Chapter 13: The Laplace Transform in Circuit Analysis

EEL 3112c – Circuits-II

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Topics to be Covered in this Chapter

- In this chapter we will discuss:
  - Circuit elements in the *s* domain
  - Circuit analysis in the *s* domain
  - Applications
  - The transfer function
  - The transfer function in partial fraction expansion
  - The impulse response
  - The transfer function and the steady-state sinusoidal response
- We will cover sections 13.1 – 13.5, 13.7

Introduction

- The Laplace transform has several characteristics that make it an attractive tool in circuit analysis
  - It transforms a set of linear constant-coefficient differential equations into a set of linear polynomial equations
    - Easier to manipulate
- The Laplace transform transforms variables from the time domain to another domain we call the s-domain
  - The s-domain is a frequency domain
- The Laplace transform will lead us to the concept of the **transfer function**
  - Transfer function for a particular circuit is the ratio of the Laplace transform of its output to the Laplace transform of its input

Circuits Elements in the *s* Domain

- The procedure for developing an *s*-domain equivalent circuit for each circuit element is simple
  - First, we write the time-domain equation that relates the terminal voltage to the terminal current
  - Next, we take the Laplace transform of the time-domain equation
    - This step generates an algebraic relationship between the *s*-domain current and voltage
- Notes:
  - The dimension of a transformed voltage is volt-seconds, and the dimension of a transformed current is ampere-seconds
  - Voltage-to-current ratio in the *s* domain carries the dimension of volts per ampere
  - An impedance in the *s* domain is measured in ohms
  - Admittance is measured in siemens (1/ohms)

A Resistor in the s Domain

$$V = \mathscr{L}\{v\} \quad \text{and} \quad I = \mathscr{L}\{i\}.$$

Time domain **s domain**

$$v = Ri. \xrightarrow[\text{Transform}]{\text{Laplace}} V = RI.$$

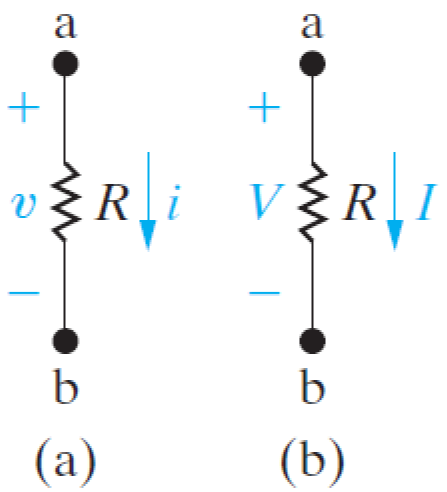


Figure 13.1 ▲ The resistance element. (a) Time domain. (b) Frequency domain.

An Inductor in the s Domain

$$V = \mathscr{L}\{v\} \quad \text{and} \quad I = \mathscr{L}\{i\}.$$

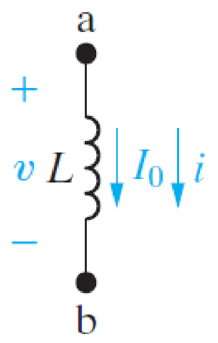


Figure 13.2 ▲ An inductor of L henrys carrying an initial current of I_0 amperes.

Time domain

s domain

$$v = L\frac{di}{dt} \xrightarrow[\textbf{Transform}]{\textbf{Laplace}} V = L[sI - i(0^-)] = sLI - LI_0.$$

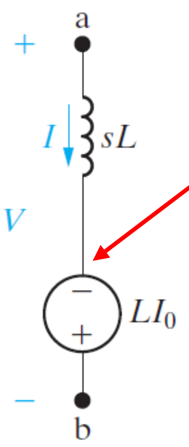


Figure 13.3 ▲ The series equivalent circuit for an inductor of L henrys carrying an initial current of I_0 amperes.

Note that the polarity marks on the voltage source LI_0 agree with the minus sign in s domain equation

We can also rearrange the s domain equation as follows:

$$I = \frac{V + LI_0}{sL} = \frac{V}{sL} + \frac{I_0}{s}$$

The circuit that represents this equation consists of an impedance of sL ohms in parallel with an independent current source of I_0/s ampere-seconds

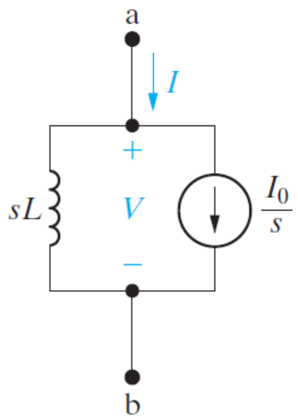
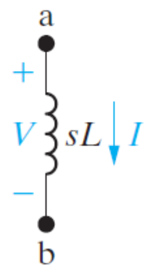


Figure 13.4 ▲ The parallel equivalent circuit for an inductor of L henrys carrying an initial current of I_0 amperes.

If the initial energy stored in the inductor is 0 (i.e. $I_0 = 0$), the s-domain equivalent circuit of the inductor (Fig. 13.3 & 13.4) reduces to an inductor with an impedance of sL ohms



A Capacitor in the s Domain

$$V = \mathscr{L}\{v\} \quad \text{and} \quad I = \mathscr{L}\{i\}.$$

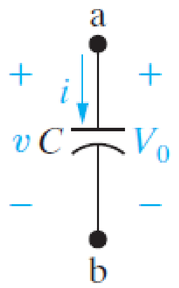


Figure 13.6 ▲ A capacitor of C farads initially charged to V_0 volts.

Time domain

$$i = C\frac{dv}{dt}$$

Laplace Transform

***s* domain**

$$I = C[sV - v(0^-)] = sCV - CV_0.$$

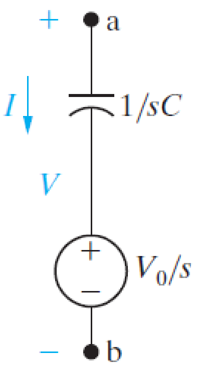


Figure 13.8 ▲ The series equivalent circuit for a capacitor initially charged to V_0 volts.

Rearranging the s domain equation, we get:

$$V = \left(\frac{1}{sC}\right)I + \frac{V_0}{s}$$

Note that the polarity marks on the voltage source V_0/s agree with the sign in the s domain equation

The s domain equation:

$$I = sCV - CV_0,$$

The circuit that represents this equation consists of an impedance of $1/sC$ ohms in parallel with an independent voltage source of CV_0 ampere-seconds

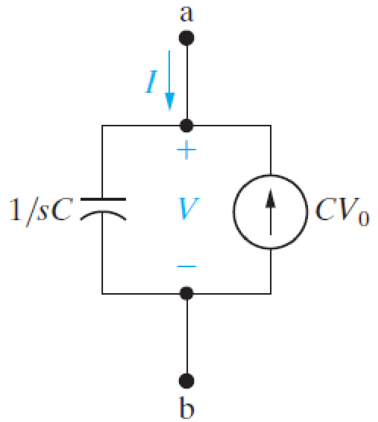
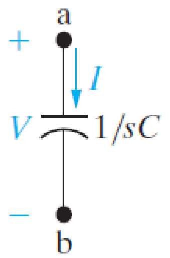


