Chapter 11: Balanced Three-Phase Circuits

EEL 3112c – Circuits-II

Dr. Suleiman Alsweiss

ECE Department

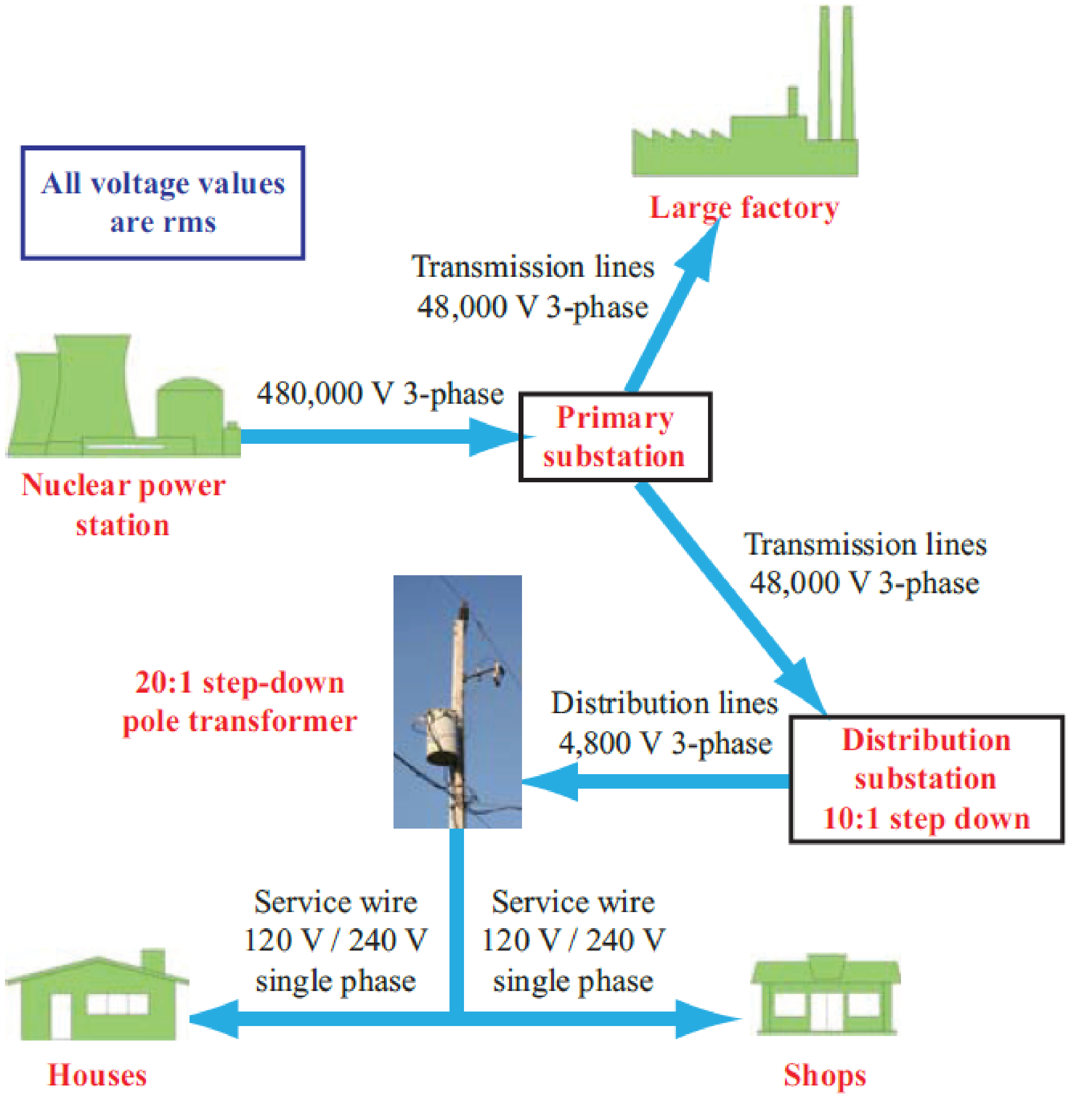
Florida Polytechnic University

Topics to be Covered in this Chapter

- In this chapter we will discuss:
  - Balanced three-phase voltages
  - Three-phase voltage sources
  - Analysis of Wye-Wye circuits
  - Analysis of Wye-Delta circuits
  - Power calculations in balanced three-phase circuits
- We will cover sections 11.1 – 11.5

Introduction

- Between the power generating station and a residence or shop, power is transferred across ***transmission lines*** in a form known as ***three-phase power***
- What is three-phase power and why is it used?
- The intent of the present chapter is to answer these questions and to provide the tools for analyzing ***three-phase*** networks



Introduction

- **Generating, transmitting, distributing, and using** large blocks of electric power is accomplished with three-phase circuits
- The comprehensive analysis of such systems is a field of study in its own right
- For economic reasons, three-phase systems are usually designed to operate in the balanced state
  - Thus, in this introductory treatment, we can justify considering only balanced circuits
- The analysis of unbalanced three-phase circuits, which you will encounter if you study electric power in later courses, relies heavily on an understanding of balanced circuits

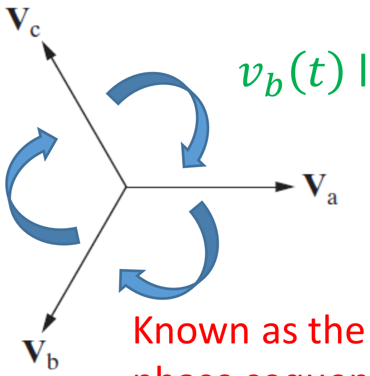
Why 3-Phase Circuits?

- Single-phase circuits are generally used to power residential homes and smaller business buildings
- Three-phase circuits are common in large industrial buildings that require a more significant amount of electrical power
- The advantage of implementing a three-phase circuit is cost-savings
  - Three-phase circuits are more efficient when it comes to transferring of electrical power; therefore, they are capable of powering much more at a lower cost
- In theory we can have more than three phases for electric circuits
  - 6- or 9- phases are possible
- There is no real engineering advantage of having more than 3 phases
  - It actually adds complexity to the design without adding much to the efficiency
  - It's a lot like how tripods have clear advantages over monopods, but a 6-legged, or 9-legged device isn't going to do anything useful for your photography

Balanced Three-Phase Voltages

- A set of **balanced** three-phase voltages consists of three sinusoidal voltages that have identical amplitudes and frequencies but are out of phase with each other by **exactly 120°**
  - Standard practice is to refer to the three phases as **a, b, and c**, and to use the a-phase as the reference phase
  - The three voltages are referred to as the **a-phase voltage**, the **b-phase voltage**, and the **c-phase voltage**
- Only two possible phase relationships can exist between the a-phase voltage and the b- and c-phase voltages

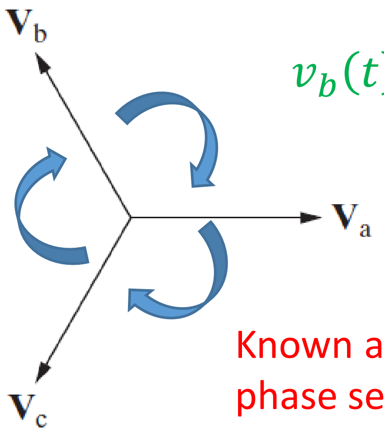
$$\mathbf{V}_a = V_m \angle 0^\circ,$$
$$\mathbf{V}_b = V_m \angle -120^\circ,$$
$$\mathbf{V}_c = V_m \angle +120^\circ,$$



$v_b(t)$ lags $v_a(t)$

Known as the **abc** (or +ve) phase sequence

$$\mathbf{V}_a = V_m \angle 0^\circ,$$
$$\mathbf{V}_b = V_m \angle +120^\circ,$$
$$\mathbf{V}_c = V_m \angle -120^\circ.$$



$v_b(t)$ leads $v_a(t)$

Known as the **acb** (or -ve) phase sequence

The phase sequence is the clockwise order of subscripts around V_a

Balanced Three-Phase Voltages – cont.

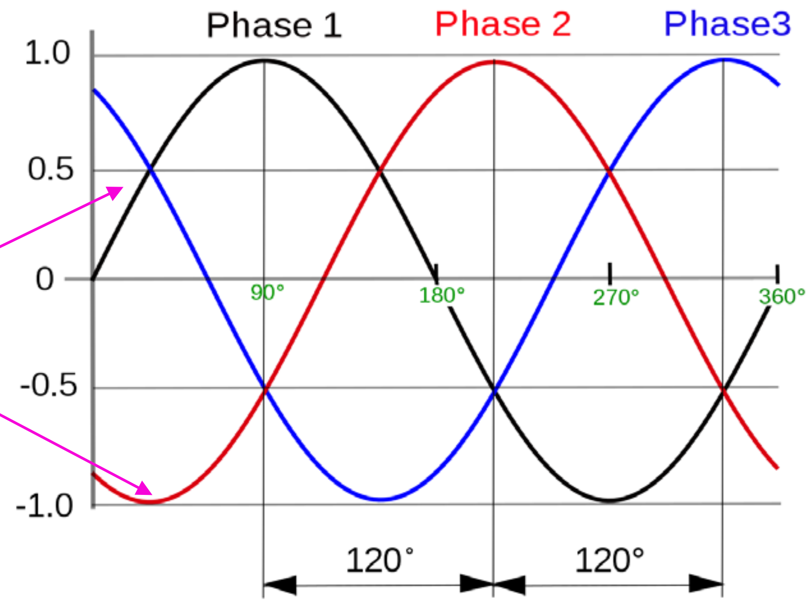
- An important note about **balanced three-phase** voltages is that the sum of the voltages is zero by design

$$\mathbf{V}_\mathrm{a} + \mathbf{V}_\mathrm{b} + \mathbf{V}_\mathrm{c} = 0.$$

- Because the sum of the phasor voltages is zero, then the sum of the instantaneous voltages is also zero

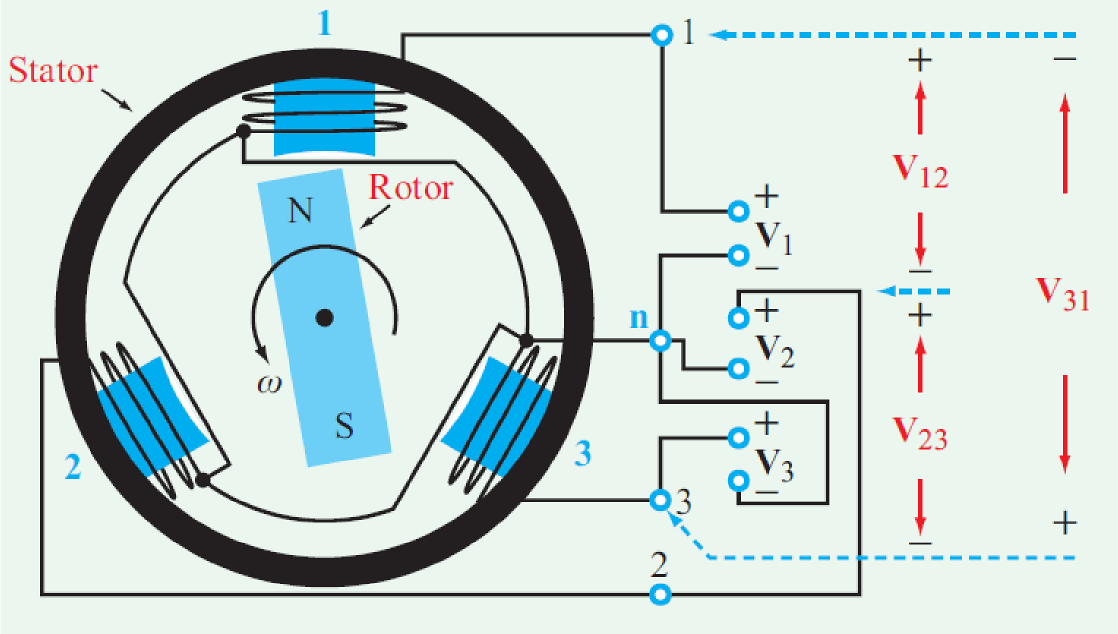
$$v_\mathrm{a} + v_\mathrm{b} + v_\mathrm{c} = 0.$$

0.5 + 0.5 + (-1) = 0



Balanced Three-Phase Generators

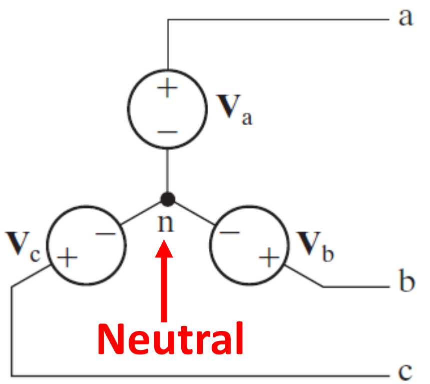
- The generator consists of a rotating electromagnet, called the ***rotor***, and three separate stationary coils distributed evenly around a circular tube called the ***stator***
  - The rotor is spun around by a turbine or some other external force
  - The three coils are arranged 120° apart over the circumference of the stator
- As the electromagnet rotates, its magnetic field induces a sinusoidal voltage at the terminals of each of the three coils
  - The coils are stationary with respect to the rotating electromagnet, so the frequency of the voltage induced in each winding is the same



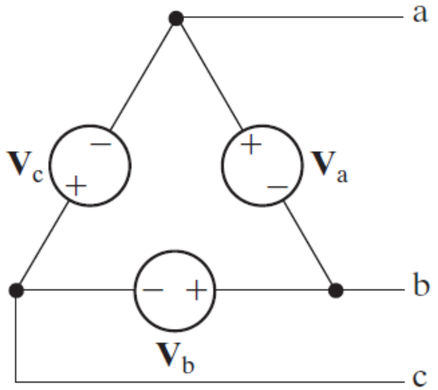
Three-Phase Voltage Sources – cont.

