}\right) = \frac{2}{(s+a)^3}$$

Applying the Laplace Transform

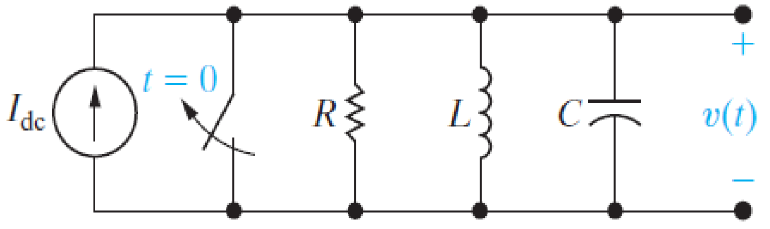


Figure 12.16 ▲ A parallel *RLC* circuit.

- Consider the circuit shown here
  - We assume that no initial energy is stored in the circuit at the instant when the switch is opened
  - The problem is to find the time-domain expression for $v(t)$ when $t \geq 0$
- We begin by writing the equation $v(t)$ must satisfy
  - We can use node-voltage with single node (at the top pf the circuit)
  - Sum all the currents away from that node at the top

Laplace Transform

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}u(t).$$

- We use the unit step function to denote that the current source is part of the circuit only at $t \geq 0$

$$\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_{dc}\left(\frac{1}{s}\right),$$

- To go from the time domain equation to the s domain equation we need to use the properties of the Laplace transform in table 12.2
- $R, L, C, I_{dc}, and\ v(0^-)$ are all known parameters
  - $v(0^-) = 0$ because the initial energy stored in the circuit is 0

$$V(s)\left(\frac{1}{R} + \frac{1}{sL} + sC\right) = \frac{I_{dc}}{s},$$

- Now, we have reduced the problem to solving an algebraic equation

$$V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}.$$

- After finding $V(s)$, we can find $v(t)$ using the inverse Laplace Transform

$$v(t) = \mathscr{L}^{-1}\{V(s)\}.$$

Poles & Zeros of $F(s)$

- In rational functions where $F(s) = N(s)/D(s)$
  - The roots of the numerator polynomial $N(s)$ are called the **zeros** of $F(s)$
    - They are the values of s at which $F(s)$ becomes zero
  - The roots of the denominator polynomial $D(s)$ are called the **poles** of $F(s)$
    - They are the values of s at which $F(s)$ becomes infinitely large
- In what follows, you may find that being able to visualize the poles and zeros of $F(s)$ as points on a complex s plane is helpful
  - A complex plane is needed because the roots may be complex
- In the complex s plane, we use the horizontal axis to plot the real values of s and the vertical axis to plot the imaginary values of s
- *Example:*

$$F(s) = \frac{10(s+5)(s+3-j4)(s+3+j4)}{s(s+10)(s+6-j8)(s+6+j8)}$$

Figure 12.17 shows the poles and zeros plotted on the s plane, where **X**'s represent poles and **O**'s represent zeros.

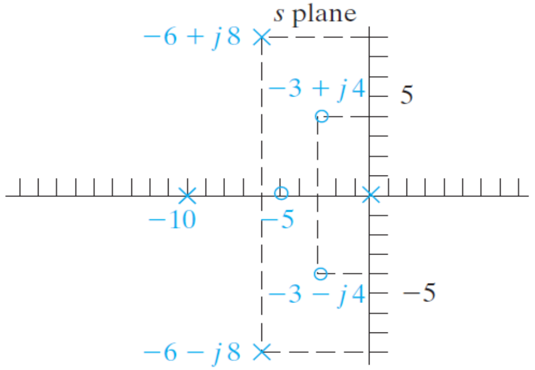


Figure 12.17 ▲ Plotting poles and zeros on the s plane.

Inverse Laplace Transform

- The expression for $V(s)$ in the previous slide is a **rational** function of s
  - A ratio of two polynomials in s
- In fact, for linear circuits, the s-domain expressions for the unknown voltages and currents are always rational functions of s
- If we can inverse-transform rational functions of s, we can solve for the time-domain expressions for the voltages and currents
- A straight-forward and systematic technique for finding the inverse transform of a rational function exists
- In general, we need to find the inverse transform of a function that has the form:

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}$$

The coefficients a and b are real constants, and the exponents m and n are positive integers.