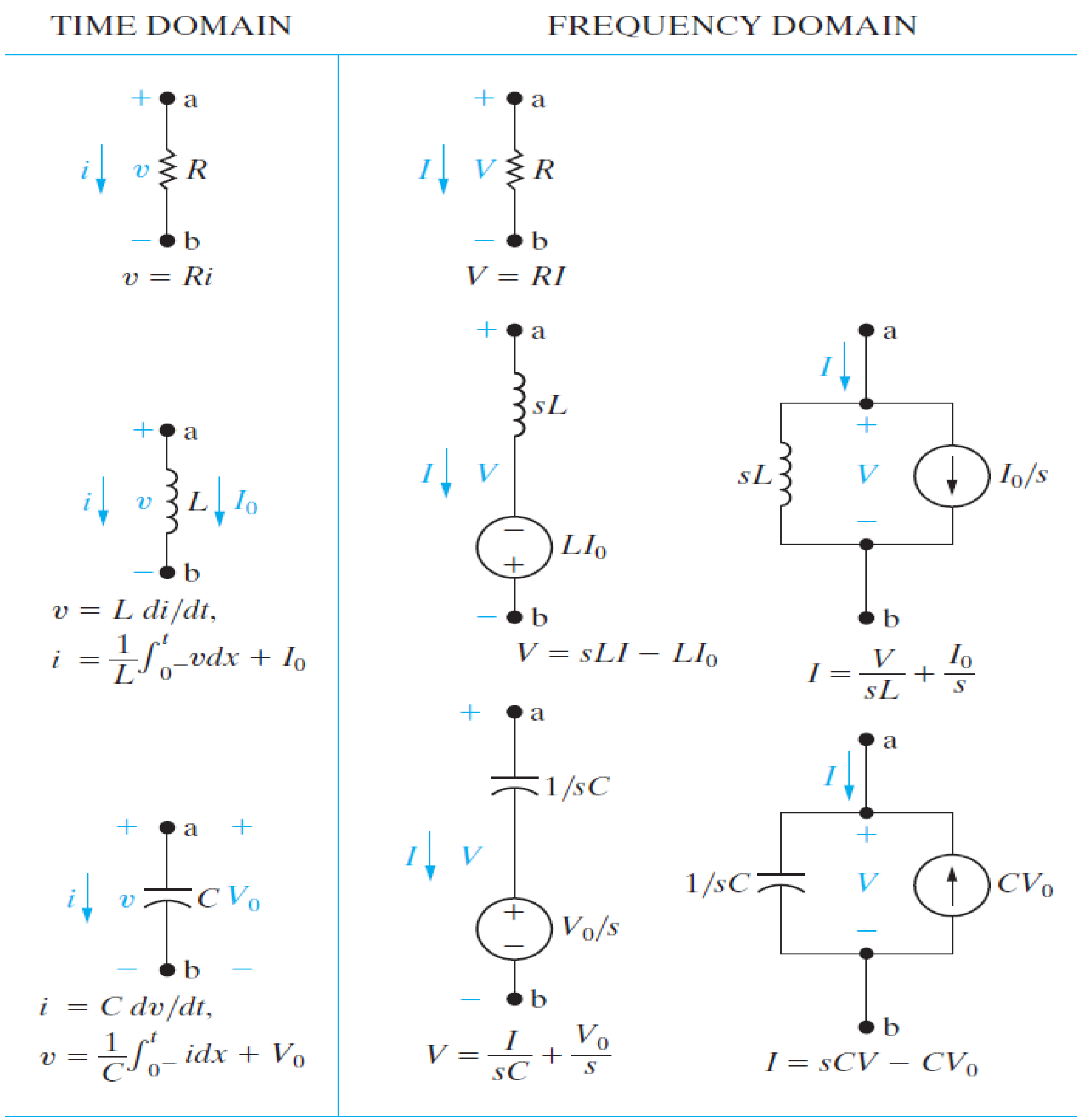
Figure 13.7 ▲ The parallel equivalent circuit for a capacitor initially charged to V_0 volts.

If the initial energy stored in the capacitor is 0 (i.e. $V_0 = 0$), the s-domain equivalent circuit of the capacitor (Fig. 13.7 & 13.8) reduces to a capacitor with an impedance of $1/sC$ ohms



Summary of Circuits Elements in the *s* Domain

TABLE 13.1 **Summary of the *s*-Domain Equivalent Circuits**



Notes on Circuit Analysis in the *s* Doamin

- The rules for combining impedances and admittances in the *s* domain are the same as those for frequency-domain circuits
  - Thus series-parallel simplifications and Δ-to-Y conversions also are applicable to *s*-domain analysis
- In addition, Kirchhoff's laws apply to *s*-domain currents and voltages
  - The algebraic sum of the currents at a node is zero in the time domain, and the algebraic sum of the transformed currents is also zero
  - A similar statement holds for the algebraic sum of the transformed voltages around a closed path
- Analytical and simplification techniques first introduced with resistive circuits, such as mesh-current and node-voltage methods and source transformations, can be used in the *s* domain as well

How to Analyze a Circuit in the *s*-Domain

1. Replacing each circuit element with its *s*-domain equivalent
   - The initial energy in *L* or *C* is taken into account by adding independent source in series or parallel with the element impedance
2. Writing & solving algebraic equations using the same circuit analysis techniques developed for resistive networks
   - All the techniques we studied previously apply to the analysis in the *s*-domain
3. Obtaining the *t*-domain solutions by inverse Laplace transform
   - You need to be familiar with partial fraction expansion
   - **Ch12** addressed this issue in details

Applications: The Natural Response of RC Circuit

- For the circuit shown here, the capacitor is initially charged to V_0 volts, and we want to find the time-domain expression for i, and v
- **Solution:**
- Let us start by finding i
- In transferring the circuit to the s domain, we have a choice of two equivalent circuits for the charged capacitor
  - Because we are interested in the current i, the series-equivalent circuit is more convenient

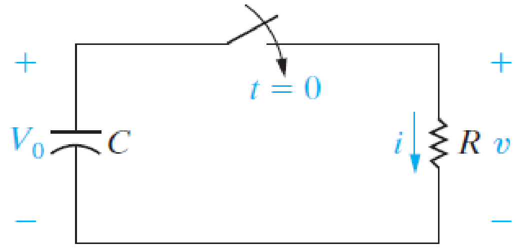


Figure 13.10 ▲ The capacitor discharge circuit.

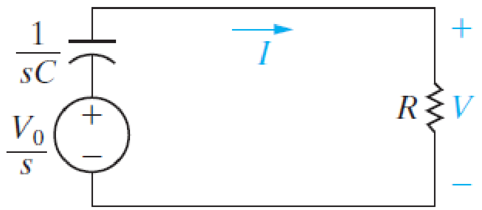


Figure 13.11 ▲ An s-domain equivalent circuit for the circuit shown in Fig. 13.10.

