Chapter 9: Sinusoidal Steady-State Analysis

EEL 3112c – Circuits-II

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Topics to be Covered in this Chapter

- In this chapter we will discuss:
  - Sinusoidal sources
  - Sinusoidal response
  - Phasor transformation
  - Phasor diagrams
  - General circuit analysis in the frequency domain
  - Transformers
- We will cover sections 9.1-9.10
  - Sections 9.7, 9.8, 9.9 will be briefly discussed
  - They are repetition of what we studied in Ch4 in circuits-I
  - Please refer to review material posted on course website related to Ch4

Introduction

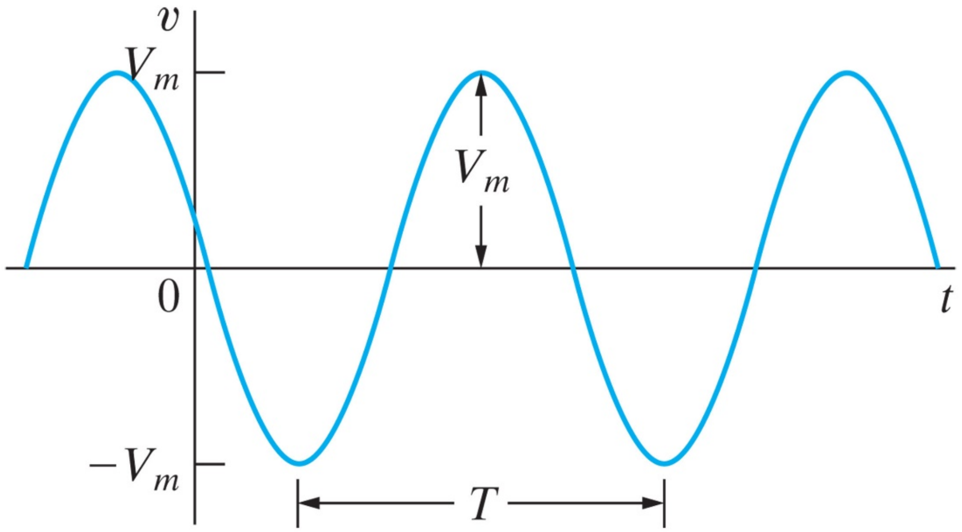
- Thus far, we have focused on circuits with constant sources
- In this chapter, we will start considering circuits energized by time-varying voltage or current sources
  - In particular, we are interested in sources in which the value of the voltage or current varies sinusoidally
- The subsequent chapters of this book are largely based on a thorough understanding of the techniques needed to analyze circuits driven by sinusoidal sources
- Fortunately, circuit analysis and simplification techniques we studied earlier (Circuits-I) are still **very helpful** and **apply** here
- The difference is that we will be working with complex numbers

The Sinusoidal Source

- A **sinusoidal voltage (or current) source** (independent or dependent) produces a voltage (or current) that varies sinusoidally with time
- We can express a sinusoidally varying function with either the sine function or the cosine function
  - Although either works equally well, we will use the cosine function for our discussion

$$\boldsymbol{v = V_m \cos(\omega t + \phi)}$$

V_m is the maximum amplitude of the sinusoidal voltage



- Note that the above sinusoidal function is periodic
  - Repeats at regular intervals
- One parameter of interest is the length of time required for the sinusoidal function to pass through all its possible values
  - This time is referred to as the **period** of the function and is denoted by $\boldsymbol{T}$ (measured in **seconds**)
- Reciprocal of $\boldsymbol{T}$ gives the number of cycles per second or the frequency ($\boldsymbol{f}$)

$$\boldsymbol{f = \frac{1}{T}}$$

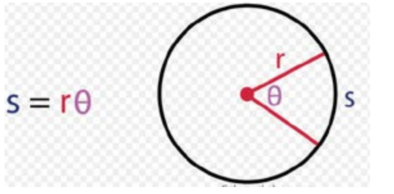
  - The unit for f is Hertz (Hz) which is $1/sec$

The Sinusoidal Source – cont.

- Omega (ω), the coefficient of t in the sinusoidal voltage equation represents the angular frequency of the sinusoidal function

$$\omega = 2\pi f = \frac{2\pi}{T} \text{ (radians/second)}$$

- The angular frequency indicates how fast an object (or a signal) is rotating
  - How many radians the angle changes per unit time ($\omega = d\theta/dt$)
  - A measure of the angular velocity or angular displacement per unit time
- So if we say that a signal completes one cycle/second, this means it has a frequency of 1 Hz
- This is equivalent to moving 2π radians/seconds
  - A circle circumference is 2π radians
  - A radian is a unit of angle at the center of a circle whose arc is equal in length to the radius of that circle

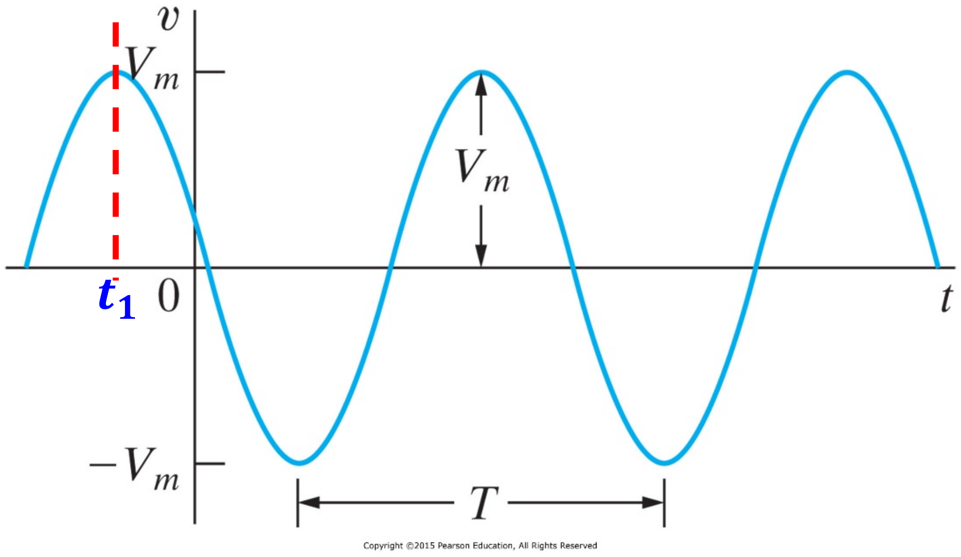


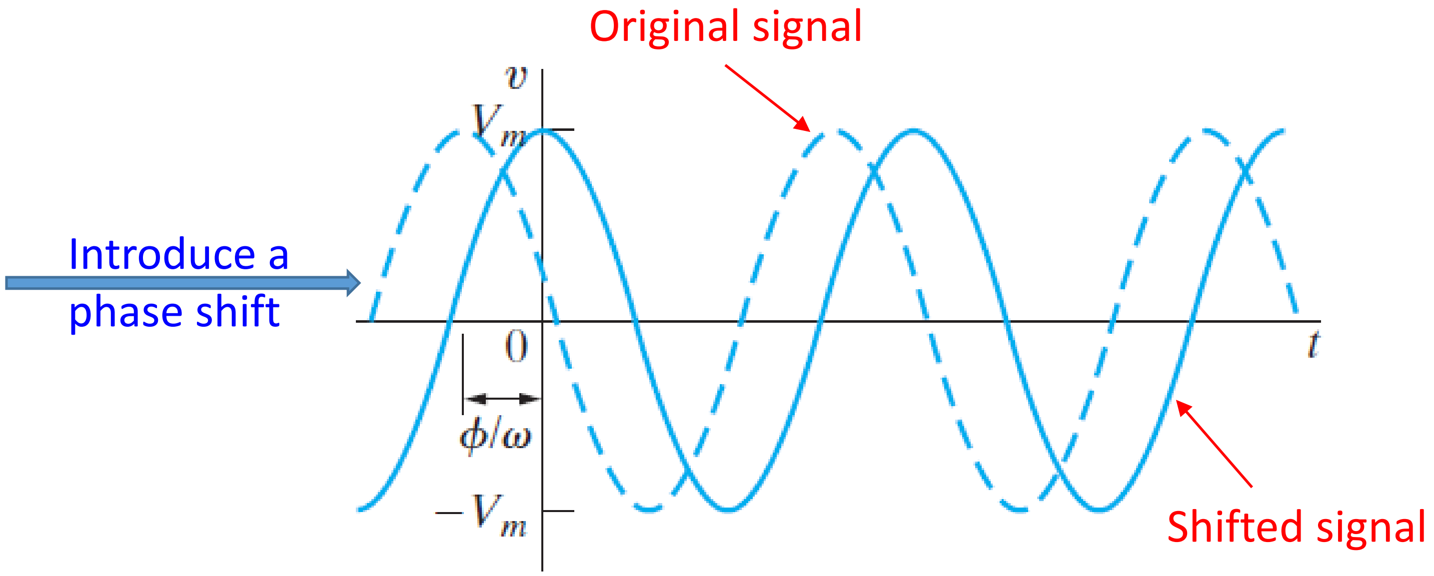
The Sinusoidal Source – cont.

- The angle phi (ϕ) is known as the phase angle of the sinusoidal voltage
  - It determines the value of the sinusoidal function at $t = 0$, and therefore it fixes the point on the periodic wave at which we start measuring time
  - Changing the phase angle ϕ shifts the sinusoidal function along the time axis but has no effect on either V_m or ω
    - It just changes the location of the zero crossing of the time axis by the signal
- Because ωt and ϕ are added together in the argument of the sinusoidal function, they should carry the same units
  - ϕ is normally given in degrees and ωt are expressed in radians
  - ωt must be converted from radians to degrees before we can add the two quantities

$$\boxed{(\text{number of degrees}) = \frac{180^\circ}{\pi}(\text{number of radians})}$$

The Sinusoidal Source – cont.





If we want to shift this signal to the right so that a peak of the signal happens at $t = 0$, we need to introduce a phase shift to the right, that will shift the signal by that amount

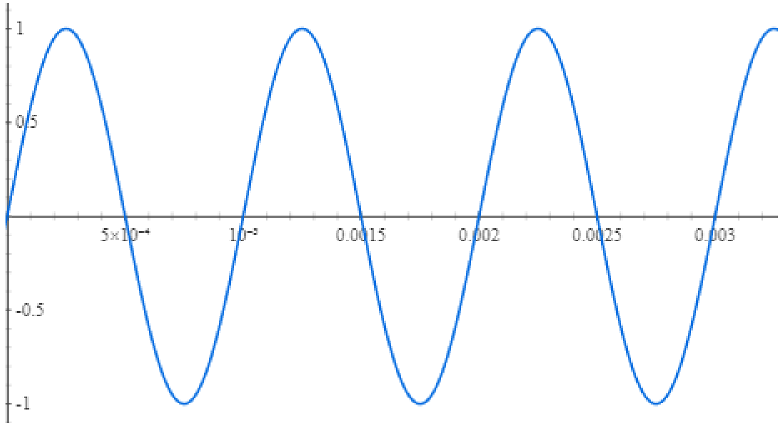
- So we want to shift the original signal (v) t_1 seconds to the right
- Remember that we need to have angles in the argument of the cosine: $\boldsymbol{v = V_m \cos(\omega t + \phi)}$
- Thus, if we know the shift in time units we can convert it to degrees via multiplying it by ω (in degrees) for that signal
- If we know that shift in degrees, and we want to see how many time units it corresponds to, we divide by ω (in degrees)

Note also that if ϕ is positive, the sinusoidal function shifts to the left, whereas if ϕ is negative, the function shifts to the right

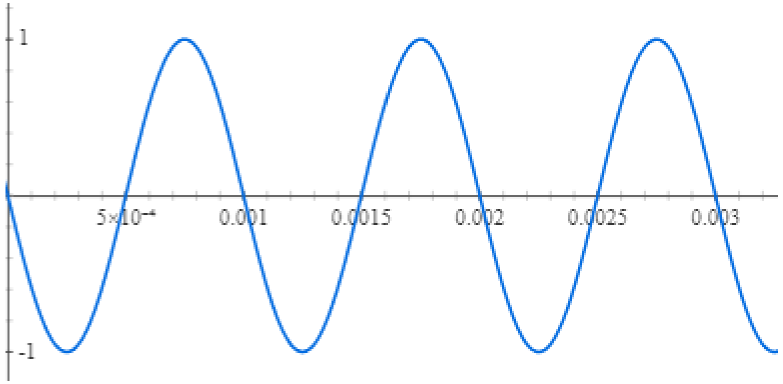
Example

- If you were asked to design a circuit that will mute a machine that generates sound (noise) in the form of sine waves with 1 kHz frequency. What can you do?

