Exploring the Underlying Geometry of Kaluza-Klein Dimensional Reduction

Peter Ferguson

DAMTP, Centre for Mathematical Sciences,
University of Cambridge,
Wilberforce Road, Cambridge, CB3 0WA
pf346@cam.ac.uk

Abstract. This essay is an investigation of the geometrical aspects of the Kaluza-Klein mechanism on fiber bundles. In particular, principal $G-$bundles are considered briefly, with a view to possible extensions of the mechanism to transitive Lie algebroids and maybe even further. On the journey we amass a wealth of categorical, homological, and geometric notions useful in many areas of physics.

Keywords: Kaluza-Klein Geometry · Short Exact Sequence · Atiyah Sequence · Transitive Lie Algebroids

Table of Contents

1 Introduction

Kaluza-Klein theory has its beginnings in 1921, when Kaluza unified electromagnetism—a U(1)−gauge theory—and gravity in a single unified field theory. The unified theory exists in higher dimensions, i.e. it is a (4+1) dimensional theory rather than the usual (3+1) dimensional theories we are used to. The fascinating part is that the unified field theory is simply a gravitational theory in (4+1) dimensions. That is, it consists of (4+1) dimensional metrics, curvatures, field equations, etc of general relativity. Specifically, in this essay we will focus our attention on the most important aspect of a gravitational theory, that being the metric. In that vein let us briefly discuss the metric Kaluza posed and why it is important. Kaluza found that the (4+1) dimensional metric decomposes such that the entries are dimensionless and that the metric explicitly separates into an underlying (3+1) dimensional metric and other terms. The dimensionlessness of these quantities allows us to compactify one of the dimensions. Unsurprisingly we choose the extra dimension not involved in this (3+1) metric, resulting in what is essentially a (3+1) dimensional field theory. What Kaluza found was that under standard general relativity analysis, i.e. application of the Einstein equation, yields field equations which contain both the standard (3+1) gravity and also the field equations of electrodynamics (which come from the other terms in the decomposed metric).

Clearly the notions of geometry of the spacetime is an imperative part of Kaluza-Klein theory. This essay is intended to be an exploration of the underlying geometric notions of dimensional reduction via Kaluza-Klein. D. Betounes demonstrated in [1], that it is possible to describe Kaluza-Klein theory on principal $G-$bundles by looking at the Atiyah sequence—although it is never mentioned by name in the paper—which is a sequence of associated vector bundles. These bundles are related by a short exact sequence used to decompose the $G-$bundle's tangent bundle quotiented by the G group action. What is interesting here is that it can be shown that splittings of this sequence are in exact correspondence with the $G-$bundle connections, which form the mathematical stage upon which modern gauge theories are constructed.

An interesting fact about the Atiyah sequence is that one can endow it with some extra structure—namely a Lie bracket and a map to the tangent bundle of the base manifold—making it into a Lie algebroid. J. Pradines created Lie algebroids in [2], in 1967 as the infinitesimal counterpart to smooth groupoids, a categorification of groups discussed briefly in Section 4; However, they have been studied in many different contexts over the years since, exhibiting their naturalness as a independent structure. Lie algebroids are a generalisation of both Lie algebras and the tangent bundle and as such, one might expect that they are found throughout physics. Indeed one would be correct, using the link to the tangent bundle one can construct an analogous theory of pseudo-Riemannian geometry on Lie algebroids. Thus the structures are seeing use in modern physics areas such as twistor theory, gauge theory and in a more generalised form—are the basic structure of generalised geometry which is a structure which is inherently T-dual—in string theory.

A basic description of what is means to be T-dual in the simplest case is as follows, suppose you have a closed string embedded in a torodial geometry with string length scale ℓ_s and the torodial geometry has length scale R. Then a T-dual space is the torodial geometry with length scale ℓ^2/R. This exchanges the momentum and winding number quantum numbers of the string. An amazing fact is that these two are physically equivalent! This fascinating symmetry is only found in some quantum field theories and string theories.

Generalised geometry—from the physics point of view—seeks to place the B-field gauge transformation—the Kalb-Ramond field of string theory—and diffeomorphisms of a manifold on equal footing. This is achieved by replacing the tangent bundle of the manifold with a direct sum of the tangent and cotangent bundle. This combines $1-$forms and vector fields into a single structure commonly called generalised vectors (although this is name is not uncommon in other situations, in fact we will use the term to describe the "vector fields" of a Lie algebroid). It turns out that this theory is inherent covariant under T-duality and, as such, is a natural structure on which to consider string theories which have some T-duality. One such theory describes a closed string in which the background has a non-vanishing $3-$form flux (a generalistion of a magnetic field), called the $H-$flux or the Neveu-Schwarz $3-$form flux. These fluxes are geometric in nature but, after applying some T-dualities, the corresponding fluxes are non-geometric. At the end of the essay we have our first encounter with non-geometric quantities; However, the two are not related but the notion of non-geometry leads one to think about extensions of the theory in this essay to the flux compactifications—a variant of Kaluza-Klein in string theory involving said fluxes—as further work. This might seem like tenuous thread linking the two, but there are is a greater association than this due to generalised geometry. The structure generalises the tangent bundle of a space as above but also equips it with a bracket which is a generalisation of Lie brackets obtained by dropping the Jacobi identity. This structure is an example of what is called a Courant algebroid, which again is a generalisation of a Lie bialgebroid, which again again is a generalisation of Lie algebroid, a major feature of the essay. The second generalisation, i.e. from a Lie algebroid to a Lie bialgebroid is very simple combining a Lie algebroid with its dual. Then first generalisation is much less trivial. Given such a relation, one might expect that some extension of the content of the essay may apply in the above situation, this is discussed later in Section 5.1

This is all a very general overview of the relationship of some of the notions in the essay—i.e. Lie algebroids and Atiyah sequences—and modern physics. More specifically in the essay we discuss the relationship between these two notions, namely the fact that, transitive Lie algebroids generalise Atiyah bundles. The nice thing about these structures is that they have well defined differential calculi, connection theory and splitting theory. Which are exactly the ingredients needed to explore the Kaluza-Klein mechanism à la [1].

The background material necessary for the reader to be at least familiar with is:

(I) Differential geometry to at least the level of pseudo-Riemannian geometry. Any course to the level of Part III Differential Geometry or some advanced General Relativity courses should suffice

(II) Lie theory including Lie groups, Lie algebras and their role in physics, particularly gauge theory. Any course similar in level to the Part III Symmetries, Fields and Particles course will more than suffice.

(III) Bosonic String Theory, as one has already seen in the introduction, some casual qualitative remarks are made referencing some string theory but this is not a prerequisite. Although the remarks are intended for those at least familiar with basic bosonic string theory to around the level of Part III String Theory.

(IV) Topology, as usual when talking about geometry there are some topological ideas floating around. We assume some introductory topology but as with string theory, the ideas only appear in non-essential remarks.

The layout of this essay is as follows:

Section 2: Introduces the category theory needed to understand the some of the proofs throughout the essay, the main features are functors and the pushout. The homological theory background is presented and discusses complexes, cohomologies and exact sequences. Then finally fibered manifolds are introduced and used to recast the usual differential geometric definitions of vector bundles, connection, etc.

Section 3: Contains my exposition of the ideas portrayed in the foundational paper of this essay by [1]. We develop a Kaluza-Klein type metric by developing a functor which acts on a short exact sequence of vector bundles. Then we introduce the Atiyah sequence as the prototypical sequence one applies these ideas to.

Section 4: Introduces the theory of Lie algebroids. In particular, it is shown that transitive Lie algebroids are essentially a natural extension of the notion of the Atiyah sequence. Then we develop the differential geometry on transitive Lie algebroids.

1.1 Acknowledgement

I have used the books of D.Bleecker [3], J. Lee [4] and K. MacKenzie [5] as general reference texts. Section 1 has used various sources including M. Gualtieri's thesis [6], the papers [7] and [8] as qualitative reference texts for the string theory and Lie algebroid generalisations. Although the exposition of the sections 2.1, 2.2 are my own the definitions are largely based on the appendices of the book [9]. Section 2.3 is my own recasting of definitions from various sources, Section 3 is my reading of the theory in the paper by D. Betounes[1] and Section 4 is my own distillation of the relevant facts from the paper [10]. The proofs of lemmas, propositions and theorems contained in the essay are my own, some are slightly different takes on standard proofs though they may well exist elsewhere in the literature. The only exceptions to this are Lemmas 3 and 5, and Proposition 2, which are largely the same as found in the literature.

2 Background

2.1 Some Basic Categorical Notions

This introductory category theory subsection does not pretend to be at all rigorous or complete, but rather takes an approach more aligned with that of the physicists modus operandi: i) know about the general idea behind the mathematical theory, ii) see that it may be applied somewhere to make some calculation/proof easier and iii) take the concepts we need to apply on a semi-rigorous basis. Note that ii) and iii) are not strictly sequential in practice and so will not be so here.

Category theory started as an alternative to set theory. The first category theorists noticed that in describing separate abstract structures we often employ the same basic principles. That is to say, we take a collection of objects (sets) impose some "conditions" on them (properties of groups, vector spaces, Lie algebras...) and look at how we transform between objects within this set ((homo)morphisms). Category theorists concentrate on the structure in a abstract sense without considering particular objects, as opposed to set theory where one focuses on specific objects.

Let's dip our feet into a little basic category theory without explicit definitions... except the ones that we will find useful. First we take a collection of structures called a **category**. All structures of the specified type belong to this category. For example, **Set** is the category of all sets. Next we want to know how objects within this category relate to each other. This is the second datum of a category, a collection of maps that preserve the structure called **morphisms**. For the set example morphisms are the maps between sets. The morphisms are also required to obey two conditions: i) associativity, so we can compose the morphisms and they satisfy the usual associative identity, ii) categories are unital, i.e. there exists a right and left unit morphism that coincide, then that is it we have a category. To someone familiar with abstract algebra this may remind them of a monoid, and indeed it is a many object extension of a monoid. These extensions are often accompanied by the application of the suffix -oid (as seen later in the case of groups and groupoids in Section 4), and so a category is sometimes (only half in jest) referred to as a monoidoid.

The usefulness of category theory comes down to its effectiveness at describing the overall mathematical structure of a class of objects. To map between categories we use the concept of a functor.

Definition 1. *A **functor** $\mathcal{F} : C \to D$ is a map between categories C and D that maps objects $X \in C$ to objects $\mathcal{F}(X) \in D$ and morphisms $\rho : X \to Y$ in C to morphisms $\mathcal{F}(\rho) : \mathcal{F}(X) \to \mathcal{F}(Y)$ in D, such that $\mathcal{F}$ preserves the identity, $\mathcal{F}(1_x) = 1_{\mathcal{F}(x)}$ and composition of morphisms $\mathcal{F}(\rho \circ \varrho) = \mathcal{F}(\rho) \circ \mathcal{F}(\varrho)$*

A particularly important functor for our purposes is the

Example 1. **Tangent bundle functor**, T : **Diff** $\to$ **VectB** mapping from the category of smooth manifolds **Diff** to the category of smooth vector bundles which we denote **VectB**. This map should be familiar (although maybe not in this sense) as the differential/pushforward map from differential geometry.

The functoriality of this map is manifested through the chain rule: Suppose $f : \mathcal{M} \to \mathcal{N}$, $h : \mathcal{N} \to \mathcal{O}$ are smooth, then $T_p f : T_p\mathcal{M} \to T_{f(p)}\mathcal{N}$ for all $p \in \mathcal{M}$ and $T_p(h \circ f) = T_{f(p)} h \circ T_p f$ (from the chain rule of the pushforward, i.e. $(h \circ f)_* = h_* \circ f_*$).

Sometimes a functor may switch the direction of the arrow of the morphism it acts on. These functors are called **contravariant** functors and the usual case, which preserves arrow direction, are called **covariant** functors. So Definition 1 is a definition of the latter and the following example explains what is meant by the former.

Example 2. An important example (for our purposes) is the second symmetric power functor, Sym^2, which maps from **Diff** to the **category of symmetric bilinear forms on smooth manifolds**. This is an example of a contravariant functor since it acts on the morphisms through the pullback.

Carrying on from Example 1, $\mathrm{Sym}^2(\mathcal{M})$ is the symmetric bilinear forms on $\mathcal{M}$. So if $g \in \mathrm{Sym}^2(\mathcal{N})$ then $g : \Gamma T\mathcal{N} \times \Gamma T\mathcal{N} \to \mathbb{R}$ and for a, $b \in \Gamma T\mathcal{N}$

$$g(a, b) = g(b, a)$$

and the functor Sym^2 acts via the pullback,

$$\mathrm{Sym}^2(f) = f^* : \mathrm{Sym}^2(\mathcal{N}) \to \mathrm{Sym}^2(\mathcal{M});$$
$$(\mathrm{Sym}^2(f)(g))(c, d) = (f^*g)(c, d) = g(f_*c, f_*d)$$

for c, $d \in \Gamma T\mathcal{M}$. For a composition of the morphisms f and h from Example 1,

$$\mathrm{Sym}^2(h \circ f) = (h \circ f)^* = f^* \circ h^* = \mathrm{Sym}^2(f) \circ \mathrm{Sym}^2(h)$$

Thus Sym^2 is a contravariant functor.