- The ratio $N(s)/D(s)$ is called a proper rational function if $m > n$, and improper rational function if $m \leq n$
- Only a proper rational function can be expanded as a sum of partial fractions

Partial Fraction Expansion: Distinct Real Roots of D(s)

- Illustration example: Find the inverse Laplace transform for: $F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)}$
- Answer:
  - $F(s)$ is a proper function, so we can use partial fraction expansion

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

To find the value of K_1 we multiply both sides by s and then evaluate both sides at $s = 0$:

$$\left.\frac{96(s+5)(s+12)}{(s+8)(s+6)}\right|_{s=0} \equiv K_1 + \left.\frac{K_2 s}{s+8}\right|_{s=0} + \left.\frac{K_3 s}{s+6}\right|_{s=0},$$

$$\frac{96(5)(12)}{8(6)} \equiv \boxed{K_1 = 120.}$$

To find the value of K_2 we multiply both sides by $s+8$ and then evaluate both sides at $s = -8$:

$$\left.\frac{96(s+5)(s+12)}{s(s+6)}\right|_{s=-8}$$

$$\equiv \left.\frac{K_1(s+8)}{s}\right|_{s=-8} + K_2 + \left.\frac{K_3(s+8)}{(s+6)}\right|_{s=-8},$$

$$\frac{96(-3)(4)}{(-8)(-2)} = \boxed{K_2 = -72.}$$

Then, to find the value of K_3 we multiply both sides by $s+6$ and then evaluate both sides at $s = -6$:

$$\left.\frac{96(s+5)(s+12)}{s(s+8)}\right|_{s=-6} = \boxed{K_3 = 48.}$$

Partial Fraction Exp.: Distinct Real Roots of D(s) – cont.

- Illustration example: - cont.
  - Now that we found all the needed K values, we have:

$$\frac{96(s+5)(s+12)}{s(s+8)(s+6)} \equiv \frac{120}{s} + \frac{48}{s+6} - \frac{72}{s+8}.$$

  - At this point, testing the result to protect against computational errors is a good idea
    - As we already mentioned, a partial fraction expansion creates an identity; thus both sides must be the same for all s values
    - The choice of test values is completely open; hence we choose values that are easy to verify. Choosing $s = -5$

$$\frac{120}{-5} + \frac{48}{1} - \frac{72}{3} = -24 + 48 - 24 = 0,$$

  - Now confident that the numerical values of the various K values are correct, we proceed to find the inverse transform from the tables:

$$\mathscr{L}^{-1}\left\{\frac{96(s+5)(s+12)}{s(s+8)(s+6)}\right\} = (120 + 48e^{-6t} - 72e^{-8t})u(t)$$

Partial Fraction Expansion: Repeated Real Roots of D(s)

- Illustration example: Find the inverse Laplace transform for: $F(s) = \dfrac{100(s+25)}{s(s+5)^3}$
- Answer:
  - $F(s)$ is a proper function, so we can use partial fraction expansion
  - The fact that we have $(s+5)^{\mathbf{3}}$ in the denominator leads to repeated roots

$$\frac{100(s+25)}{s(s+5)^3} = \frac{K_1}{s} + \frac{K_2}{(s+5)^3} + \frac{K_3}{(s+5)^2} + \frac{K_4}{s+5}$$

We find K_1 as previously described:

$$K_1 = \left.\frac{100(s+25)}{(s+5)^3}\right|_{s=0}$$

$$= \frac{100(25)}{125} = \boxed{20}$$

To find K_2 we multiply both sides by $(s+5)^3$ and then evaluate both sides at $s = -5$:

$$\left.\frac{100(s+25)}{s}\right|_{s=-5} = \left.\frac{K_1(s+5)^3}{s}\right|_{s=-5} + K_2 + K_3(s+5)|_{s=-5}$$

$$+ K_4(s+5)^2\Big|_{s=-5},$$

$$\frac{100(20)}{(-5)} = K_1 \times 0 + K_2 + K_3 \times 0 + K_4 \times 0$$

$$= \boxed{K_2 = -400.}$$

To find K_3 we multiply both sides by $(s+5)^3$, next we differentiate both sides **once** with respect to s and then evaluate at $s = -5$:

$$\frac{d}{ds}\left[\frac{100(s+25)}{s}\right]_{s=-5} =$$

$$\frac{d}{ds}\left[\frac{K_1(s+5)^3}{s}\right]_{s=-5} + \frac{d}{ds}[K_2]_{s=-5}$$

$$+ \frac{d}{ds}[K_3(s+5)]_{s=-5} + \frac{d}{ds}[K_4(s+5)^2]_{s=-5}$$

$$100\left[\frac{s-(s+25)}{s^2}\right]_{s=-5} = \boxed{K_3 = -100}$$

Partial Fraction Exp.: Repeated Real Roots of D(s) – cont.

- Illustration example: - cont.
  - To find K_4 we first multiply both sides by $(s+5)^3$, next we differentiate both sides **twice** with respect to s and then evaluate both sides at $s=-5$
    - Thus, $k_4=-20$
  - Now that we found all the needed K values, we have:

$$\frac{100(s+25)}{s(s+5)^3}=\frac{20}{s}-\frac{400}{(s+5)^3}-\frac{100}{(s+5)^2}-\frac{20}{s+5}.$$

  - At this point, testing the result to protect against computational errors is a good idea
  - The inverse transform is:

$$\mathscr{L}^{-1}\left\{\frac{100(s+25)}{s(s+5)^3}\right\}=[20-200t^2e^{-5t}-100te^{-5t}-20e^{-5t}]u(t).$$

Differentiate using the Quotient Rule which states that $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ is $\frac{g(x)\frac{d}{dx}[f(x)]-f(x)\frac{d}{dx}[g(x)]}{g(x)^2}$

$$\frac{d}{dx}f(g(x))=f'(g(x))\cdot g'(x).$$

Partial Fraction Exp.: Distinct Complex Roots of D(s)

- The only difference between finding the coefficients associated with distinct complex roots and finding those associated with distinct real roots is that the algebra in the former involves complex numbers
- Illustration example: Find the inverse Laplace transform for: $F(s) = \dfrac{100(s+3)}{(s+6)(s^2+6s+25)}$
  - $F(s)$ is a proper function, so we can use partial fraction expansion
  - To perform partial fraction expansion, we need to find the roots of the quadratic equation in the denominator $D(s) \rightarrow s^2 + 6s + 25 = (s+3-j4)(s+3+j4)$
  - With the denominator in factored form, we proceed as before:

$$\frac{100(s+3)}{(s+6)(s^2+6s+25)} \equiv$$

$$\frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4}.$$

- To find $K_1, K_2,$ and K_3 we use the same process as before:

$$K_1 = -12$$
$$K_2 = 6 - j8 = 10e^{-j53.13°}$$
$$K_3 = 6 + j8 = 10e^{j53.13°}$$

- Usually complex roots appears in complex pairs and the K coefficients associated with these pairs are themselves conjugates

Partial Fraction Exp.: Distinct Complex Roots of D(s) – cont.

- Illustration example: - cont.
  - Now that we found all the needed K values, we have:

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)} = \frac{-12}{s+6} + \frac{10\underline{/-53.13^\circ}}{s+3-j4} + \frac{10\underline{/53.13^\circ}}{s+3+j4}.$$

  - At this point, testing the result to protect against computational errors is a good idea
  - The inverse transform is:

$$\mathscr{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = (-12e^{-6t} + 10e^{-j53.13^\circ}e^{-(3-j4)t} + 10e^{j53.13^\circ}e^{-(3+j4)t})u(t).$$

    - Adding the two complex exponential terms:

$$10e^{-j53.13^\circ}e^{-(3-j4)t} + 10e^{j53.13^\circ}e^{-(3+j4)t} = 10e^{-3t}\left(e^{j(4t-53.13^\circ)} + e^{-j(4t-53.13^\circ)}\right) = 20e^{-3t}\cos(4t - 53.13^\circ).$$

$$\Rightarrow \quad \mathscr{L}^{-1}\left\{\frac{100(s+3)}{(s+6)(s^2+6s+25)}\right\} = [-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)]u(t).$$

Partial Fraction Exp.: Repeated Complex Roots of D(s)

- We handle repeated complex roots in the same way that we did repeated real roots
  - The only difference is that the algebra involves complex numbers
- Recall that complex roots always appear in conjugate pairs and that the coefficients associated with a conjugate pair are also conjugates, so that only half the Ks need to be evaluated

- Please check illustration example in the book page 449 & 450

Inverse Laplace Transform – cont.