- There are two ways of interconnecting the separate phase windings to form a three-phase source
  - The wye (**Y**) or delta (**Δ**) configuration

Y-connected source



Δ-connected source



If the impedance of each phase winding is not negligible, we place the winding impedance (usually inductive) in series with an ideal sinusoidal voltage source

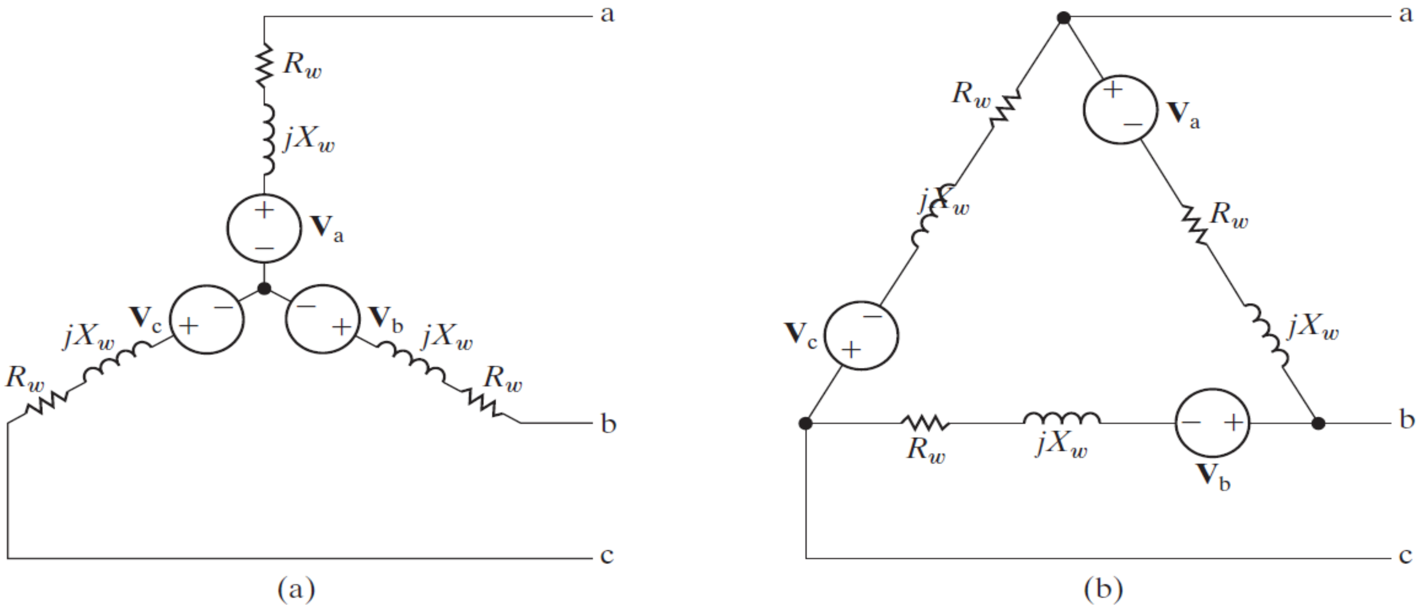


Figure 11.5 ▲ A model of a three-phase source with winding impedance: (a) a Y-connected source; and (b) a Δ-connected source.

Three-Phase Voltage Sources – cont.

- Because three-phase sources and loads can be either Y-connected or Δ-connected, the basic circuit in Fig. 11.1 represents four different configurations:

Source	Load
Y	Y
Y	Δ
Δ	Y
Δ	Δ

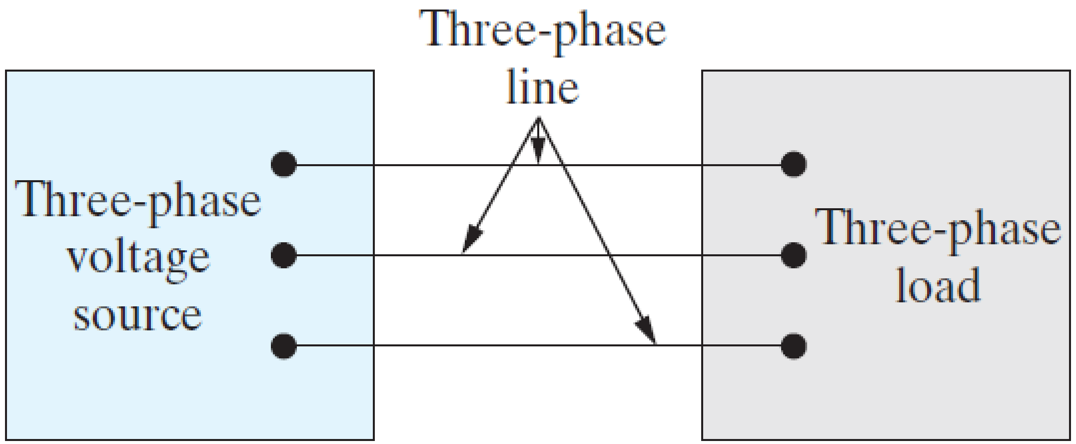


Figure 11.1 ▲ A basic three-phase circuit.

- We begin by analyzing the **Y-Y** circuit
- The remaining three arrangements can be reduced to a **Y-Y** equivalent circuit, so analysis of the **Y-Y** circuit is the key to solving all balanced three-phase arrangements

Analysis of the Wye-Wye Circuit

- This figure illustrates a general Y-Y circuit
  - We included a fourth conductor that connects the source neutral to the load neutral

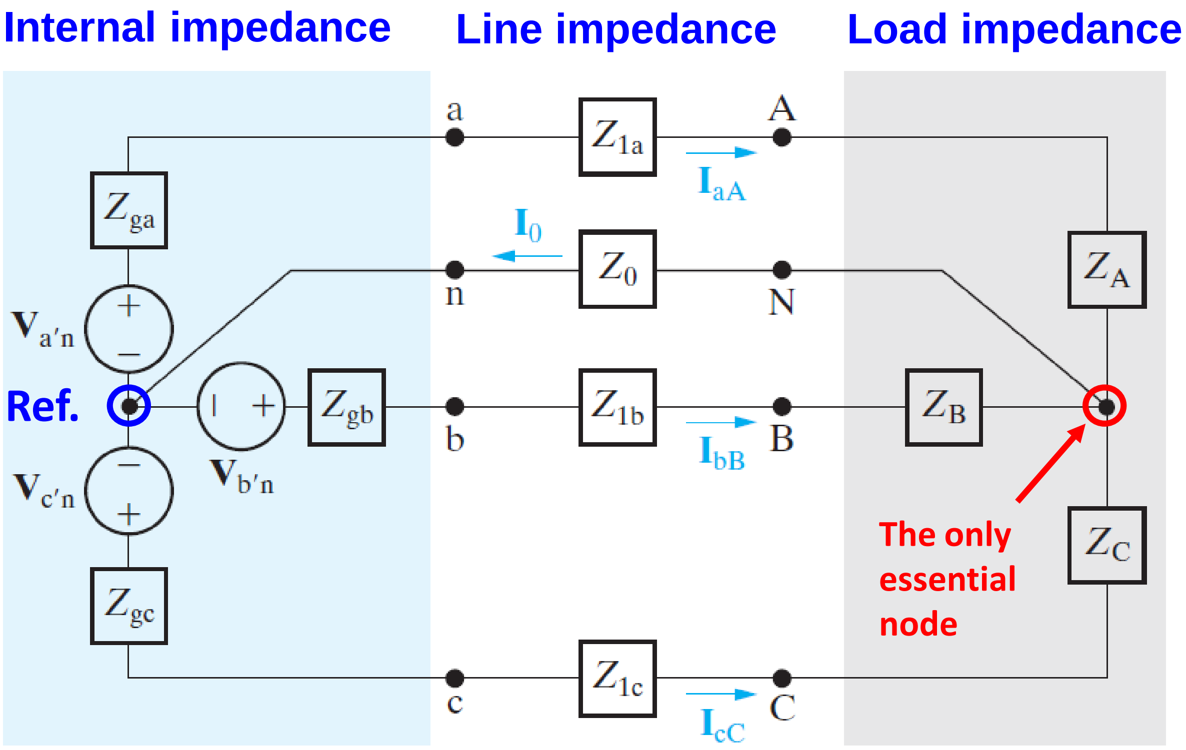


Figure 11.6 ▲ A three-phase Y-Y system.

- The node-voltage method uses the **essential nodes** of the circuit
- Essential node is a node where three or more circuit elements join

- We can describe this circuit with a single node-voltage equation
- Using the **source neutral as the reference node** and letting $\mathbf{V}_N$ denote the node voltage between the nodes **N** and **n**, we find that the node-voltage equation is

$$\mathbf{I}_0 = \mathbf{I}_{aA} + \mathbf{I}_{bB} + \mathbf{I}_{cC}, \Rightarrow$$

$$\frac{\mathbf{V}_N}{Z_0} = \frac{\mathbf{V}_{a'n} - \mathbf{V}_N}{Z_{ga} + Z_{1a} + Z_A} + \frac{\mathbf{V}_{b'n} - \mathbf{V}_N}{Z_{gb} + Z_{1b} + Z_B} + \frac{\mathbf{V}_{c'n} - \mathbf{V}_N}{Z_{gc} + Z_{1c} + Z_C}$$

- It can be rearranged as:

$$\boxed{\frac{\mathbf{V}_N}{Z_0} + \frac{\mathbf{V}_N - \mathbf{V}_{a'n}}{Z_A + Z_{1a} + Z_{ga}} + \frac{\mathbf{V}_N - \mathbf{V}_{b'n}}{Z_B + Z_{1b} + Z_{gb}} + \frac{\mathbf{V}_N - \mathbf{V}_{c'n}}{Z_C + Z_{1c} + Z_{gc}} = 0.}$$

- This is the general equation for any circuit of the **Y-Y** configuration

Analysis of the Wye-Wye Circuit – cont.

- The general equation described in previous slide can be simplified if we consider the following from the definition of a **balanced three-phase circuit**

1. The voltage sources form a set of balanced three-phase voltages. In Fig. 11.6, this means that $\mathbf{V}_{a'n}$, $\mathbf{V}_{b'n}$, and $\mathbf{V}_{c'n}$ are a set of balanced three-phase voltages.
2. The impedance of each phase of the voltage source is the same. In Fig. 11.6, this means that $Z_{ga} = Z_{gb} = Z_{gc}$.
3. The impedance of each line (or phase) conductor is the same. In Fig. 11.6, this means that $Z_{1a} = Z_{1b} = Z_{1c}$.
4. The impedance of each phase of the load is the same. In Fig. 11.6, this means that $Z_A = Z_B = Z_C$.