(a)

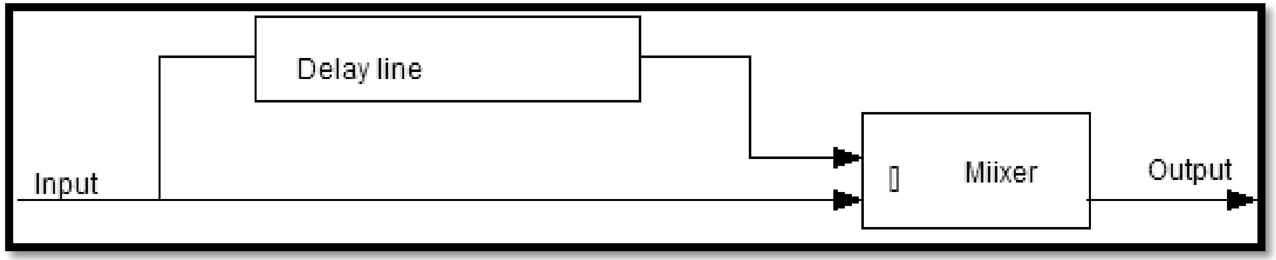


(b)



- What do you think will happen if we add original noise signal shown in (a) to the signal shown in (b)?
- How can we generate signal (b)?

- Signal (b) is in fact signal (a) shifted by $180°$
- In time units, this is a shift of $\frac{180°}{360° \times (1 \times 10^3)} = 0.5 \times 10^{-3} = 0.5\ ms$
- This means that signal (b) is in fact signal (a) time delayed by $0.5\ ms$



The Sinusoidal Source – cont.

- Another important characteristic of the sinusoidal voltage (or current) is its root mean square **(rms)** value
  - The **rms** value is the value of the equivalent direct (non varying) voltage or current which would provide the same energy to a circuit as the sinusoidal wave used
  - For example, the power supply in the US is 120V ac, this means that the sinusoidal rms voltage from the wall socket of a US home is capable of producing the same average positive power as 120 volts of steady DC voltage
- The rms value of a **sinusoidal function** is defined as the square **r**oot of the **m**ean value of the **s**quared function

$$\boxed{V_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi)\, dt.} \Longrightarrow V_{rms} = \frac{V_m}{\sqrt{2}}}$$

  - From this equation we note that we obtain the mean value of the squared voltage by integrating v^2 over one period (that is from t_0 to $t_0 + T$) and then dividing by the range of integration (T)

Note that if we want the rms value for a current waveform, we will use the expression for i instead of v

The Sinusoidal Source – cont.

- In case you are interested in the intermediate steps for calculating the **rms** value of $v(t)$

$$\int_{t_o}^{t_o+T} V_m^2 \cos^2(\omega t + \phi)\, dt = V_m^2 \int_{t_o}^{t_o+T} \frac{1}{2} + \frac{1}{2}\cos(2\omega t + 2\phi)\, dt$$

$$= \frac{V_m^2}{2}\left\{ \int_{t_o}^{t_o+T} dt + \int_{t_o}^{t_o+T} \cos(2\omega t + 2\phi)\, dt \right\}$$

$$= \frac{V_m^2}{2}\left\{ T + \frac{1}{2\omega}\left[\sin(2\omega t + 2\phi)\ \Big|_{t_o}^{t_o+T} \right] \right\}$$

$$= \frac{V_m^2}{2}\left\{ T + \frac{1}{2\omega}\left[\sin(2\omega t_o + 4\pi + 2\phi) - \sin(2\omega t_o + 2\phi) \right] \right\}$$

$$= V_m^2\left(\frac{T}{2}\right) + \frac{1}{2\omega}(0) = V_m^2\left(\frac{T}{2}\right)$$

$$v_{rms} = \sqrt[2]{\left(\frac{1}{T}\right)\left[V_m^2\left(\frac{T}{2}\right)\right]} = \frac{V_m}{\sqrt{2}}$$

Example 9.2

A sinusoidal voltage is given by the expression $v = 300\cos(120\pi t + 30^\circ)$.

a) What is the period of the voltage in milliseconds?

b) What is the frequency in hertz?

c) What is the magnitude of v at $t = 2.778$ ms?

d) What is the rms value of v?

Solution

a) From the expression for v, $\omega = 120\pi$ rad/s. Because $\omega = 2\pi/T$, $T = 2\pi/\omega = \frac{1}{60}$ s, or 16.667 ms.

b) The frequency is $1/T$, or 60 Hz.

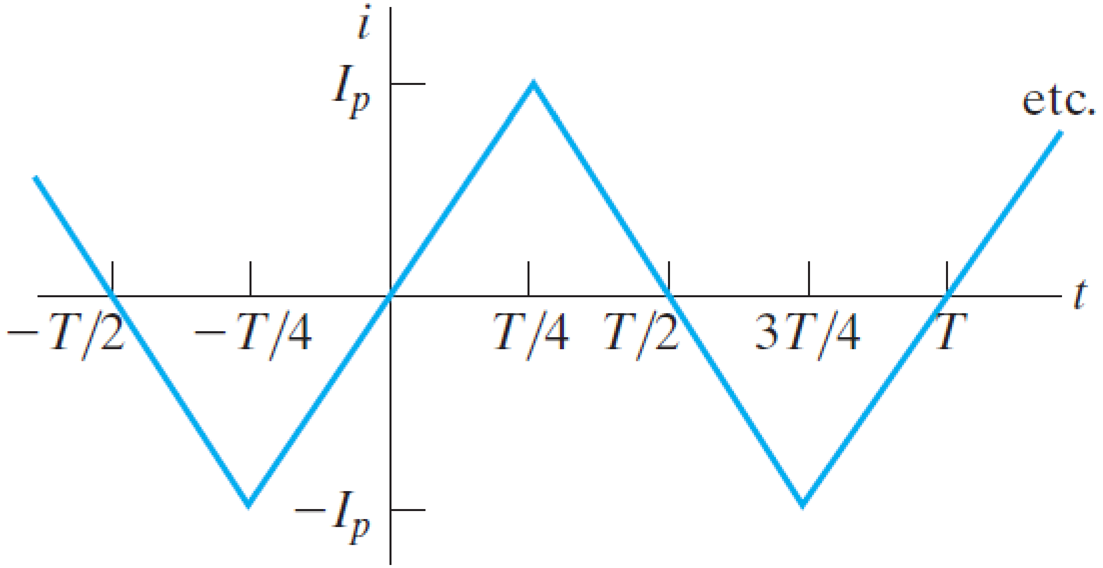
c) From (a), $\omega = 2\pi/16.667$; thus, at $t = 2.778$ ms, ωt is nearly 1.047 rad, or 60°. Therefore, $v(2.778\text{ ms}) = 300\cos(\underbrace{60^\circ + 30^\circ}) = 0$ V.

Change the units to degrees for both quantities

d) $V_{rms} = 300/\sqrt{2} = 212.13$ V.

Example 9.4

Calculate the rms value of the periodic triangular current shown in Fig. 9.3. Express your answer in terms of the peak current I_p.



$$I_{\text{rms}} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} i^2 dt}\,.$$

- We can note that the area under the curve of the squared current equals 4 times the area under the curve for the interval 0 to $T/4$

$$\int_{t_0}^{t_0+T} i^2 dt = 4\int_0^{T/4} i^2 dt.$$

- The analytical expression for i in the interval 0 to $T/4$ is

$$i = \frac{4I_p}{T}t, \quad 0 < t < T/4.$$

Line eqn.: $y - y_i = \left[\frac{y_2 - y_1}{x_2 - x_1}\right](x - x_i)$

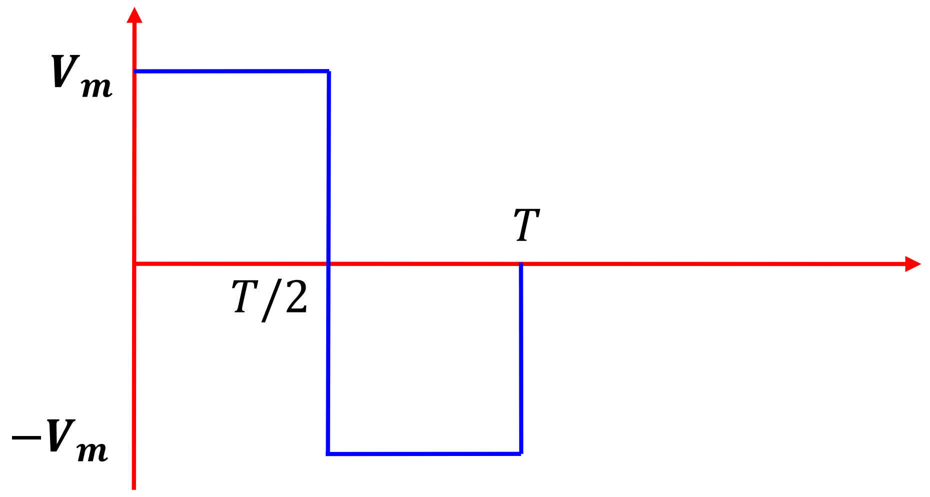
$$\int_{t_0}^{t_0+T} i^2 dt = 4\int_0^{T/4} \frac{16I_p^2}{T^2}t^2 dt = \frac{I_p^2 T}{3}.$$

The total area

$$i_{\text{mean}} = \frac{1}{T}\frac{I_p^2 T}{3} = \frac{1}{3}I_p^2. \longrightarrow I_{\text{rms}} = \frac{I_p}{\sqrt{3}}.$$

Additional Example: The square wave

- Calculate the rms value of the periodic square voltage shown here
  - Sometimes called the bipolar pulse

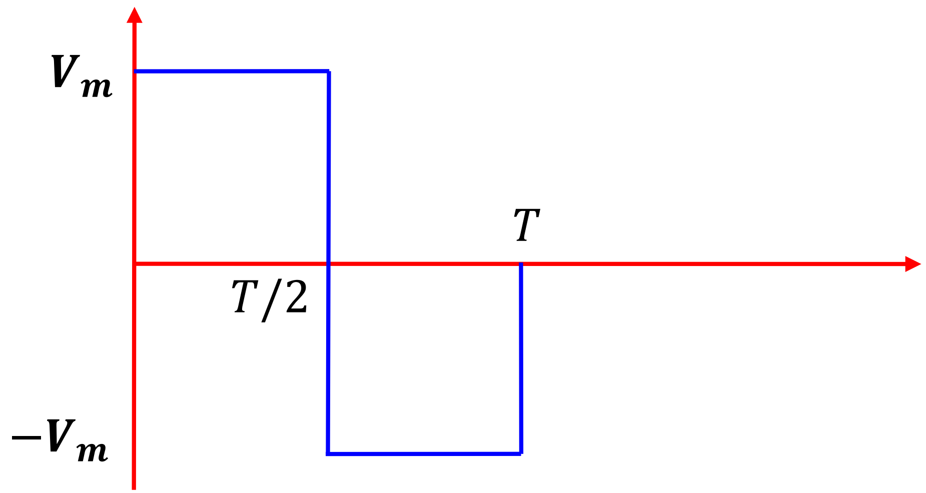


Take a minute to think about it

Additional Example: The square wave

- Calculate the rms value of the periodic square voltage shown here
  - Sometimes called the bipolar pulse