Summing the voltages around the mesh generates the expression:

$$\frac{V_0}{s} = \frac{1}{sC}I + RI$$

Solving the equation for I:

$$I = \frac{CV_0}{RCs + 1} = \frac{V_0/R}{s + (1/RC)}$$

Taking the inverse Laplace transform:

$$i = \frac{V_0}{R}e^{-t/RC}u(t)$$

Equivalent to the expression for current derived by the classical methods discussed in Chapter 7

After we have found i, the easiest way to determine v is simply to apply Ohm's law:

$$v = Ri = V_0e^{-t/RC}u(t).$$

An alternative approach to find v without first finding i is to use parallel equivalent circuit in the s domain, and then use node-voltage analysis

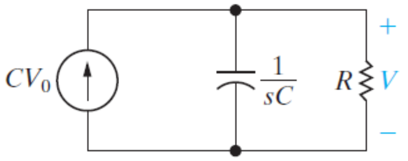


Figure 13.12 ▲ An s-domain equivalent circuit for the circuit shown in Fig. 13.10.

Applications: Step Response of Parallel *RLC* Circuit

- For the parallel RLC circuit shown here, we need to find the expression for i_L after we change the position of the switch

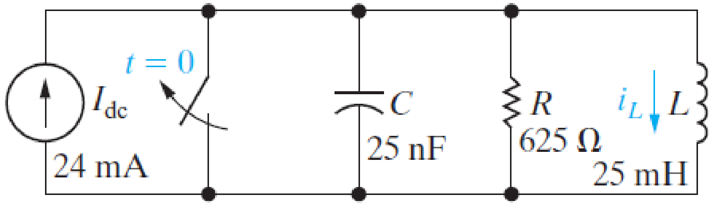


Figure 13.13 ▲ The step response of a parallel *RLC* circuit.

- **Solution:**

- Let us start by constructing the *s* domain equivalent circuit

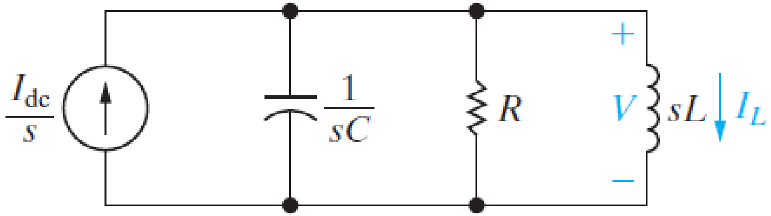


Figure 13.14 ▲ The *s*-domain equivalent circuit for the circuit shown in Fig. 13.13.

An independent source can be transformed from the time domain to the frequency domain simply by determining the Laplace transform of its time-domain function.
Here, opening the switch results in a step change in the current applied to the circuit. Therefore the *s*-domain current source is

$$\mathscr{L}\{I_{dc}u(t)\}, \text{ or } I_{dc}/s.$$

To find I_L, we first solve for V and then use: $I_L = \frac{V}{sL}$ **(*)**

Summing the currents away from the top node generates the expression:

$$sCV + \frac{V}{R} + \frac{V}{sL} = \frac{I_{dc}}{s}$$

Solving the above equation for V yields:

$$V = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}$$

Substituting the expression for V in **(*)** yields:

$$I_L = \frac{I_{dc}/LC}{s[s^2 + (1/RC)s + (1/LC)]}$$

Applications: Step Response of Parallel *RLC* Circuit – cont.

- Substituting the numerical values of R, L, C, and I_{dc} into the last eqn. yields:

$$I_L = \frac{384 \times 10^5}{s(s^2 + 64{,}000s + 16 \times 10^8)} \xrightarrow{\text{Factor the quadratic term in the denominator for partial fraction expansion}} I_L = \frac{384 \times 10^5}{s(s + 32{,}000 - j24{,}000)(s + 32{,}000 + j24{,}000)}$$

- Now we need the inverse Laplace transform to find $i(t)$
  - To do that we need partial fraction expansion

$$I_L = \frac{K_1}{s} + \frac{K_2}{s + 32{,}000 - j24{,}000} + \frac{K_2^*}{s + 32{,}000 + j24{,}000}$$

$$K_1 = \frac{384 \times 10^5}{16 \times 10^8} = 24 \times 10^{-3},$$

Multiply both sides of the equation by s and substitute $s = 0$

$$K_2 = \frac{384 \times 10^5}{(-32{,}000 + j24{,}000)(j48{,}000)}$$

$$= 20 \times 10^{-3} \underline{/126.87^\circ}.$$

$$\Rightarrow \quad i_L = [24 + 40e^{-32{,}000t} \cos(24{,}000t + 126.87^\circ)]u(t)\text{mA}$$

- We can test the s-domain expression for I_L by checking to see whether the final-value theorem predicts the correct value for i_L at $t = \infty$.
  - All the poles of except for the first-order pole at the origin, lie in the left half of the s plane, so the theorem is applicable
- We know from the behavior of the circuit that after the switch has been open for a long time, the inductor will short-circuit the current source. Therefore, the final value of i_L must be 24 mA. The limit of sI_L as $s \rightarrow 0$ is:

$$\lim_{s \to 0} sI_L = \frac{384 \times 10^5}{16 \times 10^8} = 24 \text{ mA}$$

Applications: Transient Response of Parallel *RLC* Circuit

- If we replaced the dc source in Fig. 13.13 with a sinusoidal current source
  - The new current source is: $i_g = I_m \cos \omega t$ A
    - Where $I_m = 24\,mA$, and $\omega = 40{,}000\,rad/s$

$u(t)$ will not change anything since the integral limits are from 0 to ∞

- Solution:

Constant

- Find Laplace transform for the sinusoidal source: $\int_0^\infty I_m \cos(\omega t) u(t) e^{-st} dt$

We can write I_g as: $I_g = \dfrac{sI_m}{s^2 + \omega^2}$ From Table 12.1

The voltage across the parallel elements is:

$$sCV + \frac{V}{R} + \frac{V}{sL} = I_g$$

$$V = \frac{(I_g/C)s}{s^2 + (1/RC)s + (1/LC)}$$

Substituting the I_g equation into the V equation yields:

$$V = \frac{(I_m/C)s^2}{(s^2 + \omega^2)[s^2 + (1/RC)s + (1/LC)]}$$

To find I_L, we can use: $I_L = \dfrac{V}{sL}$

$$I_L = \frac{V}{sL} = \frac{(I_m/LC)s}{(s^2 + \omega^2)[s^2 + (1/RC)s + (1/LC)]}$$

Perform partial fraction expansion and inverse Laplace transform:

$$I_L = \frac{384 \times 10^5 s}{(s^2 + 16 \times 10^8)(s^2 + 64{,}000s + 16 \times 10^8)}$$

$$I_L = \frac{K_1}{s - j40{,}000} + \frac{K_1^*}{s + j40{,}000} + \frac{K_2}{s + 32{,}000 - j24{,}000} + \frac{K_2^*}{s + 32{,}000 + j24{,}000}$$

$$i(t) = (\underbrace{15 \sin 40{,}000t}_{\text{SS Response}} - \underbrace{25e^{-32{,}000t} \sin 24{,}000t}_{\text{Transient Response, decays as time pass}})u(t)\ \text{mA}$$

Note on the Previous Two Examples

- In the previous two examples, the results we obtained are:
  - For DC source: $i_L = [24 + 40e^{-32{,}000t}\cos(24{,}000t + 126.87^\circ)]u(t)\text{mA}$
  - For AC source: $i(t) = (15\sin 40{,}000t - 25e^{-32{,}000t}\sin 24{,}000t)u(t)\text{ mA}$
- For DC source, the frequency in the cos argument is the damping frequency of the underdamped system presented by the circuit
  - s_1 and s_2 form complex conjugates
- This frequency is the frequency of oscillation of the system in the transient state before the response stabilizes
  - The source in the circuit is DC so the current final value should be DC as well
- Note that for the case when we have an AC source in the circuit, the argument of the first sin function (steady state response) has the ω of the source and the argument of the second sin function (transient response) has the ω representing the damping frequency