◀ Conditions for a balanced three-phase circuit

- There is no restriction on the impedance of a neutral conductor (Z_0); its value has no effect on whether the system is balanced

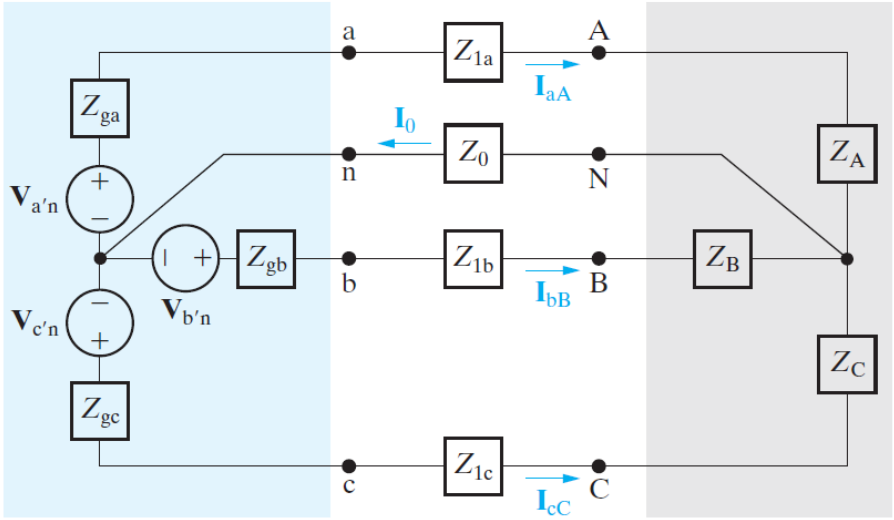


Figure 11.6 ▲ A three-phase Y-Y system.

Analysis of the Wye-Wye Circuit – cont.

- Thus we may rewrite the node-voltage equation of the balanced three-phase circuit as follows:

$$\mathbf{V}_{\mathrm{N}}\left(\frac{1}{Z_0}+\frac{3}{Z_\phi}\right)=\frac{\mathbf{V}_{\mathrm{a'n}}+\mathbf{V}_{\mathrm{b'n}}+\mathbf{V}_{\mathrm{c'n}}}{Z_\phi}, \qquad (11.6)$$

where

$$Z_\phi = Z_{\mathrm{A}} + Z_{\mathrm{1a}} + Z_{\mathrm{ga}} = Z_{\mathrm{B}} + Z_{\mathrm{1b}} + Z_{\mathrm{gb}} = Z_{\mathrm{C}} + Z_{\mathrm{1c}} + Z_{\mathrm{gc}}.$$

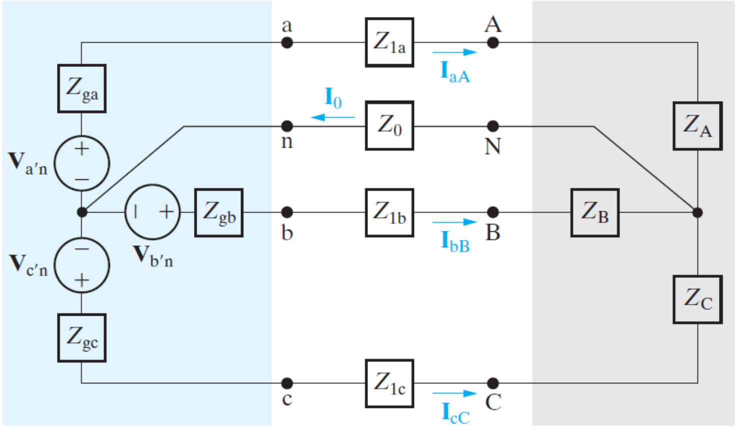


Figure 11.6 ▲ A three-phase Y-Y system.

- The right-hand side of Eq. 11.6 is zero, because by hypothesis the numerator is a set of balanced three-phase voltages (summation equals zero) and Z_Φ is not zero
  - The only possible value of $\mathbf{V}_{\mathrm{N}}$ that satisfies Eq. 11.6 is $\mathbf{V}_{\mathrm{N}} = \mathbf{0}$
- Thus, if $\mathbf{V}_{\mathrm{N}}$ is zero, there is no difference in potential between the source neutral node **n**, and the load neutral node **N**
  - Consequently, the current in the neutral conductor is zero

Neutral line is both short ($v = 0$) and open ($i = 0$).

Analysis of the Wye-Wye Circuit – cont.

- We now turn to the effect that balanced conditions have on the three line currents

$\mathbf{I}_{aA}, \mathbf{I}_{bB}, \mathbf{I}_{cC}$ are called the line currents

$$\mathbf{I}_{aA} = \frac{\mathbf{V}_{a'n} - \mathbf{V}_N}{Z_A + Z_{1a} + Z_{ga}} = \frac{\mathbf{V}_{a'n}}{Z_\phi},$$

$$\mathbf{I}_{bB} = \frac{\mathbf{V}_{b'n} - \mathbf{V}_N}{Z_B + Z_{1b} + Z_{gb}} = \frac{\mathbf{V}_{b'n}}{Z_\phi},$$

$$\mathbf{I}_{cC} = \frac{\mathbf{V}_{c'n} - \mathbf{V}_N}{Z_C + Z_{1c} + Z_{gc}} = \frac{\mathbf{V}_{c'n}}{Z_\phi}.$$

$$\mathbf{I}_o = \mathbf{I}_{aA} + \mathbf{I}_{bB} + \mathbf{I}_{cC}.$$

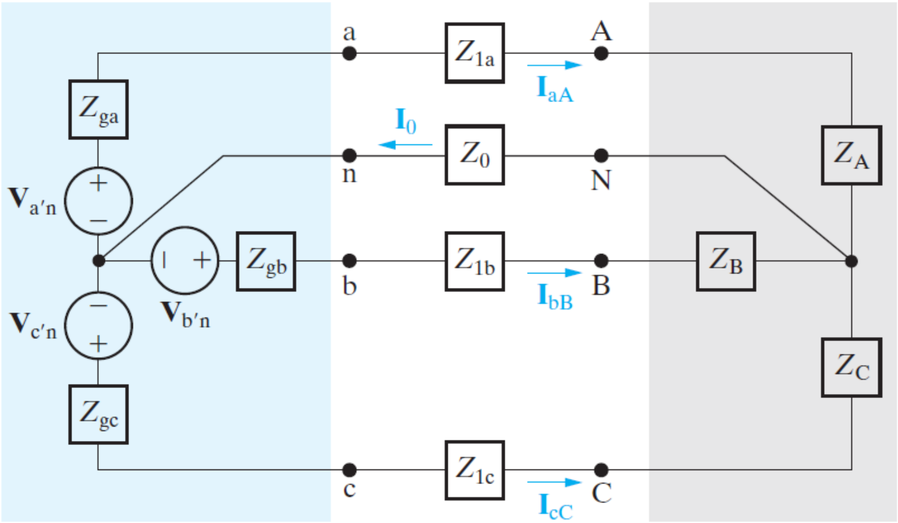


Figure 11.6 ▲ A three-phase Y-Y system.

- We see that the three line currents form a balanced set of three-phase currents
  - The current in each line is equal in amplitude and frequency and is 120° out of phase with the other two line currents
  - Thus, if we calculate the current $\mathbf{I}_{aA}$ and we know the phase sequence, we have a shortcut for finding $\mathbf{I}_{bB}$ and $\mathbf{I}_{cC}$
    - Same thing applies to find the b- and c-phase source voltages from the a-phase source voltage

Analysis of the Wye-Wye Circuit – cont.

- We can use the line current $\mathbf{I}_{aA}$ equation to construct an equivalent circuit for the a-phase of the balanced Y-Y circuit
  - From this equation, the current in the a-phase conductor line is simply the voltage generated in the a-phase winding of the generator divided by the total impedance in the a-phase of the circuit
- Thus we can now construct the **single-phase equivalent circuit** of a balanced three phase circuit as shown below

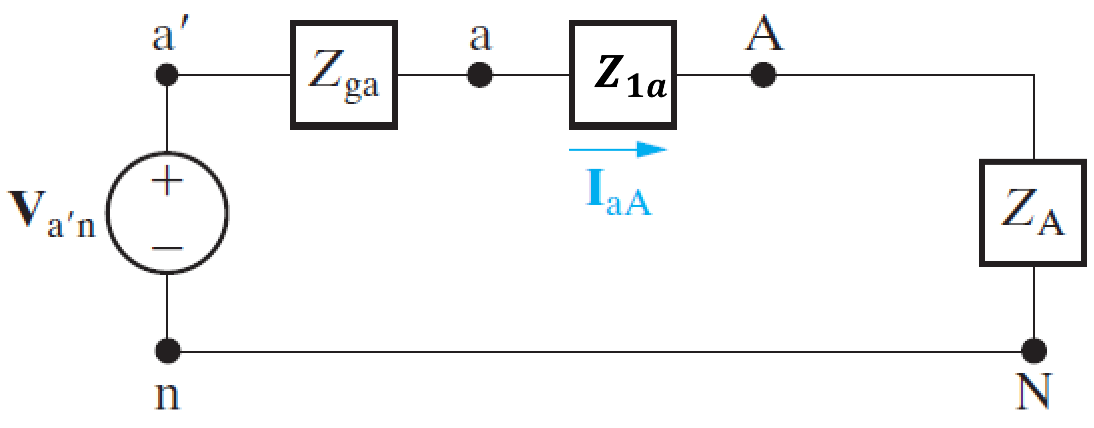


Figure 11.7 ▲ A single-phase equivalent circuit.

- Note that the neutral conductor has been replaced by a short circuit because as we discussed earlier $\mathbf{V}_n = \mathbf{V}_N$ which makes them the same node
- Because of the established relationships between phases, once we solve this circuit, we can easily write down the voltages and currents in the other two phases
- Just note that the current in the neutral conductor of the single-phase equivalent circuit is $\mathbf{I}_{aA}$ while the current in the neutral conductor of the three-phase circuit is $\mathbf{I}_0$

Analysis of the Wye-Wye Circuit – cont.

- Once we know the line current in Fig. 11.7, calculating any voltages of interest is relatively simple
- Also, of particular interest is the relationship between the line-to-line voltages ($\mathbf{V}_{\mathrm{AB}}$, $\mathbf{V}_{\mathrm{BC}}$, and $\mathbf{V}_{\mathrm{CA}}$) and the line-to-neutral voltages ($\mathbf{V}_{\mathrm{AN}}$, $\mathbf{V}_{\mathrm{BN}}$, and $\mathbf{V}_{\mathrm{CN}}$)
  - We establish this relationship at the load terminals, but our observations also apply at the source terminals

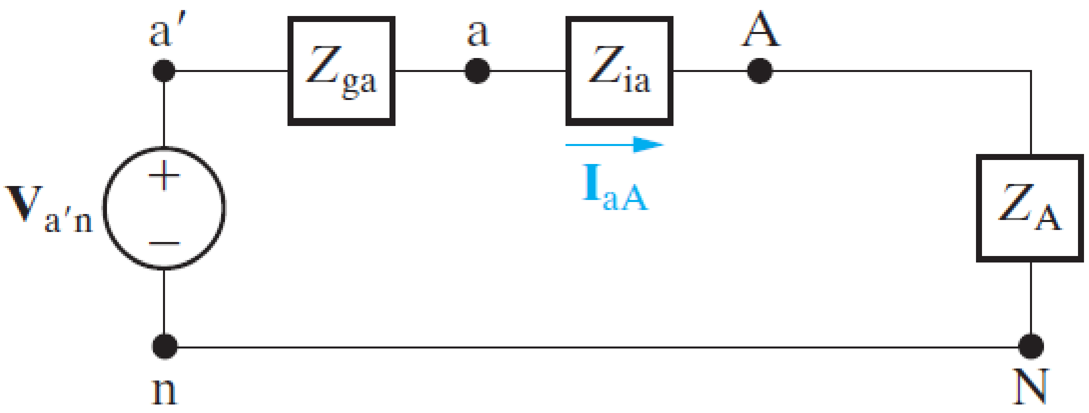


Figure 11.7 ▲ A single-phase equivalent circuit.