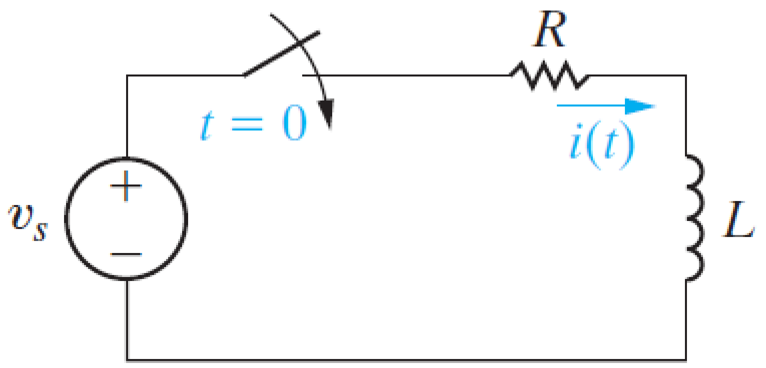
$$V_{rms} = \sqrt{\frac{1}{T}\left[\int_0^{\frac{T}{2}} V_m^2 dt + \int_{\frac{T}{2}}^{T} (-V_m)^2 dt\right]}$$



$$V_{rms} = \sqrt{\frac{1}{T}\left[V_m^2 \left(\frac{T}{2}\right) + V_m^2 \left(\frac{T}{2}\right)\right]}$$

$$V_{rms} = \sqrt{\frac{1}{T}[V_m^2 T]} = \sqrt{V_m^2} = V_m$$

The Sinusoidal Response



- Let us consider the following RL circuit:
- v_s is a sinusoidal voltage $v_s = V_m \cos(\omega t + \phi)$
- Assuming no energy was initially stored in the circuit, we can derive an expression for current as: $L\dfrac{di}{dt} + Ri = V_m \cos(\omega t + \phi)$
- From previous knowledge of diff. eqn., the solution for i is:

$$i = \underbrace{\frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\phi - \theta) e^{-(R/L)t}} + \underbrace{\frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi - \theta)},$$

Where θ is an angle defined as: $\theta = \tan^{-1}\left(\frac{\omega L}{R}\right)$

The transient component because as time elapses and t becomes larger, this term becomes infinitesimal

The steady-state component because it exists as long as the switch remains closed and the source continues to supply the sinusoidal voltage

The Sinusoidal Response – cont.

- In **this chapter**, we develop technique to **calculate the steady-state response** directly, thus avoiding the problem of solving the differential equation
  - However, in using this technique we forfeit obtaining either the transient component or the total response, which is the sum of the transient and steady-state components
- Focusing on the steady-state solution, it is important to remember the following:
  - The steady-state solution is a sinusoidal function
  - The frequency of the response signal is identical to the frequency of the source signal
  - The maximum amplitude of the steady-state response, in general, differs from the maximum amplitude of the source
    - In our circuit, for the source it is V_m while for the steady-state it is $\frac{V_m}{\sqrt{R^2+\omega^2L^2}}$
  - The phase angle of the response signal, in general, differs from the phase angle of the source
    - For the circuit being discussed, the phase angle of the current is $\phi - \theta$ and the voltage source is ϕ

Math Review: Complex Numbers

- The complex number $\boldsymbol{z}$ can be expressed in several ways
  - The Cartesian or rectangular form for $\boldsymbol{z}$

The conjugate of z will be $z^* = x - jy$

$$\boldsymbol{z = x + jy}$$

- *Where* $j = \sqrt{-1}$, *and* x *and* y are numbers referred to as the real and imaginary part of z respectively

$$x = Re\{z\}, \qquad y = Im\{z\}$$

- The Polar form of $\boldsymbol{z}$

$$\boldsymbol{z = re^{j\theta}}$$

- *Where* r is the magnitude of z, and θ is the angle or phase of z
- $r = \sqrt{x^2 + y^2}$ & $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $x = rcos(\theta)$ & $y = rsin(\theta)$

Math Review: Complex Numbers – cont.

- Complex addition and subtraction

$$A = x + jy \qquad\qquad B = w + jz$$

$$A + B = (x + w) + j(y + z)$$

$$A - B = (x - w) + j(y - z)$$

- Addition and subtraction of complex numbers must be done in the rectangular form

- Complex multiplication and division

$$Z_1 \times Z_2 = A_1 \times A_2 \angle \theta_1 + \theta_2$$

$$\frac{Z_1}{Z_2} = \left(\frac{A_1}{A_2}\right) \angle \theta_1 - \theta_2$$

- The division of complex numbers is best carried out using polar form
    - If carried out in the rectangular form, we need to multiply top and bottom by the conjugate of the denominator

The Phasor

- The **phasor** is a complex number that carries the amplitude and phase angle information of a sinusoidal function
- The phasor concept is rooted in Euler's identity

$$e^{\pm j\theta} = \cos(\theta) \pm j\sin(\theta)$$

$$\cos\theta = \Re\{e^{j\theta}\},$$ The real part

$$\sin\theta = \Im\{e^{j\theta}\},$$ The imaginary part

- Since we chose the cosine function to represent the sinusoidal waveforms

$$v = V_m\cos(\omega t + \phi) = V_m\Re\left(e^{j(\omega t+\phi)}\right) = V_m\Re\left(e^{j\omega t}e^{j\phi}\right) = \Re\left(V_m e^{j\phi}e^{j\omega t}\right)$$

$$\mathbf{V} = V_m e^{j\phi} = \mathcal{P}\{V_m\cos(\omega t + \phi)\},$$

This quantity is a complex number that carries the amplitude and phase information of the given sinusoidal function

This notation is " the phasor transform of the sinusoidal function v that transforms it from the time domain to the complex number domain", also called the frequency domain

- We can have a rectangular and angle form of a phasor

$$\boxed{\mathbf{V} = V_m\cos\phi + jV_m\sin\phi.}$$

Rectangle representation

$$\boxed{1\angle\phi^\circ = 1e^{j\phi}.}$$

Angle representation

Inverse Phasor Transform

- After we learned how to find the phasor transform of the sinusoidal function, it is useful to learn how we can reverse the process
  - Going from phasor to sinusoidal
  - Example: $\boldsymbol{V} = 100\angle - 26°$, then the expression for v is $v = 100\cos(\omega t\ - 26°)$

- Thus, the step of going from the phasor transform to the time-domain expression is referred to as *finding the inverse phasor transform*

$$\boxed{\mathcal{P}^{-1}\{V_m e^{j\phi}\} = \Re\{V_m e^{j\phi} e^{j\omega t}\}}$$

  - This equation indicates that to find the inverse phasor transform, we multiply the phasor by $e^{j\omega t}$ and then extract the real part of the product
    - We only deal with the real part because we want the cosine term only

Inverse Phasor Transform – cont.

- The phasor transform, along with the inverse phasor transform, allows you to go back and forth between the time representation and the phasor representation
- The phasor transform is useful in circuit analysis because it reduces the task of finding the maximum amplitude and phase angle of the steady-state sinusoidal response to the algebra of complex numbers
- The phasor transform is also useful in circuit analysis because it applies directly to the sum of sinusoidal functions
- If $v = v_1 + v_2 + \ ... + v_n$, where all the voltages are sinusoidal voltages of the same frequency, then $\boldsymbol{V} = \boldsymbol{V}_1 + \boldsymbol{V}_2 + \cdots + \boldsymbol{V}_n$

Example 9.5

If $y_1 = 20\cos(\omega t - 30^\circ)$ and $y_2 = 40\cos(\omega t + 60^\circ)$, express $y = y_1 + y_2$ as a single sinusoidal function. Solve by using the phasor concept.

Answer:

We can solve the problem by using phasors as follows: Because

$$y = y_1 + y_2,$$

then, from Eq. 9.24,

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2$$

$$= 20\angle -30^\circ + 40\angle 60^\circ$$

$$= (17.32 - j10) + (20 + j34.64)$$

$$= 37.32 + j24.64 = 44.72\angle 33.43^\circ$$

Once we know the phasor $\mathbf{Y}$, we can write the corresponding trigonometric function for y by taking the inverse phasor transform:

$$y = \mathcal{P}^{-1}\{44.72e^{j33.43}\} = \Re\{44.72e^{j33.43}e^{j\omega t}\}$$

$$= 44.72\cos(\omega t + 33.43^\circ).$$

The superiority of the phasor approach for adding sinusoidal functions should be apparent. Note that it requires the ability to move back and forth between the polar and rectangular forms of complex numbers.

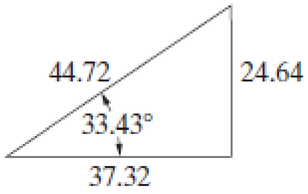
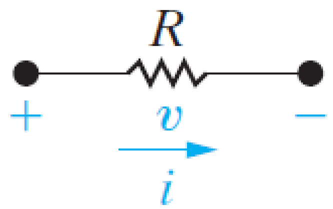


Figure 9.6 ▲ A right triangle used in the solution for y.

Passive Circuit Elements in the Freq. Domain



- The V-I relationship for a **resistor**:
  - From Ohm's law, if the current in a resistor varies sinusoidally with time, that is: $i = I_m \cos(\omega t + \theta_i)$, then the voltage at the terminal of the resistor is:

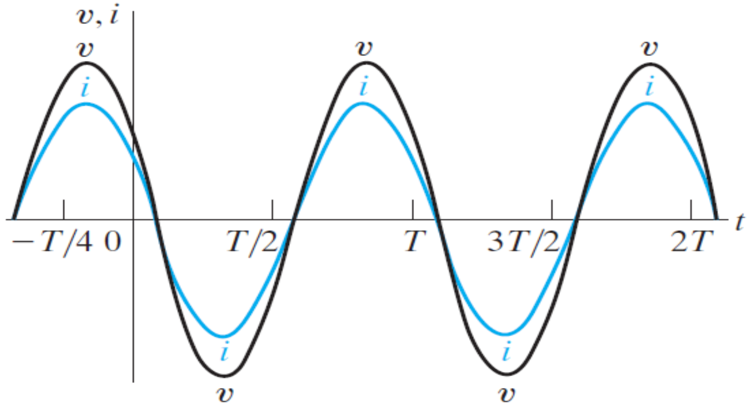
$$v = R[I_m \cos(\omega t + \theta_i)] = RI_m[\cos(\omega t + \theta_i)]$$

  - Where I_m is the maximum current, and θ_i is the phase angle of the current
- The phasor transform of this voltage is: $\mathrm{V} = RI_m e^{j\theta_i} = RI_m\angle\theta_i = R\mathbf{I}$

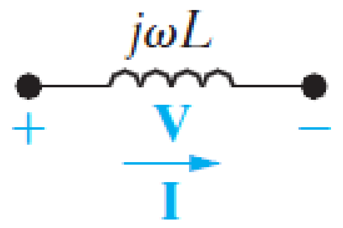
This is the phasor representation of the sinusoidal current **I**

  - This states that the phasor voltage across the resistor terminals are simply the resistance times the phasor current

There will be **NO phase shift** between the current and the voltage (they are in phase)



Passive Circuit Elements in the Freq. Domain – cont.