The final categorical concept needed is that of a pushout, which is the dual to a (categorical) pullback. This is used only once in the essay so we will keep this brief and provide a less technical definition than one might find elsewhere,

Definition 2. *The pushout of a commutative square is the unique function $f : X \to Z$ described by the following commutative diagram,*

$$\begin{tikzcd}
Y \arrow[r, "b_Y"] \arrow[d, "a_Y"'] & B \arrow[d, "b_X"'] \arrow[ddr, bend left, "b_Z"] & \\
A \arrow[r, "a_X"] \arrow[drr, bend right, "a_Z"'] & X \arrow[dr, dashed, "f"] & \\
& & Z
\end{tikzcd}$$

Note that from here on in the $\circ$ composition symbol is dropped and only used if not it is not obvious. The remaining category theory required is explained throughout the essay to keep this section short. Even though the section is short, the examples are crucial tools used in theorems throughout.

2.2 Basic Homological and Cohomological Methods

Homological algebra is a mathematical tool used in many branches of mathematics and mathematical physics. In the language of category theory, when one investigates homological aspects of a construction what one is really looking at is the functorial properties of complexes.

Definition 3. *A **complex** is a sequence of abelian groups and group homomorphisms such that composition of consecutive maps in the following sequence vanish identically,*

$$\ldots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \ldots$$

i.e. $g \circ f = 0$ for all pairs of consecutive morphisms in the sequence.

Homology theory is to commutative algebras as representation theory is to groups. The role of the pair of representation space and homomorphism map, which are the central objects of study in representation theory, is played by different pair in homology theory:

Definition 4. *A* ***chain complex*** *is a pair* $(C_\bullet, d_\bullet)$ *(which represents a complex) of abelian groups* C_i *and group homomorphisms* d_i *called* ***boundary operators*** *(or* ***differentials****) which decrease the index of* C_i *(i.e.* $d_i : C_{i+1} \to C_i$*). So the complex is written:*

$$\dots \xrightarrow{d_n} C_n \longrightarrow \dots \longrightarrow C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_0} 0$$

i.e. $d_i \circ d_{i+1} = 0 \quad \forall i.$

Cohomology theory is defined similarly and should be fairly familiar to the reader through topics such as the de Rham cohomology (see examples 3 and 4).

Definition 5. *A* ***cochain complex*** *is a complex* $(C^\bullet, d^\bullet)$ *where the maps are called* ***coboundary operators*** *(or again* ***differentials****), which increase the index of* C^i *in the complex (i.e.* $d^i : C^i \to C^{i+1}$*). So the complex is written:*

$$0 \xrightarrow{d^0} C^1 \longrightarrow \dots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \longrightarrow \dots$$

i.e. $d^{i+1} \circ d^i = 0 \quad \forall i.$

It is common to endow the groups with additional structure (generally promoting the groups to modules over a ring) then, of course the differentials must be the correct structure preserving maps.

Example 3. The **de Rham cochain complex** is a complex of the spaces of $k-$differential forms on a smooth manifold $\mathcal{M}$, $\Omega^k(\mathcal{M})$ (a module over $C^\infty(\mathcal{M})$), where the differentials are the exterior derivatives $d^k : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$:

$$0 \xrightarrow{d^0} \Omega^1(\mathcal{M}) \xrightarrow{d^1} \Omega^2(\mathcal{M}) \longrightarrow \dots$$

Essentially a (co)chain complex is a notationally convenient way of stating a relationship between the kernels and images of each of the (co)boundary operators. Specifically, if the relation of (co)boundary operators g, f is $g \circ f = 0$ then we have $\mathrm{Im}(f) \subseteq \mathrm{Ker}(g)$.
These spaces have sufficiently nice structure (in the group case these are normal subgroups) that allows us to investigate their quotient space (in the group cases these are called (co)homology groups) that measure how close the relation is to equality.

Definition 6. *The* $n-th$ ***homology*** *and* ***cohomology groups***[a] *of chain and cochain complexes* $(C_\bullet, d_\bullet)$ *and* $(C^\bullet, d^\bullet)$ *are defined as,*

$$H_n(C_\bullet) = \frac{\mathrm{Ker}\,(d_n)}{\mathrm{Im}\,(d_{n+1})}, \qquad H^n(C^\bullet) = \frac{\mathrm{Ker}\,(d^n)}{\mathrm{Im}\,(d^{n-1})}$$

respectively. Elements of the numerator of these groups are said to be ***(co)cycles*** *(or* ***exact****) and for the denominator the elements are* ***(co)boundaries*** *(or* ***closed****), i.e., the* $n-th$ *(co)homology is an equivalence class of* $n-$*(co)cycles modulo* $n-$*(co)boundaries.*

[a] In the case when the complex has more structure, say a complex of modules for example, then we would speak of the homology module. But for convenience we simply refer to the homology of a complex as its homology group.

Homology theory has many applications in fields such as differential topology and mathematical physics, since homology groups can be associated to topological spaces. Then in the case of manifolds we interpret the $n-$cycles as $n-$dimensional submanifolds and $n-$boundaries as submanifolds which are a boundary to a $(n+1)-$dimensional submanifold. Then the $n-$th homology is essentially the submanifolds which are not the boundary of any $(n+1)-$dimensional manifold and thus represent a type of $n-$dimensional holes.

Example 4. The $n-$th **de Rham cohomology** is the $n-$th cohomology group of the de Rham cochain from example 3. That is,

$$H^n(\mathcal{M}) := \frac{\operatorname{Ker}(d^n : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M}))}{\operatorname{Im}(d^{n-1} : \Omega^{k-1}(\mathcal{M}) \to \Omega^k(\mathcal{M}))}$$

Which is the exact $n-$forms modulo the closed $n-$forms on $\mathcal{M}$.

In this essay (co)homology will not be used extensively, instead it is really only necessary to define exact sequences and to give some insight into the discussion of some small details that are mostly omitted for brevity.

Definition 7. *An **exact sequence** is a (co)chain complex in which all of its (co)homology groups are trivial groups.*

Now onto the final two definitions of this definition dense homological algebra section.

Definition 8. *A **short exact sequence** is a complex of the form,*

$$A \overset{\alpha}{\hookrightarrow} B \overset{\beta}{\twoheadrightarrow} C$$

which is also exact. Often this is written as,

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0.$$

The 0*s represent the zero object of the category and the associated arrow is a unique zero morphism.*

Thus a short exact sequence (SES) conveys three pieces of information about the maps, α and β: (i) α is injective, (ii) β is surjective and (iii) $\operatorname{Ker}\beta = \operatorname{Im}\alpha$.

Definition 9. *A short exact sequence* $A \overset{\alpha}{\hookrightarrow} B \overset{\beta}{\twoheadrightarrow} C$ *is said to be split if the following diagram commutes,*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "1"] & B \arrow[r, "\beta"] \arrow[d, "\eta"] & C \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & A \arrow[r] & A \oplus C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

where the bottom line of maps are just the canonical embedding and projection maps. $\eta : B \to A \oplus C$ *is an isomorphism which is structure preserving (for whatever extra structure is endowed to the complex).*

Alternatively, if the following equations are satisfied then the sequence is split:

$$\begin{aligned} \beta\sigma &= 1_C & \varrho\alpha &= 1_A \\ \beta\alpha &= 0 & \varrho\sigma &= 0 \\ & \alpha\varrho + \beta\sigma = 1_B. \end{aligned} \tag{1}$$

For some $\sigma : C \to B$ a section of β and $\varrho : B \to A$ a retraction of α.

The essence of the definition is that there exists a decomposition of the B space into two pieces such that the maps α and β act like the standard embedding and projections respectively. We now prove two lemmas which are integral to the rest of the essay. Both relate to the question of when a SES is a split SES (SSES).

Lemma 1. *A short exact sequence $A \overset{\iota}{\hookrightarrow} B \overset{\beta}{\twoheadrightarrow} C$ is automatically split if $\iota : A \to B$ is the inclusion map, that is $A \subset B$.*

Proof. The inclusion map ι is, of course, injective. That is, there exists a left inverse of ι called a (global) retraction map $\varrho : B \twoheadrightarrow A$. Since ϱ is a left inverse, it has a right inverse ι and is therefore surjective.

Using the first isomorphism theorem and the surjectivity of ϱ gives $B/\operatorname{Ker}(\rho) \cong \varrho(B) \cong A$. Thus we can uniquely decompose as $B \cong \varrho(B) \oplus \operatorname{Ker}(\varrho) \cong \iota(A) \oplus \operatorname{Ker}(\varrho)$. This is unique since $\varrho(B) \cap \operatorname{Ker}(\varrho) = \{0_A\}$. Now since β is surjective and using exactness (i.e. $\operatorname{Ker}(\beta) = \iota(A)$ or $\beta \circ \iota = 0$) we have

$$C \cong \beta(B) \cong \beta(\iota(A) \oplus \operatorname{Ker}(\varrho)) \cong 0 \oplus \beta(\operatorname{Ker}(\varrho))$$

So we have $\beta(\operatorname{Ker}(\varrho)) \cong C$. This tells us that there exists a (global) section $\sigma : C \to B$ such that $\beta \circ \sigma = 1_C$ with $\sigma(C) \cong \operatorname{Ker}(\varrho)$.

Thus the sequence is split since $B \cong \iota(A) \oplus \sigma(C)$, so $\varrho \oplus \beta : B \to A \oplus C$ is an isomorphism as in the sense of Definition 9. □

This is an weaker statement of what is known as the splitting lemma, which is not proven here as we shall not use its full power. The key thing to note is that if the SES has a global retraction or a global section then it splits. This is not so surprising from the above proof.

The second lemma uses the first to prove a result about the specific case where we endow the complex with a vector space structure.

Lemma 2. *A short exact sequence $U \overset{\alpha}{\hookrightarrow} V \overset{\beta}{\twoheadrightarrow} W$ of vector spaces is automatically split.*

Proof. First isomorphism theorem on α immediately gives $U/\operatorname{Ker}(\alpha) \cong \alpha(U) \subset V$. Thus if $1-$forms a basis B of U then we can induce a basis on $\alpha(U)$, call it $\alpha(B)$. We can always extend the basis $\alpha(B)$ to a basis on V, say $\alpha(B) \sqcup C$, using Gram-Schmidt. Then since α is injective we can form a surjective map $\varrho : \alpha(B) \sqcup C \to B$ such that,

$$\varrho(v) \mapsto \begin{cases} u \in B, & \text{if } v \in \alpha(B) \sqcup \{0\} \\ 0, & \text{if } v \in \{0\} \sqcup C. \end{cases}$$

i.e. $\varrho(\alpha(B) \sqcup C) = B$. Then by extending this map to all of V by linearity gives a global retraction $\varrho : V \to U$ such that $\varrho \circ \alpha = 1_U$. Thus the SES is split. □

The combination of Lemma 2 and the remarks after Lemma 1 allow us to almost trivially prove the following important theorem.[1]

Theorem 1. *Every SES of vector bundles over a common base is split.*

Proof. Suppose we have a SES of vector bundles over a common base manifold $\mathcal{M}$,

$$\begin{tikzcd} E \arrow[r, hook, "\alpha"] \arrow[dr, "\pi_E"'] & F \arrow[r, two heads, "\beta"] \arrow[d, "\pi_F"] & G \arrow[dl, "\pi_G"] \\ & \mathcal{M} & \end{tikzcd}$$

Since α and β must be structure preserving this tells us that they are vector bundle morphisms so $\alpha_m : E_m \to F_m$ and $\beta_m : F_m \to G_m$ for all $m \in \mathcal{M}$ (i.e. the maps preserve the fibers).

Therefore the SES of vector bundles induces a SES of vector spaces

$$E_m \xhookrightarrow{\alpha_m} F_m \xrightarrow{\beta_m} G_m$$

at every point $m \in M$. We know from lemma 2 that each of these is split. So at every $m \in \mathcal{M}$ we have a retraction map $\rho_m : F_m \to E_m$ such that $\rho_m \circ \alpha_m = 1_{E_m}$. That is to say, there exists a fiber preserving map $\varrho : F \to E$ such that $\varrho \circ \alpha = 1_E$, which is a global retraction therefore the sequence is split. □

2.3 Fibered Manifolds

We assume familiarity with the concept of fiber bundles and use this subsection to generalise the notion, recall some important results and establish notation that will be carried throughout the essay. However we will also assume basic constructions which are not presented here such as associated bundles and hence also pull-back bundles.

