Chapter 8: Natural & Step Response of *RLC* Circuits

EEL 3112c – Circuits-II

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Topics to be Covered in this Chapter

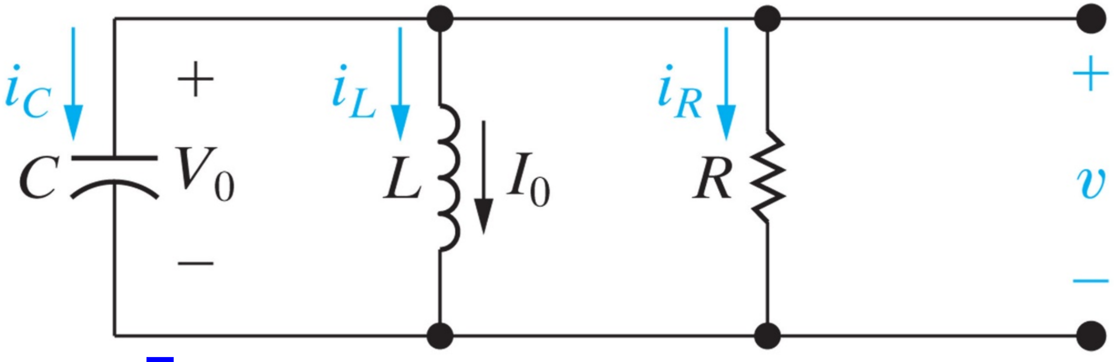
- In this chapter we will discuss:
  - Natural and step response of parallel and series *RLC* circuits
    - Introduction
    - 2^{nd} order differential equation
    - Damping scenarios
    - General response
- Chapter Objectives:
  - Be able to determine the natural response & the step response of parallel *RLC* circuits
  - Be able to determine the natural response & the step response of series *RLC* circuits

- We will cover sections 8.1-8.4

Introduction

- In this chapter we will discuss the natural & step response of circuits containing both inductors and capacitors
  - The parallel *RLC* circuit
  - The series *RLC* circuit
- **Natural response:**
  - The response of a circuit to **the sudden removal** of the constant voltage or current source in the circuit
  - Analyze the current and voltage that arise in the discharge path
- **Step Response:**
  - The response of a circuit to **the sudden application** of a constant voltage or current source is referred to as the **step response** of the circuit

Natural Response of Parallel RLC Circuits



- The first step in finding the natural response of the circuit shown here is to derive the differential equation that the voltage v must satisfy
  - We choose to find the voltage first, because it is the same for each component
  - After that, a branch current can be found by using the current-voltage relationship for the branch component
- Summing the currents away from the top node where each current is represented as a function of the unknown voltage v (node voltage analysis) we get the following set of equations

$$\frac{v}{R} + \frac{1}{L}\int_0^t v d\tau + I_0 + C\frac{dv}{dt} = 0$$

Differentiate with respect to t to eliminate the integral, we get:

$$\frac{1}{R}\frac{dv}{dt} + \frac{v}{L} + C\frac{d^2v}{dt^2} = 0$$

Divide by capacitance and rearrange:

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

This makes it a 2nd order diff. eqn. which is different than what we studied in Ch7

General Solution of 2nd -Order Differential Equation

- The classical approach to solving $\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$ is to assume that the solution is of exponential form

$$v = Ae^{st}$$

  - A & s are unknown constants
- The rational behind this assumption
  - The expressions of the waveforms derived in Ch7 were exponential
    - Physical quantities related to C or L will exponentially increase or decrease
  - The equation we are trying to solve states that a constant multiplied by the 2nd derivative of the solution plus a constant multiplied by the 1st derivative of the solution plus a constant multiplied by the solution it self sums to zero
    - This can occur only if higher orders of the derivative of the solution have the same form as the solution
    - This is true for exponential signals
      - $\frac{de^{st}}{dt} = se^{st}$
    - So that when we add the solution and the derivatives they can sum up to zero using the right constants

General Solution of 2nd -Order Diff. Eqn. – cont.

- Thus, if the hypothesized solution is Ae^{st}, we can substitute that into the 2nd order differential equation and get:

$$As^2e^{st} + \frac{As}{RC}e^{st} + \frac{A}{LC}e^{st} = 0 \;\rightarrow\; Ae^{st}\left(s^2 + \frac{s}{RC} + \frac{1}{LC}\right) = 0$$

This is called the characteristic equation because the roots of this equation determine the mathematical character of the solution $v(t)$

- The roots of the characteristics equation $\left(s^2 + \frac{s}{RC} + \frac{1}{LC} = 0\right)$ can be found as:

$$\boxed{\begin{array}{l} ax^2 + bx + c = 0 \\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \end{array}}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}, \qquad s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

General Solution of 2nd -Order Diff. Eqn. – cont.

- If either root is substituted into the hypothesized solution $v = Ae^{st}$, the assumed solution satisfies the given differential equation

$$v = A_1 e^{s_1 t} \; and \; v = A_2 e^{s_2 t}$$

- Denoting these two solutions v_1 and v_2 respectively, we can show that their sum also is a solution

$$v = v_1 + v_2 = \underbrace{A_1 e^{s_1 t} + A_2 e^{s_2 t}}_{\text{The consensus is to consider this as the general solution}}$$

  - Constants A_1 & A_2 are determined by the initial conditions of the circuit
  - s_1 & s_2 are determined by the circuit elements values
    - R, L, and C

General Solution of 2nd -Order Diff. Eqn. – cont.

- s_1 and s_2 can be written using a notation widely used in the literature as:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2},$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2},$$

Neper frequency, parallel *RLC* circuit ▶ $$\alpha = \frac{1}{2RC},$$

Resonant radian frequency, parallel *RLC* circuit ▶ $$\omega_0 = \frac{1}{\sqrt{LC}}.$$

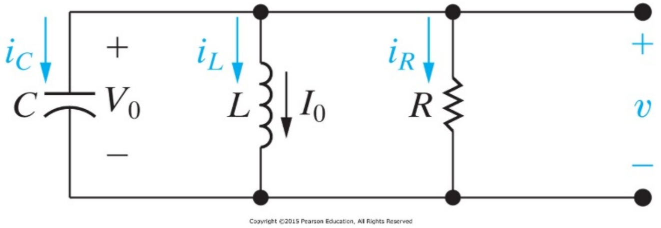


TABLE 8.1 Natural Response Parameters of the Parallel *RLC* Circuit

Parameter	Terminology	Value In Natural Response
s_1, s_2	Characteristic roots	$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
α	Neper frequency	$\alpha = \frac{1}{2RC}$
ω_0	Resonant radian frequency	$\omega_0 = \frac{1}{\sqrt{LC}}$

Also referred to as *complex frequencies*

Depends on the values of the circuit components

All units are in radians per second (***rad/sec***)

General Solution of 2nd -Order Diff. Eqn. – cont.