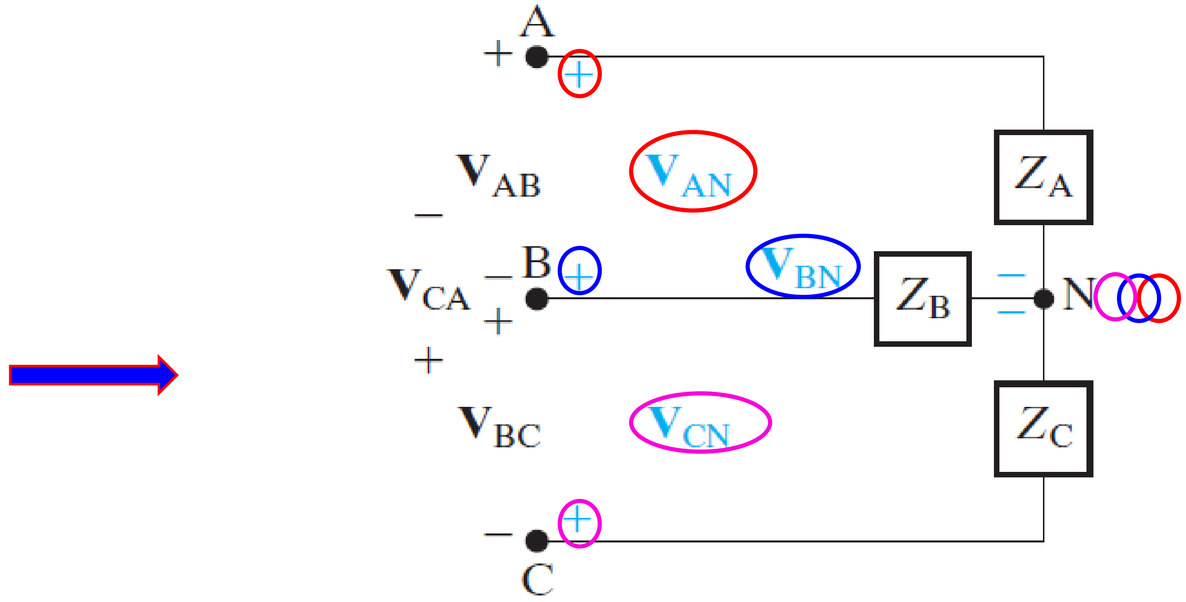


Figure 11.8 ▲ Line-to-line and line-to-neutral voltages.

Analysis of the Wye-Wye Circuit – cont.

$$\mathbf{V}_{AN} = V_\phi \angle 0^\circ,$$

$$\mathbf{V}_{BN} = V_\phi \angle -120^\circ,$$

$$\mathbf{V}_{CN} = V_\phi \angle +120^\circ,$$

A balanced three-phase circuit with +ve, or *abc*, sequence

&

$$\mathbf{V}_{AB} = \mathbf{V}_{AN} - \mathbf{V}_{BN},$$

$$\mathbf{V}_{BC} = \mathbf{V}_{BN} - \mathbf{V}_{CN},$$

$$\mathbf{V}_{CA} = \mathbf{V}_{CN} - \mathbf{V}_{AN}.$$

The double subscript notation indicates a voltage drop from the first named node to the second

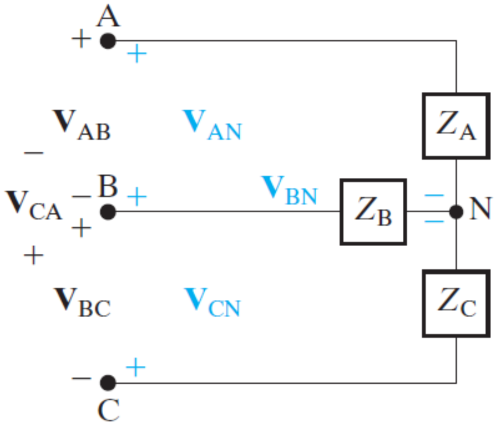


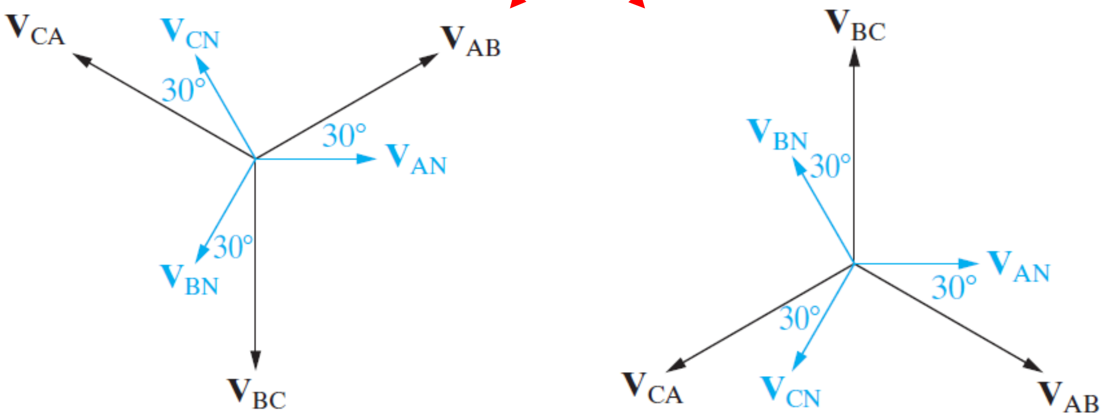
Figure 11.8 ▲ Line-to-line and line-to-neutral voltages.

$$\mathbf{V}_{AB} = V_\phi \angle 0^\circ - V_\phi \angle -120^\circ = \sqrt{3}V_\phi \angle 30^\circ,$$

$$\mathbf{V}_{BC} = V_\phi \angle -120^\circ - V_\phi \angle 120^\circ = \sqrt{3}V_\phi \angle -90^\circ,$$

$$\mathbf{V}_{CA} = V_\phi \angle 120^\circ - V_\phi \angle 0^\circ = \sqrt{3}V_\phi \angle 150^\circ.$$

Note that for +ve phase sequence the line voltages leads the phase voltages, while for –ve phase sequence, line voltages lag the phase voltages



Phasor diagram showing line-to-line and line-to-neutral voltages for a balanced system. (a) *abc sequence*, (b) *acb sequence*

Analysis of the Wye-Wye Circuit – cont.

- Proof: $\boxed{\mathbf{V}_{AB} = V_\phi \angle 0^\circ - V_\phi \angle -120^\circ = \sqrt{3} V_\phi \angle 30^\circ,}$

$$V_{AB} = V_\phi \angle 0^\circ - V_\phi \angle -120^\circ$$

$$= V_\phi \cos(0^\circ) + j V_\phi \sin(0^\circ) - [V_\phi \cos(-120^\circ) + j \overset{(V_\phi)}{} \sin(-120^\circ)]$$

$\circledast$ $\cos(0^\circ) = 1$, $\sin(0^\circ) = 0$.

$\circledast$ $\cos(-120^\circ) = -\frac{1}{2}$, $\sin(-120^\circ) = -\frac{\sqrt{3}}{2}$

$$\Rightarrow V_{AB} = V_\phi (1) + j \underbrace{\cancel{V_\phi (0)}}_{0} + V_\phi \left(\tfrac{1}{2}\right) + j V_\phi \frac{\sqrt{3}}{2}$$

$$V_{AB} = V_\phi + 0 + \frac{V_\phi}{2} + j V_\phi \frac{\sqrt{3}}{2} = \frac{3 V_\phi}{2} + j V_\phi \frac{\sqrt{3}}{2}$$

$\circledast$ change it to polar format;

$$r = \sqrt{\left(\tfrac{3}{2} V_\phi\right)^2 + \left(\frac{V_\phi \sqrt{3}}{2}\right)^2} = \sqrt{3}\, V_\phi$$

$$\theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2} V_\phi}{\frac{V_\phi}{2}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

$$\Rightarrow \boxed{V_{AB} = \sqrt{3}\, V_\phi \angle 30^\circ}$$

Analysis of the Wye-Wye Circuit – cont.

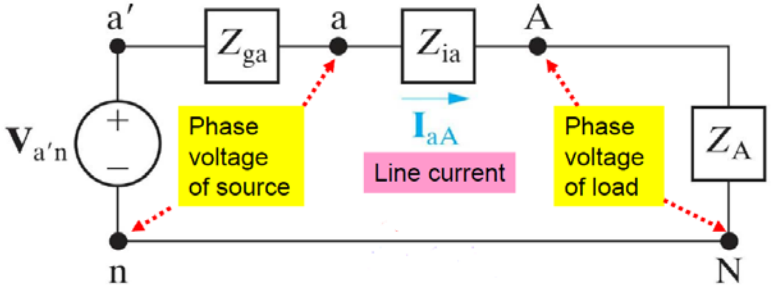
- From the set of equations in previous slide, we can cay that:
  - The magnitude of the line-to-line voltage is $\sqrt{3}$ times the magnitude of the line-to-neutral voltage
  - The line-to-line voltages form a balanced three-phase set of voltages
  - The set of line-to-line voltages leads the set of line-to-neutral voltages by 30°

$$\mathbf{V}_{AB} = V_\phi \angle 0^\circ - V_\phi \angle -120^\circ = \sqrt{3} V_\phi \angle 30^\circ,$$

$$\mathbf{V}_{BC} = V_\phi \angle -120^\circ - V_\phi \angle 120^\circ = \sqrt{3} V_\phi \angle -90^\circ,$$

$$\mathbf{V}_{CA} = V_\phi \angle 120^\circ - V_\phi \angle 0^\circ = \sqrt{3} V_\phi \angle 150^\circ.$$

- Summary of terminologies before we continue:
  - **Line voltage** refers to the voltage across any pair of lines
  - **Phase voltage** refers to the voltage across a single phase
  - **Line current** refers to the current in a single line
  - **Phase current** refers to current in a single phase



- In a **Y**-connection, line current and phase current are identical
- Finally, the Greek letter phi (Φ) is widely used in the literature to denote a per-phase quantity ($\mathbf{V}_\phi$, $\mathbf{I}_\phi$, Z_ϕ, P_ϕ, and Q_ϕ)

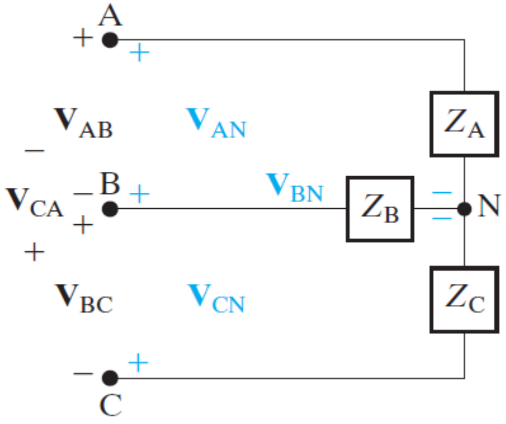


Figure 11.8 ▲ Line-to-line and line-to-neutral voltages.

Example 11.1

A balanced three-phase Y-connected generator with positive sequence has an impedance of $0.2 + j0.5\ \Omega/\phi$ and an internal voltage of $120\ V/\phi$. The generator feeds a balanced three-phase Y-connected load having an impedance of $39 + j28\ \Omega/\phi$. The impedance of the line connecting the generator to the load is $0.8 + j1.5\ \Omega/\phi$. The a-phase internal voltage of the generator is specified as the reference phasor.

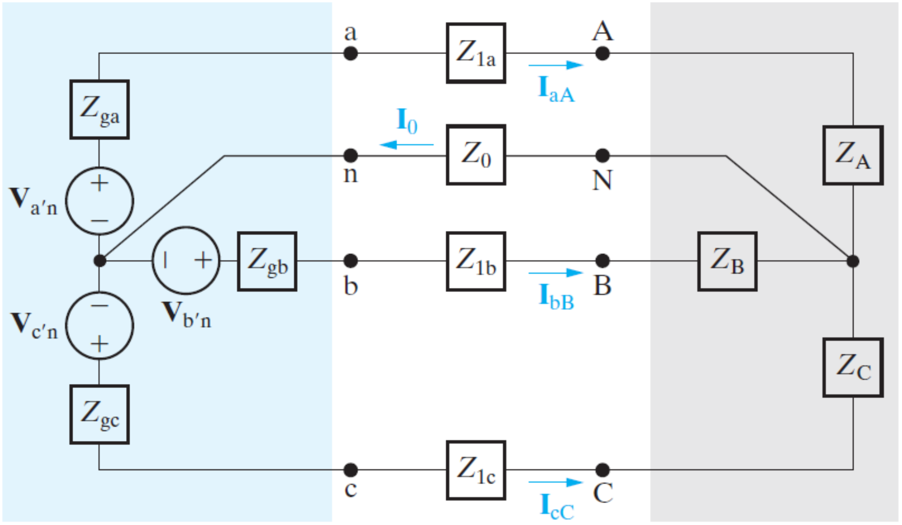


Figure 11.6 ▲ A three-phase Y-Y system.

a) Construct the a-phase equivalent circuit of the system.

b) Calculate the three line currents $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$.

c) Calculate the three phase voltages at the load, $\mathbf{V}_{AN}$, $\mathbf{V}_{BN}$, and $\mathbf{V}_{CN}$.

d) Calculate the line voltages $\mathbf{V}_{AB}$, $\mathbf{V}_{BC}$, and $\mathbf{V}_{CA}$ at the terminals of the load.

e) Calculate the phase voltages at the terminals of the generator, $\mathbf{V}_{an}$, $\mathbf{V}_{bn}$, and $\mathbf{V}_{cn}$.

f) Calculate the line voltages $\mathbf{V}_{ab}$, $\mathbf{V}_{bc}$, and $\mathbf{V}_{ca}$ at the terminals of the generator.

g) Repeat (a)–(f) for a negative phase sequence.