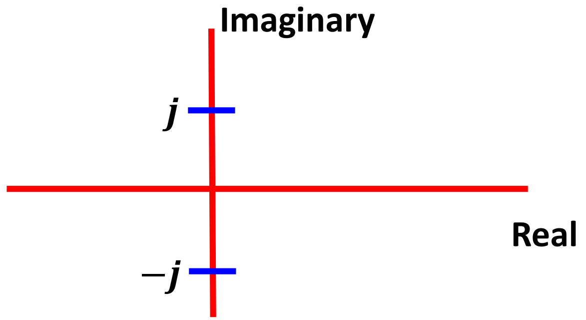


- The V-I relationship for an **inductor**:

$$v = L\frac{di}{dt} = -\omega L I_m \sin(\omega t + \theta_i)$$, Given that $i = I_m \cos(\omega t + \theta_i)$

- Rewrite the above equation using the cosine function to follow convention:

$$v = -\omega L I_m \cos(\omega t + \theta_i - 90^\circ).$$



- The phasor representation of the voltage is then:

$$\mathbf{V} = -\omega L I_m e^{j(\theta_i - 90^\circ)}$$

$$= -\omega L I_m e^{j\theta_i} e^{-j90^\circ}$$

$$e^{-j90^\circ} = \cos 90^\circ - j \sin 90^\circ = -j.$$

$$= j\omega L I_m e^{j\theta_i}$$

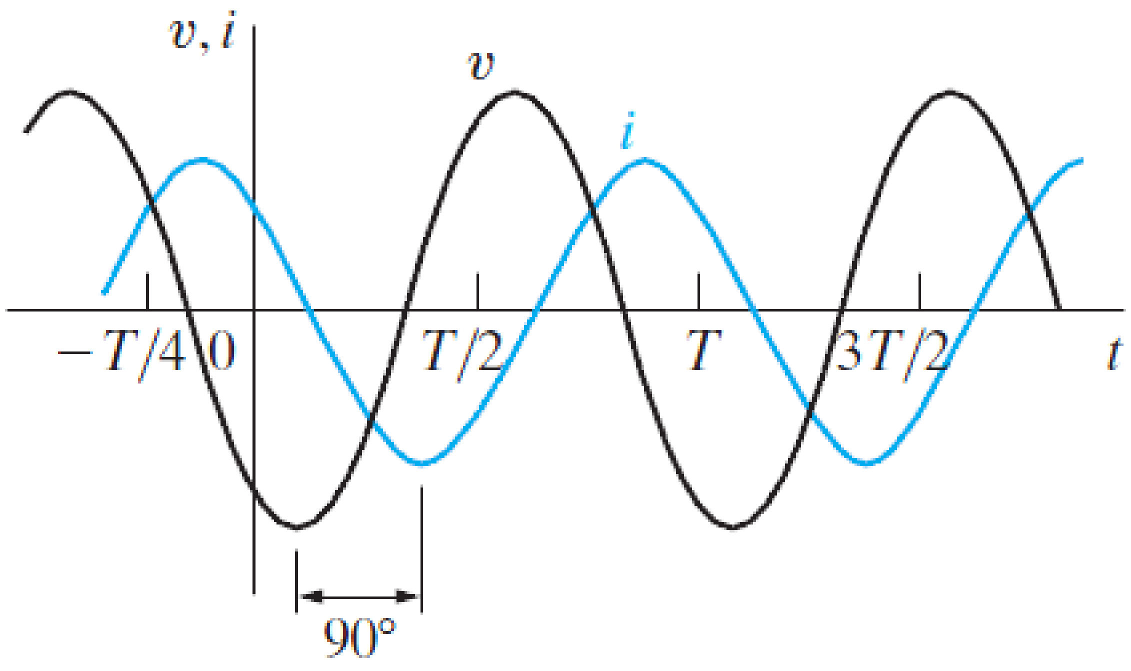
$$= j\omega L \mathbf{I}.$$

$$\mathbf{V} = (\omega L \angle 90^\circ) I_m \angle \theta_i$$

$$= \omega L I_m \angle (\theta_i + 90)^\circ,$$

Passive Circuit Elements in the Freq. Domain – cont.

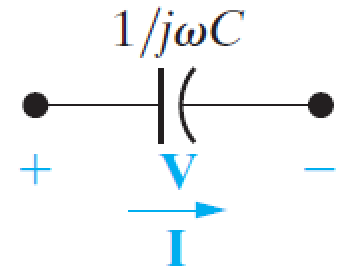
- The V-I relationship for an **inductor**: - cont.



- The voltage and current are out of phase by exactly $90°$
- In particular, the voltage leads the current by or, equivalently, the current lags behind the voltage by $90°$
- To better illustrate the concept, in the figure shown here, the voltage reaches its negative peak exactly $90°$ before the current reaches its negative peak. The same observation can be made with respect to the zero-going-positive crossing or the positive peak.
- We can also express the phase shift in time units. A phase shift of $90°$ corresponds to one-fourth of a period; hence the voltage leads the current by $T/4$ seconds

$$\frac{(90°)}{\left(360° \times \frac{1}{T}\right)} = T/4$$

Passive Circuit Elements in the Freq. Domain – cont.



- The V-I relationship for a **capacitor**:

$$i = C\frac{dv}{dt}$$

- Given that $v = V_m\cos(\omega t + \theta_v)$, then:

$$i = -\omega C V_m \sin(\omega t + \theta_v)$$
$$i = -\omega C V_m \cos(\omega t + \theta_v - 90°)$$

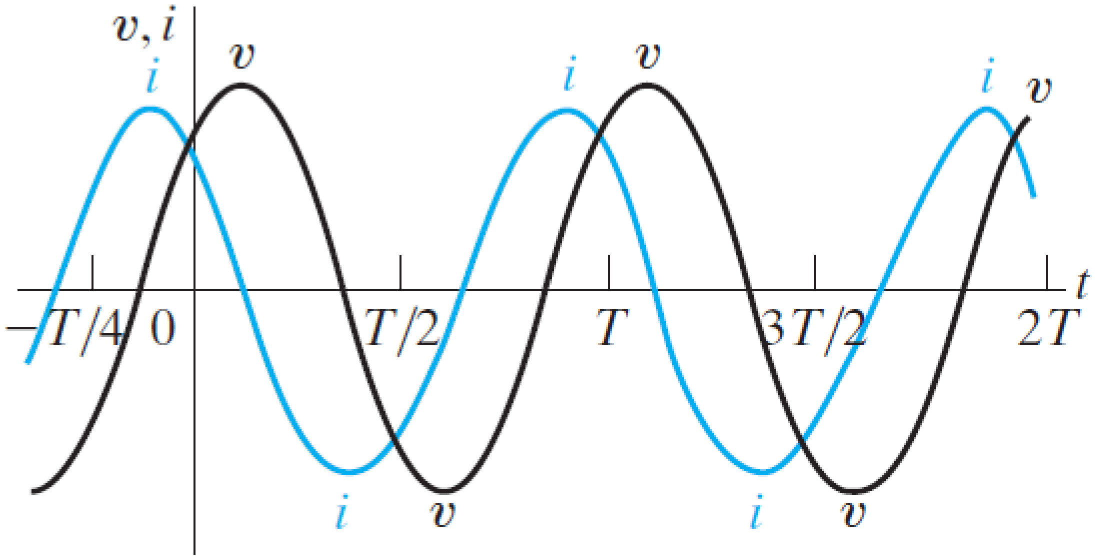
- Using **phasor** representation:

$$\boldsymbol{I} = -\omega C V_m e^{j\theta_v} e^{-j90°} = j\omega C V_m e^{j\theta_v} = j\omega C \boldsymbol{V}$$

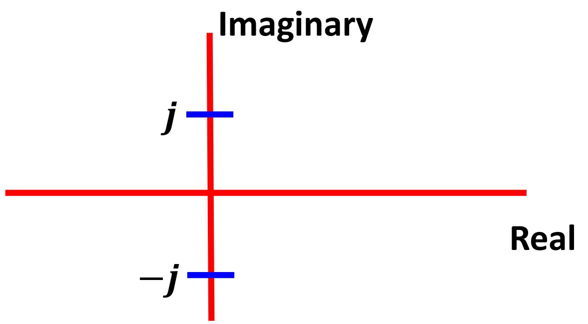
- Thus, $\boldsymbol{I} = j\omega C \boldsymbol{V}$

- If we want to solve for the voltage as a function of current:

$$\boldsymbol{V} = \frac{1}{j\omega C}\boldsymbol{I} = \frac{1}{\omega C}\angle -90°\; I_m\angle\theta_i = \frac{I_m}{\omega C}\angle(\theta_i - 90°)$$



We can note that the current leads the voltage by 90°



Passive Circuit Elements in the Freq. Domain – cont.

- Impedance & Reactance:
  - From our discussion on passive circuit elements in freq. domain, we can summarize it as follows using phasor representation:
    - For resistors: $\boldsymbol{V} = R\boldsymbol{I}$
    - For inductors: $\boldsymbol{V} = j\omega L\boldsymbol{I}$
    - For capacitors: $\boldsymbol{V} = \frac{1}{j\omega C}\boldsymbol{I}$
  - They are all of the form $\boldsymbol{V} = \boldsymbol{Z}\boldsymbol{I}$, where $\boldsymbol{Z}$ represents the impedance of the circuit element
  - Impedance in the frequency domain is the quantity analogous to resistance, inductance, and capacitance in the time domain

- Impedance is measured in **ohms**
- The imaginary part of impedance is called the **reactance**

TABLE 9.1 Impedance and Reactance Values

Circuit Element	Impedance	Reactance
Resistor	R	—
Inductor	$j\omega L$	ωL
Capacitor	$j(-1/\omega C)$	$-1/\omega C$

Impedance when we have mutual inductance is $j\omega M$

Kirchhoff's Laws in the Frequency Domain

- Let us begin by some assumptions:
  - v_1 to v_n represents the voltages around a closed path in a circuit
  - The circuit is operating in a sinusoidal steady state (SS)
- KVL states: $v_1 + v_2 + \cdots + v_n = 0$, which in sinusoidal SS becomes complex

$$V_{m_1}\cos(\omega t + \theta_1) + V_{m_2}\cos(\omega t + \theta_2) + \cdots + V_{m_n}\cos(\omega t + \theta_n) = 0.$$

We now use Euler's identity to write Eq. 9.37 as

$$\Re\{V_{m_1}e^{j\theta_1}e^{j\omega t}\} + \Re\{V_{m_2}e^{j\theta_2}e^{j\omega t}\} + \cdots + \Re\{V_{m_n}e^{j\theta_n}e^{j\omega t}\}$$

which we rewrite as

$$\Re\{V_{m_1}e^{j\theta_1}e^{j\omega t} + V_{m_2}e^{j\theta_2}e^{j\omega t} + \cdots + V_{m_n}e^{j\theta_n}e^{j\omega t}\} = 0.$$

Factoring the term $e^{j\omega t}$ from each term yields

$$\Re\{(V_{m_1}e^{j\theta_1} + V_{m_2}e^{j\theta_2} + \cdots + V_{m_n}e^{j\theta_n})e^{j\omega t}\} = 0,$$

or

$$\Re\{(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n)e^{j\omega t}\} = 0.$$

But $e^{j\omega t} \neq 0$, so

$$\boxed{\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0,}$$

When ω *or* t are equal to 0, the exponential is 1 and then it grows to ∞

KVL in the frequency domain

Kirchhoff's Laws in the Frequency Domain – cont.

- KCL states: $i_1 + i_2 + \cdots + i_n = 0$, which in sinusoidal SS becomes complex
- Following the same analysis we did for KVL, we reach

KCL in the frequency domain ▶ $$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0,$$

- You may use all the techniques developed for analyzing resistive circuits to find phasor currents and voltages
  - The basic circuit analysis and simplification tools covered in Chapters 2 – 4 can all be used to analyze circuits in the frequency domain

Combining Impedances in Series and Parallel

Series:

$$\mathbf{V}_{ab} = Z_1\mathbf{I} + Z_2\mathbf{I} + \cdots + Z_n\mathbf{I}$$
$$= (Z_1 + Z_2 + \cdots + Z_n)\mathbf{I}.$$

$$Z_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{I}} = Z_1 + Z_2 + \cdots + Z_n.$$

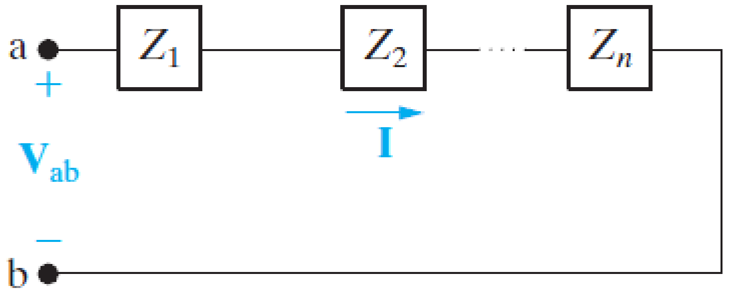


Figure 9.14 ▲ Impedances in series.