- The nature of the roots s_1 & s_2 depends on the values of α & ω_0. Thus, there are three possible outcomes

$\boxed{\omega_0^2 < \alpha^2}$

- Roots are real & distinct
- **Overdamped response**

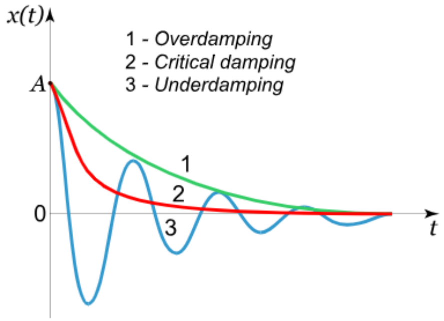
$\boxed{\omega_0^2 > \alpha^2}$

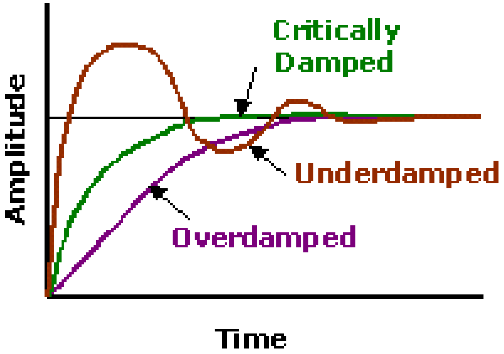
- Roots are complex
- s_1 and s_2 form complex conjugates
- **Underdamped response**

$\boxed{\omega_0^2 = \alpha^2}$

- Roots are real
- s_1 and s_2 are equal
- **Critically damped response**

- As we shall see, damping affects the way the voltage response reaches its final (or steady-state) value





Example 8.1

a) Find the roots of the characteristic equation that governs the transient behavior of the voltage shown in Fig. 8.5 if $R = 200\ \Omega$, $L = 50$ mH, and $C = 0.2\ \mu$F.

b) Will the response be overdamped, underdamped, or critically damped?

c) Repeat (a) and (b) for $R = 312.5\ \Omega$.

d) What value of R causes the response to be critically damped?

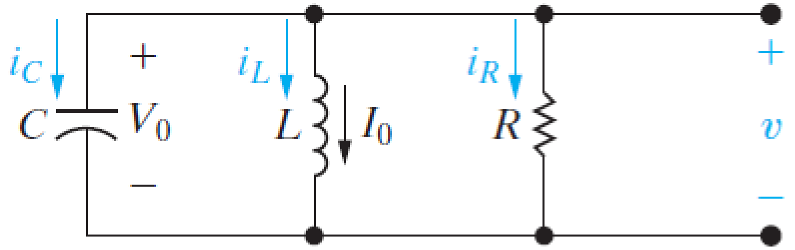


Figure 8.5 ▲ A circuit used to illustrate the natural response of a parallel *RLC* circuit.

Example 8.1 – cont.

a) For the given values of R, L, and C,

$$\alpha = \frac{1}{2RC} = \frac{10^6}{(400)(0.2)} = 1.25 \times 10^4 \text{ rad/s},$$

$$\omega_0^2 = \frac{1}{LC} = \frac{(10^3)(10^6)}{(50)(0.2)} = 10^8 \text{ rad}^2/\text{s}^2.$$

From Eqs. 8.14 and 8.15,

$$s_1 = -1.25 \times 10^4 + \sqrt{1.5625 \times 10^8 - 10^8}$$

$$= -12{,}500 + 7500 = -5000 \text{ rad/s},$$

$$s_2 = -1.25 \times 10^4 - \sqrt{1.5625 \times 10^8 - 10^8}$$

b) The voltage response is overdamped because $\omega_0^2 < \alpha^2$.

c) For $R = 312.5\ \Omega$,

$$\alpha = \frac{10^6}{(625)(0.2)} = 8000 \text{ rad/s},$$

$$\alpha^2 = 64 \times 10^6 = 0.64 \times 10^8 \text{ rad}^2/\text{s}^2.$$

As ω_0^2 remains at 10^8 rad^2/s^2,

$$s_1 = -8000 + j6000 \text{ rad/s},$$

$$s_2 = -8000 - j6000 \text{ rad/s}.$$

(In electrical engineering, the imaginary number $\sqrt{-1}$ is represented by the letter j, because the letter i represents current.)

In this case, the voltage response is underdamped since $\omega_0^2 > \alpha^2$.

d) For critical damping, $\alpha^2 = \omega_0^2$, so

$$\left(\frac{1}{2RC}\right)^2 = \frac{1}{LC} = 10^8,$$

or

$$\frac{1}{2RC} = 10^4,$$

and

$$R = \frac{10^6}{(2 \times 10^4)(0.2)} = 250\ \Omega.$$

Forms of the Natural Response of Parallel *RLC* Circuit

- So far we have seen that the behavior of a second-order *RLC* circuit depends on the values of $s_1$and s_2
  - Depend on the circuit parameters *R*, *L*, and *C*
- Therefore, to find the natural response
  - Calculate the values of *R*, *L*, *and C*
  - Determine whether the response is under-, over-, or critically damped
  - Find unknown coefficients, such as A_1 and A_2
    - We rely on the initial conditions of the circuit
      - Initial voltage, initial current, etc.
- In the coming slides we will analyze the natural response form for each of the three types of damping

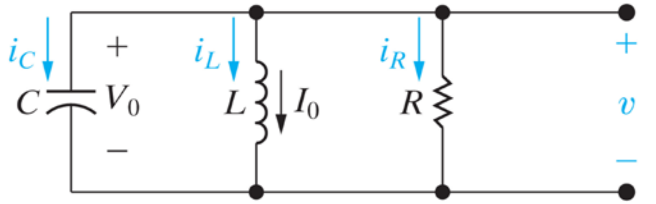
The Overdamped Voltage Response

- The solution for the voltage is of the form: $\boxed{\boldsymbol{v = A_1 e^{s_1 t} + A_2 e^{s_2 t}}}$
  - s_1 & s_2 are the roots of the characteristics equation
  - The constants A_1 & A_2 are determined by the initial conditions
    - Specifically from the values of $v(0^+)$ and $dv(0^+)/dt$ which in turn are determined from the initial voltage on the capacitor (V_0), and the initial current in the inductor (I_0)
- How we do that is as follows:

$v(0^+) = A_1 + A_2,$ By substituting $t = 0$ in the solution eqn.

$\frac{dv(0^+)}{dt} = s_1 A_1 + s_2 A_2.$ By deriving the solution eqn. and substituting $t = 0$

- Given that we can find the values of s_1 & s_2 from the *R, L, C* values in the circuit, all what we need to do to find A_1 & A_2 is to find $v(0^+)$ & $\frac{dv(0^+)}{dt}$
  - 2 equations & 2 unknowns (A_1 & A_2)
- The value of $v(0^+)$ is the initial voltage on the capacitor V_0
- The value of $\frac{dv(0^+)}{dt}$ is derived from the initial current in the capacitor: $\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C}$
- We need to find $i_C(0^+)$ by applying ***KCL*** on the upper node of the reference circuit



$$i_c(0^+) + \frac{V_0}{R} + I_0 = 0$$

$$i_R = \frac{V_0}{R}$$

$$i_c(0^+) = -\frac{V_0}{R} - I_0$$

Example 8.2

For the circuit in Fig. 8.6, $v(0^+) = 12$ V, and $i_L(0^+) = 30$ mA.

a) Find the initial current in each branch of the circuit.

b) Find the initial value of dv/dt.

c) Find the expression for $v(t)$.