Example 11.1 – cont.

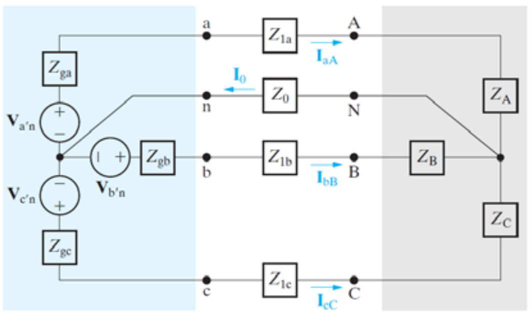


Figure 11.6 ▲ A three-phase Y-Y system.

Solution

a) Figure 11.10 shows the single-phase equivalent circuit.

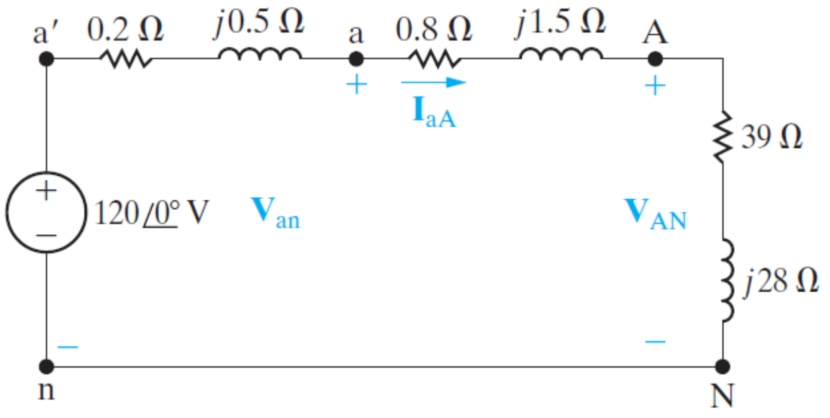


Figure 11.10 ▲ The single-phase equivalent circuit for Example 11.1.

b) The a-phase line current is

$$\mathbf{I}_{aA} = \frac{120\,\angle 0°}{(0.2 + 0.8 + 39) + j(0.5 + 1.5 + 28)}$$

$$= \frac{120\,\angle 0°}{40 + j30}$$

$$= 2.4\,\angle -36.87° \text{ A}.$$

For a positive phase sequence,

$$\mathbf{I}_{bB} = 2.4\,\angle -156.87° \text{ A},$$

$$\mathbf{I}_{cC} = 2.4\,\angle 83.13° \text{ A}.$$

c) The phase voltage at the A terminal of the load is

$$\mathbf{V}_{AN} = (39 + j28)(2.4\,\angle -36.87°)$$

$$= 115.22\,\angle -1.19° \text{ V}.$$

$$\mathbf{V}_{AN} = \mathbf{I}_{aA} \times Z_A$$

For a positive phase sequence,

$$\mathbf{V}_{BN} = 115.22\,\angle -121.19° \text{ V},$$

$$\mathbf{V}_{CN} = 115.22\,\angle 118.81° \text{ V}.$$

d) For a positive phase sequence, the line voltages lead the phase voltages by 30°; thus

$$\mathbf{V}_{AB} = (\sqrt{3}\,\angle 30°)\mathbf{V}_{AN}$$

$$= 199.58\,\angle 28.81° \text{ V},$$

$$\mathbf{V}_{BC} = 199.58\,\angle -91.19° \text{ V},$$

$$\mathbf{V}_{CA} = 199.58\,\angle 148.81° \text{ V}.$$

Example 11.1 – cont.

e) The phase voltage at the a terminal of the source is

$$\mathbf{V}_{an} = 120 - (0.2 + j0.5)(2.4\,\angle{-36.87^\circ})$$
$$= 120 - 1.29\,\angle{31.33^\circ}$$
$$= 118.90 - j0.67$$
$$= 118.90\,\angle{-0.32^\circ}\ \text{V}.$$

$$\mathbf{V}_{an} = \mathbf{V}_{\phi} - \mathbf{I}_{aA} \times (\mathbf{Z}_{ga})$$

For a positive phase sequence,

$$\mathbf{V}_{bn} = 118.90\,\angle{-120.32^\circ}\ \text{V},$$
$$\mathbf{V}_{cn} = 118.90\,\angle{119.68^\circ}\ \text{V}.$$

f) The line voltages at the source terminals are

$$\mathbf{V}_{ab} = (\sqrt{3}\,\angle{30^\circ})\mathbf{V}_{an}$$
$$= 205.94\,\angle{29.68^\circ}\ \text{V},$$
$$\mathbf{V}_{bc} = 205.94\,\angle{-90.32^\circ}\ \text{V},$$
$$\mathbf{V}_{ca} = 205.94\,\angle{149.68^\circ}\ \text{V}.$$

Repeat previous analysis for Negative phase sequence

g) Changing the phase sequence has no effect on the single-phase equivalent circuit. The three line currents are

$$\mathbf{I}_{aA} = 2.4\,\angle{-36.87^\circ}\ \text{A},$$
$$\mathbf{I}_{bB} = 2.4\,\angle{83.13^\circ}\ \text{A},$$
$$\mathbf{I}_{cC} = 2.4\,\angle{-156.87^\circ}\ \text{A}.$$

The phase voltages at the load are

$$\mathbf{V}_{AN} = 115.22\,\angle{-1.19^\circ}\ \text{V},$$
$$\mathbf{V}_{BN} = 115.22\,\angle{118.81^\circ}\ \text{V},$$
$$\mathbf{V}_{CN} = 115.22\,\angle{-121.19^\circ}\ \text{V}.$$

For a negative phase sequence, the line voltages lag the phase voltages by 30°:

$$\mathbf{V}_{AB} = (\sqrt{3}\,\angle{-30^\circ})\mathbf{V}_{AN}$$
$$= 199.58\,\angle{-31.19^\circ}\ \text{V},$$
$$\mathbf{V}_{BC} = 199.58\,\angle{88.81^\circ}\ \text{V},$$
$$\mathbf{V}_{CA} = 199.58\,\angle{-151.19^\circ}\ \text{V}.$$

Note the difference between +ve and –ve phase sequence

Example 11.1 – cont.

The phase voltages at the terminals of the generator are

$$\mathbf{V}_{an} = 118.90\,\underline{/-0.32^\circ}\text{ V},$$

$$\mathbf{V}_{bn} = 118.90\,\underline{/119.68^\circ}\text{ V},$$

$$\mathbf{V}_{cn} = 118.90\,\underline{/-120.32^\circ}\text{ V}.$$

The line voltages at the terminals of the generator are

$$\mathbf{V}_{ab} = (\sqrt{3}\,\underline{/-30^\circ})\mathbf{V}_{an}$$

$$= 205.94\,\underline{/-30.32^\circ}\text{ V},$$

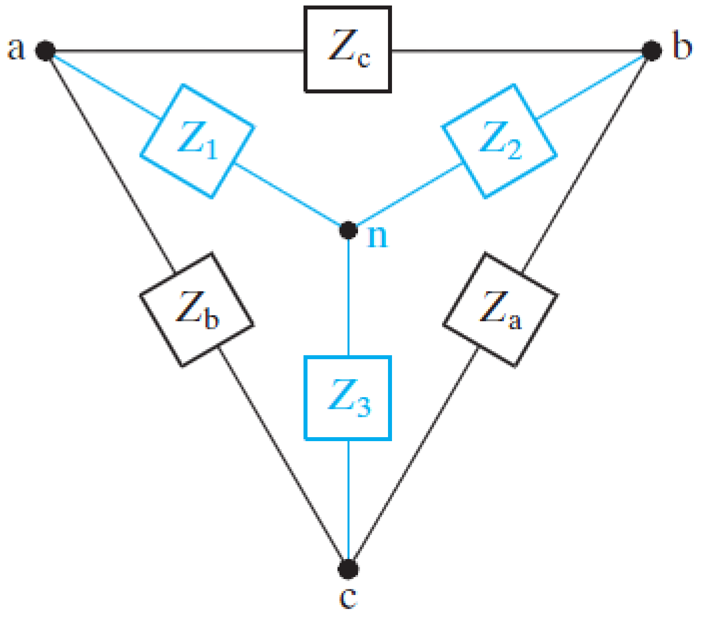
$$\mathbf{V}_{bc} = 205.94\,\underline{/89.68^\circ}\text{ V},$$

$$\mathbf{V}_{ca} = 205.94\,\underline{/-150.32^\circ}\text{ V}.$$

Analysis of the Wye-Delta Circuit – cont.

- If the load in a three-phase circuit is connected in a **delta**, it can be transformed into a wye by using the delta-to-wye transformation discussed in **Section 9.6**
- When the load is balanced, the impedance of each leg of the wye is one third the impedance of each leg of the delta

From section 9.6, we know that:



From $\Delta - to - Y$:

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c},$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c},$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}.$$

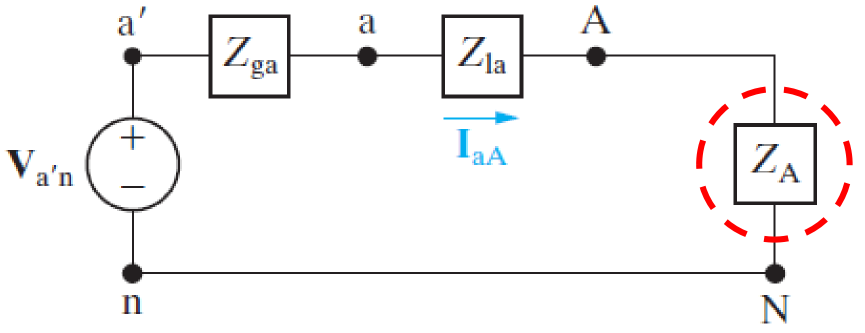
Because it is a balanced 3-phase circuit, $Z_a = Z_b = Z_c$

$$Z_Y = \frac{(Z_\Delta \times Z_\Delta)}{Z_\Delta + Z_\Delta + Z\_\Delta}$$

$$Z_Y = \frac{Z_\Delta^2}{3Z_\Delta} \longrightarrow \boxed{Z_Y = \frac{Z_\Delta}{3},}$$

Analysis of the Wye-Delta Circuit – cont.

- After the load has been replaced by its **Y** equivalent, and the source is already **Y** connected, the a-phase can be modeled by the single phase equivalent circuit shown below



$$\boldsymbol{Z_A = \frac{Z_\Delta}{3}}$$

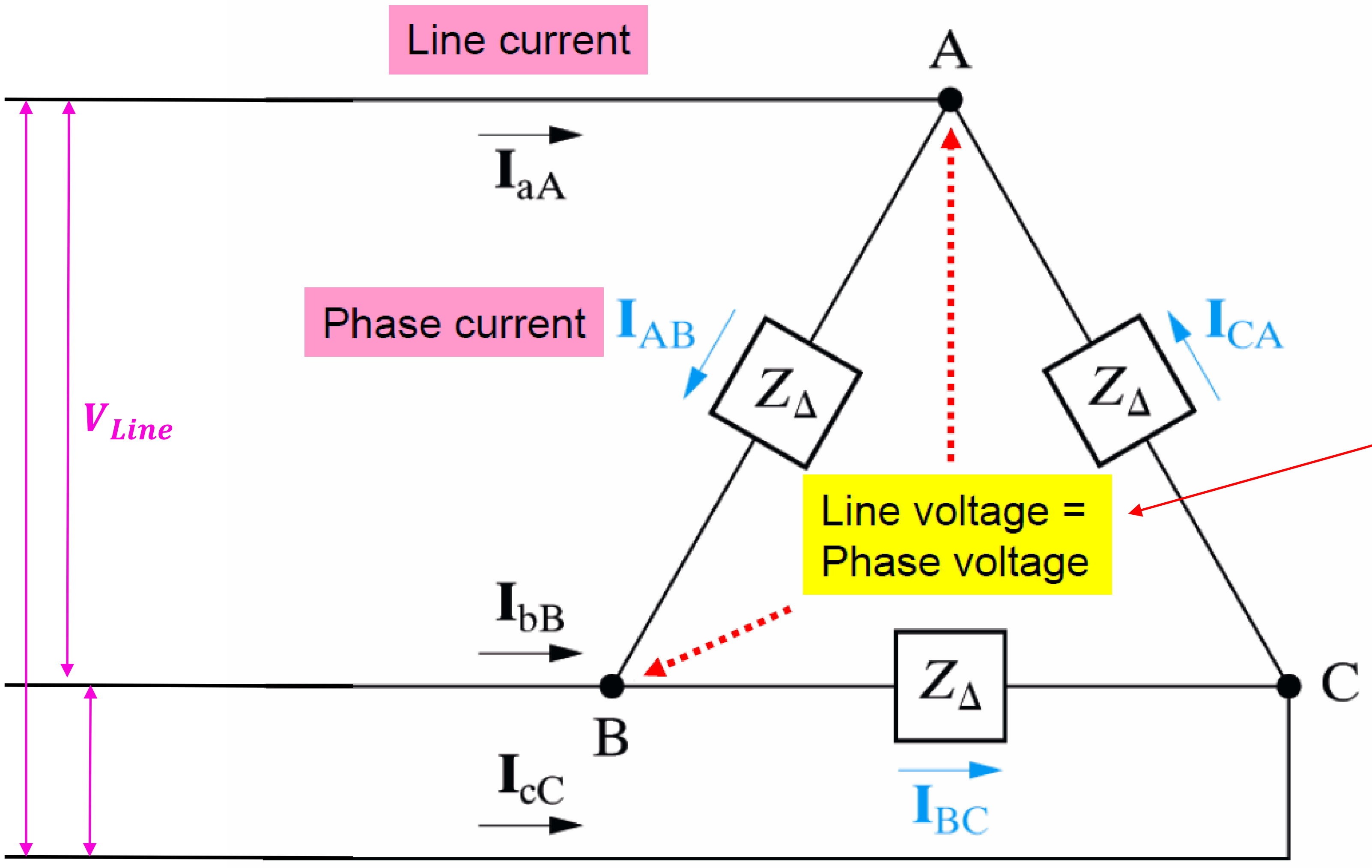
Figure 11.11 ▲ A single-phase equivalent circuit.