Fibered manifolds (FMs) are rather simple structures that generalise fiber bundles of smooth manifolds to bundles which do not carry the local trivialisation condition. Basically they assign to each point of a smooth base manifold $\mathcal{M}$ a fiber which is not necessarily typical (see Definition 13 for definition of typical). That is to say, not only do the fibers not have to be the same space, in fact they need not even carry the same topology. The map which assigns these fibers is a smooth surjective submersion which actually acts as a definition.

Definition 10. *A smooth surjective submersion, π, referred to as the projection (or sometimes fibration), $\pi : \mathcal{N} \to \mathcal{M}$ from the total space $\mathcal{N}$ to the base space $\mathcal{M}$, both smooth manifolds, is called a* ***fibered manifold****.*

Definition 11. *A* ***fibered chart*** *on a fibered manifold, $\pi : \mathcal{N} \to \mathcal{M}$, is a chart (V, ϕ) on $\mathcal{N}$ which induces a unique chart (U, φ) on $\mathcal{M}$, such that $\pi(V) = U$ and $\varphi \circ \pi = \mathrm{pr}_1 \circ \phi$ i.e.*

[1] Although in Section 2.3.2 we introduce vector bundles, this is mainly to establish notation. Knowledge of these structures is assumed as a prerequisite.

the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathbb{R}^m \times \mathbb{R}^{n-m} \arrow[d, "\mathrm{pr}_1"] \\ U \arrow[r, "\varphi"] & \mathbb{R}^m \end{tikzcd}$$

*where $n = dim(\mathcal{N})$ and $m = dim(\mathcal{M})$. A collection of these charts covering $\mathcal{N}$ is called a **fibered atlas**.*

The uniqueness of the induced charts is a consequence of the surjectivity of the projection π. To see this, suppose two chart maps φ and $\tilde{\varphi}$ satisfy the above commutative diagram then, $(\varphi - \tilde{\varphi}) \circ \pi = 0$ but π is surjective and thus right-cancellative therefore uniqueness of φ is immediate.

Of course there are many interesting properties of FMs but later we will have a specific interest in the following property,

Proposition 1. *A surjection $\pi : \mathcal{N} \to \mathcal{M}$ is a fibered manifold if and only if each point of the total space $\mathcal{N}$ admits a local section of π.*

Proof. First consider the converse and suppose there exists at each point $n \in \mathcal{N}$ a local section $\sigma : M \subset \mathcal{M} \to N \subset \mathcal{N}$, such that, $\pi\sigma = 1_{\mathcal{M}}$. Then using the functoriality of the tangent map, T, we have $T_n\pi : T_nN \to T_{\pi(n)}M$ satisfying $T_n\pi T_{\pi(n)}\sigma = T_{\pi(n)}1_{\mathcal{M}}$. So $T_n\pi$ has a local right inverse, $T_{\pi(n)}\sigma$ and therefore is locally surjective, that is π is a surjective submersion. Now the forward implication is a simple reverse of this argument. □

Note that this condition has no implications for the existence of global sections. Generally when it comes to global sections the choice of fibration is massively important. For example, if the fibrations have a single Lie group structure (as they do in principal $G-$bundles) then global sections are admitted if and only if the bundle is trivial; However, for vector space fibrations (vector bundles) there are infinitely many global sections in general... think vector fields. We will not discuss what is causing the obstruction here; However, in Section 2.2 we developed the beginnings of cohomology theory, which is needed to answer the question (specifically the Čech cohomology is needed but this is not discussed in the essay).

2.3.1 Ehresmann Connections

Familiar to the reader should be the notion of a connection on a fiber bundle, this is generalised here to the case of FMs.

Definition 12. *A connection on a fibered manifold $\pi : \mathcal{N} \to \mathcal{M}$ is called an **Ehresmann connection**, it corresponds to a choice of horizontal distribution $H\mathcal{N} \subseteq \pi^*T\mathcal{M}$ in the following short exact sequence of vector bundles,*

$$V\mathcal{N} \hookrightarrow T\mathcal{N} \xrightarrow{\pi_*} \pi^*T\mathcal{M}.$$

Where $V\mathcal{N} = \mathrm{Ker}(\pi_*)$ is the vertical distribution. So a connection is a section of the map π_*, $\sigma : \pi^*T\mathcal{M} \to T\mathcal{N}$ such that $\pi_*\sigma = 1$ which induces a splitting $T\mathcal{N} = V\mathcal{N} \oplus H\mathcal{N}$ with $H\mathcal{N} = \mathrm{Im}\,\sigma$.

These connections are probably the most crucial tool in the essay! They will crop up throughout, being applied to vector bundles, principal bundles and Lie algebroids. On these structures we require further structure preserving conditions on the connection. For example, on vector bundles we require $\mathbb{R}-$linearity, for the other two see Definitions 16 and 24 respectively.

2.3.2 Vector Bundles

Definition 13. *A* ***vector bundle*** *is a fibered manifold $\pi : \mathcal{N} \to \mathcal{M}$ and a fixed vector space V (called the* ***typical fiber****), such that the total space is locally trivial. That is, for induced chart $(U_\alpha, \varphi_\alpha)$ on $\mathcal{M}$ we have a fiber preserving diffeomorphism ψ_α such that,*

$$\begin{tikzcd} W_\alpha \arrow[r, "\psi_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times V \arrow[dl, "\mathrm{pr}_1"] \\ U_\alpha & \end{tikzcd}$$

commutes for all $\{W_\alpha\}$ in the fibered atlas. i.e. $\pi = \mathrm{pr}_1 \circ \psi_\alpha \quad \forall \alpha$. The collection $\{W_\alpha, \psi_\alpha\}$ is called the ***bundle atlas****.*

The name bundle atlas applies in the more general case of a fiber bundle, in which the fibers are not some fixed vector space but instead are just some general smooth manifold V. In the vector bundle case the bundle atlas has a structure group $GL(V)$, which means the transition functions (which are maps induced by the bundle atlas explicitly defined below) take value in this group. It is also often the case that one can reduce this to a $G-$structure, if the transition functions only take values in a subgroup $G \subseteq GL(V)$.

Definition 14. *On the intersection of two chart neighbourhoods, $W_\alpha \cap W_\beta$ in the bundle atlas $\{W_\alpha, \psi_\alpha\}$ of the vector bundle defined above, we can define the (unique)* ***transition functions*** *of the bundle, $\varphi_{\alpha\beta} : W_\alpha \cap W_\beta \to GL(V)$ such that,*

$$\psi_\alpha \circ \psi_\beta^{-1}(b, v) = (b, \varphi_{\alpha\beta}(b)(v)).$$

If these functions satisfy the ***cocycle conditions*** *(note that $\varphi_{\alpha\beta}\varphi_{\beta\alpha} = 1_{W_\alpha}$):*

$$\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$$

defined on $W_\alpha \cap W_\beta \cap W_\gamma$. they are called a ***cocycle of transition functions*** *for the open cover $\{W_\alpha\}$.*

It is a well known result[2] that given such a cocycle of transition functions, one can construct a vector bundle which is unique up to isomorphism. This allows us to define a unique vector bundle simply by stating what the fibers are and how they glue together. Here we shall just provide the fibers and assume they have some natural notion of gluing.

2.3.3 Principal Bundles in Gauge Theory

In the interest of brevity this section will merely establish what we mean by principal bundles and introduce some requisite results. For an introduction by example see Section 2.5 of former part III student M. Vákár's bachelor thesis [11], where he discusses in detail the U(1)−principal bundle gauge theory that describes electromagnetism. In part III Symmetries, Fields and Particles it is shown that to impose local symmetries on the Lagrangian of a field theory one must introduce a covariant derivative coupling the field to a gauge group potential where one then creates a gauge-invariant Lagrangian from this derivative. This construction has a natural geometric description in terms of connections on principal $G-$bundles over spacetime where G is the gauge group governing the internal symmetry.

[2] Those unfamiliar should see Chapter 6 in [4]

Definition 15. *A smooth **principal** $G-$**bundle** is a locally trivial fibered manifold $\pi : P \to \mathcal{M}$ and a Lie group G with smooth, fiber preserving, transitive, free right action $R : P \times G \to P$ on P which maps $(p, g) \mapsto R_g p = pg$.*

Fiber preserving simply means that on restriction to a fiber the image remains in that fiber, i.e. $\mathrm{Im}(R_g|_{P_p}) \subseteq P_p$, $\forall g \in G$; free action means all stabilizers are trivial; i.e. $gp = p \iff g = e_G$, $\forall p \in P$ and transitive means there is a single orbit, i.e. $R_G p = P$, $\forall p \in P$. The transitivity of the action tells us that the fibers are isomorphic to G.

Lemma 3. *The quotient bundle $\frac{P}{G}$ is diffeomorphic to the base $\mathcal{M}$.*

Proof. Factoring π over the orbit map $q : P \to P/G$ gives a smooth map $h : \mathcal{M} \to \frac{P}{G}$ with $h\pi = q$. This map is unique and bijective since π and q are smooth surjections[a].

[a] We have implicitly assumed some properties—and will continue to throughout the essay—first the $G-$action is proper and so by the quotient manifold theorem Theorem (15.3.4) of [12] the quotient map is smooth submersion. Second the universal property of the quotient map (1.2.3) ibidem.

It will be more useful much later but we note here since we are on the top of quotient bundles, that any quotient map $q : E \to \frac{E}{F}$ of a vector bundle $\pi_E : E \to P$ associated to a principal $G-$bundle $P :\to \mathcal{M}$, has the following commutative square.

$$\begin{tikzcd} E \arrow[r, "\pi_E"] \arrow[d, "q"', two heads] & P \arrow[d, "\pi", two heads] \\ \frac{E}{G} \arrow[r, "\pi_{\frac{E}{G}}"'] & M \end{tikzcd} \qquad \iff \qquad \pi_E \pi = \pi_{\frac{E}{G}} q \tag{2}$$

Connections on principal $G-$bundles are called principal connections and are defined as follows,

Definition 16. *A **principal connection** is an Ehresmann connection on a $G-$bundle which is $G-$**invariant**, that is the horizontal subspaces H_p for each $T_pP = V_pP \oplus H_pP$ that satisfies, $H_{pg} = (R_g)_* H_p$.*

$G-$invariance can also be defined using a commutative diagram, which we will use but it is instructive to consider the diagram for specific maps, so we construct the diagrams when needed. Given such a connection one can define a unique Lie algebra valued $1-$form whose kernel is the horizontal distribution and which is the Maurer-Cartan form on the vertical distribution.

Definition 17. *A **connection** $1-$**form** $\omega^H \in \Lambda^1 P \otimes \mathfrak{g} = \Omega^1(P, \mathfrak{g})$ is a Lie algebra $\mathfrak{g}-$valued $1-$form such that $\mathrm{Ker}(\omega^H) = HP$ for $G-$invariant horizontal distribution HP and supp$(\omega^H) = VP$ for the vertical distribution VP, where it acts as*

$$\omega^H_g = \mathrm{ad}(g^{-1}) R^*_{g^{-1}} \omega^H_e, \ \ \forall g \in G,$$

where $\omega^H_e = 1 : T_eG \to \mathfrak{g}$ and $\mathrm{ad} : G \otimes \mathfrak{g} \to \mathfrak{g}$ is the usual adjoint action on $\mathfrak{g}$. $G-$invariance of this corresponds to the equation

$$\omega^H((R_g)_* v) = (R_g)_* \omega^H(v)$$

for all $v \in \mathfrak{g}$ and $g \in G$.

3 Kaluza-Klein Metrics using Split Short Exact Sequences

The reader might question the relationship between an abstract homological tool and Kaluza-Klein theory. However strange it may seem at the outset, this approach is actually rather natural if one understands the intuition behind SESs.

Essentially a SES $A \hookrightarrow B \twoheadrightarrow C$ tells us that the object B is somehow composed of—at least in part—the objects A and C. In some loose sense this is an extension of what it means for an inner product space to be composed of orthogonal subspaces. Later it will be the case that we have SESs endowed with an inner product structure and on these this will be exactly the intuition; However, for now we will have to satisfy ourselves with the idea that A is, at least partly, a subobject of B that fills some of the space, and the quotient B/A at least contains C.

So why SSESs? Well the splitting eases the part of our mind that is asking just exactly how much of this quotient object does C take up and how much of A is contained in B? It is clear from the definition of SSES that C is in fact all that is left when we quotient by A, which is now completely a subobject of B (i.e. A contains nothing not in B). So in the inner product space case A and C will be orthogonal complements that span the space minimally.