Parallel:

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n,$$

$$\frac{1}{Z_{ab}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}.$$

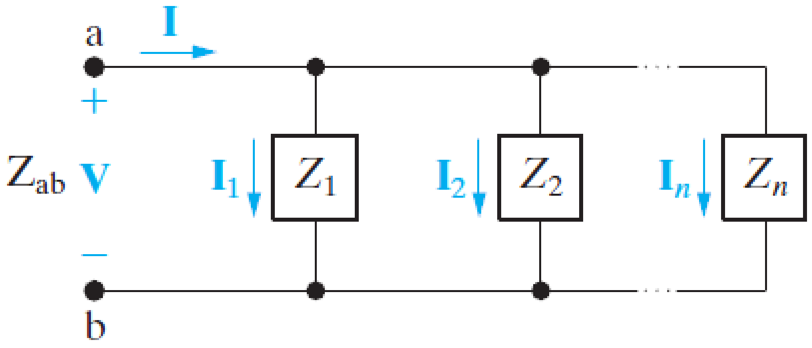


Figure 9.17 ▲ Impedances in parallel.

Example 9.6

A 90 Ω resistor, a 32 mH inductor, and a 5 μF capacitor are connected in series across the terminals of a sinusoidal voltage source, as shown in Fig. 9.15. The steady-state expression for the source voltage v_s is $750 \cos(5000t + 30°)$ V.

a) Construct the frequency-domain equivalent circuit.

b) Calculate the steady-state current i by the phasor method.

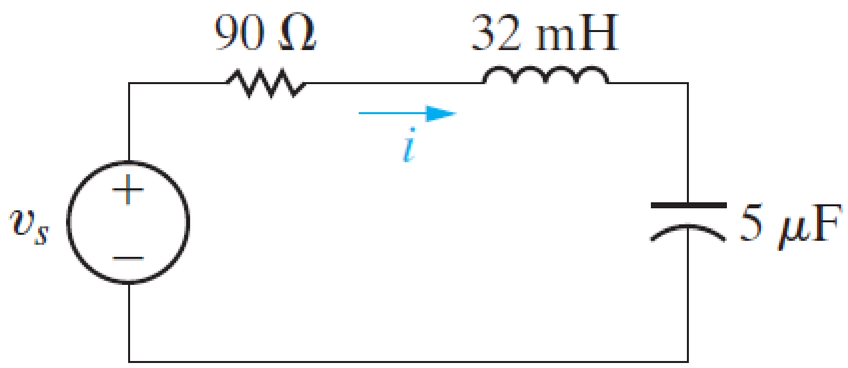


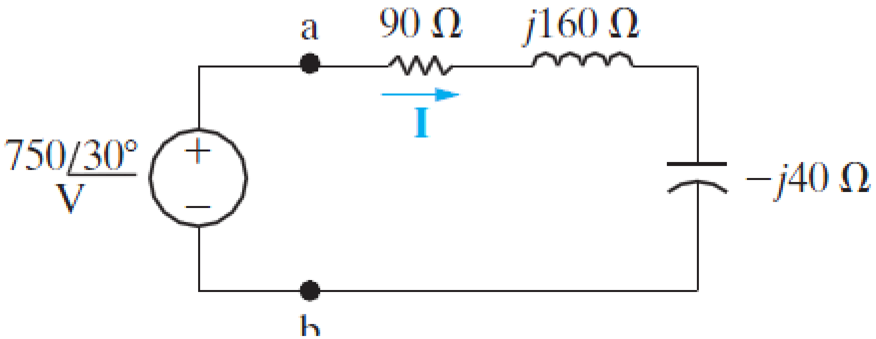
Figure 9.15 ▲ The circuit for Example 9.6.

Answer:

a) $Z_L = j\omega L = j(5000)(32 \times 10^{-3}) = j160\ \Omega,$

$$Z_C = j\frac{-1}{\omega C} = -j\frac{10^6}{(5000)(5)} = -j40\ \Omega.$$

$$\mathbf{V}_s = 750\angle 30°\ \text{V}.$$



b) $Z_{ab} = 90 + j160 - j40$

$$= 90 + j120 = 150\angle 53.13°\ \Omega.$$

$$\mathbf{I} = \frac{750\angle 30°}{150\angle 53.13°} = 5\angle -23.13°\ \text{A}.$$

$$i = 5\cos(5000t - 23.13°)\ \text{A}.$$

Delta-to-Wye Transformations

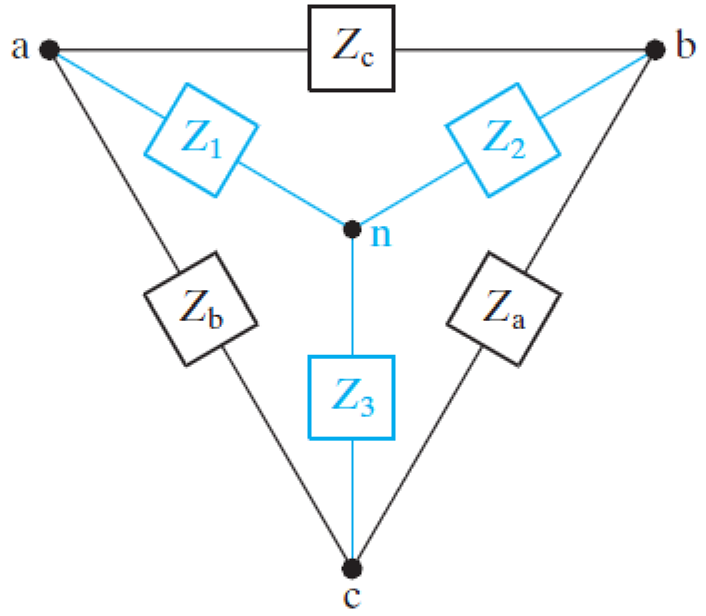
- The previous discussion we had on this topic in Chapter 3 still applies here
  - We just replace the symbol R with Z

From $\Delta - to - Y$:

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c},$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c},$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}.$$



From Y – to – Δ:

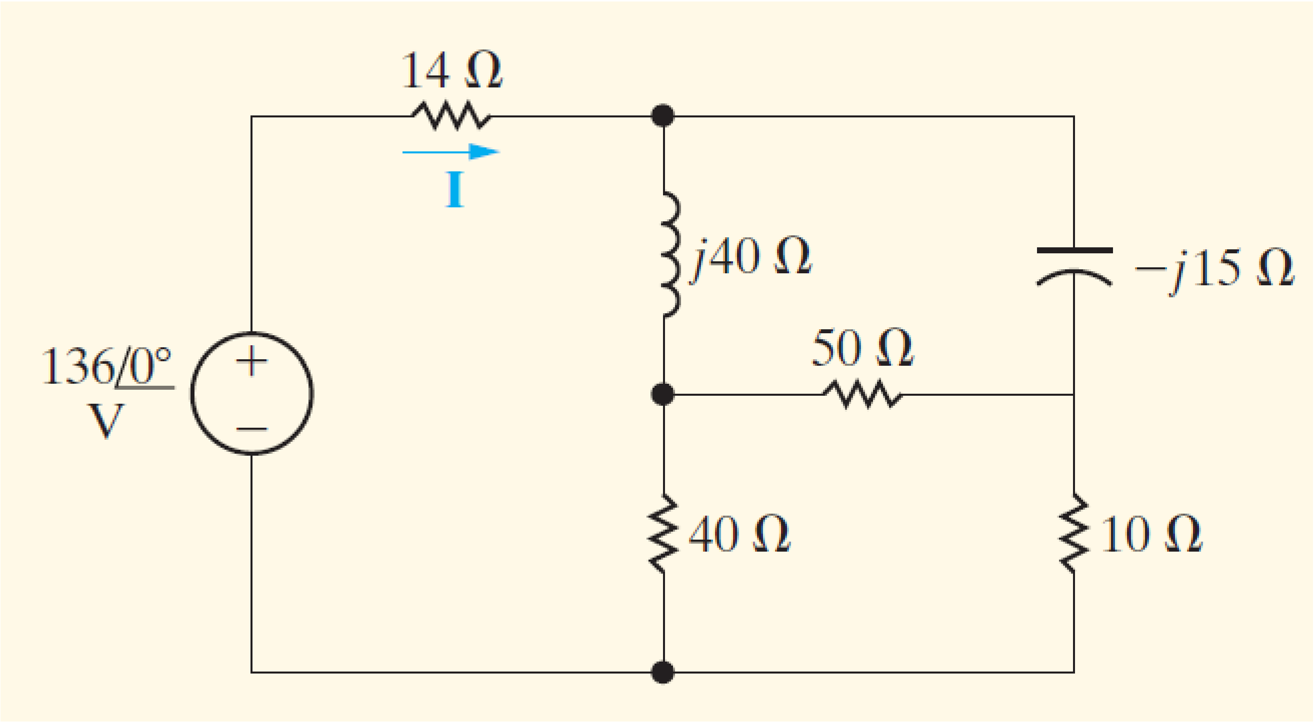
$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1},$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2},$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}.$$

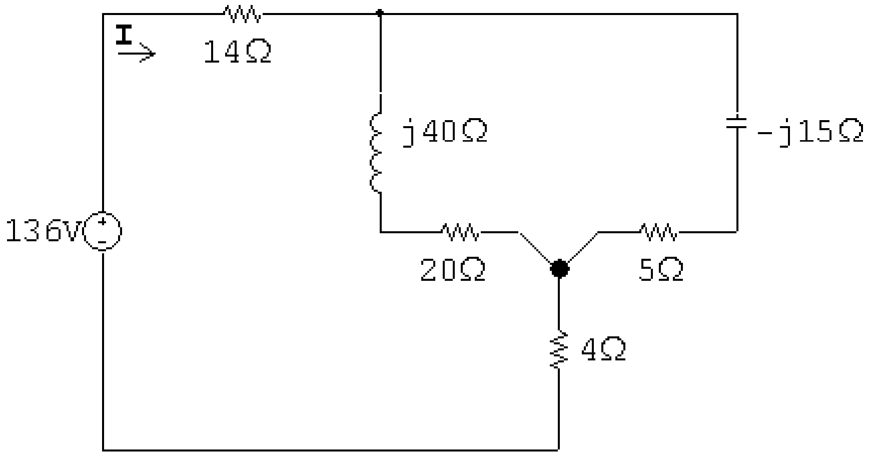
Assessment Problem 9.9

- Use a $\Delta - to - Y$ transformation to find **I** in the circuit shown below



Assessment Problem 9.9 – cont.

- After replacing the delta connection made up of the $50\ \Omega, 40\ \Omega, and\ the\ 10\ \Omega$ with its equivalent wye, the circuit becomes as shown below



- The circuit is further simplified by combining the parallel branches:

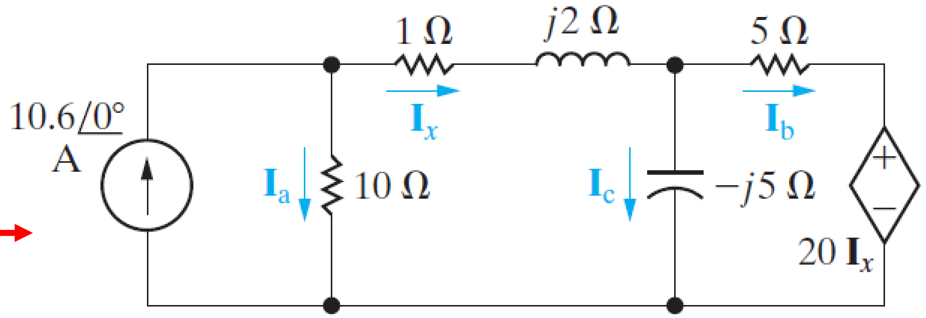
$$(20 + j40)||(5 - j15) = (12 - j16)\Omega$$

One way to work with this expression is to multiply denominator and numerator with the complex conjugate of the denominator and then simplify the expression. Or you can change the numbers to polar format, which is probably easier.