- We use this circuit to calculate the line currents ($\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$)
  - Then we can use the line currents to find the current in each leg of the original Δ load

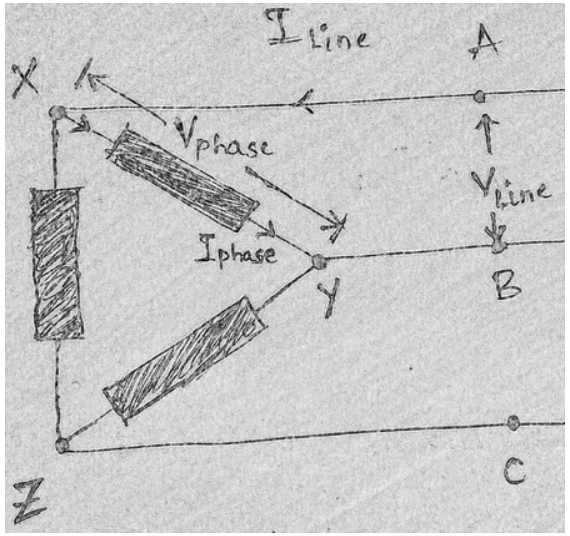
$$\boxed{\mathbf{I}_{aA} = \frac{\mathbf{V}_{a'n}}{Z_{ga} + Z_{1a} + Z_A}}$$

$$\boxed{\mathbf{V}_{AN} = \mathbf{I}_{aA} Z_A}$$

Analysis of the Wye-Delta Circuit



By design, in a delta system, the two points required to measure a line voltage also happen to be the same two points connected across the phase.



Analysis of the Wye-Delta Circuit – cont.

- The relationship between the line currents and the currents in each leg of the delta connected load can be derived using the circuit shown below
  - When a load is connected in Δ, the current in each leg of the Δ is the phase current, and the voltage across each leg is the phase voltage

$\mathbf{I}_{AB}$, $\mathbf{I}_{CA}$, and $\mathbf{I}_{BC}$ are phase currents

$\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$ are line currents

$$\mathbf{I}_{AB} = I_\phi \angle 0^\circ,$$
$$\mathbf{I}_{BC} = I_\phi \angle -120^\circ,$$
$$\mathbf{I}_{CA} = I_\phi \angle 120^\circ.$$

$$|\mathbf{V}_{AB}| = |\mathbf{V}_{BC}| = |\mathbf{V}_{CA}| = V_\phi,$$
$$|\mathbf{I}_{AB}| = |\mathbf{I}_{BC}| = |\mathbf{I}_{CA}| = I_\phi,$$

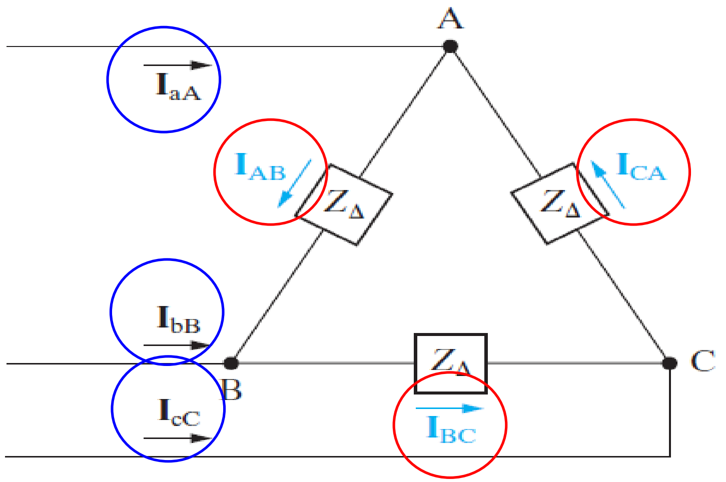


Figure 11.12 ▲ A circuit used to establish the relationship between line currents and phase currents in a balanced Δ load.

Analysis of the Wye-Delta Circuit – cont.

- In writing these equations, we arbitrarily selected $\mathbf{I}_{AB}$ as the reference phasor
- We can write the line currents in terms of the phase currents by direct application of Kirchhoff's current law

Relationships between line currents and phase currents in a Δ connected load

$$
\begin{aligned}
\mathbf{I}_{aA} &= \mathbf{I}_{AB} - \mathbf{I}_{CA} \\
&= I_\phi \angle 0^\circ - I_\phi \angle 120^\circ \\
&= \sqrt{3} I_\phi \angle -30^\circ, \\
\mathbf{I}_{bB} &= \mathbf{I}_{BC} - \mathbf{I}_{AB} \\
&= I_\phi \angle -120^\circ - I_\phi \angle 0^\circ \\
&= \sqrt{3} I_\phi \angle -150^\circ, \\
\mathbf{I}_{cC} &= \mathbf{I}_{CA} - \mathbf{I}_{BC} \\
&= I_\phi \angle 120^\circ - I_\phi \angle -120^\circ \\
&= \sqrt{3} I_\phi \angle 90^\circ.
\end{aligned}
$$

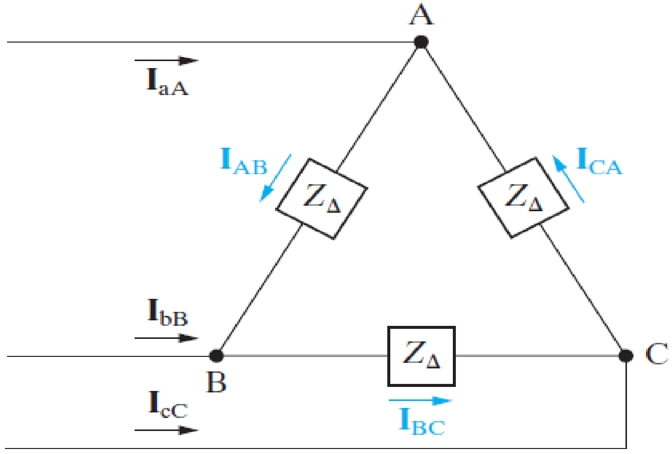


Figure 11.12 ▲ A circuit used to establish the relationship between line currents and phase currents in a balanced Δ load.

Applying KCL on nodes A, B, and C will lead to these equations

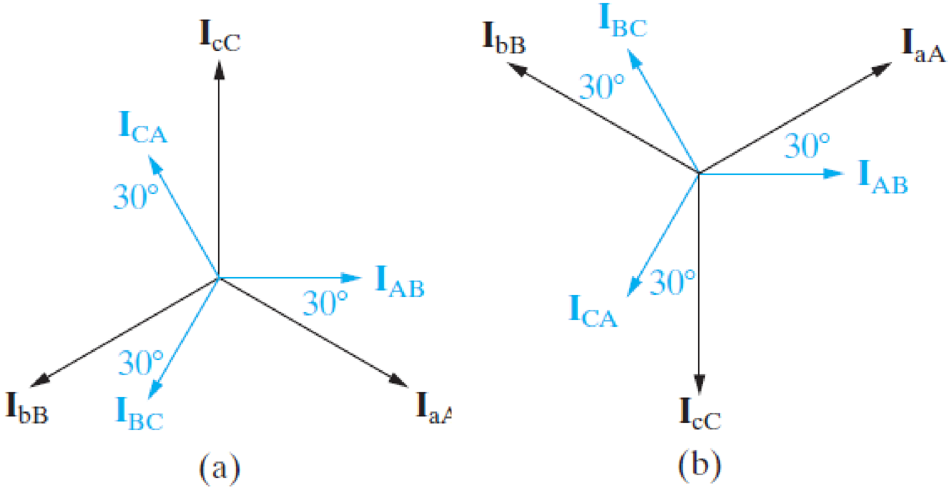


Figure 11.13 ▲ Phasor diagrams showing the relationship between line currents and phase currents in a Δ-connected load. (a) The positive sequence. (b) The negative sequence.

Example 11.2

The Y-connected source in Example 11.1 feeds a Δ-connected load through a distribution line having an impedance of $0.3 + j0.9\ \Omega/\phi$. The load impedance is $118.5 + j85.8\ \Omega/\phi$. Use the a-phase internal voltage of the generator as the reference.

a) Construct a single-phase equivalent circuit of the three-phase system.

b) Calculate the line currents $\mathbf{I}_{aA}$, $\mathbf{I}_{bB}$, and $\mathbf{I}_{cC}$.

c) Calculate the phase voltages at the load terminals.

d) Calculate the phase currents of the load.

e) Calculate the line voltages at the source terminals.

Example 11.1

A balanced three-phase Y-connected generator with positive sequence has an impedance of $0.2 + j0.5\ \Omega/\phi$ and an internal voltage of $120\ V/\phi$.

Example 11.2 – cont.

Solution

a) Figure 11.14 shows the single-phase equivalent circuit. The load impedance of the Y equivalent is

$$\frac{118.5 + j85.8}{3} = 39.5 + j28.6\ \Omega/\phi.$$

$$Z_Y = \frac{Z_\Delta}{3}$$

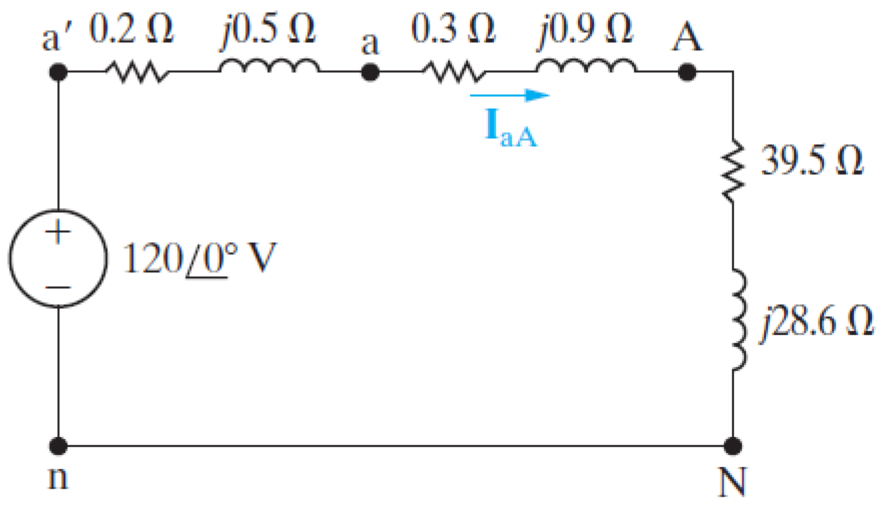


Figure 11.14 ▲ The single-phase equivalent circuit for Example 11.2.

b) The a-phase line current is

$$\mathbf{I}_{aA} = \frac{120\angle 0^\circ}{(0.2 + 0.3 + 39.5) + j(0.5 + 0.9 + 28.6)}$$

$$= \frac{120\angle 0^\circ}{40 + j30} = 2.4\angle -36.87^\circ\ \text{A}.$$

Hence

$$\mathbf{I}_{bB} = 2.4\angle -156.87^\circ\ \text{A},$$

$-120°$

$$\mathbf{I}_{cC} = 2.4\angle 83.13^\circ\ \text{A}.$$

$+120°$

Example 11.2 – cont.