Applications: The Use of Thévenin's Equivalent

- The problem is to find the capacitor current that results from closing the switch in the circuit shown here
  - The energy stored in the circuit prior to closing is zero
- Solution:

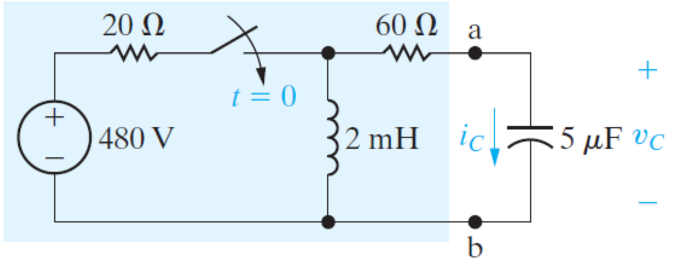


Figure 13.17 ▲ A circuit to be analyzed using Thévenin's equivalent in the s domain.

- To find i_C we first construct the s-domain equivalent circuit and then find the Thévenin equivalent of this circuit with respect to the terminals of the capacitor
- The Thévenin voltage is the open-circuit voltage across terminals a, b. Under open-circuit conditions, there is no voltage across the 60 Ω resistor

$$V_{\text{Th}} = \frac{(480/s)(0.002s)}{20 + 0.002s} = \frac{480}{s + 10^4}$$

Voltage divider between the 2 mH inductor and the 20 Ω resistor

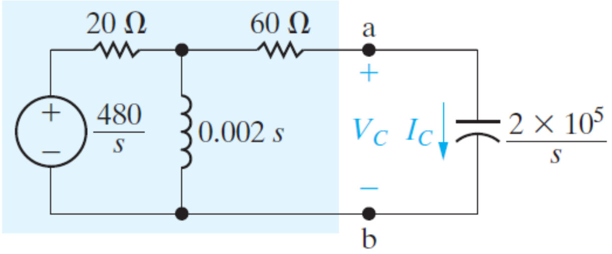


Figure 13.18 ▲ The s-domain model of the circuit shown in Fig. 13.17.

- The Thévenin impedance seen from terminals a, and b:

$$Z_{\text{Th}} = 60 + \frac{0.002s(20)}{20 + 0.002s} = \frac{80(s + 7500)}{s + 10^4}$$

$$I_C = \frac{480/(s + 10^4)}{[80(s + 7500)/(s + 10^4)] + [(2 \times 10^5)/s]}$$

From this, we use partial fraction expansion and inverse Laplace transform to find $i_c(t)$

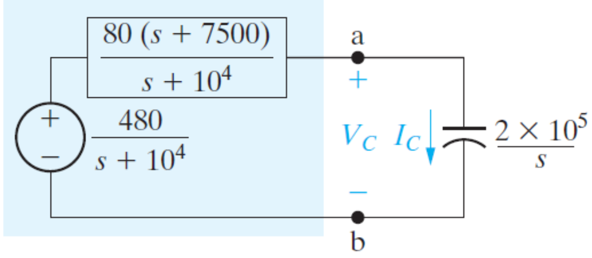


Figure 13.19 ▲ A simplified version of the circuit shown in Fig. 13.18, using a Thévenin equivalent.

Notes on Applications

- Section 13.3 in the book contains additional examples on the applications of the Laplace transform in circuit analysis
  - It is important that you practice these examples
    - Superposition, mesh currents, etc.

- It is obvious that the step of finding the inverse Laplace transform depends on the partial fraction expansion of the rational function in the s-domain
  - Being familiar with partial fraction expansion is extremely important to use the Laplace transform in circuit analysis

The Transfer Function

- The **transfer function** is defined as the s-domain ratio of the Laplace transform of the output (response) to the Laplace transform of the input (source)
  - If a circuit has multiple independent sources, we can find the transfer function for each source and use superposition to find the response to all sources

Definition of a transfer function ▶ $$H(s) = \frac{Y(s)}{X(s)},$$

- Note that the transfer function depends on what is defined as the output (response) signal of the circuit, and the input signal is the power source used
- $H(s)$ is always a rational function of s
  - The poles of $H(s)$ must lie in the left half of the $s - plane$
    - This is the condition for stability (a response to a bounded source)
  - Zeros can lie in either the right half or the left half of the $s - plane$

Example 13.1: Derive the Transfer Function of a Circuit

The voltage source v_g drives the circuit shown in Fig. 13.31. The response signal is the voltage across the capacitor, v_o.

a) Calculate the numerical expression for the transfer function.

b) Calculate the numerical values for the poles and zeros of the transfer function.

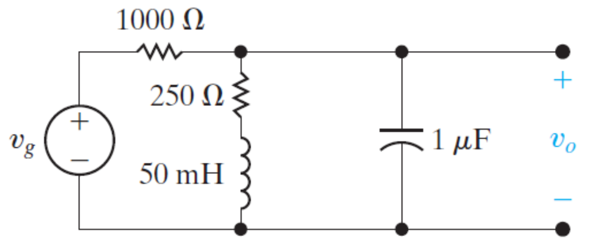


Figure 13.31 ▲ The circuit for Example 13.1.

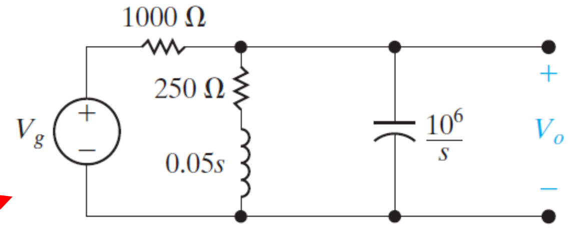


Figure 13.32 ▲ The *s*-domain equivalent circuit for the circuit shown in Fig. 13.31.

Solution

a) The first step in finding the transfer function is to construct the *s*-domain equivalent circuit, as shown in Fig. 13.32. By definition, the transfer function is the ratio of V_o/V_g, which can be computed from a single node-voltage equation. Summing the currents away from the upper node generates

$$\frac{V_o - V_g}{1000} + \frac{V_o}{250 + 0.05s} + \frac{V_o s}{10^6} = 0.$$

Solving for V_o yields

$$V_o = \frac{1000(s + 5000)V_g}{s^2 + 6000s + 25 \times 10^6}.$$

Hence the transfer function is

$$H(s) = \frac{V_o}{V_g}$$

$$= \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}.$$

b) The poles of $H(s)$ are the roots of the denominator polynomial. Therefore

$$p_1 = -3000 - j4000,$$

$$p_2 = -3000 + j4000.$$

The zeros of $H(s)$ are the roots of the numerator polynomial; thus $H(s)$ has a zero at

$$z_1 = -5000.$$

Example 13.2: Analyze the Transfer Function of a Circuit

The circuit in Example 13.1 (Fig. 13.31) is driven by a voltage source whose voltage increases linearly with time, namely, $v_g = 50tu(t)$.

a) Use the transfer function to find v_o.

b) Identify the transient component of the response.

c) Identify the steady-state component of the response.

d) Sketch v_o versus t for $0 \le t \le 1.5$ ms.