TABLE 12.3 Four Useful Transform Pairs

Pair Number	Nature of Roots of $D(s), i.e. Poles$	$F(s)$	$f(t)$
1	Distinct real	$\frac{K}{s+a}$	$Ke^{-at}u(t)$
2	Repeated real	$\frac{K}{(s+a)^2}$	$Kte^{-at}u(t)$
3	Distinct complex	$\frac{K}{s+\alpha-j\beta}+\frac{K^*}{s+\alpha+j\beta}$	$2\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta)u(t)$
4	Repeated complex	$\frac{K}{(s+\alpha-j\beta)^2}+\frac{K^*}{(s+\alpha+j\beta)^2}$	$2t\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta)u(t)$

Note: In pairs 1 and 2, K is a real quantity, whereas in pairs 3 and 4, K is the complex quantity $\lvert K\rvert \angle\theta$.

Partial Fraction Exp.: Improper Rational Functions

- The ratio $N(s)/D(s)$ is called improper rational function if $m \leq n$
- An improper rational function can always be expanded into a polynomial plus a proper rational function
  - One way of doing that is long division of $N(s)$ by $D(s)$ until the remainder is a proper function
  - The polynomial is then inverse-transformed into impulse functions and derivatives of impulse functions
  - The proper rational function is inverse-transformed by the techniques we discussed earlier
- Illustration example: Find the inverse Laplace transform for:

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

This is an improper rational function. Using division, we can write it as

$$F(s) = s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20},$$

Using the partial fraction expansion techniques we discussed earlier

$$F(s) = s^2 + 4s + 10 - \frac{20}{s + 4} + \frac{50}{s + 5}.$$

The inverse transform will be:

$$f(t) = \frac{d^2\delta(t)}{dt^2} + 4\frac{d\delta(t)}{dt} + 10\delta(t) - (20e^{-4t} - 50e^{-5t})u(t).$$

Initial- & Final-Value Theorems

- The initial- and final-value theorems are useful because they enable us to determine from $F(s)$ the behavior of $f(t)$ at $t = 0$ & ∞
  - Hence we can check the initial and final values of $f(t)$ to see if they conform with known circuit behavior, before actually finding the inverse transform of $F(s)$
  - This theorem is valid only if the poles of $F(s)$ lie in the left half of the s plane
    - This will guarantee that the system (circuit) is causal & stable

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s), \qquad (12.93)$$

◀ **Initial value theorem**

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s). \qquad (12.94)$$

◀ **Final value theorem**

- To prove, we start with the operational transform of the first derivative:

$$\mathscr{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0^-) = \int_{0^-}^{\infty} \frac{df}{dt} e^{-st}\, dt.$$

Please follow the rest of the proof in the book

Application of Initial- & Final-Value Theorems

- If we go back to this example:
  - We assume that no initial energy is stored in the circuit at the instant when the switch is opened
    - Thus, we know that $v(0^-) = v(0^+) = 0$
    - We also know that $v(\infty) = 0$ because the ideal inductor is a perfect short circuit across the dc current source

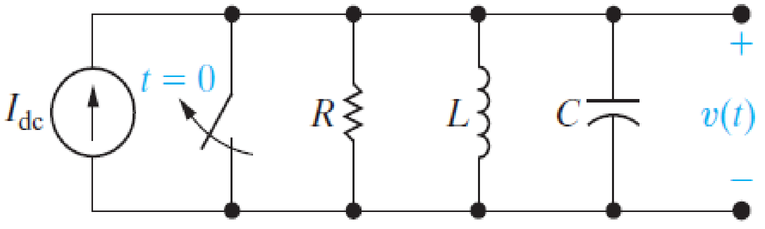


Figure 12.16 ▲ A parallel *RLC* circuit.