Note that the line voltages are $\mathbf{V_{AB}}$, $\mathbf{V_{BC}}$, and $\mathbf{V_{CA}}$

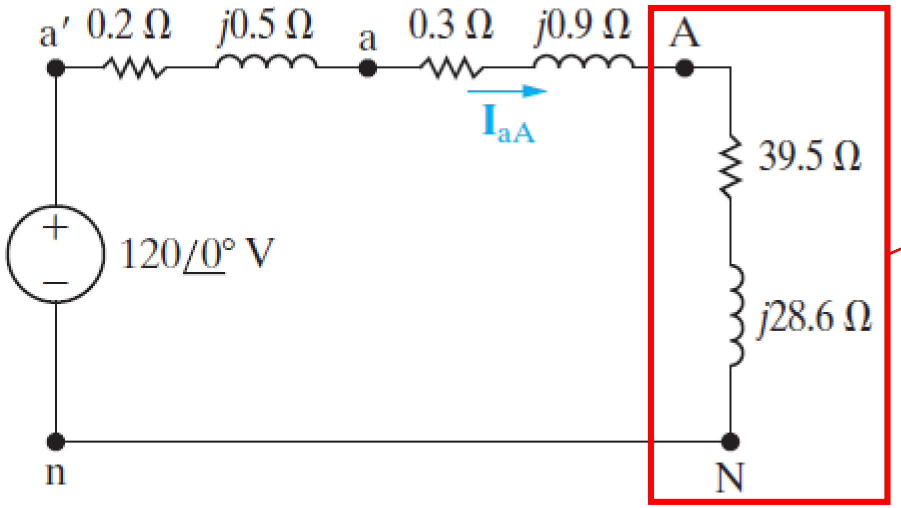


Figure 11.14 ▲ The single-phase equivalent circuit for Example 11.2.

Note that V_{AN} is the phase voltage of the Y-connected load. This is not what we want. We want the phase voltage of the Δ-connected load. Thus, we need to find the line voltage from V_{AN} then this line voltage will be equal to the phase voltage of the Δ-connected load.

c) Because the load is Δ connected, the phase voltages are the same as the line voltages. To calculate the line voltages, we first calculate $\mathbf{V}_{AN}$:

$$\mathbf{V}_{AN} = (39.5 + j28.6)(2.4 \angle{-36.87^\circ})$$

$$= 117.04 \angle{-0.96^\circ} \text{ V}.$$

Because the phase sequence is positive, the line voltage $\mathbf{V}_{AB}$ is

$$\mathbf{V}_{AB} = (\sqrt{3} \angle{30^\circ})\mathbf{V}_{AN}$$

$$= 202.72 \angle{29.04^\circ} \text{ V}.$$

Therefore

$$\mathbf{V}_{BC} = 202.72 \angle{-90.96^\circ} \text{ V},$$

$$\mathbf{V}_{CA} = 202.72 \angle{149.04^\circ} \text{ V}.$$

We know these relationships from the Y-Y 3-phase circuits discussed earlier

Example 11.2 – cont.

d) The phase currents of the load may be calculated directly from the line currents:

$$\mathbf{I}_{\mathrm{AB}} = \left(\frac{1}{\sqrt{3}}\angle 30^\circ\right)\mathbf{I}_{\mathrm{aA}}$$

$$= 1.39\angle -6.87^\circ \text{ A}.$$

Once we know $\mathbf{I}_{\mathrm{AB}}$, we also know the other load phase currents:

$$\mathbf{I}_{\mathrm{BC}} = 1.39\angle -126.87^\circ \text{ A},$$

$$\mathbf{I}_{\mathrm{CA}} = 1.39\angle 113.13^\circ \text{ A}.$$

Note that we can check the calculation of $\mathbf{I}_{\mathrm{AB}}$ by using the previously calculated $\mathbf{V}_{\mathrm{AB}}$ and the impedance of the Δ-connected load; that is,

$$\mathbf{I}_{\mathrm{AB}} = \frac{\mathbf{V}_{\mathrm{AB}}}{Z_\phi} = \frac{202.72\angle 29.04^\circ}{118.5 + j85.8}$$

$$= 1.39\angle -6.87^\circ \text{ A}.$$

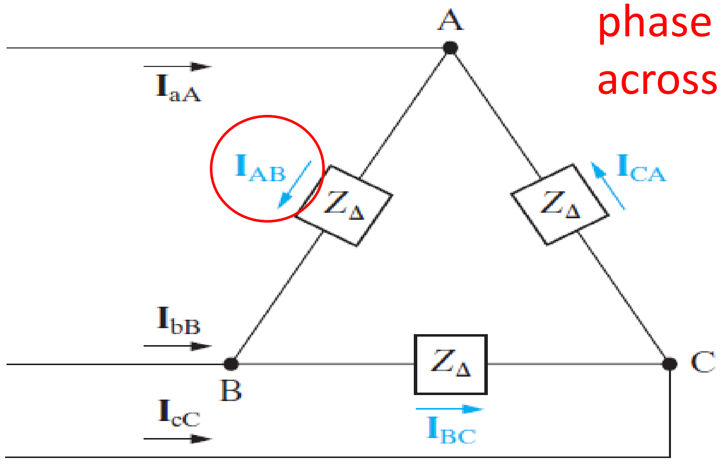


Figure 11.12 ▲ A circuit used to establish the relationship between line currents and phase currents in a balanced Δ load.

When a load is connected in Δ, the current in each leg of the Δ is the phase current, and the voltage across each leg is the phase voltage

$$\mathbf{I}_{\mathrm{AB}} = I_\phi\angle 0^\circ,$$

$$\mathbf{I}_{\mathrm{BC}} = I_\phi\angle -120^\circ,$$

$$\mathbf{I}_{\mathrm{CA}} = I_\phi\angle 120^\circ.$$

$$\mathbf{I}_{\mathrm{aA}} = \mathbf{I}_{\mathrm{AB}} - \mathbf{I}_{\mathrm{CA}}$$

$$= I_\phi\angle 0^\circ - I_\phi\angle 120^\circ$$

$$= \sqrt{3}I_\phi\angle -30^\circ,$$

$$\mathbf{I}_{\mathrm{bB}} = \mathbf{I}_{\mathrm{BC}} - \mathbf{I}_{\mathrm{AB}}$$

$$= I_\phi\angle -120^\circ - I_\phi\angle 0^\circ$$

$$= \sqrt{3}I_\phi\angle -150^\circ,$$

$$\mathbf{I}_{\mathrm{cC}} = \mathbf{I}_{\mathrm{CA}} - \mathbf{I}_{\mathrm{BC}}$$

$$= I_\phi\angle 120^\circ - I_\phi\angle -120^\circ$$

$$= \sqrt{3}I_\phi\angle 90^\circ.$$

Example 11.2 – cont.

e) To calculate the line voltage at the terminals of the source, we first calculate $\mathbf{V}_{an}$. Figure 11.14 shows that $\mathbf{V}_{an}$ is the voltage drop across the line impedance plus the load impedance, so

$$\mathbf{V}_{an} = (39.8 + j29.5)(2.4 \angle -36.87^\circ)$$
$$= 118.90 \angle -0.32^\circ \text{ V}.$$

The line voltage $\mathbf{V}_{ab}$ is

$$\mathbf{V}_{ab} = (\sqrt{3} \angle 30^\circ)\mathbf{V}_{an},$$

or

$$\mathbf{V}_{ab} = 205.94 \angle 29.68^\circ \text{ V}.$$

Therefore

$$\mathbf{V}_{bc} = 205.94 \angle -90.32^\circ \text{ V},$$
$$\mathbf{V}_{ca} = 205.94 \angle 149.68^\circ \text{ V}.$$

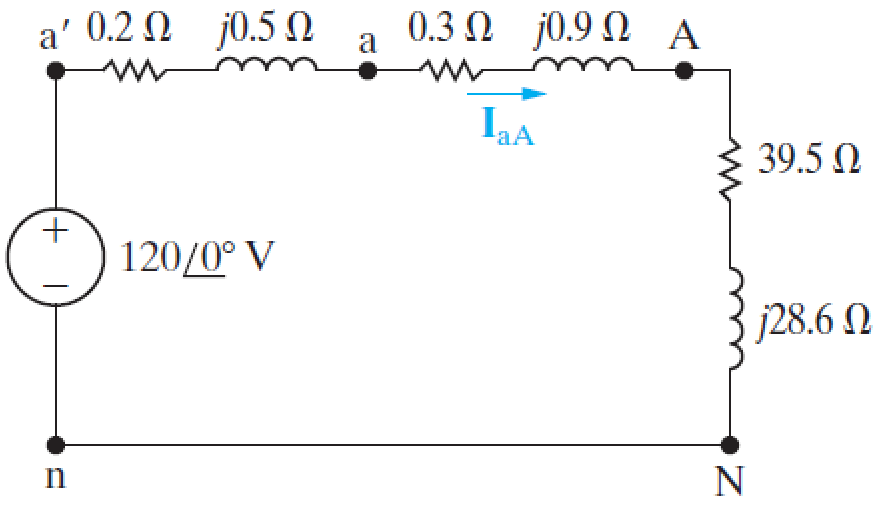


Figure 11.14 ▲ The single-phase equivalent circuit for Example 11.2.

Power Calculation in Balanced 3-Phase Circuits

- The average power delivered to Z_A is:

$$P_A = V_\phi I_\phi \cos\theta_\phi,$$

$$\begin{cases} V_\phi \equiv |\mathbf{V}_{AN}| = V_L/\sqrt{3}, \\ I_\phi \equiv |\mathbf{I}_{aA}| = I_L, \\ \theta_\phi \equiv \angle V_\phi - \angle I_\phi = \angle Z_A. \end{cases}$$

V_L is the line voltage

(rms value)

I_L is the line current

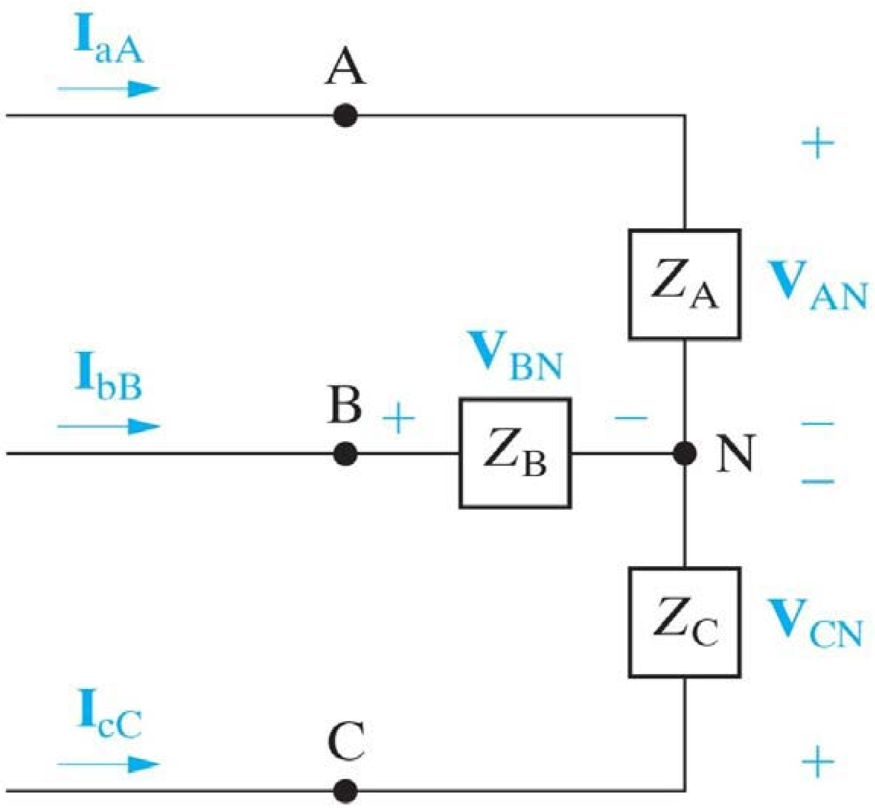


Figure: 11-15

- The total power delivered to the Y-Load is:

$$P_{tot} = 3P_A = 3V_\phi I_\phi \cos\theta_\phi = \sqrt{3}V_L I_L \cos\theta_\phi.$$

Power Calculation in Balanced 3-Phase Circuits – cont.