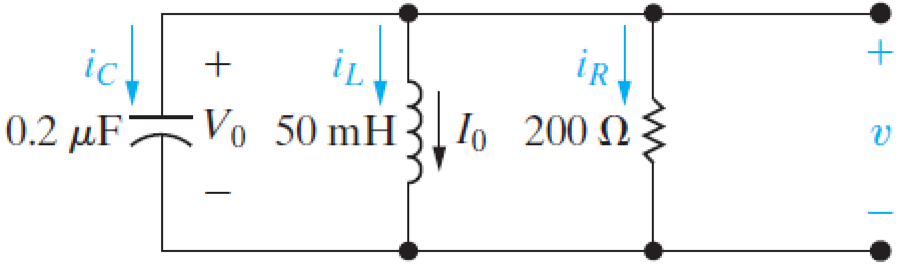


Figure 8.6 ▲ The circuit for Example 8.2.

Solution:

a) The inductor prevents an instantaneous change in its current, so the initial value of the inductor current is 30 mA:

$$i_L(0^-) = i_L(0) = i_L(0^+) = 30 \text{ mA}.$$

The capacitor holds the initial voltage across the parallel elements to 12 V. Thus the initial current in the resistive branch, $i_R(0^+)$, is 12/200, or 60 mA. Kirchhoff's current law requires the sum of the currents leaving the top node to equal zero at every instant. Hence

$$i_C(0^+) = -i_L(0^+) - i_R(0^+)$$
$$= -90 \text{ mA}.$$

b) Because $i_C = C(dv/dt)$,

$$\frac{dv(0^+)}{dt} = \frac{-90 \times 10^{-3}}{0.2 \times 10^{-6}} = -450 \text{ kV/s}.$$

c) The roots of the characteristic equation come from the values of R, L, and C. For the values specified and from Eqs. 8.14 and 8.15 along with 8.16 and 8.17,

$$s_1 = -1.25 \times 10^4 + \sqrt{1.5625 \times 10^8 - 10^8}$$
$$= -12{,}500 + 7500 = -5000 \text{ rad/s},$$
$$s_2 = -1.25 \times 10^4 - 2\ \overline{1.5625 \times 10^8 - 10^8}$$
$$= -12{,}500 - 7500 = -20{,}000 \text{ rad/s}.$$

Because the roots are real and distinct, we know that the response is overdamped and hence has the form of Eq. 8.18. We find the co-efficients A_1 and A_2 from Eqs. 8.23 and 8.24. We've already determined s_1, s_2, $v(0^+)$, and $dv(0^+)/dt$, so

$$12 = A_1 + A_2,$$

$$-450 \times 10^3 = -5000A_1 - 20{,}000A_2.$$

We solve two equations for A_1 and A_2 to obtain $A_1 = -14$ V and $A_2 = 26$ V. Substituting these values into Eq. 8.18 yields the overdamped voltage response:

$$v(t) = (-14e^{-5000t} + 26e^{-20{,}000t}) \text{ V}, \quad t \geq 0.$$

The Underdamped Voltage Response

- When $\omega_0^2 > \alpha^2$, the roots of the characteristic equation are complex, and the response is underdamped
  - s_1 & s_2 are complex conjugates
- For convenience, we express the roots s_1 & s_2 as

$$s_1 = -\alpha + \sqrt{-\left(\omega_0^2 - \alpha^2\right)}$$

$$= -\alpha + j\sqrt{\left(\omega_0^2 - \alpha^2\right)}$$

$$= -\alpha + j\omega_d$$

$$s_2 = -\alpha - j\omega_d$$

Where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ and is called the damped radian frequency or damping frequency

To examine the solution in more details:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$s_1 = -\alpha + j\omega_d \quad s_2 = -\alpha - j\omega_d$$

$$v(t) = A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$$

$$v(t) = A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

Recall: $e^{\pm j\theta} = \cos(\theta) \pm j\sin(\theta)$

Substituting

$$v(t) = A_1 e^{-\alpha t}\left(\cos(\omega_d t) + j\sin(\omega_d t)\right) + A_2 e^{-\alpha t}\left(\cos(\omega_d t) - j\sin(\omega_d t)\right)$$

Rearranging yields

$$v(t) = e^{-\alpha t}\begin{bmatrix}(A_1 + A_2)\cos(\omega_d t) + \\ j(A_1 - A_2)\sin(\omega_d t)\end{bmatrix}$$

$$\boxed{v(t) = B_1 e^{-\alpha t}\cos(\omega_d t) + B_2 e^{-\alpha t}\sin(\omega_d t)}$$

Underdamped Voltage Response

The Underdamped Voltage Response – cont.

- We determine B_1 & B_2 using the initial conditions similar to what we did with A_1 & A_2

$$v(0^+) = V_0 = B_1$$ Substitute $t = 0$ in the expression of $v(t)$

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned}$$

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = -\alpha B_1 + \omega_d B_2$$ Use the product and chain rule to derive $v(t)$ expression and substitute $t = 0$ in the derivative of $v(t)$

- $\alpha \; and \; \omega_d$ are determined by circuit elements parameters
  - α is the damping factor or damping coefficient and it determines how quickly the oscillation subsides
  - ω_d is the damped radian frequency

Example 8.4

In the circuit shown in Fig. 8.8, $V_0 = 0$, and $I_0 = -12.25$ mA.

a) Calculate the roots of the characteristic equation.

b) Calculate v and dv/dt at $t = 0^+$.

c) Calculate the voltage response for $t \geq 0$.

d) Plot $v(t)$ versus t for the time interval $0 \leq t \leq 11$ ms.

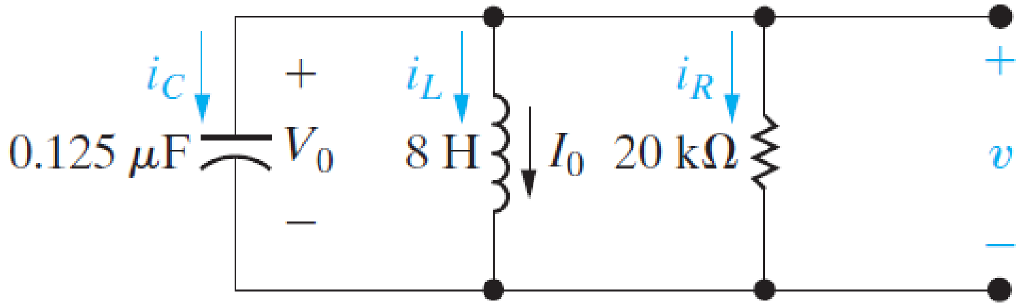


Figure 8.8 ▲ The circuit for Example 8.4.

Solution

a) Because

$$\alpha = \frac{1}{2RC} = \frac{10^6}{2(20)10^3(0.125)} = 200 \text{ rad/s},$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{\frac{10^6}{(8)(0.125)}} = 10^3 \text{ rad/s},$$

we have $\omega_0^2 > \alpha^2$.