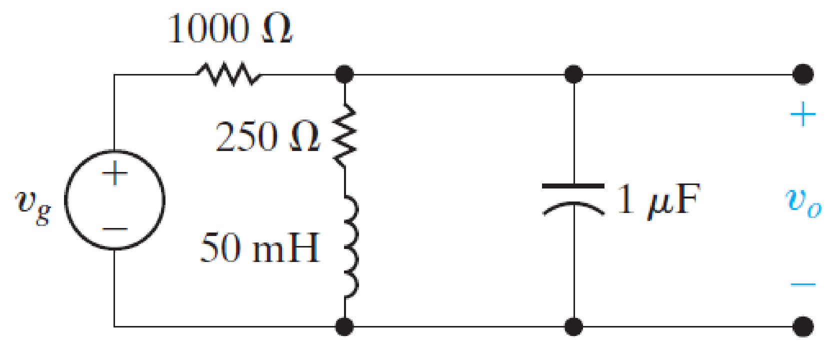


Figure 13.31 ▲ The circuit for Example 13.1.

Example 13.2: Analyze the Transfer Function of a Circuit – cont.

Solution

a) From Example 13.1,

$$H(s) = \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}.$$

The transform of the driving voltage is $50/s^2$; therefore, the s-domain expression for the output voltage is

$$V_o = \frac{1000(s + 5000)}{(s^2 + 6000s + 25 \times 10^6)} \frac{50}{s^2}.$$

The partial fraction expansion of V_o is

$$V_o = \frac{K_1}{s + 3000 - j4000} + \frac{K_1^*}{s + 3000 + j4000} + \frac{K_2}{s^2} + \frac{K_3}{s}.$$

$K_1 = 5\sqrt{5} \times 10^{-4} \angle 79.70°$; $K_1^* = 5\sqrt{5} \times 10^{-4} \angle -79.70°$

$K_2 = 10, \quad K_3 = -4 \times 10^{-4}.$

The time-domain expression for v_o is

$$v_o = [10\sqrt{5} \times 10^{-4} e^{-3000t} \cos(4000t + 79.70°) + 10t - 4 \times 10^{-4}]u(t) \text{ V}.$$

b) The transient component of v_o is

$$10\sqrt{5} \times 10^{-4} e^{-3000t} \cos(4000t + 79.70°).$$

Note that this term is generated by the poles $(-3000 + j4000)$ and $(-3000 - j4000)$ of the transfer function.

c) The steady-state component of the response is

$$(10t - 4 \times 10^{-4})u(t).$$

These two terms are generated by the second-order pole (K/s^2) of the driving voltage.

d) Figure 13.33 shows a sketch of v_o versus t. Note that the deviation from the steady-state solution $10{,}000t - 0.4$ mV is imperceptible after approximately 1 ms.

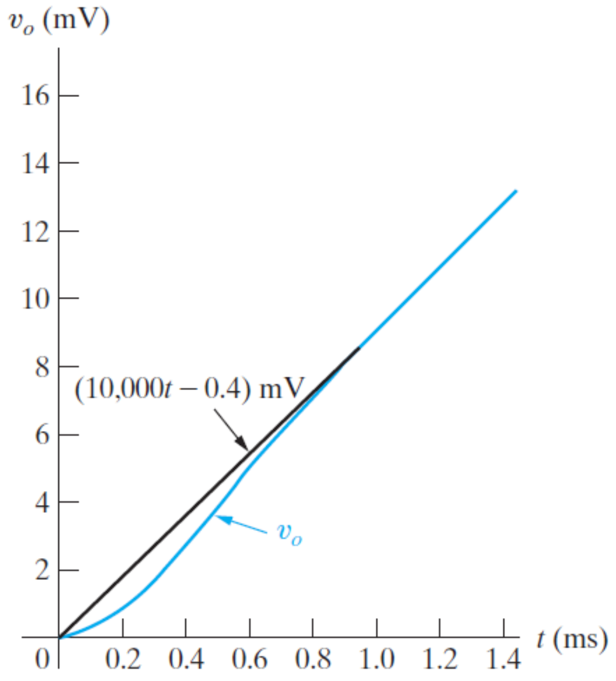
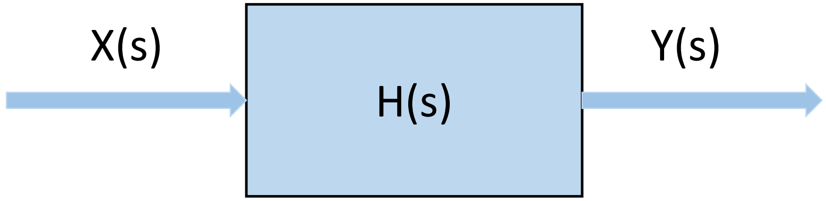


Figure 13.33 ▲ The graph of v_o versus t for Example 13.2.

The Impulse Response



$$H(s) = \frac{Y(s)}{X(s)} = \frac{\mathcal{L}\{\text{output of a system/circuit}\}}{\mathcal{L}\{\text{input to a system/circuit}\}}$$

Note – to calculate a transfer function for a circuit or other system, there must be no initial stored energy (e.g. – the initial conditions must be zero).

The Impulse Response – cont.

- Suppose the input to the system whose transfer function is $H(s)$ is a unit impulse, $\delta(t)$:

$$X(s) = \mathcal{L}\{\delta(t)\} = 1$$
$$\Rightarrow \qquad Y(s) = H(s)X(s) = H(s)$$
$$\Rightarrow \qquad y(t) = \mathcal{L}^{-1}\{H(s)\}$$

- In this case, the inverse Laplace transform of the transfer function is the unit impulse response of the system
- The impulse response of a system is very useful to identify its properties

The Transfer Function and the SS Sinusoidal Response

- Once we have computed a circuit's transfer function, we no longer need to perform a separate phasor analysis of the circuit to determine its steady state response
  - Instead, we use the transfer function to relate the steady state response to the excitation source
- Assume that $x(t) = Acos(\omega t + \Phi)$, then we can use $H(s)$ to find the *SS* sinusoidal response of a circuit as:

Steady-state sinusoidal response computed using a transfer function ▶

$$y_{ss}(t) = A|H(j\omega)| \cos[\omega t + \phi + \theta(\omega)],$$

Please follow proof in the book

  - Where A is the amplitude, $|H(j\omega)|$ is the magnitude of $H(s)|_{s=j\omega}$, and $\theta(\omega)$ is the phase angle of $H(s)$

Example 13.4: Using $H(s)$ to Find the SS Sinusoidal Response

The circuit from Example 13.1 is shown in Fig. 13.46. The sinusoidal source voltage is $120\cos(5000t + 30^\circ)$ V. Find the steady-state expression for v_o.