- The reactive powers of one phase and the entire Y-Load are:

$$\begin{cases} Q_\phi = V_\phi I_\phi \sin\theta_\phi, \\ Q_{tot} = 3V_\phi I_\phi \sin\theta_\phi = \sqrt{3} V_L I_L \sin\theta_\phi. \end{cases}$$

- The complex powers of one phase and the entire Y-Load are:

$$\begin{cases} S_\phi = P_\phi + jQ_\phi = V_\phi I_\phi e^{j\theta_\phi} = \mathbf{V}_\phi \mathbf{I}_\phi^*; \\ S_{tot} = 3S_\phi = 3V_\phi I_\phi e^{j\theta_\phi} = \sqrt{3} V_L I_L e^{j\theta_\phi}. \end{cases}$$

$V_\phi I_\phi e^{j\theta_\phi} = V_\phi e^{j\theta_v} I_\phi e^{-j\theta_i}$

From **Ch10** we know that $I_\phi e^{-j\theta_i} = \mathrm{I}_\phi^*$

Power Calc. in Balanced 3-Phase Circuits – cont.

- If the load is Δ-connected, the calculation of power - reactive or complex - is basically the same as that for a Y-connected load
- Figure below shows a Δ-connected load, along with its pertinent currents and voltages. The power associated with each phase is

$$|\mathbf{V}_{AB}| = |\mathbf{V}_{BC}| = |\mathbf{V}_{CA}| = V_\phi, \quad \& \quad |\mathbf{I}_{AB}| = |\mathbf{I}_{BC}| = |\mathbf{I}_{CA}| = I_\phi,$$

$$P_A = P_B = P_C = P_\phi = V_\phi I_\phi \cos\theta_\phi.$$

Average power

$$\theta_{vAB} - \theta_{iAB} = \theta_{vBC} - \theta_{iBC} = \theta_{vCA} - \theta_{iCA} = \theta_\phi,$$

- If we let V_L and I_L represent the **rms** magnitudes of the line voltage and current, then

$$P_T = 3P_\phi = 3V_\phi I_\phi \cos\theta_\phi$$

Total power delivered to a balanced Δ-connected load

$$= 3V_L\left(\frac{I_L}{\sqrt{3}}\right)\cos\theta_\phi$$

$$= \sqrt{3}V_L I_L \cos\theta_\phi.$$

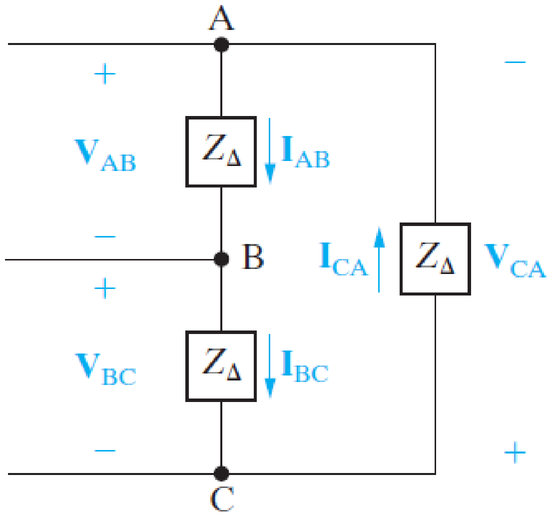
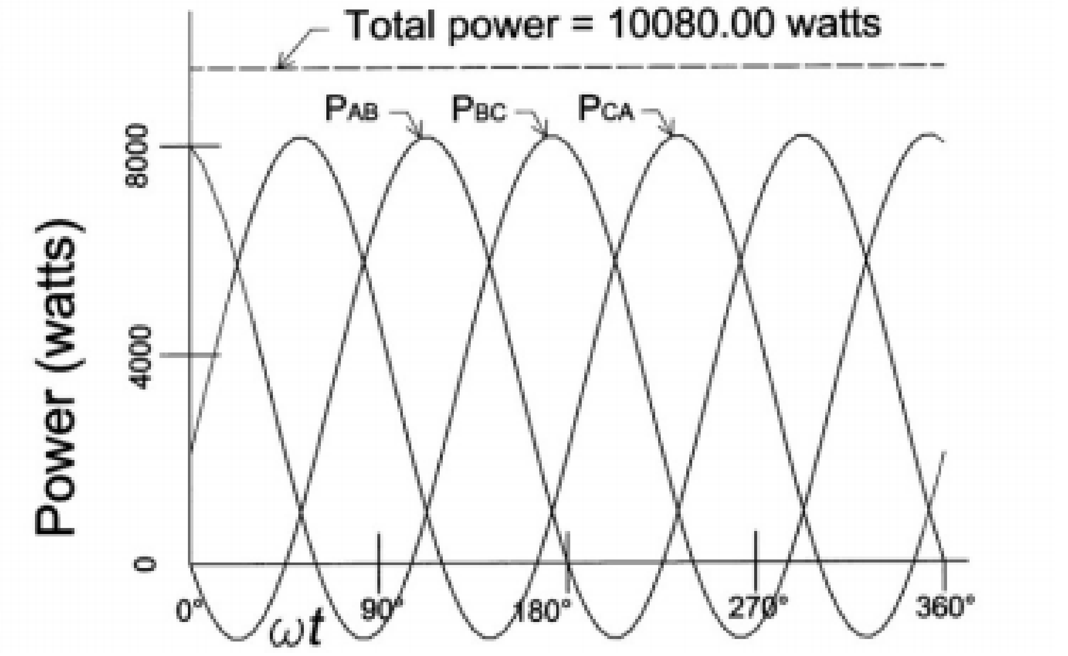


Figure 11.16 ▲ A Δ-connected load used to discuss power calculations.

Power Calculation in Balanced 3-Phase Circuits – cont.

- Although we are primarily interested in average, reactive, and complex power calculations, the computation of the total instantaneous power is also important
- This plot shows an example of the instantaneous power generated by a balanced 3-phase circuit
- It is significant to note that the value of total instantaneous power throughout the cycle is constant
  - In other words, the sum of the powers of the three phases is calculated to be exactly a constant value, in this instance 10,080.00 watts



- This power is invariant with time
- Thus the torque developed at the shaft of a 3-phase motor is constant, which in turn means less vibration in the machinery powered by 3-phase motors
- This is another **advantage** of using 3-phase circuits

Example 11.3: Calc. Power in a 3-Phase Y-Y Circuit

a) Calculate the average power per phase delivered to the Y-connected load of Example 11.1.

b) Calculate the total average power delivered to the load.

c) Calculate the total average power lost in the line.

d) Calculate the total average power lost in the generator.

e) Calculate the total number of magnetizing vars absorbed by the load.

f) Calculate the total complex power delivered by the source.

Example 11.1

A balanced three-phase Y-connected generator with positive sequence has an impedance of $0.2 + j0.5\ \Omega/\phi$ and an internal voltage of $120\ V/\phi$. The generator feeds a balanced three-phase Y-connected load having an impedance of $39 + j28\ \Omega/\phi$. The impedance of the line connecting the generator to the load is $0.8 + j1.5\ \Omega/\phi$. The a-phase internal voltage of the generator is specified as the reference phasor.

Example 11.3: Calc. Power in a 3-Phase Y-Y Circuit

Solution

a) We need to calculate the average power per phase delivered to the load

$$P_A = P_B = P_C = P_\phi = V_\phi I_\phi \cos\theta_\phi,$$
$$\theta_\phi = \theta_{vA} - \theta_{iA} = \theta_{vB} - \theta_{iB} = \theta_{vC} - \theta_{iC}.$$

θ_Φ is the phase angle between the phase voltage and the current

From Example 11.1, $V_\phi = 115.22$ V, $I_\phi = 2.4$ A, and $\theta_\phi = -1.19 - (-36.87) = 35.68^\circ$. Therefore

$$P_\phi = (115.22)(2.4)\cos 35.68^\circ$$
$$= 224.64 \text{ W}.$$

The power per phase may also be calculated from $I_\phi^2 R_\phi$, or

$$P_\phi = (2.4)^2 \underbrace{(39)}_{\text{The real part of the load impedance}} = 224.64 \text{ W}.$$

From Example 11.1 we know that:

b) The a-phase line current is

$$\mathbf{I}_{aA} = \frac{120\angle 0^\circ}{(0.2 + 0.8 + 39) + j(0.5 + 1.5 + 28)}$$
$$= \frac{120\angle 0^\circ}{40 + j30}$$
$$= 2.4\angle -36.87^\circ \text{ A}.$$

c) The phase voltage at the A terminal of the load is

$$\mathbf{V}_{AN} = (39 + j28)(2.4\angle -36.87^\circ)$$
$$= 115.22\angle -1.19^\circ \text{ V}.$$

Example 11.3 – cont.

Solution - cont.

b) The total average power delivered to the load is $P_T = 3P_\phi = 673.92$ W. We calculated the line voltage in Example 11.1, so we may also use Eq. 11.36:

$$P_T = \sqrt{3}(199.58)(2.4)\cos 35.68^\circ$$
$$= 673.92 \text{ W.}$$

$$P_T = 3\left(\frac{V_L}{\sqrt{3}}\right)I_L \cos\theta_\phi$$
$$= \sqrt{3}V_L I_L \cos\theta_\phi.$$

From Example 11.1 we know the line voltage:

d) For a positive phase sequence, the line voltages lead the phase voltages by 30°; thus

$$\mathbf{V}_{AB} = (\sqrt{3}\,\underline{/30^\circ})\mathbf{V}_{AN}$$
$$= 199.58\,\underline{/28.81^\circ} \text{ V,}$$
$$\mathbf{V}_{BC} = 199.58\,\underline{/-91.19^\circ} \text{ V,}$$
$$\mathbf{V}_{CA} = 199.58\,\underline{/148.81^\circ} \text{ V.}$$

c) The total average power lost in the line is:

$$P_{line} = 3(2.4)^2(0.8) = 13.824 \text{ W.}$$

d) The total average power lost in the generator is:

$$P_{gen} = 3(2.4)^2(0.2) = 3.456 \text{ W.}$$

The 0.8 Ω and 0.2 Ω are the real part of the line and generator impedance

Example 11.3 – cont.

Solution - cont.

e) The total number of magnetizing vars absorbed by the load is

$$Q_T = \sqrt{3}(199.58)(2.4)\sin 35.68^\circ$$
$$= 483.84 \text{ VAR}.$$

$$Q_\phi = V_\phi I_\phi \sin\theta_\phi,$$

$$Q_T = 3Q_\phi = \sqrt{3}V_L I_L \sin\theta_\phi.$$

Total reactive power in a balanced three-phase load

f) The total complex power associated with the source is

$$S_T = 3S_\phi = -3\underbrace{(120)}_{V_\Phi}\underbrace{(2.4)\angle 36.87^\circ}_{I_\Phi^*}$$

$$S_\phi = P_\phi + jQ_\phi = \mathbf{V}_\phi \mathbf{I}_\phi^*,$$

The minus sign indicates that the internal power and magnetizing reactive power are being delivered to the circuit.

$$= -691.20 - j518.40 \text{ VA}.$$

We know from example 11.1 that the phase voltage of the source is $120\angle 0°$, and the phase current is the line current which equals $2.4\angle -36.87°$

Also check examples 11.5

Example 11.4

a) Calculate the total complex power delivered to the Δ-connected load of Example 11.2.

b) What percentage of the average power at the sending end of the line is delivered to the load?

Solution

a) Using the a-phase values from the solution of Example 11.2, we obtain

$$\mathbf{V}_\phi = \mathbf{V}_{AB} = 202.72 \underline{/29.04^\circ} \text{ V},$$

$$\mathbf{I}_\phi = \mathbf{I}_{AB} = 1.39 \underline{/-6.87^\circ} \text{ A}.$$

Using Eqs. 11.52 and 11.53, we have

$$S_T = 3(202.72 \underline{/29.04^\circ})(1.39 \underline{/6.87^\circ})$$

$$= 682.56 + j494.21 \text{ VA}.$$

b) The total power at the sending end of the distribution line equals the total power delivered to the load plus the total power lost in the line; therefore

$$P_{\text{input}} = 682.56 + 3(2.4)^2(0.3)$$

$$= 687.74 \text{ W}.$$

The percentage of the average power reaching the load is 682.56/687.74, or 99.25%. Nearly 100% of the average power at the input is delivered to the load because the impedance of the line is quite small compared to the load impedance.

Also, please check example 11.5

Summary of Topics Covered in this Chapter

- In this chapter we discussed:
  - Balanced three-phase voltages
  - Three-phase voltage sources
  - Analysis of Wye-Wye circuits
  - Analysis of Wye-Delta circuits
  - Power calculations in balanced three-phase circuits
- We covered sections 11.1 – 11.5
- Next chapter (Ch12) we will talk about the Laplace transform
  - Introduction to Laplace Transform