Therefore, the response is underdamped. Now,

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{10^6 - 4 \times 10^4} = 100\sqrt{96}$$

$$= 979.80 \text{ rad/s},$$

$$s_1 = -\alpha + j\omega_d = -200 + j979.80 \text{ rad/s},$$

$$s_2 = -\alpha - j\omega_d = -200 - j979.80 \text{ rad/s}.$$

Example 8.4 – cont.

b) Because v is the voltage across the terminals of a capacitor, we have

$$v(0) = v(0^+) = V_0 = 0.$$

Because $v(0^+) = 0$, the current in the resistive branch is zero at $t = 0^+$. Hence the current in the capacitor at $t = 0^+$ is the negative of the inductor current:

$$i_C(0^+) = -(-12.25) = 12.25 \text{ mA}.$$

Therefore the initial value of the derivative is

$$\frac{dv(0^+)}{dt} = \frac{(12.25)(10^{-3})}{(0.125)(10^{-6})} = 98{,}000 \text{ V/s}.$$

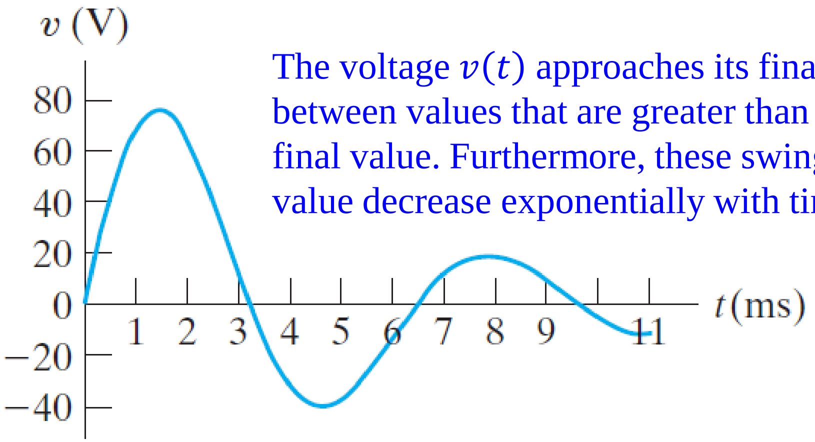
c) From Eqs. 8.30 and 8.31, $B_1 = 0$ and

$$B_2 = \frac{98{,}000}{\omega_d} \approx 100 \text{ V}.$$

Substituting the numerical values of α, ω_d, B_1, and B_2 into the expression for $v(t)$ gives

$$v(t) = 100e^{-200t} \sin 979.80t \text{ V}, \quad t \geq 0.$$

d)



The voltage $v(t)$ approaches its final value, alternating between values that are greater than and less than the final value. Furthermore, these swings about the final value decrease exponentially with time.

The Critically Damped Voltage Response

- When a circuit is critically damped, the response is on the verge of oscillating
  - The two roots of the characteristic equation are real and equal

$$s_1 = s_2 = -\alpha = -\frac{1}{2RC}$$

- When the roots of the characteristic equation are equal, the solution for the differential equation involves a simple exponential term plus the product of a linear and an exponential term

$$\boxed{v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}}$$

- D_1 & D_2 are determined from the initial conditions

Substitute $t = 0$ in the expression of $v(t)$

$$v(0^+) = V_0 = D_2$$

Use the product rule to derive $v(t)$ expression and substitute $t = 0$ in the derivative of $v(t)$

$$\frac{dv(0^+)}{dt} = \frac{i_c(0^+)}{C} = D_1 - \alpha D_2$$

- You will rarely encounter critically damped systems in practice, largely because ω_0 must equal α exactly
- Both of these quantities depend on circuit parameters, and in a real circuit it is very difficult to choose component values that satisfy an exact equality relationship

Example 8.5

a) For the circuit in Example 8.4 (Fig. 8.8), find the value of R that results in a critically damped voltage response.

b) Calculate $v(t)$ for $t \geq 0$.

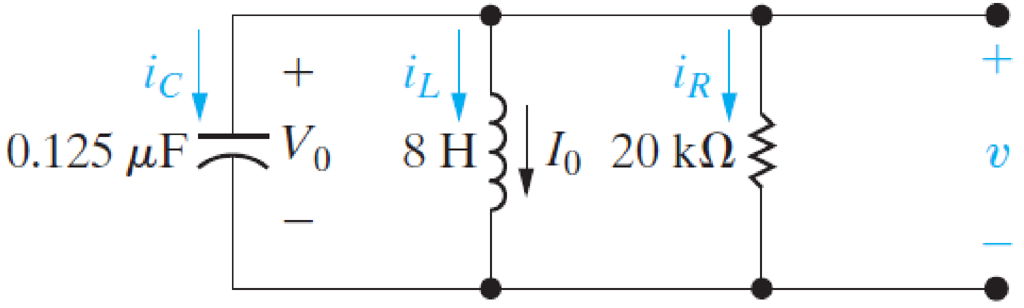


Figure 8.8 ▲ The circuit for Example 8.4.

Solution

a) From Example 8.4, we know that $\omega_0^2 = 10^6$. Therefore for critical damping,

For critically damped systems we know that $\alpha = \omega_0$

$$\alpha = 10^3 = \frac{1}{2RC},$$

or

$$R = \frac{10^6}{(2000)(0.125)} = 4000\ \Omega.$$

b) From the solution of Example 8.4, we know that $v(0^+) = 0$ and $dv(0^+)/dt = 98{,}000$ V/s. From Eqs. 8.35 and 8.36, $D_2 = 0$ and $D_1 = 98{,}000$ V/s.

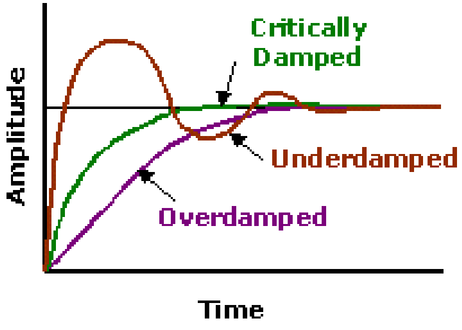
Substituting these values for a, D_1, and D_2 into Eq. 8.34 gives

$$v(t) = 98{,}000te^{-1000t}\ \text{V}, \quad t \geq 0.$$

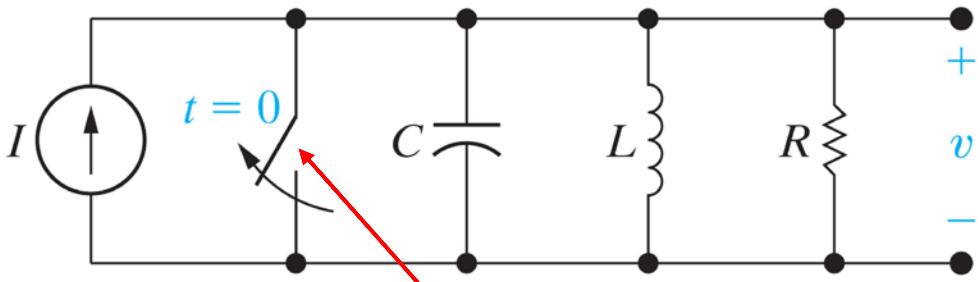
Overdamped vs. Underdamped

- When specifying the desired response of a second order system, you may want to reach the final value in the shortest time possible, and you may not be concerned with small oscillations about that final value. If so, you would design the system components to achieve an **underdamped response**

- On the other hand, you may be concerned that the response not exceed its final value, perhaps to ensure that components are not damaged. In such a case, you would design the system components to achieve an **overdamped response**, and you would have to accept a relatively slow rise to the final value
  - In an overdamped system, the response approaches its final value without ringing (oscillating) or in what is sometimes described as a “sluggish” manner