Kaluza-Klein theory is concerned with the dimensional reduction of higher dimensional universes in such a way that the physics of this universe reproduce those of our own universe after the reduction. So a natural first question is how should we model these higher dimensional universes so that reduction leaves us with an acceptable model of our own universe? Geometrically the most tangible answer is to use a space that, at least locally, looks like our universe with somehow extended. This immediately suggests that the use of some sort of topological bundle. It is common in the literature to take a number of different, but related geometries, e.g. principal fiber bundles, associated bundles and homogeneous spaces. In the interest of generality we shall assume that the higher dimensional universe is a FM over a base manifold which is our universe. This construction can be restricted to all of the above spaces by imposing some constraints. In fact, the main objects of our consideration are sequences of vector bundles defined on the FM since they force a splitting of SESs by Theorem 1. This splitting allows us to isolate the directions which we do not consider a part of our own universe. From this we shall also obtain a fiber metric on the tangent space of the FM and hence a metric on the space itself. This allows us to investigate the geometry of our own universe through the higher dimensional geometry vis-à-vis Kaluza-Klein.

3.1 Kaluza-Klein Mechanism

The mechanism is the name we give to a theorem which will be the main focus of this section. We begin with a base manifold $\mathcal{M}$ over which we have a FM $\mathcal{N}$ projected onto the base by the surjective submersion π. Then over the FM we have a SES of vector bundles:

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[dr, "\pi_A"'] & B \arrow[r, two heads, "\beta"] \arrow[d, "\pi_B"] & C \arrow[dl, "\pi_C"] \\ & \mathcal{N} \arrow[d, "\pi"] & \\ & \mathcal{M} & \end{tikzcd} \tag{3}$$

We shall assume that A is a subbundle of B, thus by Lemma 1 we know the sequence is automatically split.

If we are going to assume that A is a subbundle, then why do we restrict ourselves to vector bundles? We just stated that one of the reasons for using vector bundles is that they automatically split the sequence. We shall see in the applications of this section that Kaluza-Klein has a nice form if we require a complex of associated vector bundles. In fact, we also extend these to Lie algebroids which have an underlying vector bundle structure.

Since the sequence is split then there exists a global section of $\beta : B \twoheadrightarrow C$, say $\sigma : C \hookrightarrow B$. On the fibers C_n and B_n over $n \in \mathcal{N}$ this gives the constraint that $\beta_n \sigma_n = 1_{\mathcal{N}}$. Using this we can decompose any element of B as $B = A \oplus \sigma(C)$. Written explicitly, an element b_n of the fiber B_n for any $n \in \mathcal{N}$ is such that,

$$b_n = (b_n - \sigma_n \beta_n b_n) + \sigma_n \beta_n b_n, \tag{4}$$

where the term is parentheses is the $A-$part of b_n since we have subtracted the projection on to the $C-$part $\sigma_n \beta_b b_n$. It is intuitively clear that if there is a decomposition of the whole space then there is an induced decomposition on the sections, i.e. $\Gamma(B) = \Gamma(A) \times \Gamma(C)$.

As previously discussed we want to understand the geometry of the structure $\mathcal{N}$ to do this we are going to introduce a metric structure on $\mathcal{N}$ just as we are used to from general relativity. But how should we define the metric? The answer will be the subject of the rest of the section and the motivation for the choice is discussed briefly at the end of the section. To this end let's consider how the second symmetric power functor acts on the sequence in (3):

$$\mathrm{Sym}^2(A) \xleftarrow{\iota^*} \mathrm{Sym}^2(B) \xleftarrow{\beta^*} \mathrm{Sym}^2(C) \tag{5}$$

How much of the structure of the original complex is preserved? First let's look at the vector bundle structure. Well we know that we can uniquely define a vector bundle given a cocycle of transition functions. Take for example, the vector bundle $\pi_A : A \to \mathcal{N}$ and suppose it is defined by the cocycle of transition functions $\varphi_{\alpha\beta} : W_\alpha \cap W_\beta \to GL(V)$ for the cover $\{W_\alpha\}$ and typical fiber V. Consider the action of the functor Sym^2 on the cocycle condition (see Definition 14) for diffeomorphisms $\varphi_{\alpha\beta}$;

$$\begin{aligned} \mathrm{Sym}^2(\varphi_{\alpha\delta}) = \mathrm{Sym}^2(\varphi_{\alpha\beta}\varphi_{\beta\delta}) \implies \varphi^*_{\alpha\delta} = \varphi^*_{\beta\delta}\varphi^*_{\alpha\beta} &\implies (\varphi^{-1}_{\delta\alpha})^* = (\varphi^{-1}_{\delta\beta})^*(\varphi^{-1}_{\beta\alpha})^* \\ &\implies \mathrm{Sym}^2(\varphi^{-1}_{\delta\alpha}) = \mathrm{Sym}^2(\varphi^{-1}_{\delta\beta})\,\mathrm{Sym}^2(\varphi^{-1}_{\beta\alpha}) \end{aligned} \tag{6}$$

The second line indicates that the inverse of the transition functions form a cocycle under the action of the pull back. Thus we have a cocycle of transition functions $\varphi^{-1}_{\alpha\beta} : \mathrm{Sym}^2(W_\alpha) \cap \mathrm{Sym}^2(W_\beta) \to \mathrm{Sym}^2(GL(V))$ for the cover $\{\mathrm{Sym}^2(W_\alpha)\}$ which uniquely determines a vector bundle $\pi_{\mathrm{Sym}^2(A)} : \mathrm{Sym}^2(A) \to \mathcal{N}$. So we retain the vector bundle structure, but what of the split exactness? Consider the action of Sym^2 on the commutative diagram in Definition 9:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "1"] & B \arrow[r, "\beta"] \arrow[d, "\eta"] & C \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & A \arrow[r] & A \oplus C \arrow[r] \arrow[d, "\mathrm{Sym}^2"] & C \arrow[r] & 0 \\
\mathrm{Sym}^2(0) & \mathrm{Sym}^2(A) \arrow[l] & \mathrm{Sym}^2(B) \arrow[l, "\iota^*"'] & \mathrm{Sym}^2(C) \arrow[l, "\beta^*"'] & \mathrm{Sym}^2(0) \arrow[l] \\
\mathrm{Sym}^2(0) & \mathrm{Sym}^2(A) \arrow[l] \arrow[u, "1"] & \mathrm{Sym}^2(A \oplus C) \arrow[l] \arrow[u, "\eta^*"] & \mathrm{Sym}^2(C) \arrow[l] \arrow[u, "1"] & \mathrm{Sym}^2(0) \arrow[l]
\end{tikzcd}$$

All is not well here! There is a problem, because the action of the functor on the morphisms (the pullback) is not additive, i.e. $\mathrm{Sym}^2(\alpha+\beta) = (\alpha+\beta)^* \neq \alpha^* + \beta^* = \mathrm{Sym}^2(\alpha) + \mathrm{Sym}^2(\beta)$. This means the functor Sym^2 does not preserve direct sums since $\mathrm{Sym}^2(A \oplus B) \ncong \mathrm{Sym}^2(A) \oplus \mathrm{Sym}^2(B)$. Thus the split exactness of the first set of commutative squares (as in Definition 9) is not preserved onto the second set by the functor. Nor even exactness, since the zero object is not preserved $\mathrm{Sym}^2(0) \neq 0$. This is because the bilinear form could be degenerate so there is no unique zero morphism as required in Definition 8. Since our goal is to preserve split exactness it is clear that we must require the functor is additive!

We could have noted that the sequence was not exact just by heeding Theorem 1. We just proved that this is a sequence of vector bundles and that it is not split thus is can not be exact. Another simple way to see why this is the case is to return to the intuitive dimensionality argument. Exactness of the original sequence implies that if $dim(A) = a$ and $dim(C) = c$ then $dim(B) = a+c$. Let's now look at the second sequence. The dimension of the second symmetric power of A is $\binom{a+1}{2}$. To see this consider a vector space V with basis $\{e_1, ..., e_a\}$ (i.e. $dim(V) = a$), then the basis of $\mathrm{Sym}^2(V)$ is $\{e_i \otimes e_j - e_j \otimes e_i | i \leq j\} = \{e_i e_j | i \leq j\}$ and this of course has dimension $\frac{a(a+1)}{2} = \binom{a+1}{2}$. Thus if the sequence is to be exact it must be the case that

$$\begin{aligned} dim(\mathrm{Sym}^2(B)) &= dim(\mathrm{Sym}^2(A)) + dim(\mathrm{Sym}^2(C)), \\ \binom{a+c+1}{2} &= \binom{a+1}{2} + \binom{c+1}{2}, \end{aligned}$$

which holds if and only if $a = c = 0$ an almost inconceivably uninteresting case.

So we have determined that this functor does not preserve exactness but preserves the vector bundle structure. The non-exactness is a consequence of the fact that the Sym^2 functor satisfies neither of the additive functor properties, that is preserving zero objects and also direct sums, so if we want to find a space of metrics which is split we need to restrict this sequence somehow. I.e we need to look at some subfunctor of Sym^2 which preserves zero objects and direct sums. Adding some restrictions is not a problem since we have not yet imposed that the symmetric bilinear forms must be non-degenerate, a defining requirement of a pseudo-Riemannian metric. So we form a subbundle of metrics[3] of A, B and C by simply restricting the respective Sym^2 bundle to the case where the forms are non-degenerate on the fibers. This subbundle is given by the action of the subfunctor $\mathfrak{M}$ (a functor which gives a subset of Sym^2 defined by some restriction is known as a subfunctor of Sym^2.) such that, $\mathfrak{M}(A) \subset \mathrm{Sym}^2(A)$ under this fiberwise non-degeneracy restriction. Then we define fiber (bundle) metrics as,

Definition 18. *A* ***fiber metric*** *on the bundle A is a map $g^A \in \Gamma \mathfrak{M} A$.*

The $\mathfrak{M}$ subfunctor acts exactly as Sym^2 except it also preserves zero objects, this functor is not additive. What further restrictions can we make so that we guarantee split exactness? Let's proceed by investigating the nature of the hindrance to the functor's additivity. Consider the action of a $\mathrm{Sym}^2(C)$ metric g, on two elements $b, \tilde{b} \in B$ after being pulled back using $(\alpha+\delta)^*$ where $\alpha, \delta : B \to C$:

$$\begin{aligned} ((\alpha+\delta)^* g)(b, \tilde{b}) &= g((\alpha+\delta)b, (\alpha+\delta)\tilde{b}) \\ &= ((\alpha^* + \delta^*)g)(b, \tilde{b}) + g(\alpha b, \delta \tilde{b}) + g(\delta b, \alpha \tilde{b}) \end{aligned}$$

So to achieve additivity we should restrict the action of the functor Sym^2 so that the target space of Sym^2 is such that the following holds,

$$g(\delta b, \alpha \tilde{b}) = 0, \qquad \forall\, b, \tilde{b} \in B. \tag{7}$$

[3] We are rather indiscriminate in our use of the term here. Often by a metric we simply mean a section of the Sym^2 space.

It is not immediately clear how to do this in general, so let's look at our specific case. Take the SES $A \xhookrightarrow{\iota} B \xrightarrow{\beta} C$, we know we have $B \cong A \oplus C$ and the decomposition is governed by the following equation (cf. (4)),

$$\iota\varrho + \sigma\beta = 1_B \tag{8}$$

Henceforth, 1 shall replace the identity of a space unless it is necessary to be more explicit. (8) is precisely the equation that we wish to preserve under action of our functor. So let's act on it by the pullback and suppose that $\mathfrak{g} \in \mathrm{Sym}^2(B)$ and $b, \tilde{b} \in B$:

$$\begin{aligned} 1^*\mathfrak{g}(b,\tilde{b}) &= ((\iota\varrho + \sigma\beta)^*\mathfrak{g})(b,\tilde{b}) \\ &= (((\iota\varrho)^* + (\sigma\beta)^*)\mathfrak{g})(b,\tilde{b}) + \mathfrak{g}(\iota\varrho b, \sigma\beta\tilde{b}) + \mathfrak{g}(\sigma\beta b, \iota\varrho\tilde{b}). \end{aligned}$$

So the equivalent of (7) here is,

$$\mathfrak{g}(\sigma\beta b, \iota\varrho\tilde{b}) = 0, \qquad \forall\, b, \tilde{b} \in B. \tag{9}$$

Note that $\iota\varrho$ and $\sigma\beta$ act as projections onto A and C respectively, and thus (9) implies that we require that the action of a metric in B vanishes when acting across the two constituent spaces A and C, i.e. A and C are orthogonal with respect to $\mathfrak{g}$. This can be imposed by requiring that the metric $\mathfrak{g}$ be non-degenerate on either A or C. Take A for example,

$$\mathfrak{g}|_A \quad \text{is non-degenerate} \iff \mathrm{Ker}\,\mathfrak{g}|_A = \{0\}, \tag{10}$$

i.e. if an entry has no A component then the action of the metric restricted to A vanishes. We will call the space of $\mathfrak{M}(B)$ metrics which are non-degenerate on restriction to A, $\mathfrak{M}_A(B)$. Note that this is the only additive morphism equation required to define a SSES, the other morphism equations from (1) in Definition 9 are composite not additive. So if we want to construct a subfunctor of Sym^2 which preserves split exact sequences, we only need it to be additive over the split space. Thus the functor must have knowledge of not only the split space but our choice of which subspace is required to be non-degenerate. It is for this reason we choose to define the following (very contrived) two argument map. The first argument, takes our chosen non-degenerate subspace, which then chooses which functor is applied to the second argument.