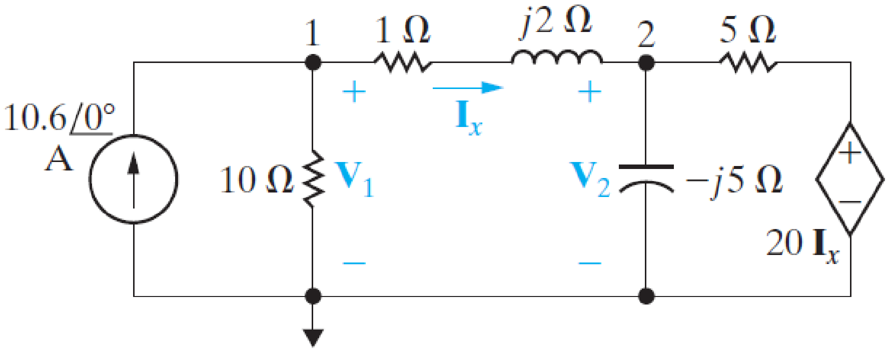
$$\text{Therefore} \quad \mathbf{I} = \frac{136\underline{/0^\circ}}{14 + 12 - j16 + 4} = 4\underline{/28.07^\circ}\ \text{A}$$

Example 9.11: **The Node Voltage**

Use the node-voltage method to find the branch currents $\mathbf{I}_a$, $\mathbf{I}_b$, and $\mathbf{I}_c$ in the circuit shown in Fig. 9.34.



Answer:



We use node voltage analysis

Summing the currents away from node 1 yields

$$-10.6 + \frac{\mathbf{V}_1}{10} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{1 + j2} = 0.$$

Multiplying by $1 + j2$ and collecting the coefficients of $\mathbf{V}_1$ and $\mathbf{V}_2$ generates the expression

$$\mathbf{V}_1(1.1 + j0.2) - \mathbf{V}_2 = 10.6 + j21.2.$$

Summing the currents away from node 2 gives

$$\frac{\mathbf{V}_2 - \mathbf{V}_1}{1 + j2} + \frac{\mathbf{V}_2}{-j5} + \frac{\mathbf{V}_2 - 20\mathbf{I}_x}{5} = 0.$$

The controlling current $\mathbf{I}_x$ is

$$\mathbf{I}_x = \frac{\mathbf{V}_1 - \mathbf{V}_2}{1 + j2}.$$

Substituting this expression for $\mathbf{I}_x$ into the node 2 equation, multiplying by $1 + j2$, and collecting coefficients of $\mathbf{V}_1$ and $\mathbf{V}_2$ produces the equation

$$-5\mathbf{V}_1 + (4.8 + j0.6)\mathbf{V}_2 = 0.$$

The solutions for $\mathbf{V}_1$ and $\mathbf{V}_2$ are

$$\mathbf{V}_1 = 68.40 - j16.80 \text{ V},$$
$$\mathbf{V}_2 = 68 - j26 \text{ V}.$$

Now from $\mathbf{V_1}$ & $\mathbf{V_2}$, we can find all the other currents in the circuit

$$\mathbf{I}_a = \frac{\mathbf{V}_1}{10}$$

$$\mathbf{I}_x = \frac{\mathbf{V}_1 - \mathbf{V}_2}{1 + j2}$$

$$\mathbf{I}_b = \frac{\mathbf{V}_2 - 20\mathbf{I}_x}{5}$$

$$\mathbf{I}_c = \frac{\mathbf{V}_2}{-j5}$$

Note

- Sections 9.7, 9.8, and 9.9 are not going to be discussed in details in class
  - They are similar to material covered in Ch3 & Ch4 in circuits-I
- Please make sure to read these sections and work out the examples provided
- I assigned some homework problems related to these sections
- You can refer to review material posted on course website

- Please check example 9.12, and assessment problems 9.12, 9.13

Phasor Diagrams

- When we are using the phasor method to analyze the steady-state sinusoidal operation of a circuit, a diagram of the phasor currents and voltages provides an alternative way to look at the behavior of these quantities
- A phasor diagram shows the magnitude and phase angle of each phasor quantity in the complex-number plane
  - Phase angles are measured counterclockwise from the positive real axis
  - Magnitudes are measured from the origin of the axes

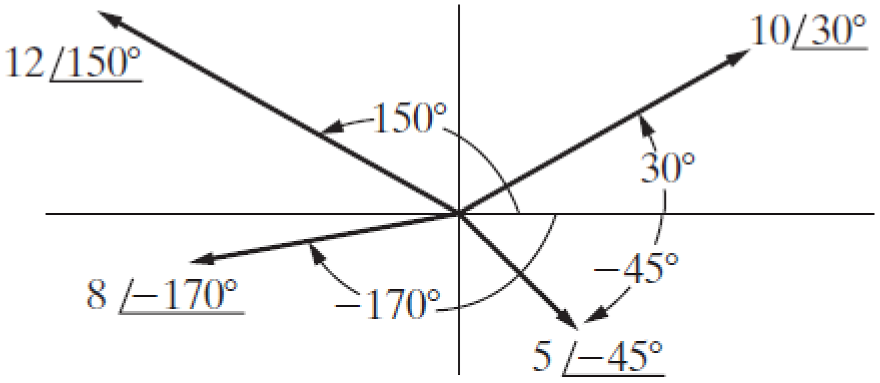


Figure 9.48 ▲ A graphic representation of phasors.

Example 9.15: Phasor Diagrams to Analyze a Circuit

For the circuit in Fig. 9.50, use a phasor diagram to find the value of R that will cause the current through that resistor, i_R, to lag the source current, i_s, by 45° when $\omega = 5$ krad/s.

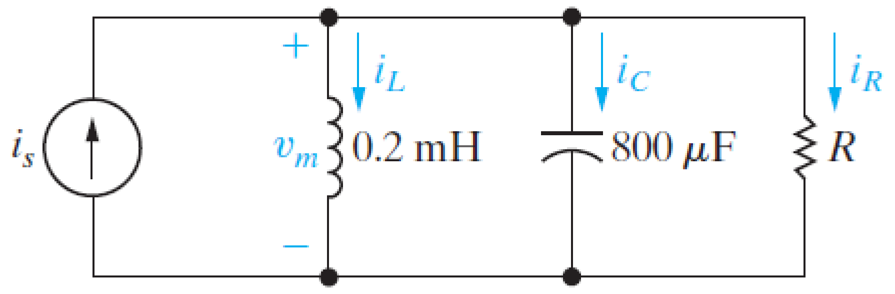


Figure 9.50 ▲ The circuit for Example 9.15.

Solution

By Kirchhoff's current law, the sum of the currents $\mathbf{I}_R$, $\mathbf{I}_L$, and $\mathbf{I}_C$ must equal the source current $\mathbf{I}_s$. If we assume that the phase angle of the voltage $\mathbf{V}_m$ is zero, we can draw the current phasors for each of the components. The current phasor for the inductor is given by

$$\mathbf{I}_L = \frac{V_m \angle 0^\circ}{j(5000)(0.2 \times 10^{-3})} = V_m \angle -90^\circ,$$

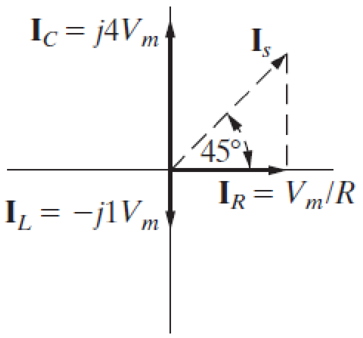
whereas the current phasor for the capacitor is given by

$$\mathbf{I}_C = \frac{V_m \angle 0^\circ}{-j/(5000)(800 \times 10^{-6})} = 4V_m \angle 90^\circ,$$

and the current phasor for the resistor is given by

$$\mathbf{I}_R = \frac{V_m \angle 0^\circ}{R} = \frac{V_m}{R} \angle 0^\circ.$$

These phasors are shown in Fig. 9.51. The phasor diagram also shows the source current phasor, sketched as a dotted line, which must be the sum of the current phasors of the three circuit components and must be at an angle that is 45° more positive than the current phasor for the resistor. As you can see, summing the phasors makes an isosceles triangle, so the length of the current phasor for the resistor must equal $3V_m$. Therefore, the value of the resistor is $\frac{1}{3}\ \Omega$.



Example 9.16: Phasor Diagrams to Analyze a Circuit

The circuit in Fig. 9.52 has a load consisting of the parallel combination of the resistor and inductor. Use phasor diagrams to represent $\mathbf{V}_s$ as a function of $\mathbf{V}_L$ and branch currents

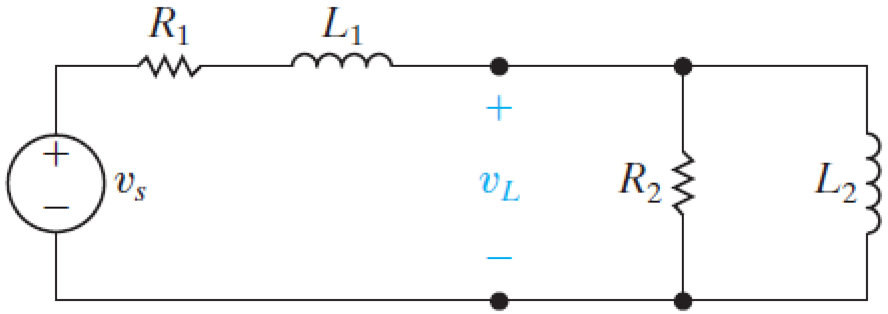


Figure 9.52 ▲ The circuit for Example 9.16.

Solution Figure 9.53 shows the frequency-domain equivalent of the circuit shown in Fig. 9.52. We added the phasor branch currents $\mathbf{I}$, $\mathbf{I}_a$, and $\mathbf{I}_b$ to Fig. 9.53 to aid discussion.

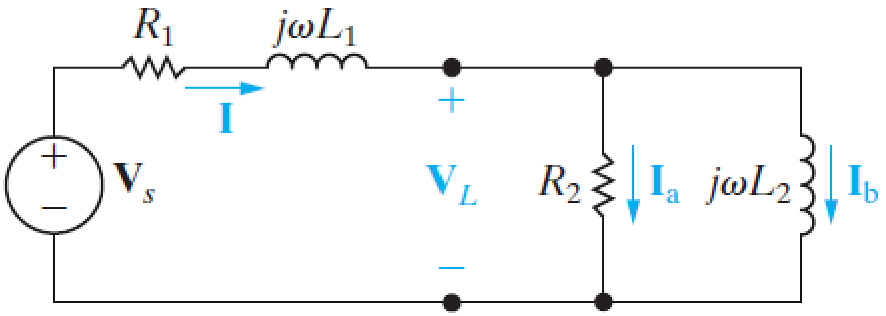


Figure 9.53 ▲ The frequency-domain equivalent of the circuit in Fig. 9.52.

Example 9.16: Phasor Diagrams to Analyze a Circuit

Solution - cont.

Relating the phasor diagram to the circuit shown in Fig. 9.53 reveals the following points:

a) Because we are holding the amplitude of the load voltage constant, we choose $\mathbf{V}_L$ as our reference.

For convenience, we place this phasor on the positive real axis.

b) We know that $\mathbf{I}_a$ is in phase with $\mathbf{V}_L$ and that its magnitude is $|\mathbf{V}_L|/R_2$. (On the phasor diagram, the magnitude scale for the current phasors is independent of the magnitude scale for the voltage phasors.)

c) We know that $\mathbf{I}_b$ lags behind $\mathbf{V}_L$ by 90° and that its magnitude is $|\mathbf{V}_L|/\omega L_2$.

d) The line current $\mathbf{I}$ is equal to the sum of $\mathbf{I}_a$ and $\mathbf{I}_b$.

e) The voltage drop across R_1 is in phase with the line current, and the voltage drop across $j\omega L_1$ leads the line current by 90°.

f) The source voltage is the sum of the load voltage and the drop along the line; that is, $\mathbf{V}_s = \mathbf{V}_L + (R_1 + j\omega L_1)\mathbf{I}$.