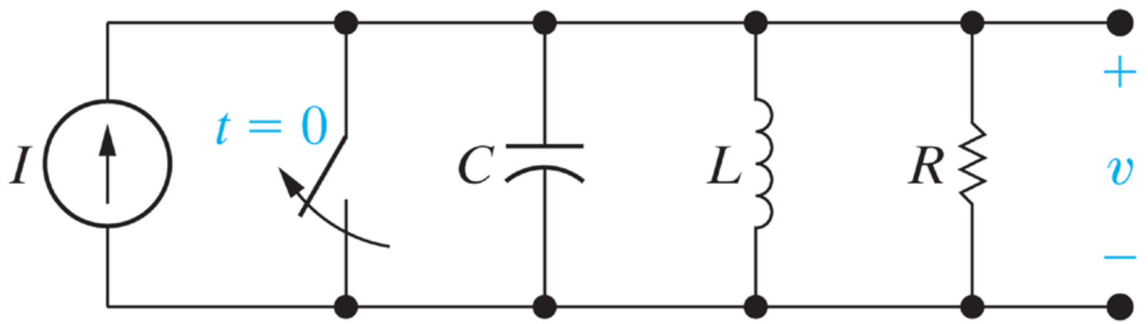
The Step Response of Parallel *RLC* Circuit



Note that when the switch is closed the current source is short circuit, but when we open the switch we include the current source in the circuit

- Change the circuit scenario
  - Add a DC source to the circuit
    - Assume this source is suddenly applied to the circuit
    - Energy may or may not be stored in capacitor or inductor
    - DC source serves as a forcing function
- We need to find the step response
  - Finding the step response of a parallel *RLC* circuit involves finding the voltage across the parallel branches or the current in the individual branches of the circuit
- Solve for the current in the inductor branch
  - It will not approach zero as t increases
  - It remains constant during the circuit state transition $\left(i_L(0^-) = i_L(0^+)\right)$
  - As the circuit approaches a steady state (t >> 0), inductor becomes a short circuit
    - The inductor current equals DC source current

The Step Response of Parallel *RLC* Circuit – cont.



- Apply KCL to the parallel loop (at the top left loop):

$$i_L + i_R + i_C = I$$

$$i_L + \frac{v}{R} + C\frac{dv}{dt} = I$$

- We know that:

We are interested in i_L so we rewrite the equation in terms of i_L

$$v = L\frac{di_L}{dt} \rightarrow \frac{dv}{dt} = L\frac{d^2 i_L}{dt^2}$$

- Substitute into equation from KCL:

$$i_L + \frac{L}{R}\frac{di_L}{dt} + LC\frac{d^2 i_L}{dt^2} = I$$

- Rearrange:

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

- This is a second order differential equation
- Very similar to the differential equation for the natural response
- **We have two ways to solve this equation**
  - **The indirect approach**
  - **The direct approach**

The Step Response of Parallel *RLC* Circuit – cont.

- General comments on the solution of $$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$
  - This is a 2nd order differential equation with a constant forcing function
  - Solution to this equation is based on combining the homogenous solution (the natural response) and the forcing function
    - The homogeneous solution will die out (transient)
    - Leaves the forcing function as the driving influence on the circuit
- Thus, the general solution will take the form below

$$i = I_f + \left\{\begin{matrix}\text{function of the same form}\\ \text{as the natural response}\end{matrix}\right\}, \qquad \text{Or} \qquad v = V_f + \left\{\begin{matrix}\text{function of the same form}\\ \text{as the natural response}\end{matrix}\right\}$$

  - I_f *and* V_f represent the final value of the response function
  - It is a 2nd order equation, it requires two initial conditions to complete the solution

The Step Response of Parallel RLC Circuit – cont.

- We can solve for i_L indirectly by first finding the voltage $v(t)$
  - Return to the initial equation for i_L: $i_L + \frac{v}{R} + C\frac{dv}{dt} = I$
  - Rewrite the previous equation in terms of v:

$$\frac{1}{L}\int_0^t v\,d\tau + \frac{v}{R} + C\frac{dv}{dt} = I$$

  - Differentiate with respect to time:

$$\frac{v}{L} + \frac{1}{R}\frac{dv}{dt} + C\frac{d^2v}{dt^2} = 0$$

I is a constant, so when we differentiate $\left(\frac{dI}{dt}\right)$ it becomes zero

  - Rearrange:

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

2nd order differential equation similar to the natural response

The Step Response of Parallel RLC Circuit – cont.

- Based on the damping properties of the circuit, we will have three possible solutions (as before)

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$v(t) = B_1 e^{-\alpha t}\cos(\omega_d t) + B_2 e^{-\alpha t}\sin(\omega_d t)$$

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

- If we substitute these three equation into $i_L + \frac{v}{R} + C\frac{dv}{dt} = I$, we get the following:

$A_1', A_2', B_1', B_2', D_1', and\ D_2'$ are constants

$$i_L = I + A_1' e^{s_1 t} + A_2' e^{s_2 t},$$

$$i_L = I + B_1' e^{-\alpha t}\cos\omega_d t + B_2' e^{-\alpha t}\sin\omega_d t,$$

$$i_L = I + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t},$$

The Step Response of Parallel *RLC* Circuit – cont.

$$\boxed{\begin{aligned} i_L &= I + A_1' e^{s_1 t} + A_2' e^{s_2 t}, \\ i_L &= I + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t, \\ i_L &= I + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t}, \end{aligned}}$$

- The $A_1', A_2', B_1', B_2', D_1', and\ D_2'$ constants can be found in two ways:
  - **The indirect method**
    - Find them in terms of the voltage solution constants
    - When solving for the coefficients one must account for the source current for t > 0^+ due to the source in the circuit for t > 0
    - This approach is complicated
  - **The direct method**
    - Find them directly in terms of the current initial conditions $i_L(0^+)$ and $\frac{di_L(0^+)}{dt}$
    - This is an easier approach
- The following example will illustrate the technique

Example 8.6

The initial energy stored in the circuit in Fig. 8.12 is zero. At $t = 0$, a dc current source of 24 mA is applied to the circuit. The value of the resistor is 400 Ω.

a) What is the initial value of i_L?

b) What is the initial value of di_L/dt?

c) What are the roots of the characteristic equation?

d) What is the numerical expression for $i_L(t)$ when $t \geq 0$?

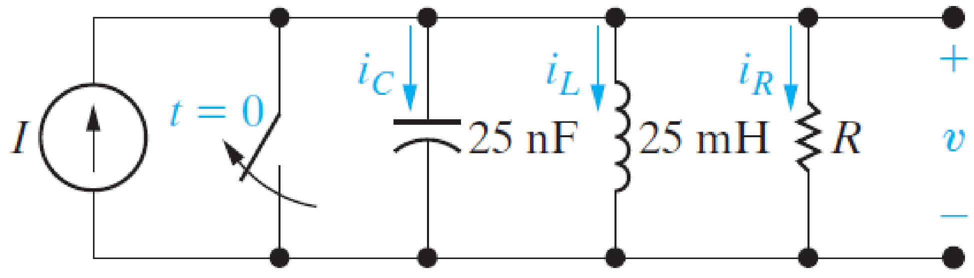


Figure 8.12 ▲ The circuit for Example 8.6.