Definition 19. *$\mathfrak{M}(-,-)$ is a contravariant subfunctor in the second argument of $\mathfrak{M}$ (and thus of Sym^2) on the category **VectB** such that,*

$$\mathfrak{M}(E,F) = \begin{cases} \mathfrak{M}(F) & \text{if } E \nsubseteq F, \\ \mathfrak{M}_E(F) & \text{if } E \subseteq F. \end{cases}$$

*It acts on the morphisms of **VectB** as,*

$$\mathfrak{M}(E,f) = \begin{cases} \mathfrak{M}(f) & \text{if } E \nsubseteq \mathrm{Im}(f), \\ \mathfrak{M}_E(f) & \text{if } E \subseteq \mathrm{Im}(f), \end{cases}$$

where $\mathrm{Im}(f)$ is the image of the morphism f and $\mathfrak{M}_E$ is a Sym^2 subfunctor which acts on morphisms as,

$$\mathfrak{M}_E(f) = \mathfrak{M}(\mathrm{pr}_E\, f) + \mathfrak{M}((1 - \mathrm{pr}_E)f). \tag{11}$$

and is such that $\mathfrak{M}_E(F) \subset \mathfrak{M}(F)$.

How does this functor[4] act on our SSES of vector bundles (3)? As a subfunctor of Sym^2 it also preserves the vector bundle structure. What about the splitting, is it preserved? Well let us act on the sequence with the choice that A will be our non-degenerate space, then under action of $\mathfrak{M}(A,-)$ the resulting complex is,

$$\mathfrak{M}(A) \xleftarrow{\iota^*} \mathfrak{M}_A(B) \xhookleftarrow{\beta^*} \mathfrak{M}(C). \tag{12}$$

[4] Although this is not strictly a functor, for simplicity we will refer to it as such.

A few remarks, first note that the projections onto A have a specific form now that we have a specific SSES, $\mathrm{pr}_A = \iota\varrho = 1 - \sigma\beta$. Thus we have the immediate consequence that (using $\beta\iota = 0$),

$$\begin{aligned}\mathfrak{M}_A(\iota) &= ((1-\sigma\beta)\iota)^* + (\sigma\beta\iota)^* = \iota^* \\ \mathfrak{M}_A(\beta) &= \mathfrak{M}(\beta) = \beta^*\end{aligned}$$

Secondly to see that β^* is injective, and ι^* is surjective, consider the pullback action on the defining equations for the existence of the retraction ϱ and section σ in Lemma 1:

$$\beta\sigma = 1, \qquad \varrho\iota = 1 \qquad \xrightarrow{\mathfrak{M}(A,\cdot)} \qquad \sigma^*\beta^* = 1, \qquad \iota^*\varrho^* = 1 \tag{13}$$

That is β^* is left-cancellable and ι^* is right-cancellable on all of Sym^2 and thus these also hold on the restrictions. The calculation of these two is discussed after (14). Finally $\mathfrak{M}_A(A) = \mathfrak{M}(A)$ simply by definition.

What about exactness and split exactness? Well we can check either and, by Theorem 1, the other is immediate. So we unsurprisingly choose to check that the functor preserves split exactness since this is what we constructed it to do. Consider the action of the functor on the composite morphism equations (1) from Definition 9, two of which we have actually just checked in (13):

$$\beta\iota = 0, \qquad \varrho\sigma = 0 \qquad \xrightarrow{\mathfrak{M}(A,\cdot)} \qquad \iota^*\beta^* = 0, \qquad \sigma^*\varrho^* = 0 \tag{14}$$

One needn't even calculate the action of $\mathfrak{M}(A,\cdot)$ on the composite morphisms above. Simply note that the image of these two morphisms is either A or C, and therefore, the action is the same on all four (just the Sym^2 pullback action), since the projection pr_A on an element of A is simply the identity. Now the additive morphism equation from (1):

$$\begin{aligned}\iota\varrho + \sigma\beta = 1 \qquad \xrightarrow{\mathfrak{M}(A,\cdot)} \qquad & ((1-\sigma\beta)(\iota\varrho+\sigma\beta))^* + (\sigma\beta(\iota\varrho+\sigma\beta))^* = 1 \\ (\iota\varrho)^* + (\sigma\beta)^* &= 1 \\ \mathfrak{M}_A(\iota\varrho) + \mathfrak{M}_A(\sigma\beta) &= 1 \end{aligned} \tag{15}$$

Therefore (12) is a SSES of vector bundles. So the following diagrams commute:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "1"] & B \arrow[r, "\beta"] \arrow[d, "\eta"] & C \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & A \arrow[r] & A \oplus C \arrow[r] \arrow[dd, "\mathfrak{M}(A,\cdot)"] & C \arrow[r] & 0 \\
& & & & \\
0 & \mathfrak{M}(A) \arrow[l] & \mathfrak{M}_A(B) \arrow[l, "\iota^*"'] & \mathfrak{M}(C) \arrow[l, "\beta^*"'] & 0 \arrow[l] \\
0 & \mathfrak{M}(A) \arrow[l] \arrow[u, "1"] & \mathfrak{M}(A, A\oplus C) \arrow[l] \arrow[u, "\mathfrak{M}(A,\eta)"] & \mathfrak{M}(C) \arrow[l] \arrow[u, "1"] & 0 \arrow[l]
\end{tikzcd}$$

With $\mathfrak{M}(A, A \oplus C) \cong \mathfrak{M}(A) \oplus \mathfrak{M}(C)$.

For completeness we also check exactness, which is almost trivial given the above in (13)-(15). First injectivity and surjectivity are already proved using (13). So we need only consider $\mathrm{Ker}\,\iota^* =$

$\operatorname{Im}\beta^*$? Well equations in (14) imply that $\operatorname{Ker}\iota^* \supseteq \operatorname{Im}\beta^*$. What about the reverse inclusion? Consider some morphism x into $\mathfrak{M}_A(B)$ and suppose $x \in \operatorname{Ker}\iota^*$, i.e. $\iota^* x = 0$. Then

$$\varrho^*\iota^* x = 0 \overset{(15)}{=} (1-\beta^*\sigma^*)x,$$

therefore $x \in \operatorname{Im}\beta^*$. Thus the reverse inclusion holds also so the sequence is exact.

We know that a splitting on the total space of a bundle induces one on its fibers, so let's find the form of the metrics in the fibers of $\mathfrak{M}_A(B)$. Suppose $\mathfrak{g} \in \Gamma\mathfrak{M}_A(B)$ then the splitting induced by $\mathfrak{M}(A,\cdot)$ is,

$$\mathfrak{g} = (1-\sigma\beta)^* g^A + \beta^* g^C. \tag{16}$$

Here we have inserted g^A since if $b, \tilde{b} \notin A$ then $(1-\sigma\beta)^*\mathfrak{g}(b,\tilde{b}) = 0$ and also since σ^* is surjective then we can replace $\sigma^*\mathfrak{g}$ with $g^C \in \operatorname{Sym}^2(C)$.

Theorem 2. *Given a short exact sequence of vector bundles $A \overset{\iota}{\hookrightarrow} B \overset{\beta}{\twoheadrightarrow} C$, we can form the following short exact sequence of vector bundles,*

$$\mathfrak{M}(A) \overset{\iota^*}{\twoheadleftarrow} \mathfrak{M}_A(B) \overset{\beta^*}{\hookleftarrow} \mathfrak{M}(C).$$

This defines a pseudo-Riemannian fiber metric on B, $\mathfrak{g} \in \Gamma\mathfrak{M}_A(B)$ which has form as in (16) and is called the **Kaluza-Klein metric**. We also have fiber metrics $\iota^*\mathfrak{g}$ and $\sigma^*\mathfrak{g}$ on the constituent spaces A and C where ($\sigma^* : \mathfrak{M}(C) \to \mathfrak{M}_A(B)$ is the pullback of $\sigma : C \to B$ the section of β).

Proof. The split exactness has been proved above. Using (16) $\mathfrak{g}$ is immediately non-degenerate since g^A and g^C are also. Finally,

$$\iota^*\mathfrak{g} = \iota^*(1-\sigma\beta)^* g^A + \iota^*\beta^* g^C = \iota^* g^A = g^A$$
$$\sigma^*\mathfrak{g} = \sigma^*(1-\sigma\beta)^* g^A + \sigma^*\beta^* g^C = g^C$$

This construction may look somewhat contrived (and it is) but this method of defining the Kaluza-Klein metric was developed after a plethora of work in the field so the author had something to aim at. If the reader is familiar with the Riemannian geometry of the tangent bundle then they will recognise that the form of the metric resembles that of the natural Sasaki metric induced on the tangent bundle of a Riemannian manifold. Normally the Sasaki metric is induced on the tangent bundle by the metric on the base; However, the goal here is somewhat reversed: we consider a metric on the bundle first, and put it in a Sasaki metric type form and then one recovers the unique metric on the base it was induced by. This should remind the reader of the method of Kaluza: find a higher dimensional metric and from it induce the lower dimensional metric.

3.2 Some Applications

As we know, $G-$bundles are the mathematical playground of gauge theorists interested in abelian and non-abelian gauge theories such as electromagnetism and Yang-Mills. So let's apply the above generalities to the case of a $G-$bundle, $\pi : P \to \mathcal{M}$ for an arbitrary Lie group G. Immediately by definition as a FM we have $\pi_* = TP \twoheadrightarrow T\mathcal{M}$ thus we can induce a fiberwise surjective morphism $X \in T_pP \mapsto (p, \pi_* X) \in \pi^* T_{\pi(p)}\mathcal{M}$ to give the following SES of principal bundles,

$$VP \hookrightarrow TP \twoheadrightarrow \pi^* T\mathcal{M}, \tag{17}$$

with $VP = \operatorname{Ker}\pi_*$ the vertical bundle and $\pi^*T\mathcal{M}$ is the pullback bundle.

We could work with this sequence and derive a metric using Theorem 2; However, the action of the gauge fields, their potentials and curvatures results in a differential form in the Lie algebra rather than on the base manifold. Since physicists tend to work with forms on the base manifold we want to use a related sequence that has action on the base. $G-$invariance of the connection $1-$form in Definition 17 allows us to do this by "dividing" by G.

First we will consider some helpful constructions. Recall the fact that a principal $G-$bundle's transition functions can be used to construct a vector bundle provided that G has some representation on $\mathbb{R}^{dimG}$. This is defined as follows,

Definition 20. *Let $\pi : P \to \mathcal{M}$ be a principal $G-$bundle for Lie group G, with representation $\varrho : G \to \mathrm{GL}(V)$ where V is a representation vector space. Define a action of G on $P \times V$ such that, $(p, v) \mapsto (pg^{-1}, \varrho(g)v) \in P \times V$. Then the quotient bundle (orbit space) $E = (P \times V)/G$ with projection map $\pi_E : E \to \mathcal{M}x$, such that, $\pi_E(\langle p, v\rangle) = \pi(p)$ for an orbit $\langle p, v\rangle$, is called the* ***associated vector bundle****. Generally this is written $E = P \times_G V$.*

An important example for our purposes is,

Example 5. The **adjoint bundle** of a principal $G-$bundle $\pi : P \to \mathcal{M}$ is the associated bundle $P \times_G \mathfrak{g}$ over the same base $\mathcal{M}$ using the adjoint representation of G on $\mathfrak{g}$. This bundle is sometimes written as $\mathrm{ad}(P)$.

This bundle is of interest to us because,

Lemma 4. *VP from* (17) is isomorphic to the adjoint bundle $P \times \mathfrak{g}$.