Laplace Transform

$$\frac{v(t)}{R} + \frac{1}{L}\int_0^t v(x)\,dx + C\frac{dv(t)}{dt} = I_{dc}u(t).$$

$$\frac{V(s)}{R} + \frac{1}{L}\frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_{dc}\left(\frac{1}{s}\right),$$

$$V(s)\left(\frac{1}{R} + \frac{1}{sL} + sC\right) = \frac{I_{dc}}{s},$$

$$V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}.$$

Applying the initial-value theorem yields:

$$\lim_{s\to\infty} sV(s) = \lim_{s\to\infty}\frac{s(I_{dc}/C)}{s^2[1 + 1/(RCs) + 1/(LCs^2)]} = 0.$$

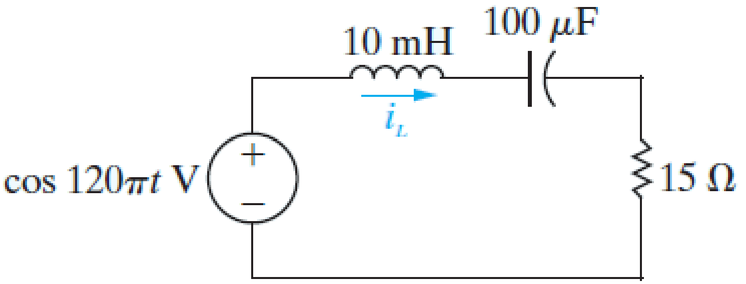
Applying the final-value theorem yields:

$$\lim_{s\to 0} sV(s) = \lim_{s\to 0}\frac{s(I_{dc}/C)}{s^2 + (s/RC) + (1/LC)} = 0.$$

Thus, the derived expression for $V(s)$ correctly predicts the initial and final values of $v(t)$

General Example

- For the circuit shown here, find an expression for $i(t)$ using the Laplace transform



$$15i_L(t) + 0.01\frac{di_L(t)}{dt} + \frac{1}{100 \times 10^{-6}}\int_0^t i_L(x)dx = \cos 120\pi t$$

$$15I_L(s) + 0.01sI_L(s) + 10^4\frac{I_L(s)}{s} = \frac{s}{s^2 + (120\pi)^2}$$

$$I_L(s) = \frac{100s^2}{[s^2 + 1500s + 10^6][s^2 + (120\pi^2)]}$$

Note that the expression for $I_L(s)$ has two complex conjugate pairs of poles, so the partial fraction expansion of $I_L(s)$ will have four terms

$$\Rightarrow \quad I_L(s) = \frac{K_1}{(s + 750 - j661.44)} + \frac{K_1^*}{(s + 750 + j661.44)} + \frac{K_2}{(s - j120\pi)} + \frac{K_2^*}{(s + j120\pi)}$$

$$K_1 = \left.\frac{100s^2}{[s + 750 + j661.44][s^2 + (120\pi)^2]}\right|_{s=-750+j661.44} = 0.07357\angle -97.89°$$

$$K_2 = \left.\frac{100s^2}{[s^2 + 1500s + 10^6][s + j120\pi]}\right|_{s=j120\pi} = 0.018345\angle 56.61°$$

$$\Rightarrow \quad i_L(t) = \underbrace{147.14e^{-750t}\cos(661.44t - 97.89°)} + \underbrace{36.69\cos(120\pi t + 56.61°)\text{ mA}}$$

This is the transient response, which will decay to zero after some time

This is the steady state response, which has the same frequency as the source and will persist as long as the source is connected

Note: Roots of Quadratic Equation

The Quadratic Formula: For $ax^2 + bx + c = 0$, the values of x which are the solutions of the equation are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Summary of Topics Covered in this Chapter

- In this chapter we discussed:
    - Definition of the Laplace transform
    - The step & the impulse function
    - Functional & operational transforms
    - Applying Laplace Transform
    - Inverse Transforms
    - Poles and Zeros of $F(s)$
    - Initial and final value theorem
- We covered sections 12.1 – 12.9
- Next chapter (Ch13) we will discuss how we can apply Laplace transform in circuit analysis
    - The Laplace Transform in Circuit Analysis