Example 8.6 – cont.

Solution

a) No energy is stored in the circuit prior to the application of the dc current source, so the initial current in the inductor is zero. The inductor prohibits an instantaneous change in inductor current; therefore $i_L(0) = 0$ immediately after the switch has been opened.

b) The initial voltage on the capacitor is zero before the switch has been opened; therefore it will be zero immediately after. Now, because $v = L di_L/dt$,

$$\frac{di_L}{dt}(0^+) = 0.$$

c) From the circuit elements, we obtain

$$\omega_0^2 = \frac{1}{LC} = \frac{10^{12}}{(25)(25)} = 16 \times 10^8,$$

$$\alpha = \frac{1}{2RC} = \frac{10^9}{(2)(400)(25)} = 5 \times 10^4 \text{ rad/s},$$

or

$$\alpha^2 = 25 \times 10^8.$$

Because $\omega_0^2 < \alpha^2$, the roots of the characteristic equation are real and distinct. Thus

$$s_1 = -5 \times 10^4 + 3 \times 10^4 = -20{,}000 \text{ rad/s},$$

$$s_2 = -5 \times 10^4 - 3 \times 10^4 = -80{,}000 \text{ rad/s}.$$

Example 8.6 – cont.

d) Because the roots of the characteristic equation are real and distinct, the inductor current response will be overdamped. Thus $i_L(t)$ takes the form of Eq. 8.47, namely,

$$\boxed{i_L = I_f + A_1' e^{s_1 t} + A_2' e^{s_2 t}.} \Longrightarrow i = I_f + \left\{ \begin{array}{c} \text{function of the same form} \\ \text{as the natural response} \end{array} \right\},$$

Hence, from this solution, the two simultaneous equations that determine A_1' and A_2' are

$$i_L(0) = I_f + A_1' + A_2' = 0,$$

$$\frac{di_L}{dt}(0) = s_1 A_1' + s_2 A_2' = 0.$$

We used the two initial conditions we found in part **a** and **b**

Solving for A_1' and A_2' gives

$$A_1' = -32 \text{ mA} \quad \text{and} \quad A_2' = 8 \text{ mA}.$$

The numerical solution for $i_L(t)$ is

$$i_L(t) = (24 - 32e^{-20{,}000t} + 8e^{-80{,}000t}) \text{ mA}, \quad t \geq 0.$$

Natural Response of Series *RLC* Circuit

- Follow the same general procedure as that for the parallel RLC natural response
  - Assume a source initially charges the inductor or capacitor and then is suddenly disconnected
- For series circuit, determine the current; then find voltages for any circuit element
- Analysis of the following circuit scenario provides the **natural response** of the series RLC circuit

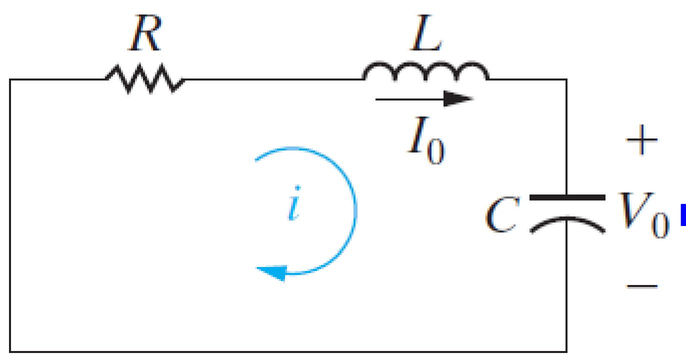


Figure 8.14 ▲ A circuit used to illustrate the natural response of a series *RLC* circuit.

Apply KVL around the closed path

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,d\tau + V_0 = 0$$

Differentiate once with respect to t :

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{1}{C}i = 0$$

Rearrange

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

2nd order diff. eqn. that looks similar to the 2nd diff. eqn. for the voltage in the parallel *RLC* circuit

Natural Response of Series *RLC* Circuit – cont.

- As before, assume solution takes the form: $i(t) = Ae^{st}$
- Substitute assumed solution into the 2nd order diff. eqn.

$$As^2e^{st} + \frac{ARs}{L}e^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st}\underbrace{\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}_{\text{The characteristic equation}} = 0$$

Find roots of characteristic eqn.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{R}{2L} \quad \& \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

α is the Neper freq. and ω_0 is the resonant radian frequency, both units are rad/s

- From the above, we can find the equations for the three damping scenarios

$$i(t) = A_1e^{s_1t} + A_2e^{s_2t} \quad \text{(overdamped)},$$

$$i(t) = B_1e^{-\alpha t}\cos\omega_d t + B_2e^{-\alpha t}\sin\omega_d t \quad \text{(underdamped)},$$

$$i(t) = D_1te^{-\alpha t} + D_2e^{-\alpha t} \quad \text{(critically damped)}.$$

Once you have obtained the natural current response, you can find the natural voltage response across any circuit element

Natural Response of Series *RLC* Circuit – cont.

These equations are derived using the exact same logic we used to derive them for the parallel *RLC* circuits

To determine A_1 & A_2 for overdamped circuits, we use:

$$i(0^+) = A_1 + A_2$$

$$\frac{di(0^+)}{dt} = s_1 A_1 + s_2 A_2$$

To determine B_1 & B_2 for underdamped circuits, we use:

$$i(0^+) = I_0 = B_1$$

$$\frac{di(0^+)}{dt} = \frac{v_c(0^+)}{L} = -\alpha B_1 + \omega_d B_2$$

To determine D_1 & D_2 for critically damped circuits, we use:

$$i(0^+) = I_0 = D_2$$

$$\frac{di(0^+)}{dt} = \frac{v_c(0^+)}{L} = D_1 - \alpha D_2$$

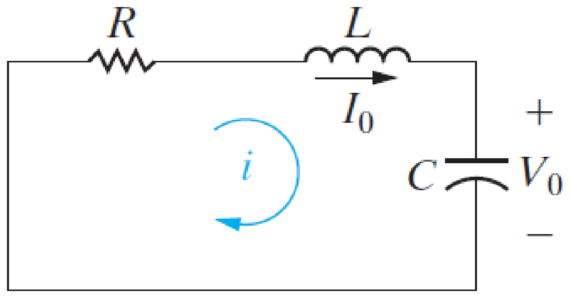


Figure 8.14 ▲ A circuit used to illustrate the natural response of a series *RLC* circuit.

Example 8.11

The 0.1 μF capacitor in the circuit shown in Fig. 8.16 is charged to 100 V. At $t = 0$ the capacitor is discharged through a series combination of a 100 mH inductor and a 560 Ω resistor.

a) Find $i(t)$ for $t \geq 0$.

b) Find $v_C(t)$ for $t \geq 0$.