Proof. First construct a map $P \times \mathfrak{g} \to TP$, such that, $(p, X) \mapsto T(R(p))_e(X)$ for $R : P \times G \to P$, the right action on P then $T(R) : TP \times TG \to TP$. Now construct the composite map $T(R) \circ (0, \iota) : P \times \mathfrak{g} \to TP$ defined by $(T(R) \circ (0, \iota))(p, X) = T(R(p))_e(X)$. Here $\iota : T_eG \to TG$ is the inclusion map and $0 : P \to TP$ is the zero section. Clearly the map is injective, since its constituent parts are. Note that $\operatorname{Im} T(R(p)) \subseteq VP$ since $T(\pi R(p)) = 0$ using the fact that R is fiber preserving. Finally using the fact that both have the same bundle rank $dim\mathfrak{g}$ the two spaces are fiberwise isomorphic. □

Note that the map $P \times \mathfrak{g} \to TP$ is indeed a section of the bundle $P \times \mathfrak{g}$ which provides some motivation for its common name, the **fundamental vector field** $X^{\#}$ where $(p, X) \mapsto X^{\#}(p)$. These vector fields on $P \times \mathfrak{g}$ coincide with the vertical tangent vector fields. Thus the complex in (17) becomes

$$\begin{tikzcd} P \times \mathfrak{g} \arrow[r, hook] \arrow[dr] & TP \arrow[r, two heads] \arrow[d] & \pi^* T\mathcal{M} \arrow[dl] \\ & P & \end{tikzcd} \tag{18}$$

Before taking quotients one should check the $G-$invariance[5] of the inclusion map $\iota : P \times \mathfrak{g} \to TP$, $(p, X) \mapsto T(R^p)_e(X)$ which equates to the following diagram commuting,

$$\begin{tikzcd} P \times \mathfrak{g} \arrow[r, "\mathrm{ad}_g"] \arrow[d, "\iota"] & P \times \mathfrak{g} \arrow[d, "\iota"] \\ TP \arrow[r, "T(R_g)"] & TP \end{tikzcd} \tag{19}$$

where ad_g represents the adjoint action on $\mathfrak{g}$ $(p, X) \mapsto (pg^{-1}, \mathrm{ad}_g X)$. To see that this commutes, simply note the functoriality of T and use $T(\mathrm{Ad}_g) = \mathrm{ad}_g$, then $T(R^{pg^{-1}})(\mathrm{ad}_g X) =$

[5] Essentially the $G-$invariance ensures we have a well-defined $G-$structure on the quotient bundle. Although we will not explicitly define quotients under $G-$action of vector bundles here, to understand why we must check $G-$invariance one should look to Proposition 3.1.1 in [5].

$T(R^{pg^{-1}}\,\mathrm{Ad}_g)_e(X)$. The final step of the calculation is some trivial rearrangement of right actions. We can now take the quotient of the map to obtain the bundle morphism,

$$\jmath : \frac{VP}{G} \cong \frac{P\times\mathfrak{g}}{G} \times_G \mathfrak{g} \cong \mathrm{ad}(P) \hookrightarrow \mathrm{At}(P) \cong \frac{TP}{G}; \qquad \langle u, X\rangle = \langle X^{\#}(p)\rangle. \tag{20}$$

Let's now look at the quotient bundle of $\pi^*T\mathcal{M}$. As before one should check the $G-$invariance before quotienting but the argument is similar to the above. So we skip to the quotienting and look at another isomorphism.

Lemma 5. *$\frac{\pi^*T\mathcal{M}}{G}$ is isomorphic to the tangent bundle of the base $T\mathcal{M}$.*

Proof. Replicating the argument in Lemma 3, factoring the differential of the projection map $T\pi : TP \to T\mathcal{M}$ over the map $TP \to \pi^*T\mathcal{M}$ described above (17) and composing the resultant diffeomorphism $h : T\mathcal{M} \to \pi^*T\mathcal{M}$ with the orbit map of $\pi^*T\mathcal{M}$, $\tilde{q} : \pi^*T\mathcal{M} \to \frac{\pi^*T\mathcal{M}}{G}$ gives a map $\tilde{q}h : T\mathcal{M} \to \frac{\pi^*T\mathcal{M}}{G}$ which is a diffeomorphism since h is injective and $\tilde{q}$ is surjective.

The combination of Lemmas 4 and 5 allows us to form the following SES which has $\mathcal{M}-$valued $1-$form connections. Note that the action of $T\pi : TP \to T\mathcal{M}$ on $\frac{TP}{G}$ preserves the equivalence classes, since $R_g : P \to P$ is fiber preserving and $T\pi$ is surjective. Thus quotienting to the map $\pi_* : \frac{TP}{G} \to T\mathcal{M}$ is a well defined bundle morphism and is surjective.

Definition 21. *The **Atiyah sequence** of vector bundles, for a principal $G-$bundle $\pi : P \to \mathcal{M}$ is,*

$$\begin{tikzcd} ad(P) \arrow[r, hook, "\jmath"] \arrow[dr] & At(P) \arrow[r, two heads, "\pi_*"] \arrow[d] & T\mathcal{M} \arrow[dl] \\ & \mathcal{M} & \end{tikzcd} \tag{21}$$

where $\mathrm{ad}(P) = P \times_G \mathfrak{g}$ *is the adjoint bundle and* $\mathrm{At}(P) = \frac{TP}{G}$.

This sequence can be equipped with Lie brackets making it a sequence of Lie algebroids which will be the topic of Section 4. In fact, in that section we show that $\mathrm{ad}(P)$ is a bundle of Lie algebras (if the base is connected then $\mathrm{ad}(P)$ becomes a Lie algebra bundle with typical fiber $\mathfrak{g}$, although this is not proved.) Thus if we have an inner product on $\mathfrak{g}$, that defines a fiber metric on $\mathrm{ad}(P)$ if it respects the equivalence relation we quotiented by, i.e. the inner product is $\mathrm{ad}-$invariant. Another nice property we will quote without proof is that since we have an $\mathrm{ad}-$invariant bundle metric, the scalar curvature associated to it is constant on the fibers[6] and thus simply projects onto the base. Well if we recall some Lie theory, we can immediately define such a metric.

Recall that the Killing form $k : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ of a Lie algebra $\mathfrak{g}$, is a symmetric bilinear form which is $G-$invariant, is non-degenerate if and only if G is semi-simple and non-negative semi-definite if G is compact. Also the combination of these last two, i.e. if G is compact and semi-simple, imposes that k is negative definite. Thus it is clear that $\kappa = -k$ will give a Riemannian metric on $\mathrm{ad}(P)$ whenever G is compact and semi-simple. Also recall that given a basis $\{T^a\}$ of $\mathfrak{g}$ then the components of the Killing form can be written as $\kappa^{ab} = -f^{ad}_c f^{bc}_d$, where f^{ab}_c are the structure constants of G. Then by (21) we have,

$$\begin{tikzcd} \mathrm{ad}(P) \arrow[r, hook, shift left, "\jmath"] & \mathrm{At}(P) \arrow[l, shift left, "\omega_\sigma"] \arrow[r, two heads, shift left, "\pi_*"] & T\mathcal{M}. \arrow[l, shift left, "\sigma"] \end{tikzcd}$$

where σ is a right splitting corresponding to a connection on P and ω_σ is its corresponding connection $1-$form (there is more discussion on why this is the case in Section 4).

[6] Proof can be found in Theorem 9.3.7 of [3]

Thus, applying Theorem 2, using this metric and a tangent bundle metric g corresponding to a metric on the base $\mathcal{M}$, we have the following metric $\mathfrak{g}$ on $\mathrm{At}(P)$ by (16),

$$\mathfrak{g} = (\omega_\sigma^* \jmath^* + (\pi_*)^* \sigma^*)\mathfrak{g} = \omega_\sigma^* \kappa + \pi^* g, \tag{22}$$

where $(\pi_*)^*\mathfrak{g}$ is the usual notion of pullback by π, denoted $\pi^* g$. What does this tell us about the $G = U(1)$ case, i.e. electromagnetism? $U(1)$ is abelian so its structure constants and thus its Killing form vanishes, making it not a poor choice for our metric. Note however, that any inner product on $\mathfrak{u}(1)$ is ad $-$invariant, so let us choose the simplest one with $\kappa = 1$. Let's now look at ω_σ: first take trivial bundle $P = \mathcal{M} \times U(1)$ then $\mathrm{ad}(P) = \mathcal{M} \times \mathfrak{u}(1)$ and $\mathrm{At}(P) = T\mathcal{M} \times \mathfrak{u}(1)$ using the identification $TG = G \times \mathfrak{g}$ for any Lie group G and its Lie algebra $\mathfrak{g}$. Therefore $\omega_\sigma : \mathrm{At}(P) \to \mathrm{ad}(P)$ is a $\mathfrak{u}(1)-$valued $1-$form, but $\mathfrak{u}(1) = \{i\theta |\, \theta \in \mathbb{R}\}$ so choosing $\{x^\mu\}$ a basis for $\mathcal{M}$ and $\{x^i\}$ basis of P, we have $\omega = A_\mu dx^\mu + \theta d\theta$, resulting in the following local form of the metric,

$$\mathfrak{g}_{ij} dx^i dx^j = g_{\mu\nu} dx^\mu dx^\nu + A_\mu A_\nu dx^\mu dx^\nu + A_\mu \theta dx^\mu d\theta + \theta A_\nu d\theta dx^\nu + \theta^2 d\theta^2, \tag{23}$$

which is exactly the metric of the original Kaluza-Klein theory! In fact, one can use standard differential geometric techniques to calculate the Riemann tensor and show that on the base we get a theory of gravity and electromagnetism. Here we merely state that when we look at the curvature of the connection (see Section 4) $F = d\omega_\sigma$, we obtain the Bianchi identity $dF = 0$ and also the Yang-Mills action $F \wedge \star F$ gives the standard electromagnetic action on the base.

4 The Atiyah Sequence and Lie Algebroids

Lie algebroids are to Lie groupoids as Lie algebras are to Lie groups. That is, they are related by Lie integration—the familiar process of going from a Lie algebra to a Lie group—but, of course, generalised to consider algebroids and groupoids. The objects with the -oid suffix are actually more general than their non-oid partners, although the suffix may suggest otherwise. Using the language of category theory, the -oids are horizontal categorifications of algebras and groups, meaning their structure is extended over many objects rather than a single object. Just as we did not explicitly define a category, our definition of a groupoid is also not very explicit, although it is precise.

Definition 22. *A **groupoid** is a (small)[a] category in which all morphisms are invertible.*

[a] Small meaning the set of objects is small enough to be a set and not a proper class.

As an illustration consider a group as a structure that naturally describes the set of automorphisms of an object, i.e. the symmetries of an object. Extending this concept to contain the set of symmetries—isomorphisms—between multiple objects is exactly that of a groupoid. Inbetween pairs of objects there are morphisms which can be composed associatively, with neutral and inverse elements, i.e. restricting to a single object returns the concept of a group.

The following makes no attempt to be precise but gives the reader an idea of the motivation for considering these objects in a physics setting. One may actually think of gauge redundacy as a result of reversing the horizontal categorification—what is unsurpisingly called vertical categorification—of the configuration groupoid whose morphisms are gauge transformations. The result of this operation is the familiar quotient space called the moduli space. Some information is retained in this process, namely the gauge invariant configurations (infinitesimally this corresponds to the 0th cohomology class), but also some is lost (the higher cohomology classes). For example, take information contained in the BRST cohomology groups, we know the BRST closed states (0th class) contain the physical field information, but after the vertical categorification we would lose the higher cohomologies and thus the information in the ghost fields.

Lie groupoids are the obvious smooth manifold variant of a groupoid—the objects are smooth manifolds. The infinitesimal version of these objects are the Lie algebroids, but just as with the non-oid Lie theory we can define a Lie algebroid without making reference to the groupoid.

Definition 23. *A **Lie algebroid** on $\mathcal{M}$ is a vector bundle $\pi : E \to \mathcal{M}$ with a **Lie bracket** $[\cdot,\cdot]_E : \mathit{\Gamma}(E) \times \mathit{\Gamma}(E) \to \mathit{\Gamma}(E)$ and a bundle morphism $\rho : E \to T\mathcal{M}$ called the **anchor map**. The Leibniz rule must hold for $X, Y \in \mathit{\Gamma}(E)$ and $f \in C^\infty(\mathcal{M})$,*

$$[X, fY]_E = \mathcal{L}_{\rho(X)}(f)Y + f[X,Y]_E, \tag{24}$$

with $\mathcal{L} : \mathit{\Gamma} T\mathcal{M} \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ the Lie derivative. Also the anchor commutes with the bracket, i.e. is a homomorphism,

$$\rho([X,Y]_E) = [\rho(X), \rho(Y)]_{T\mathcal{M}}.$$

We say a Lie algebroid is **transitive** if its anchor is a surjective, **regular** if the anchor's rank is constant and **totally intransitive** if its anchor vanishes identically. Consider the following simple examples of Lie algebroids,

Example 6. Suppose we take a manifold consists of a single point, $\mathcal{M} = \{\star\}$, then the $T\mathcal{M} = T_\star\mathcal{M}$ is simply a vector space. The anchor provides an vector space isomorphism $E \cong T\mathcal{M}$. So this Lie algebroid is simply a vector space equipped with a Lie bracket, hence it is a Lie algebra.