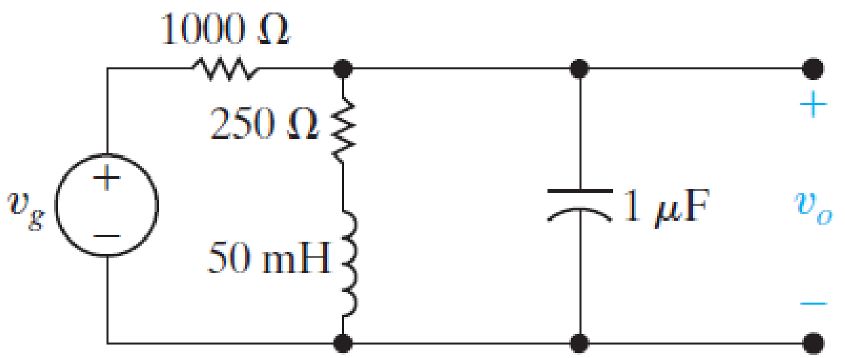


Figure 13.46 ▲ The circuit for Example 13.4.

Solution

From Example 13.1,

$$H(s) = \frac{1000(s + 5000)}{s^2 + 6000s + 25 * 10^6}.$$

The frequency of the voltage source is 5000 rad/s; hence we evaluate $H(s)$ at $H(j5000)$:

$$H(j5000) = \frac{1000(5000 + j5000)}{-25 * 10^6 + j5000(6000) + 25 \times 10^6}$$

$$= \frac{1 + j1}{j6} = \frac{1 - j1}{6} = \frac{\sqrt{2}}{6}\angle -45^\circ.$$

Then, from Eq. 13.120,

$$v_{o_{ss}} = \frac{(120)\sqrt{2}}{6}\cos(5000t + 30^\circ - 45^\circ)$$

$$= 20\sqrt{2}\cos(5000t - 15^\circ)\ \text{V}.$$

General Examples: 1

- Evaluate the following integral, using the sifting property of the impulse function

$$\int_{-10}^{10} (6t^2 + 3)\delta(t-2)dt$$

- A) 24
- B) 27
- C) 3

General Examples: 1 – cont.

- Evaluate the following integral, using the sifting property of the impulse function

$$\int_{-10}^{10} (6t^2 + 3)\delta(t-2)\,dt$$

- A) 24
- B) 27
- C) 3

We evaluate the function $\boldsymbol{\left(6t^2+3\right)}$ **@** $\boldsymbol{t=2 \;\rightarrow 6(2)^2+3=27}$

General Examples: 2

- Find the zeros and the poles of $I_1(s)$

$$I_1(s) = \frac{40(s+9)}{s(s+2)(s+12)}$$

- Poles:
  - A) s=2 rad/s, s=12 rad/sec
  - B) s=-2 rad/sec, s=-12 rad/sec
  - C) s=0 rad/sec, s=-2 rad/sec, s=-12 rad/sec
- Zeros:
  - A) s=-9 rad/sec
  - B) s=9 rad/sec
  - C) No zeros

General Examples: 2 – cont.

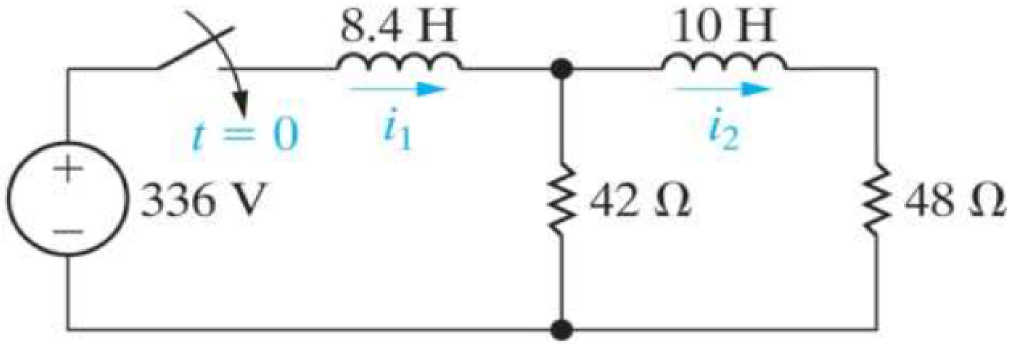
- Find the zeros and the poles of $I_1(s)$

$$I_1(s) = \frac{40(s+9)}{s(s+2)(s+12)}$$

- Poles:
  - A) s=2 rad/s, s=12 rad/sec
  - B) s=-2 rad/sec, s=-12 rad/sec
  - C) s=0 rad/sec, s=-2 rad/sec, s=-12 rad/sec
- Zeros:
  - A) s=-9 rad/sec
  - B) s=9 rad/sec
  - C) No zeros

General Examples: 3

Example:
There is no initial energy stored in this circuit. Find $i_1(t)$ and $i_2(t)$ for $t > 0$.



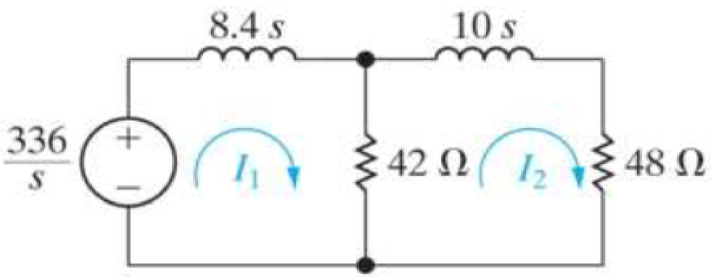
$$-\frac{336}{s} + (42 + 8.4s)I_1 - 42I_2 = 0$$

$$(10s + 90)I_2 - 42I_1 = 0 \qquad \Rightarrow \qquad I_1 = \frac{10s + 90}{42} I_2$$

Substituting, $$-\frac{336}{s} + \left[\frac{(42 + 8.4s)(10s + 90)}{42} - 42\right] I_2 = 0$$

$$\Rightarrow \qquad I_2(s) = \frac{336(42)}{s[(42 + 8.4s)(10s + 90) - 42^2} = \frac{168}{s^3 + 14s^2 + 24s}$$

$$I_1(s) = \frac{10s + 90}{42}\left[\frac{168}{s^3 + 14s^2 + 24s}\right] = \frac{40s + 360}{s^3 + 14s^2 + 24s}$$



General Examples: 3 – cont.

$$K_1 = \left.\frac{40s+360}{(s+2)(s+12)}\right|_{s=0} = 15; \qquad K_2 = \left.\frac{40s+360}{s(s+12)}\right|_{s=-2} = -14; \qquad K_3 = \left.\frac{40s+360}{s(s+2)}\right|_{s=-12} = -1$$

$$\therefore \qquad I_1(s) = \frac{15}{s} + \frac{-14}{s+2} + \frac{-1}{s+12}$$

$$i_1(t) = \mathcal{L}^{-1}\left\{\frac{15}{s} + \frac{-14}{s+2} + \frac{-1}{s+12}\right\}$$

$$= [15 - 14e^{-2t} - e^{-12t}]u(t) \text{ A}$$

The forced response is $15u(t)$ A;

The natural response is $[-14e^{-2t} - e^{-12t}]u(t)$ A.