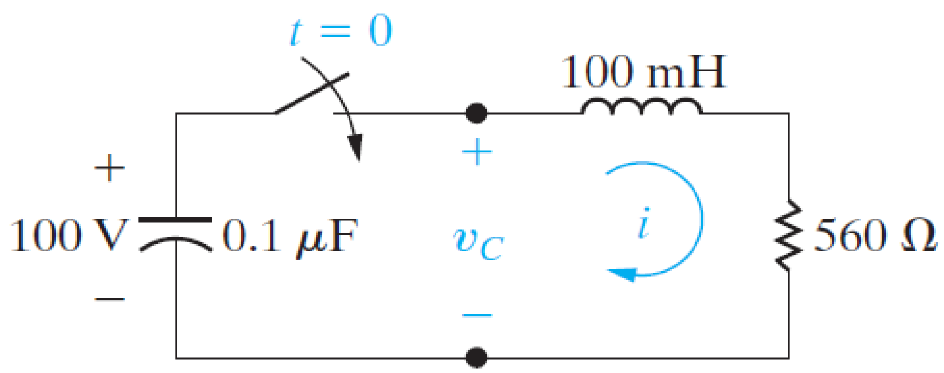


Figure 8.16 ▲ The circuit for Example 8.11.

Note that after closing the switch, we do not have a source in the circuit, so we will be finding the natural response

Solution

a) The first step to finding $i(t)$ is to calculate the roots of the characteristic equation. For the given element values,

$$\omega_0^2 = \frac{1}{LC}$$

$$= \frac{(10^3)(10^6)}{(100)(0.1)} = 10^8,$$

$$\alpha = \frac{R}{2L}$$

$$= \frac{560}{2(100)} \times 10^3$$

$$= 2800 \text{ rad/s}.$$

Example 8.11 – cont.

Next, we compare ω_0^2 to α^2 and note that $\omega_0^2 > \alpha^2$, because

$$\alpha^2 = 7.84 \times 10^6$$
$$= 0.0784 \times 10^8.$$

At this point, we know that the response is underdamped and that the solution for $i(t)$ is of the form

$$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t,$$

Where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

where $\alpha = 2800$ rad/s and $\omega_d = 9600$ rad/s. The numerical values of B_1 and B_2 come from the initial conditions. The inductor current is zero before the switch has been closed, and hence it is zero immediately after. Therefore

$$i(0) = 0 = B_1.$$

To find B_2, we evaluate $di(0^+)/dt$. From the circuit, we note that, because $i(0) = 0$ immediately after the switch has been closed, there will be no voltage drop across the resistor. Thus the initial voltage on the capacitor appears across the terminals of the inductor, which leads to the expression,

$$L\frac{di(0^+)}{dt} = V_0,$$

or

$$\frac{di(0^+)}{dt} = \frac{V_0}{L} = \frac{100}{100} \times 10^3$$
$$= 1000 \text{ A/s}.$$

Because $B_1 = 0$,

$$\frac{di}{dt} = 400 B_2 e^{-2800t}(24 \cos 9600t - 7 \sin 9600t).$$

Thus

$$\frac{di(0^+)}{dt} = 9600 B_2,$$

$$B_2 = \frac{1000}{9600} \approx 0.1042 \text{ A}.$$

The solution for $i(t)$ is

$$\boxed{i(t) = 0.1042 e^{-2800t} \sin 9600t \text{ A}, \quad t \geq 0.}$$

Example 8.11 – cont.

- B) To find $v_C(t)$ we can use the following equation:

$$v_C(t) = Ri(t) + \frac{Ldi(t)}{dt}$$

  - Using the expression for $i(t)$ from part (a) and substitute in the $v_C(t)$ equation, we get:

$$v_C(t) = (100 \cos 9600t + 29.17 \sin 9600t)e^{-2800t}\ \text{V}, \quad t \geq 0.$$

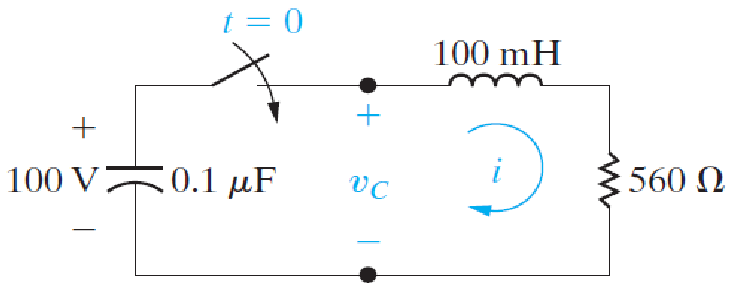
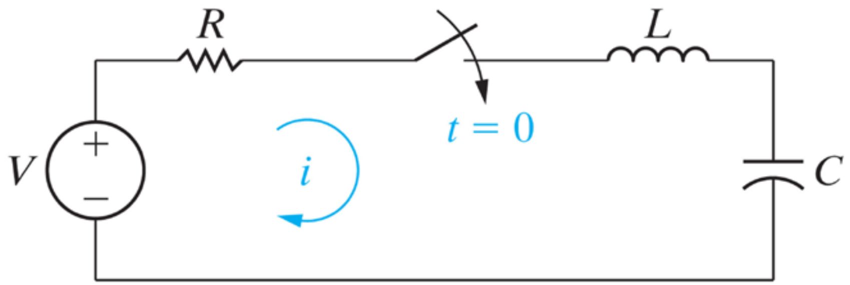


Figure 8.16 ▲ The circuit for Example 8.11.

Step Response of Series *RLC* Circuit



- Change the circuit scenario to find the step response
  - Add a DC source to the circuit
    - Assume this source is suddenly applied to the circuit
    - Energy may or may not be stored in capacitor or inductor
    - DC source serves as a forcing function
    - Focus on the case where no energy is initially stored in the circuit
- Solve for the voltage across the capacitor
  - It must remain constant during the circuit state transition $\left(v_C(0^-) = v_C(0^+)\right)$
  - As the circuit approaches steady state (t >> 0), capacitor voltage equals DC source voltage
- <u>Circuit analysis:</u>

Apply KVL to the closed path

$$Ri + L\frac{di}{dt} + v_C = V$$

Recall that: $i = C\dfrac{dv_C}{dt}$ $\quad \dfrac{di}{dt} = C\dfrac{d^2v_C}{dt^2}$

Substitute & rearrange:

$$\frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_C}{dt} + \frac{1}{LC}v_C = \frac{V}{LC}$$

2nd order diff. eqn. that looks similar to the 2nd diff. eqn. for the current in the parallel *RLC* circuit

Step Response of Series *RLC* Circuit – cont.

- The three possible solutions for v_C are as follows:

$$v_C = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t} \quad \text{(overdamped)},$$

$$v_C = V_f + B_1' e^{-\alpha t} \cos \omega_d t + B_2' e^{-\alpha t} \sin \omega_d t \quad \text{(underdamped)},$$

$$v_C = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t} \quad \text{(critically damped)},$$

◀ Capacitor voltage step response forms in series *RLC* circuits

  - v_f is the final value of v_c

- Reminder:
  - Overdamped response: $\omega_0^2 < \alpha^2$
  - Underdamped response: $\omega_0^2 > \alpha^2$
  - Critically damped response: $\omega_0^2 = \alpha^2$

Step Response of Series *RLC* Circuit – cont.