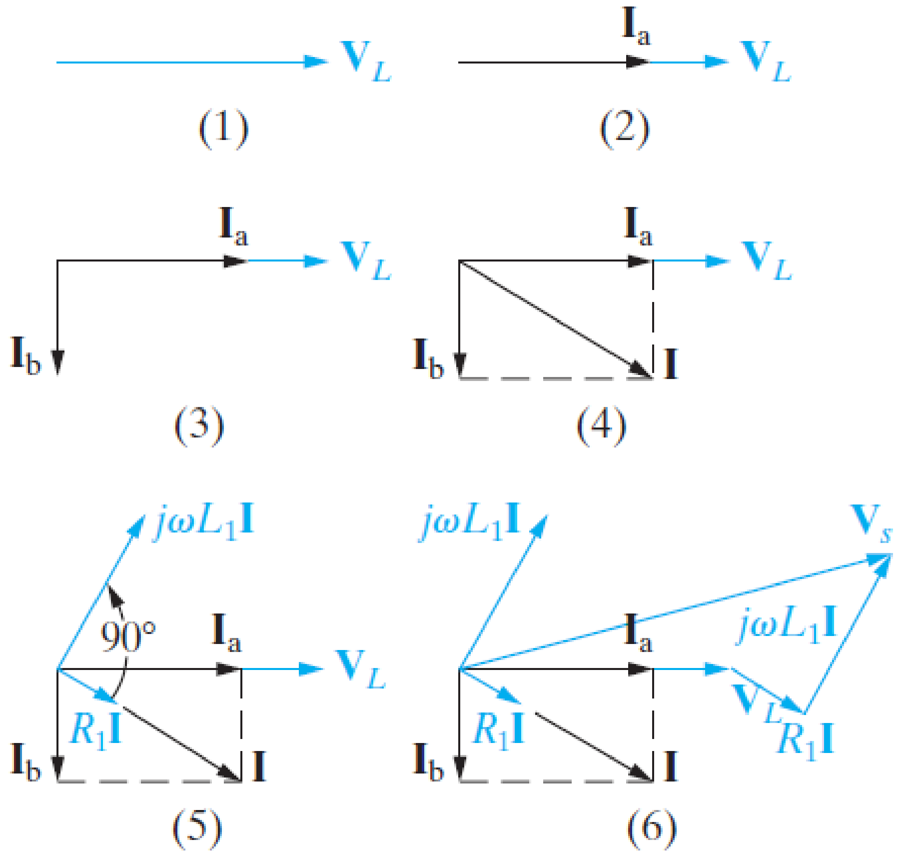


Figure 9.54 ▲ The step-by-step evolution of the phasor diagram for the circuit in Fig. 9.53.

The Transformer

- A transformer is a device that is based on magnetic coupling
- Transformers are used in both communication and power circuits
  - In communication circuits, the transformer is used to match impedances and eliminate dc signals from portions of the system
  - In power circuits, transformers are used to establish ac voltage levels that facilitate the transmission, distribution, and consumption of electrical power

A simple **transformer** is formed when two coils are wound on a single core to ensure magnetic coupling

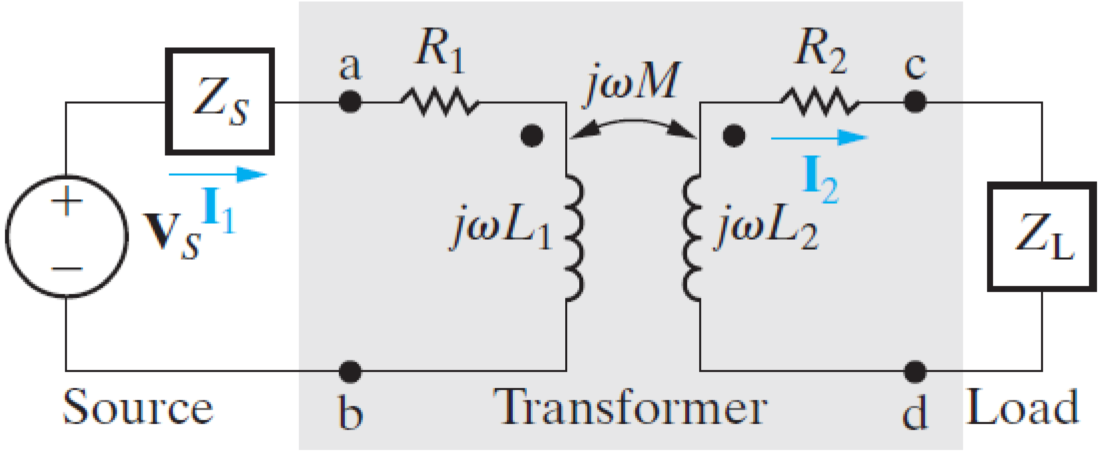


Figure 9.38 ▲ The frequency domain circuit model for a transformer used to connect a load to a source.

R_1 = the resistance of the primary winding,

R_2 = the resistance of the secondary winding,

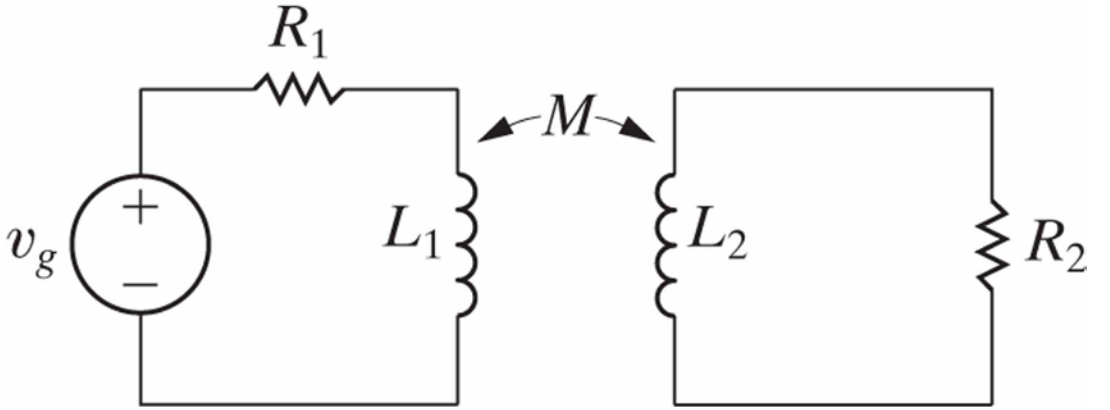
L_1 = the self-inductance of the primary winding,

L_2 = the self-inductance of the secondary winding,

M = the mutual inductance.

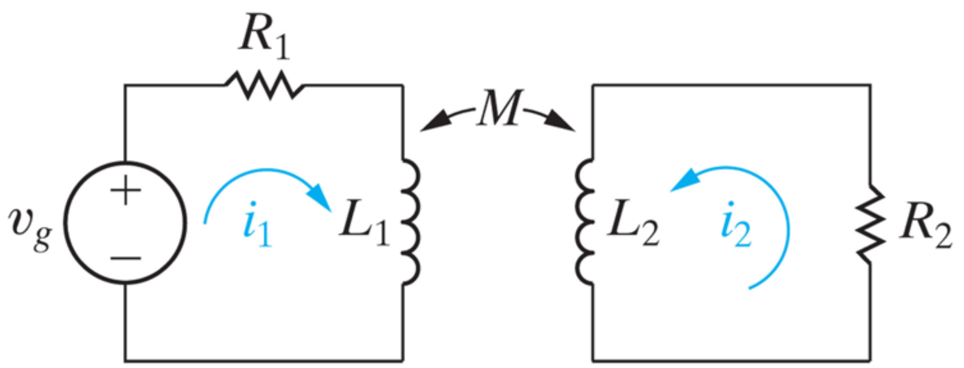
The Transformer: Mutual Inductance

- We now consider the situation in which two circuits are linked by a magnetic field (magnetically coupled)
  - Any current flowing in a an inductor will generate a magnetic field
- In this case, the voltage induced in the second circuit can be related to the time-varying current in the first circuit by a parameter known as mutual inductance
  - The self-inductances of the two coils are labeled L_1 and L_2 and the mutual inductance is labeled M
  - The double headed arrow adjacent to M indicates the pair of coils with this value of mutual inductance



The Transformer: Mutual Inductance – cont.

- The easiest way to analyze circuits containing mutual inductance is to use mesh currents
  - First select the reference direction for each coil
  - Then sum the voltages around each closed path
- Because of the mutual inductance *M*, there will be two voltages across each coil
  - Self-induced voltage
  - Mutually-induced voltage



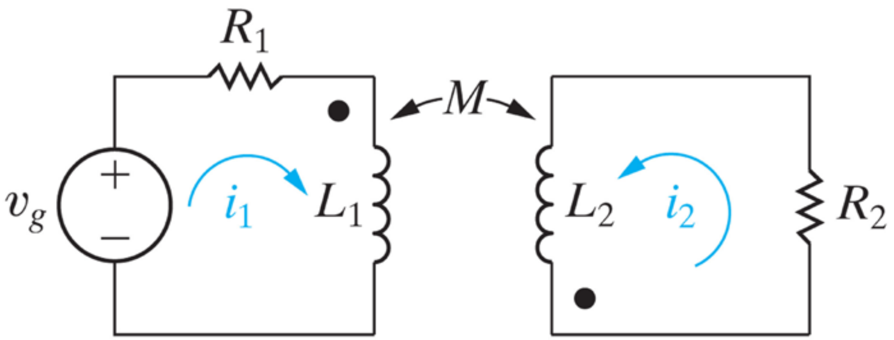
Left-hand circuit

Self-induced voltage: $L_1 \dfrac{di_1}{dt}$

Mutually-induced voltage: $M \dfrac{di_2}{dt}$

Note that the mutually induced voltage in one coil depends on the mutual inductance and the current of the other coil

The Transformer: Mutual Inductance – cont.



- Dot convention is used to determine polarity
  - Dot is placed on one terminal of each winding
  - These dots carry the sign information and allow us to draw the coils schematically rather than showing how they wrap around a core structure
- The rule for using the dot convention to determine the polarity of mutually induced voltage can be summarized as follows:
  - When the reference direction for a current enters the dotted terminal of a coil, the reference polarity of the voltage that it induces in the other coil is positive at its dotted terminal

  OR:

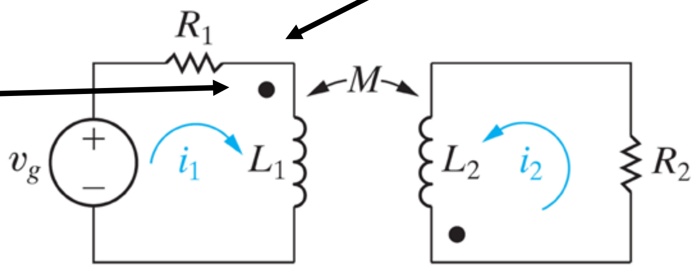
  - When the reference direction for a current leaves the dotted terminal of a coil, the reference polarity of the voltage that it induces in the other coil is negative at its dotted terminal

The Transformer: Mutual Inductance – cont.

i_1 enters the dotted terminal of coil 1

$$M\frac{di_2}{dt}$$

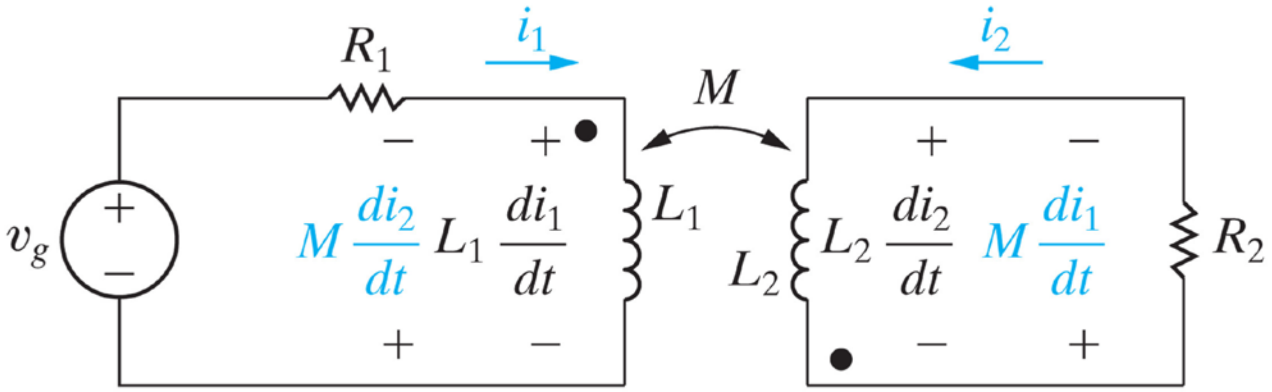
Result: reference polarity for voltage induced by current i_2, is negative at the dotted terminal of coil 1.



$$M\frac{di_1}{dt}$$

Result: reference polarity for voltage induced by current, i_1, is positive at the dotted terminal of coil 2.

i_2 leaves the dotted terminal of coil 2



$$-v_g + i_1R_1 + L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = 0$$

$$i_2R_2 + L_2\frac{di_2}{dt} - M\frac{di_1}{dt} = 0$$

The Transformer – cont.