General Examples: 3 – cont.

$$I_2(s) = \frac{168}{s(s+2)(s+12)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{K_3}{s+12}$$

$$K_1 = \left.\frac{168}{(s+2)(s+12)}\right|_{s=0} = 7; \qquad K_2 = \left.\frac{168}{s(s+12)}\right|_{s=-2} = -8.4; \qquad K_3 = \left.\frac{168}{s(s+2)}\right|_{s=-12} = 1.4$$

$$\therefore \qquad I_2(s) = \frac{7}{s} + \frac{-8.4}{s+2} + \frac{1.4}{s+12}$$

General Examples: 3 – cont.

$$i_2(t) = \mathcal{L}^{-1}\left\{\frac{7}{s} + \frac{-8.4}{s+2} + \frac{1.4}{s+12}\right\}$$

$$= [7 - 8.4e^{-2t} + 1.4e^{-12t}]u(t)\ \text{A}$$

The forced response is $7u(t)$ A;

The natural response is $[-8.4e^{-2t} - 1.4e^{-12t}]u(t)$ A.

$$i_1(t) = (15 - 14e^{-2t} - e^{-12t})u(t)A$$

$$i_2(t) = (7 - 8.4e^{-2t} + 1.4e^{-12t})u(t)A$$

Check the answers at t = 0 and t = ∞ to make sure the circuit and the equations match!

General Examples: 3 – cont.

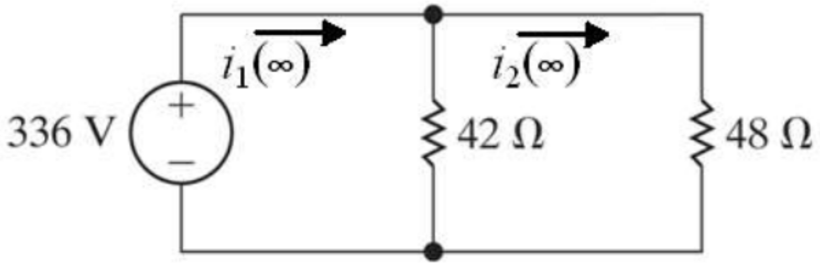
At t = 0, the circuit has no initial stored energy, so $i_1(0) = 0$ and $i_2(0) = 0$. Now check the equations:

$$i_1(0) = (15 - 14 - 1)(1) = 0$$
$$i_2(0) = (7 - 8.4 + 1.4)(1) = 0$$

And at $t \rightarrow \infty$, inductors behave as a short circuit

$$i_1(t) = (15 - 14e^{-2t} - e^{-12t})u(t)A \quad \Rightarrow \quad i_1(\infty) = 15 - 0 - 0 = 15\ \text{A}$$
$$i_2(t) = (7 - 8.4e^{-2t} + 1.4e^{-12t})u(t)A \quad \Rightarrow \quad i_2(\infty) = 7 - 0 - 0 = 7\ \text{A}$$

Draw the circuit for t = ∞ and check these solutions.



$$42 \,||\, 48 = 22.4\Omega$$

$$i_1(\infty) = \frac{336}{22.4} = 15\ \text{A(check!)}$$

$$i_2(\infty) = \frac{22.4}{48}(15) = 7\ \text{A(check!)}$$

General Examples: 3 – cont.

- We can also check the answers at $t = 0\ and\ t = \infty$ using the initial and final value theorem (IVT and SVT) in the s domain

Initial value theorem (IVT)

$$I_1(s) = \frac{40s+360}{s^3+14s^2+24s}$$
$$\lim_{t\to 0} i_1(t) = \lim_{s\to\infty} sI_1(s)$$
$$= \lim_{s\to\infty} \frac{40s^2+360s}{s^3+14s^2+24s}$$
$$= \lim_{1/s\to 0} \frac{(40/s)+(360/s^2)}{1+(14/s)+(24/s^2)}$$
$$= 0\ \text{A(check!)}$$

$$I_2(s) = \frac{168}{s^3+14s^2+24s}$$
$$\lim_{t\to\infty} i_1(t) = \lim_{s\to\infty} sI_1(s)$$
$$= \lim_{s\to\infty} \frac{168s}{s^3+14s^2+24s}$$
$$= \lim_{1/s\to 0} \frac{(168/s^2)}{1+(14/s)+(24/s^2)}$$
$$= 0\ \text{A(check!)}$$

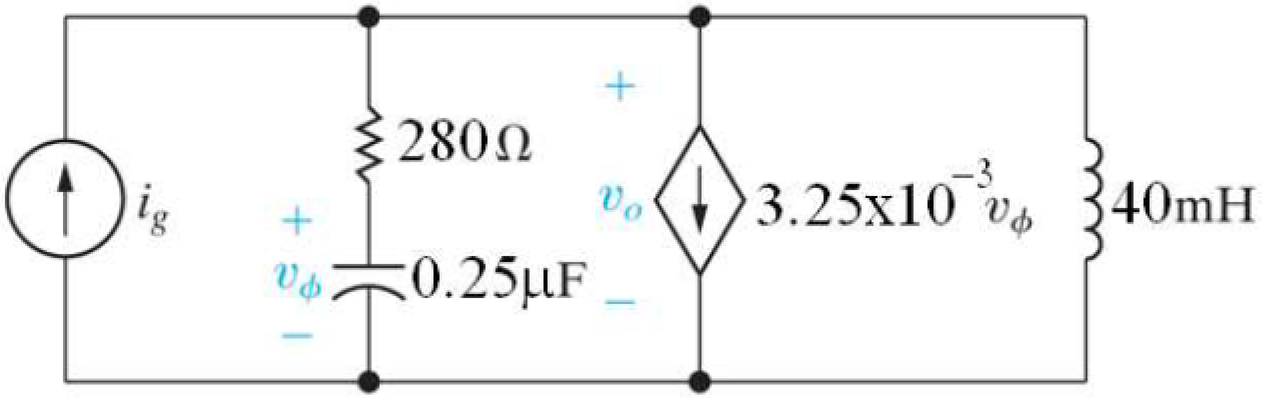
Final value theorem (FVT)

$$I_1(s) = \frac{40s+360}{s^3+14s^2+24s}$$
$$\lim_{t\to\infty} i_1(t) = \lim_{s\to 0} sI_1(s)$$
$$= \lim_{s\to 0} \frac{40s^2+360s}{s^3+14s^2+24s}$$
$$= \lim_{s\to 0} \frac{40s+360}{s^2+14s+24}$$
$$= \frac{360}{24} = 15\ \text{A(check!)}$$

$$I_2(s) = \frac{168}{s^3+14s^2+24s}$$
$$\lim_{t\to\infty} i_1(t) = \lim_{s\to 0} sI_1(s)$$
$$= \lim_{s\to 0} \frac{168s}{s^3+14s^2+24s}$$
$$= \lim_{s\to 0} \frac{168}{s^2+14s+24}$$
$$= \frac{168}{24} = 7\ \text{A(check!)}$$

General Examples: 4

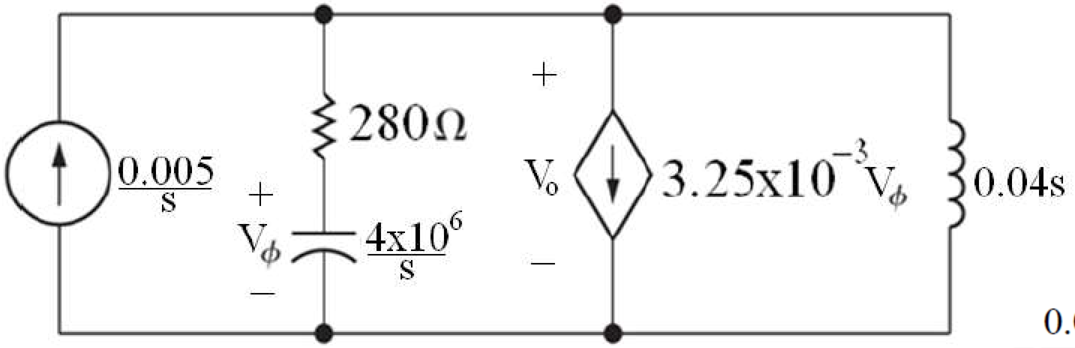
Example:
There is no initial energy stored in this circuit.
Find v_o if $i_g = 5u(t)$ mA.