Example 7. Suppose the anchor map is the identity, then the vector bundle $E = T\mathcal{M}$ and so the tangent bundle of any manifold endowed with the Lie bracket of vector fields on $\mathcal{M}$, $[\cdot,\cdot]_{T\mathcal{M}} : \Gamma T\mathcal{M} \times \Gamma T\mathcal{M} \to \Gamma T\mathcal{M}$, is a transitive Lie algebroid.

Example 8. Suppose we take the zero anchor map, i.e. a totally intransitive Lie algebroid. Then we are left with a vector bundle structure with a Lie bracket on each of its fibers, since the bracket is now $C^\infty(\mathcal{M})-$linear, i.e. $[X, fY]_E(m) = f(m)[X(m), Y(m)]_m$. Thus the fibers of the bundle are Lie algebras and thus the Lie algebroid is a bundle of Lie algebras. If there is a single Lie algebra structure, i.e. each fiber is the same Lie algebra, then this is a Lie algebra bundle.

To prove that the Atiyah sequence from (eq: Atiyah VB) is a sequence of Lie algebroids we first need the following lemma,

Lemma 6. *The sections $\Gamma(\frac{E}{G})$ are isomorphic to the $G-$invariant sections $\Gamma^G E$.*

Proof. First note that a section $X \in \Gamma E$ for vector bundle $\pi_E : E \to P$ with $G-$action $\mathcal{R}$, where $\pi : P \to \mathcal{M}$ is a principle $G-$bundle, is $G-$invariant—written as $X \in \Gamma^G(E)$—if the following diagram commutes,

$$\begin{tikzcd} G \times E \arrow[r, "\mathcal{R}"] & E \\ G \times P \arrow[u, "id_G \times X"] \arrow[r, "R"'] & P \arrow[u, "X"'] \end{tikzcd}$$

Using thepasting together the commutative diagram definitions of sections (the second and fourth square), $G-$invariance (first square), and quotient bundles (third square) we get the following diagram,

$$\begin{tikzcd} G \times E \arrow[r, two heads, "\mathcal{R}"] & E \arrow[r, "1_E"] & E \arrow[r, two heads, "q"] \arrow[d, two heads, "\pi_E"] & \frac{E}{G} \arrow[r, "1_{\frac{E}{G}}"] \arrow[d, two heads, "\pi_{\frac{E}{G}}"] & \frac{E}{G} \\ G \times P \arrow[u, hook, "id_G \times \underline{X}"] \arrow[r, two heads, "R"'] & P \arrow[u, hook, "\underline{X}"'] \arrow[r, "1_P"'] & P \arrow[r, two heads, "\pi"'] & \mathcal{M} \arrow[r, "1_{\mathcal{M}}"'] & \mathcal{M} \arrow[u, hook, "X"'] \end{tikzcd}$$

Note that the right actions for $g \in G$, $\mathcal{R}_g$ and R_g, are vector bundle isomorphisms, so the diagram tells us that $X\pi = q\underline{X} = q\mathcal{R}_g\underline{X}R_g^{-1}$. The $\mathcal{R}_g\underline{X}R_g^{-1}$ restricts q to the image of $\underline{X}$ on which it is injective, and so bijective. We use this to construct the following morphism

$$\pi^* q^{-1} : \Gamma\left(\frac{E}{G}\right) \to \Gamma^G E, \quad \underline{X} = \pi^* q^{-1} X = q^{-1} X\pi$$

Which is an isomorphism with inverse qX which acts on the preimage $\pi^{-1}\mathcal{M}$ □

Using this lemma we can construct the following Lie brackets on the Atiyah sequence.

Lemma 7. *$At(P)$ is the Atiyah Lie algebroid over $\mathcal{M}$ associated to the principal bundle $\pi : P \to \mathcal{M}$, with anchor map $\pi_* : \mathrm{At}(P) \twoheadrightarrow T\mathcal{M}$ and Lie bracket $[\cdot,\cdot] : \Gamma(\mathrm{At}(P)) \times \Gamma(\mathrm{At}(P)) \to \Gamma(\mathrm{At}(P))$ of $G-$invariant vector fields on P $\Gamma^G TP$.*

Proof. The Atiyah sequence of vector bundles in (eq: Atiyah VB) induces a SES of $C^\infty(M)-$modules of sections,

$$\Gamma(\mathrm{ad}(P)) \xhookrightarrow{\jmath} \Gamma(\mathrm{At}(P)) \xrightarrow{\pi_*} \Gamma T\mathcal{M}, \tag{25}$$

which we endow with a Lie bracket structure—with the intent to use these as the Lie brackets for our Lie algebroid structure. We choose the canonical choice of bracket on $\Gamma T\mathcal{M}$—the Lie bracket of vector fields. Using Lemma 6, $\Gamma(\mathrm{At}(P)) \cong \Gamma^G TP$ and the elements of the latter are R_g-related to themselves, meaning $T(R_g)X = XR_g$. Then the canonical bracket on ΓTP preserves this, that is the restricted bracket closes.
Before considering the bracket on $\mathrm{ad}(P)$, note that the action of the map $\pi_* : \Gamma(\mathrm{At}(P)) \twoheadrightarrow \Gamma T\mathcal{M}$ on the bracket of $\Gamma(\mathrm{At}(P))$ is a Lie algebra homomorphism. To see that this holds consider the following pushout out as in Defintion 2 of the following commutative square,

$$\begin{tikzcd}
P \arrow[r, "\underline{X}"] \arrow[d, two heads, "\pi"'] & TP \arrow[d, two heads, "q"'] \arrow[ddr, bend left, two heads, "T\pi"] & \\
M \arrow[r, "X"] \arrow[drr, bend right, "\overline{X}"'] & \frac{TP}{G} \arrow[dr, dashed, two heads, "\pi_*"] & \\
 & & T\mathcal{M}
\end{tikzcd} \implies \pi_*([X,Y]) = [\pi_* X, \pi_* Y] \quad \forall X, Y \in \Gamma(\frac{TP}{G}).$$

To see this note that the brackets on both $\Gamma^G TP$ and $\Gamma T\mathcal{M}$ close. The first bracket closes simply by showing it preserves R_g-relatedness, $T(R_g)[\underline{X},\underline{Y}] = [\underline{X},\underline{Y}]R_g$. Then using $\overline{X}\pi = T\pi\underline{X}$, implies $T\pi\underline{[X,Y]} = \overline{[X,Y]}\pi = [\overline{X},\overline{Y}]\pi$ by the closures of the bracket. But $\underline{X}$ is $\pi-$related to $\pi_*(X)$, therefore $T\pi\underline{[X,Y]} = \pi_*[X,Y]\pi = [\overline{X},\overline{Y}]\pi$. Finally since $\overline{X} = \pi_*(X)$ and π is surjective, we have that π_* is a Lie algebra homomorphism. Thus it is immediate that $\Gamma(\mathrm{ad}(P)) = \mathrm{Ker}(\pi_*)$ is a Lie subalgebra, i.e. the restriction of the bracket on $\Gamma(\mathrm{At}(P))$ to $\Gamma(\mathrm{ad}(P)$ is well-defined. Finally that the anchor π_*, is surjective implies that $\mathrm{At}(P)$ over $\mathcal{M}$ associated to $\pi : P \to \mathcal{M}$ is a transitive Lie algebroid. □

Note that this makes the SES of (eq: Atiyah VB) into a SES of Lie algebroids. Let's turn our attention to the Riemannian geometry of Lie algebroids before applying it the transitive Lie algebroid case.

4.1 Riemannian Geometry and Differential Calculus of Lie Algebroids

As fibered manifolds, the Lie algebroids carry a Ehresmann connection furnished with some extra structure preservation properties, that preserve the Lie algebroid structure; However, the following definition generalises the notion of an Ehresmann connection on Lie algebroids, which would require Lie bracket preservation (We will call these connections **ordinary**), by dropping this condition.

Definition 24. *An $\mathcal{A}-$**connection** for a Lie algebroid $(\mathcal{A}, \mathcal{M}, [\cdot,\cdot], \rho_{\mathcal{A}})$ is a $\mathbb{R}-$linear morphism on the vector bundle $\pi : E \to \mathcal{M}$, $\nabla : \Gamma E \to \Gamma(\mathcal{A}^* \otimes E)$—equivalently $\nabla : \Gamma E \times \Gamma\mathcal{A} \to \Gamma E$—which is $C^\infty(\mathcal{M})-$linear in its argument and the outcome is Leibniz—as in (24). That is for $\alpha \in \Gamma A$, $\epsilon \in \Gamma E$ and $f \in C^\infty(\mathcal{M})$,*

$$\nabla_\epsilon(f\alpha) = f\nabla_\epsilon\alpha + \mathcal{L}_{\rho(\epsilon)}(f)\alpha,$$
$$\nabla_{f\epsilon}(\alpha) = f\nabla_\epsilon\alpha.$$

This object, which is the analogue of a covariant derivative, allows us to promote a number of important quantities of standard gauge theory on principal bundles to their counterparts on Lie algebroids. We interpret these notions similarly to their counterparts on the tangent bundle but now, instead of acting on vector fields, they act on a type of generalised vector field which is related to the original vector fields by the anchor. First, a notion of curvature of a Lie algebroid is defined by adapting the normal curvature form.

Definition 25. *The composite map* $F_\nabla : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \times \Gamma(E) \to \Gamma(E)$ *such that*

$$F_\nabla(\alpha, \beta, \nu) = \nabla_\alpha \nabla_\beta \nu - \nabla_\beta \nabla_\alpha \nu - \nabla_{[\alpha,\beta]} \nu$$

is a tensor since it is $C^\infty(M)-$*trilinear. Therefore the* ***curvature*** *tensor of the Lie algebroid* $\mathcal{A}$ *or simply the* $\mathcal{A}-$*curvature is* $F_\nabla \in \Omega^2(\mathcal{A}, \mathrm{End}(E))$, *a* $2-$*form with values in* $\mathrm{End}(E)$ *given by,*

$$F_\nabla(\alpha, \beta) = [\nabla_\alpha, \nabla_\beta] - \nabla_{[\alpha,\beta]} \tag{26}$$

which is the obstruction of the connection ∇, *to be a Lie algebra homomorphism.*

Trilinearity follows by noting the first argument is linear by definition and after a little algebra (exactly as in the standard curvature tensor case) the second and third arguments follow.

Secondly, associated with a covariant derivative, ∇, there is a unique associated exterior covariant derivative d_∇, which acts on $\Gamma\left(\bigwedge^\bullet \mathcal{A} \otimes E\right)$ to form the differential graded commutative algebra known as the Chevalley-Eilenberg algebra of the Lie algebroid, $(\Omega^\bullet(A, E), d_\nabla)$. But first we need another simple notion,

Definition 26. *A Lie algebroid* ***representation*** *for* $\mathcal{A}$ *is a Lie algebroid morphism* $\phi : \mathcal{A} \to \mathcal{D}(E)$ *where* $D(E)$ *is the space of derivations on* E.

An important example for the next section is,

Example 9. The **adjoint** representation for A a transitive Lie algebroid (see (29)) is a Lie algebroid morphism $\mathrm{ad} : \mathcal{A} \to \mathcal{D}(L)$, written as

$$\mathrm{ad}(\mathfrak{X})(\ell) = \mathfrak{X} \cdot \ell,$$

for $\mathfrak{X} \in \Gamma(A)$ and $\ell \in \Gamma L$. This is defined uniquely as

$$\iota(\mathfrak{X} \cdot \ell) = [\mathfrak{X}, \iota(\ell)]_A.$$

Sometimes this representation is referred to as the standard kernel representation for reasons which will become clear in the next section.