These equations are derived using the exact same logic we used to derive them for the parallel *RLC* circuits

To determine A'_1 & A'_2 for overdamped circuits, we use:

$$v(0^+) = V_f + A'_1 + A'_2$$

$$\frac{dv(0^+)}{dt} = s_1 A'_1 + s_2 A'_2$$

To determine B'_1 & B'_2 for underdamped circuits, we use:

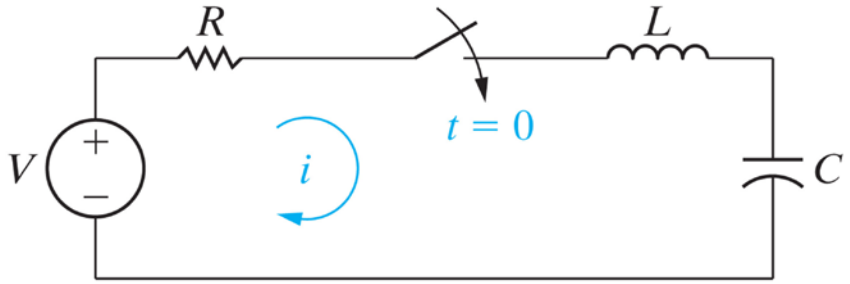
$$v(0^+) = V_f + B'_1$$

$$\frac{dv(0^+)}{dt} = \frac{i(0^+)}{C} = -\alpha B'_1 + \omega_d B'_2$$

To determine D'_1 & D'_2 for critically damped circuits, we use:

$$v(0^+) = V_f + D'_2$$

$$\frac{dv(0^+)}{dt} = \frac{i(0^+)}{C} = D'_1 - \alpha D'_2$$



Example 8.12

No energy is stored in the 100 mH inductor or the 0.4 μF capacitor when the switch in the circuit shown in Fig. 8.17 is closed. Find $v_C(t)$ for $t \geq 0$.

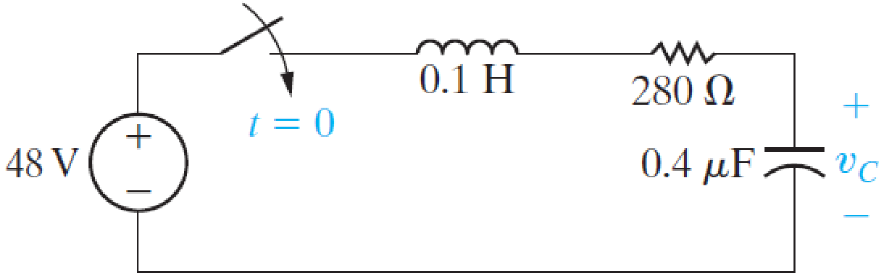


Figure 8.17 ▲ The circuit for Example 8.12.

Example 8.12

No energy is stored in the 100 mH inductor or the 0.4 μF capacitor when the switch in the circuit shown in Fig. 8.17 is closed. Find $v_C(t)$ for $t \geq 0$.

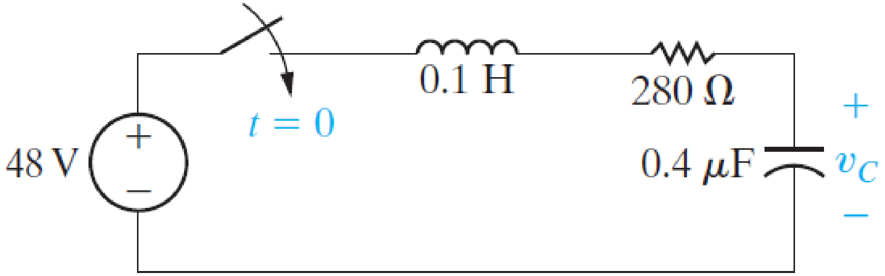


Figure 8.17 ▲ The circuit for Example 8.12.

Solution

The roots of the characteristic equation are

$$s_1 = -\frac{280}{0.2} + \sqrt{\left(\frac{280}{0.2}\right)^2 - \frac{10^6}{(0.1)(0.4)}}$$

$$= (-1400 + j4800)\text{ rad/s},$$

$$s_2 = (-1400 - j4800)\text{ rad/s}.$$

The roots are complex, so the voltage response is underdamped. Thus

$$v_C(t) = 48 + B_1'e^{-1400t}\cos 4800t$$

$$+ B_2'e^{-1400t}\sin 4800t, \quad t \geq 0.$$

No energy is stored in the circuit initially, so both $v_C(0)$ and $dv_C(0^+)/dt$ are zero. Then,

$$v_C(0) = 0 = 48 + B_1',$$

$$\frac{dv_C(0^+)}{dt} = 0 = 4800B_2' - 1400B_1'.$$

Solving for B_1' and B_2' yields

$$B_1' = -48\text{ V},$$

$$B_2' = -14\text{ V}.$$

Therefore, the solution for $v_C(t)$ is

$$v_C(t) = (48 - 48e^{-1400t}\cos 4800t$$

$$- 14e^{-1400t}\sin 4800t)\text{ V}, \quad t \geq 0.$$

Summary

TABLE 8.2 The Response of a Second-Order Circuit is Overdamped, Underdamped, or Critically Damped

The Circuit is	When	Qualitative Nature of the Response
Overdamped	$\alpha^2 > \omega_0^2$	The voltage or current approaches its final value without oscillation
Underdamped	$\alpha^2 < \omega_0^2$	The voltage or current oscillates about its final value
Critically damped	$\alpha^2 = \omega_0^2$	The voltage or current is on the verge of oscillating about its final value

TABLE 8.3 In Determining the Natural Response of a Second-Order Circuit, We First Determine Whether it is Over-, Under-, or Critically Damped, and Then We Solve the Appropriate Equations

Damping	Natural Response Equations	Coefficient Equations
Overdamped	$x(t) = A_1e^{s_1t} + A_2e^{s_2t}$	$x(0) = A_1 + A_2$; $dx/dt(0) = A_1s_1 + A_2s_2$
Underdamped	$x(t) = (B_1 \cos \omega_d t + B_2 \sin \omega_d t)e^{-\alpha t}$	$x(0) = B_1$; $dx/dt(0) = -\alpha B_1 + \omega_d B_2$, where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
Critically damped	$x(t) = (D_1t + D_2)e^{-\alpha t}$	$x(0) = D_2$, $dx/dt(0) = D_1 - \alpha D_2$

TABLE 8.4 In Determining the Step Response of a Second-Order Circuit, We Apply the Appropriate Equations Depending on the Damping

Damping	Step Response Equations[a]	Coefficient Equations
Overdamped	$x(t) = X_f + A_1' e^{s_1t} + A_2' e^{s_2t}$	$x(0) = X_f + A_1' + A_2'$; $dx/dt(0) = A_1' s_1 + A_2' s_2$
Underdamped	$x(t) = X_f + (B_1' \cos \omega_d t + B_2' \sin \omega_d t)e^{-\alpha t}$	$x(0) = X_f + B_1'$; $dx/dt(0) = -\alpha B_1' + \omega_d B_2'$
Critically damped	$x(t) = X_f + D_1' te^{-\alpha t} + D_2' e^{-\alpha t}$	$x(0) = X_f + D_2'$; $dx/dt(0) = D_1' - \alpha D_2'$

[a] where X_f is the final value of $x(t)$.

Summary of Topics Covered in this Chapter

- In this chapter we discussed:
  - Natural and step response or parallel and series RLC circuits
    - 2nd order differential equation
    - Damping scenarios
      - Over, under, and critically damped circuits
    - General response
- We covered sections 8.1-8.4
- Next chapter (Ch9) we will talk about sinusoidal steady state analysis
  - We will learn how to deal with sources that varies sinusoidally with time