N_1/N_2 is the number of turns ratio

$$\left|\frac{\mathbf{V}_1}{N_1}\right| = \left|\frac{\mathbf{V}_2}{N_2}\right|. \quad (9.76)$$

$$|\mathbf{I}_1 N_1| = |\mathbf{I}_2 N_2|. \quad (9.77)$$

If the coil voltages $\mathbf{V}_1$ and $\mathbf{V}_2$ are both positive or negative at the dot-marked terminal, use a plus sign in Eq. 9.76. Otherwise, use a negative sign.

If the coil currents $\mathbf{I}_1$ and $\mathbf{I}_2$ are both directed into or out of the dot-marked terminal, use a minus sign in Eq. 9.77. Otherwise, use a plus sign.

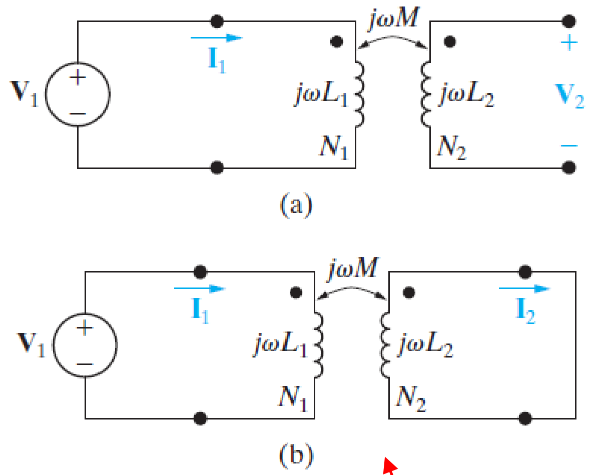


Figure 9.41 ▲ The circuits used to verify the volts-per-turn and ampere-turn relationships for an ideal transformer.

You can think of all the circuits below similar to this circuit diagram and follow the dot convention

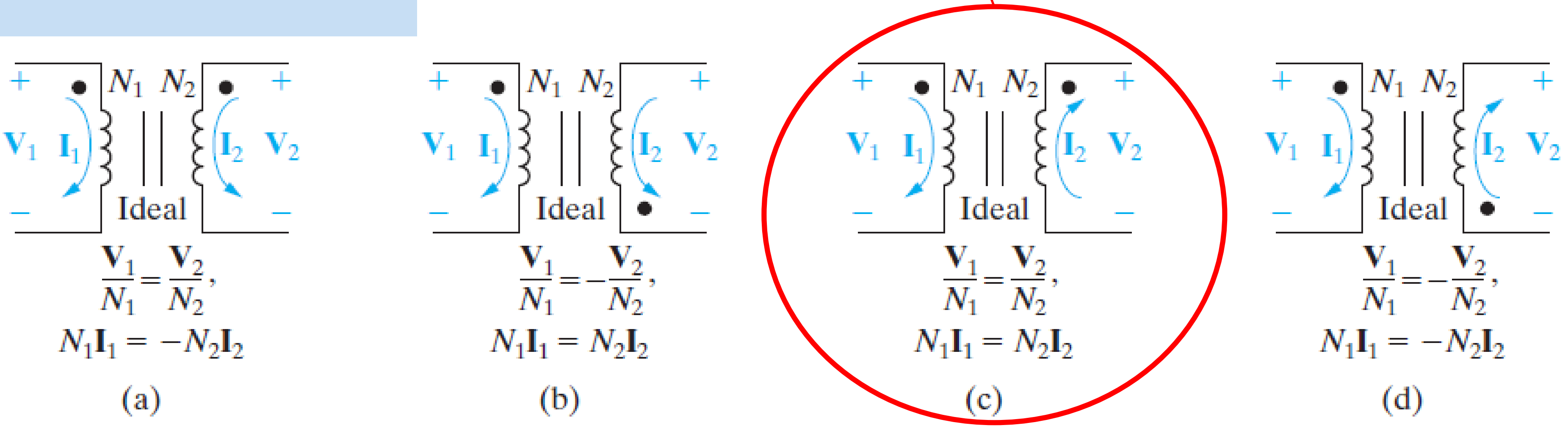


Figure 9.43 ▲ Circuits that show the proper algebraic signs for relating the terminal voltages and currents of an ideal transformer.

Example 9.14

The load impedance connected to the secondary winding of the ideal transformer in Fig. 9.45 consists of a 237.5 mΩ resistor in series with a 125 μH inductor.

If the sinusoidal voltage source (v_g) is generating the voltage $2500 \cos 400t$ V, find the steady-state expressions for: (a) i_1; (b) v_1; (c) i_2; and (d) v_2.

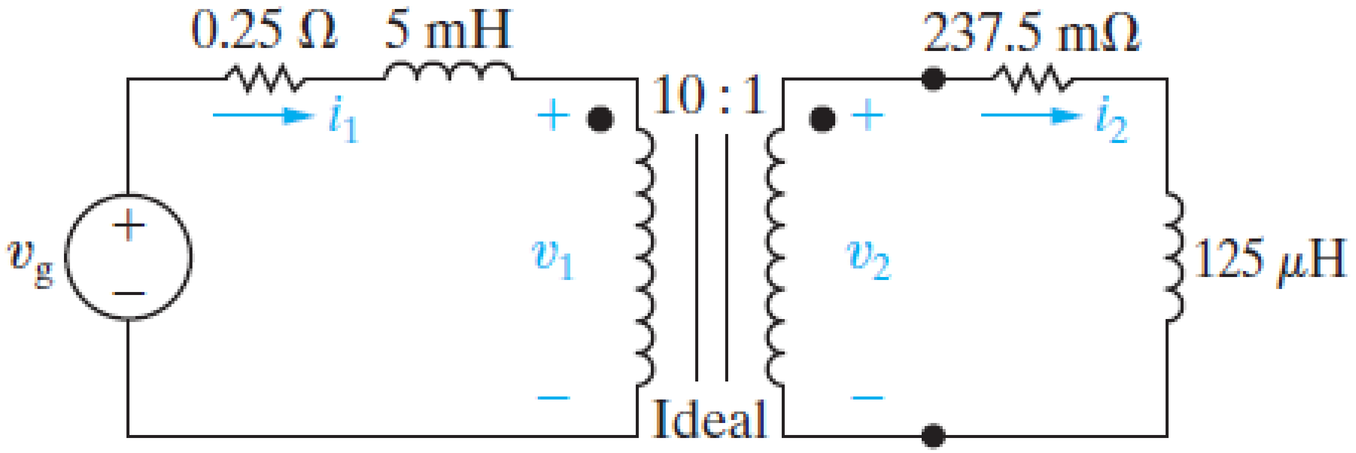


Figure 9.45 ▲ The circuit for Example 9.14.

Solution

a) We begin by constructing the phasor domain equivalent circuit. The voltage source becomes $2500\underline{/0^\circ}$ V; the 5 mH inductor converts to an impedance of $j2\ \Omega$; and the 125 μH inductor converts to an impedance of $j0.05\ \Omega$. The phasor domain equivalent circuit is shown in Fig. 9.46.

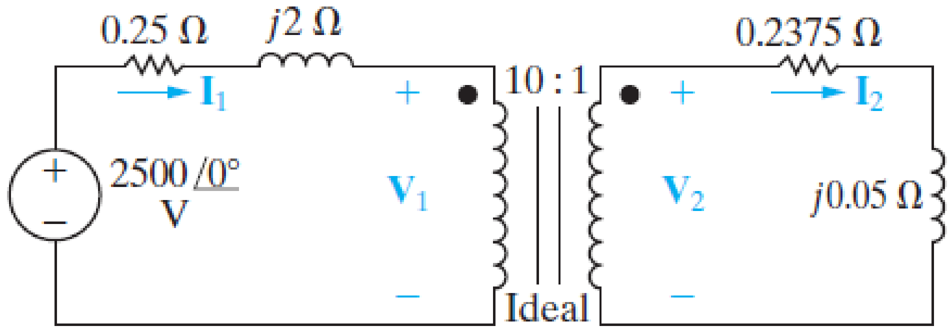


Figure 9.46 ▲ Phasor domain circuit for Example 9.14.

Example 9.14 – cont.

It follows directly from Fig. 9.46 that

$$2500\angle 0^\circ = (0.25 + j2)\mathbf{I}_1 + \mathbf{V}_1,$$

and

$$\mathbf{V}_1 = 10\mathbf{V}_2 = 10[(0.2375 + j0.05)\mathbf{I}_2].$$

Because

$$\mathbf{I}_2 = 10\mathbf{I}_1$$

we have

$$\mathbf{V}_1 = 10(0.2375 + j0.05)10\mathbf{I}_1$$
$$= (23.75 + j5)\mathbf{I}_1.$$

Therefore

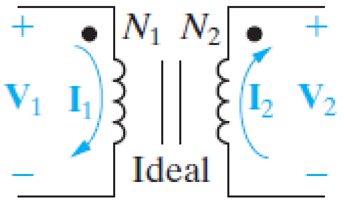
$$2500\angle 0^\circ = (24 + j7)\mathbf{I}_1,$$

or

$$\mathbf{I}_1 = 100\angle -16.26^\circ \text{ A}.$$

Thus the steady-state expression for i_1 is

$$i_1 = 100\cos(400t - 16.26^\circ) \text{ A}.$$



$$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_2},$$
$$N_1\mathbf{I}_1 = N_2\mathbf{I}_2$$

(c)

b) $\mathbf{V}_1 = 2500\angle 0^\circ - (100\angle -16.26^\circ)(0.25 + j2)$

$$= 2500 - 80 - j185$$
$$= 2420 - j185 = 2427.06\angle -4.37^\circ \text{ V}.$$

Hence

$$v_1 = 2427.06\cos(400t - 4.37^\circ) \text{ V}.$$

c) $\mathbf{I}_2 = 10\mathbf{I}_1 = 1000\angle -16.26^\circ$ A.

Therefore

$$i_2 = 1000\cos(400t - 16.26^\circ) \text{ A}.$$

d) $\mathbf{V}_2 = 0.1\mathbf{V}_1 = 242.71\angle -4.37^\circ$ V,

giving

$$v_2 = 242.71\cos(400t - 4.37^\circ) \text{ V}.$$

Summary of Topics Covered in this Chapter

- In this chapter we discussed:
  - Sinusoidal sources
  - Sinusoidal response
  - Phasor transformation
  - Phasor diagrams
  - General circuit analysis in the frequency domain
  - Transformers
- Next chapter (Ch10) we will talk about power calculations with sinusoidal sources
  - Sinusoidal Steady State Power Calculations