General Examples: 4 – cont.

- ANSWER:

Laplace transform the circuit:



$$-\frac{0.005}{s}+\frac{V_o}{280+4\times10^6/s}+3.25\times10^{-3}V_\phi+\frac{V_o}{0.04s}=0 \qquad \text{KCL at top node}$$

$$V_\phi=\frac{4\times10^6/s}{280+4\times10^6/s}V_o=\frac{4\times10^6V_o}{280s+4\times10^6} \qquad \text{voltage division}$$

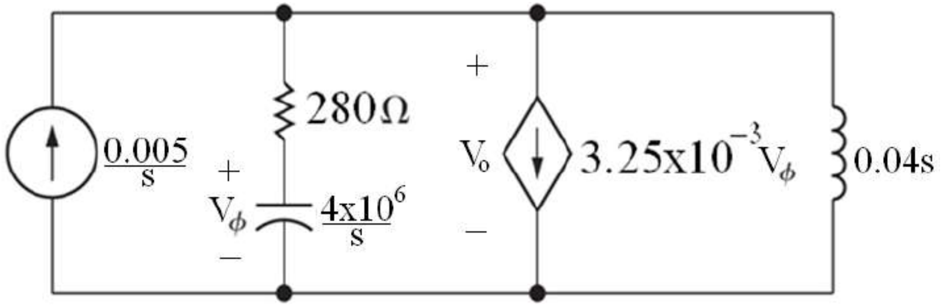
$$\therefore \qquad V_o\left[\frac{s}{280s+4\times10^6}+\frac{13{,}000}{280s+4\times10^6}+\frac{25}{s}\right]=\frac{0.005}{s}$$

$$\Rightarrow \qquad V_o\left[\frac{s^2+13{,}000s+25\left(280s+4\times10^6\right)}{s\left(280s+4\times10^6\right)}\right]=\frac{0.005}{s}$$

$$\Rightarrow \qquad V_o=\frac{1.4s+20{,}000}{s^2+20{,}000s+10^8}$$

General Examples: 4 – cont.

- Let us check the answer using IVT and FVT:



IVT

$$V_0(s) = \frac{1.4s + 20{,}000}{s^2 + 20{,}000s + 10^8}$$

$$\lim_{t \to 0} v_0(t) = \lim_{s \to \infty} sF(s)$$

$$= \lim_{s \to \infty} \frac{1.4s^2 + 20{,}000s}{s^2 + 20{,}000s + 10^8}$$

$$= \lim_{1/s \to 0} \frac{1.4 + 20{,}000/s}{1 + 20{,}000/s + 10^8/s^2}$$

$$= 1.4\ \text{V}$$

FVT

$$V_0(s) = \frac{1.4s + 20{,}000}{s^2 + 20{,}000s + 10^8}$$

$$\lim_{t \to \infty} v_0(t) = \lim_{s \to 0} sF(s)$$

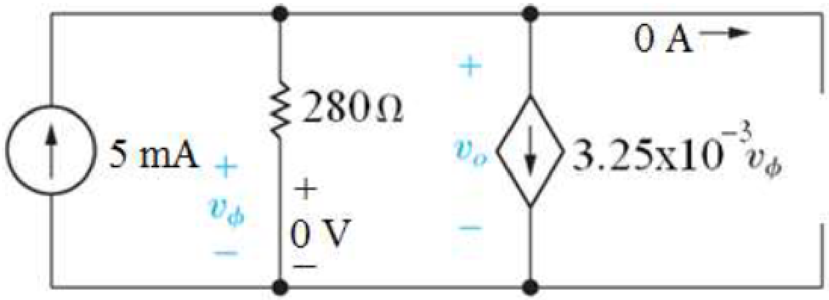
$$= \lim_{s \to 0} \frac{1.4s^2 + 20{,}000s}{s^2 + 20{,}000s + 10^8}$$

$$= \frac{0}{10^8} = 0\ \text{V}$$

General Examples: 4 – cont.

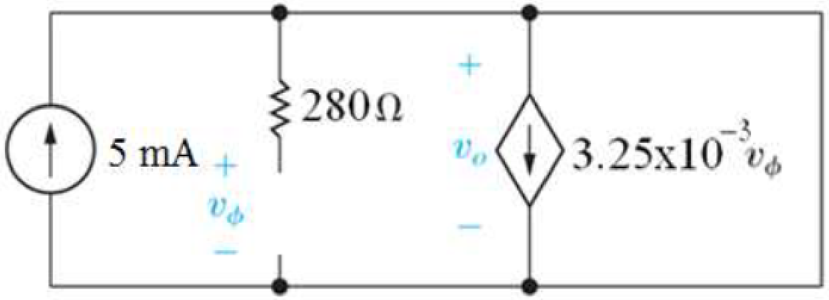
- To check the answers we got from IVT and FVT, let us consider what happens to the circuit at $t = 0$ and at $t = \infty$
  - Given that no initial energy was stored in the circuit, then at $\boldsymbol{t = 0}$ the **capacitor will behave like a short circuit** and the **inductor will behave like an open circuit**
  - At $\boldsymbol{t = \infty}$ the **capacitor will behave like an open circuit** and the **inductor will behave like a short circuit**

For t = 0



$$v_0(0) = (0.005)(280)$$
$$= 1.4 \text{ V (check!)}$$

For t $\rightarrow \infty$



$$v_0(0) = 0 \text{ V}$$
(it is the voltage across a wire!)

General Examples: 4 – cont.

$$V_0(s) = \frac{1.4s + 20{,}000}{(s+10{,}000)^2} = \frac{K_1}{(s+10{,}000)^2} + \frac{K_2}{(s+10{,}000)}$$

$$K_1 = 1.4s + 20{,}000\Big|_{s=-10{,}000} = 6000$$

$$K_2 = \frac{d}{ds}\left[1.4s + 20{,}000\right]\Big|_{s=-10{,}000} = 1.4$$

$$V_0(s) = \frac{6000}{(s+10{,}000)^2} + \frac{1.4}{(s+10{,}000)}$$

$v_0(t) = \left[6000te^{-10{,}000t} + 1.4e^{-10{,}000t}\right]u(t)$ V (see the Laplace tables)

$v_0(0) = 1.4$ V (check!)

$v_0(\infty) = 0$ V (check!)

Summary of Topics Covered in this Chapter

- In this chapter we discussed:
  - Circuit elements in the s domain
  - Circuit analysis in the s domain
  - Applications
  - The transfer function
  - The transfer function in partial fraction expansion
  - The impulse response
  - The transfer function and the steady-state sinusoidal response
- We covered sections 13.1 – 13.5, 13.7
- Next chapter (Ch14) we will talk about frequency selective circuits
  - Introduction to Frequency Selective Circuits