Using the covariant derivative as a representation we have,

Definition 27. *The* ***exterior derivative*** *associated to the covariant derivative* ∇ *is* $d_\nabla : \Omega^d(\mathcal{A}, E) \to \Omega^{d+1}(\mathcal{A}, E)$; $\epsilon \mapsto \nabla\epsilon$ *for* $\epsilon \in \Gamma(E)$ *and for* $\alpha \in \Omega^p(\mathcal{A})$;

$$d_\nabla(\alpha \otimes \epsilon) = d\alpha \otimes \epsilon + (-1)^p \alpha \wedge \nabla\epsilon$$

Then the obvious extension to the entire algebra, replaces ϵ *with a section of the algebra and* ∇ *with* d_∇. *The operator has the following explicit definition using the following Koszul-type formula (which is reminiscent of the Koszul formula in Proposition 4),*

$$\begin{aligned}(d_\nabla\beta)(\gamma_0, ..., \gamma_k) = &\sum_{i=0}^{k} (-1)^i \nabla_{\gamma_i} \beta(\gamma_0, ...\hat{\gamma}_i, ..., \gamma_k) \\ &+ \sum_{i<j} (-1)^{i+j} \beta([\gamma_i, \gamma_j], \gamma_0, ..., \hat{\gamma}_i, ..., \hat{\gamma}_j, ..., \gamma_k).\end{aligned}$$

for all $\beta \in \Omega^k(\mathcal{A}, E)$. *The hat notation means that entry is omitted.*

We can relate the curvature tensor F_∇, to the exterior derivative through the following proposition,

Proposition 2. *For all $\omega \in \Omega^\bullet(\mathcal{A}, E)$ the following holds,*

$$d_\nabla^2 \omega = F_\nabla \wedge \omega$$

Proof. Let $\alpha, \beta \in \Gamma\mathcal{A}$, $\nu \in \Gamma E$ and $d_\nabla \nu = \nabla \nu = \eta \in \Omega^1(\mathcal{A}, E)$,

$$\begin{aligned} d_\nabla \eta(\alpha, \beta) &= \nabla_\alpha \eta(\beta) - \nabla_\beta \eta(\alpha) - \omega([\alpha, \beta]) \\ &= \nabla_\alpha \nabla_\beta \nu - \nabla_\beta \nabla_\alpha \nu - \nabla_{[\alpha,\beta]} \nu \\ &= F_\nabla(\alpha, \beta)\nu = (F_\nabla \wedge \nu)(\alpha, \beta) \end{aligned}$$

Next consider the action more generally on $\theta \otimes \mu \in \Omega^k(\mathcal{A}, E)$,

$$\begin{aligned} d_\nabla^2(\theta \otimes \mu) = d_\nabla^2(\theta \otimes \mu) &= d_\nabla(d\theta \otimes \mu + (-1)^k \theta \wedge \nabla \mu) \\ &= d^2\theta \otimes \mu + (-1)^{k+1} d\theta \wedge d_\nabla \mu + (-1)^k d\theta \wedge d_\nabla \mu + \theta \wedge d_\nabla^2 \mu \\ &= \theta \wedge d_\nabla^2 \mu = \theta \wedge (F_\nabla \wedge \mu) \\ &= F_\nabla \wedge (\theta \otimes \mu) \end{aligned}$$

Thus we have the result. □

Thus if the $\mathcal{A}-$connection is **flat**, i.e. $F_\nabla = 0$, then the exterior covariant derivative d_∇ is a differential of the algebra, which is nilpotent and so defines a cohomology theory in the normal way. Note that it is isomorphic to the deRham cohomology from Example 4, if the anchor is invertible. To see why, construct an isomorphism by pulling back d_∇ applied to a form by the inverted anchor. As such, the cohomology (along with its corresponding homology) tells us about the obstructions to $d_\nabla-$exactness, we shall not discuss this further. We can also obtain the Bianchi identity using Proposition 2,

Proposition 3. *The **Bianchi identity** can be posed as follows,*

$$d_\nabla F_\nabla = 0$$

Proof. Using Proposition 2

$$\begin{aligned} d_\nabla^3 \omega = d_\nabla(F_\nabla \wedge \omega) &= d_\nabla F_\nabla \wedge \omega + F_\nabla \wedge d_\nabla \omega \\ &= d_\nabla^2(d_\nabla \omega) = F_\nabla \wedge d_\nabla \omega \end{aligned}$$

Thus $d_\nabla F_\nabla = 0$ for all $\omega \in \Omega^\bullet(\mathcal{A}, E)$ □

Let's now restrict our attention to the scenario where the $\mathcal{A}-$connections are on $\pi : \mathcal{A} \to \mathcal{M}$ itself. Finally, then we can define an analogous torsion tensor,

Definition 28. *The **torsion tensor** associated to the covariant derivative ∇ is $T_\nabla : \Gamma\mathcal{A} \times \Gamma\mathcal{A} \to \Gamma\mathcal{A}$ such that*

$$T_\nabla(\alpha, \beta) = \nabla_\alpha \beta - \nabla_\beta \alpha - [\alpha, \beta]$$

This allows us to define the analogous notion of a Levi-Civita connection. Given a (pseudo-)Riemannian fiber metric defined on the bundle of the Lie algebroid allows us to define,

Proposition 4. *The **Levi-Civita connection** ∇ on a Lie algebroid $\mathcal{A}$ with respect to a (pseudo-)Riemannian metric $g =$, is the $\mathcal{A}-$connection over the $\mathcal{A}$ bundle which satisfies* *the Koszul formula.*

$$2g(\nabla_X Y, Z) = -\nabla_Y g(Z, X) - g([Y, Z], X) + \textit{cyclic permutations}$$

Proof. Proof follows exactly as in standard differential geometry. The formula forces the connection to be metric compatible and torsion free.

Now that we have the calculus in-hand, the question is whether or not the construction is dimensionally reducible? By which we mean, we want to know if it is possible to construct a metric on the bundle which is composed of dimensionless quantities valued in the base. In true Kaluza-Klein fashion, we would like also to consider a gravitional theory, i.e. construct an Einstein tensor from the curvature, and see if it is dimensionally reducible. For this we restrict our attention to transitive Lie algebroids.

4.2 Prelude to Kaluza-Klein on Transitive Lie Algebroids

Recall that a transitive Lie algebroid $A \xrightarrow{\rho} T\mathcal{M}$ has a surjective anchor ρ. Let's look at the kernel of the anchor, $L = \operatorname{Ker}\rho$. If we evaluate ρ at a point m in the base $\mathcal{M}$ then $L_m = \operatorname{Ker}\rho_m$ is a fiber of A and can be endowed with a bracket $[a, b]_A(m) = [a(m), b(m)]_m$ for $a, b \in \Gamma(A)$. Then L_m forms a Lie alegbra called the **isotropy Lie algebra** of m. Thus L forms a bundle of Lie algebras[7] called the **isotropy**, and when equipped with the zero anchor, L forms a Lie algebroid. In fact, using the inclusion into A as a injective bundle map, L forms a Lie subalgebroid (a Lie subalgebroid is a Lie algebroid with a injective bundle map into another). Thus for any transitive Lie algebroid we have the following SES of Lie algebroids

$$L \xhookrightarrow{\iota} A \xrightarrow{\rho} T\mathcal{M}, \tag{27}$$

In particular this is a SES of vector bundles, as in (eq: Atiyah VB) therefore it splits (as a vector bundle, not necessarily a Lie algebroid). We will adopt here the terminology of [10] and call the objects associated with the L space inner objects. The reasoning is that L represents the "inner" space of A which corresponds to the infinitesimal gauge symmetries, the "inner" symmetries of spacetime. Using the splitting we define $\sigma : T\mathcal{M} \to A$, as usual as a right inverse of the surjection, in this case the anchor $\rho : A \to T\mathcal{M}$. This map is referred to as the **horizontal lift**, which is a $A-$valued $1-$form on $\mathcal{M}$ by virtue of acting on the vector fields of $\mathcal{M}$. The left splitting $\omega_\sigma : A \to L$, will be called the connection form. The curvature associated with the horizontal lift, F_σ is as in Definition 25,

$$\iota F_\sigma(X, Y) = [\sigma(X), \sigma(Y)]_A - \sigma[X, Y]_{T\mathcal{M}},$$

where $X, Y \in T\mathcal{M}$. Note that $F_\sigma(X, Y) \in \Gamma(L)$ and so in order for the curvature of the $A-$connection σ to be valued in $\operatorname{End}(A)$ we needed to add the ι. Observe that F_σ is not only dependent on σ but also on ι. As before the curvature is $C^\infty(\mathcal{M})-$linear in all its arguments and so is a tensor, in fact it is a $L-$valued $2-$form. Using ω_σ and Theorem 2 we can define what is called in the Lie algebroid literature an inner-non-degenerate metric on A $\hat{g}$[8], using a metric on the base g (i.e. a fiber metric on $T\mathcal{M}$), a inner metric h on L and a connection σ. The metric takes form as in (16),

$$\hat{g} = \rho^* g + \omega_\sigma^* h. \tag{28}$$

Recall that assigning a connection form on L does not necessarily choose a unique horizontal space to L; however, the combination of a inner non-degenerate metric and a L connection form does designate a unique ordinary connection $\nabla^\sigma : \Gamma T\mathcal{M} \to A$, which is generally called the **metric connection**. Note that replacing $T\mathcal{M}$ in the SES of (27) gives another SES of $C^\infty(\mathcal{M})-$modules

[7] In fact for a connected base $\mathcal{M}$ it is a Lie algebra bundle see [5].

[8] A metric on A which when restricted to L is a non-degenerate inner metric.

since the map $\rho : A \to \Gamma T\mathcal{M}$[9] is induced by the anchor map and so retains its surjectivity and the image is the same as the anchor.

$$L \underset{\omega_\sigma}{\overset{\iota}{\rightleftarrows}} A \underset{\nabla^\sigma}{\overset{\rho}{\rightleftarrows}} \Gamma T\mathcal{M}. \tag{29}$$

The SES is split automatically since $L \subset A$ using Lemma 1. Define the splitting of the identity on A (as in (1) where A replaces B and a minus sign is used with the retraction to adhere to convention in the literature [10]),

$$1_A = \nabla^\sigma \rho - \iota\omega_\sigma. \tag{30}$$

Let's investigate the condition in (9), that we demanded to achieve the metric (28),

$$\begin{aligned} \hat{g}(\nabla^\sigma \rho(\mathfrak{X}), \iota\omega_\sigma(\mathfrak{X})) = 0 \quad &\implies \quad \hat{g}(\mathfrak{X} + \iota\omega_\sigma(\mathfrak{X}), \iota\omega_\sigma(\mathfrak{X})) = 0 \\ &\implies \quad \hat{g}(\mathfrak{X}, \iota\omega_\sigma(\mathfrak{X})) = -h(\omega_\sigma(\mathfrak{X}), \omega_\sigma(\mathfrak{X})) \end{aligned}$$

Using a version of Riesz's representation theorem[10] as in [10], we can show that in the following situation (which subsumes our own) the information of a splitting section and an inner-non degenerate metric defines a unique ordinary connection ∇^σ such that,

$$\hat{g}(\nabla^\sigma(\mathfrak{X}), \iota(\mathfrak{L})) = 0, \quad \forall \mathfrak{X} \in A \;\; \mathfrak{L} \in L. \tag{31}$$

That is, $L \subset A$ is horizontal and orthogonal with respect to the metric to $\Gamma T\mathcal{M}$. A sketch of the argument of uniqueness is as follows, given the non-degenerate metric h on L, and an element $\mathfrak{L} \in L$, we can write the metric action this element as a $C^\infty(M)-$valued map on L. Using the variation of Riesz theorem the map has a unique action on $\omega_\sigma(\mathfrak{X})$ resulting in $h(\nabla^\sigma(\mathfrak{X}), \mathfrak{L}) = -\hat{g}(\mathfrak{X}, \iota(\mathfrak{L}))$. So if ω_σ defines a splitting, the splitting equation (30) uniquely defines an ordinary connection ∇^σ and leads to (31).

Notice so far in the work on transitive Lie algebroids we have only mentioned ordinary connections. This is because we have set out to do some Kaluza-Klein theory working off a metric defined by a splitting of a SES. It is always true that for any such splitting that we can define an ordinary connection (which is also normalised, meaning that (30) holds and the $\nabla^\sigma \rho$ term vanishes, so that $\omega\iota = -1_L$ on restriction to L.) Whereas, for $\mathcal{A}-$connections does not split the sequence unless it is ordinary, i.e. the bracket is preserved. One can measure the failure of this preservation using the following construction, called the reduced kernel endomorphism,

Definition 29. *Given a $A-$connection form $\omega \in \Omega^1(A, L)$, $\tau : L \to L$ such that,*

$$\tau = \omega\iota + 1_L, \tag{32}$$

measures the failure of the connection to be ordinary.

Using this we can build the Lie algebroid connections of Definition 24 from the ordinary connection defined by a splitting and the above endomorphism. We start off by assuming that on the transitive Lie algebroid $A \xrightarrow{\rho} \Gamma T\mathcal{M}$ we have all of the constructions of this section, that is, the metric $\hat{g}$ defined by the splitting, the connection and connection form associated to the splitting (and so are ordinary) defined "in the background". We refer to these connections as the background connections and label them $\mathring{\nabla}$, $\mathring{\omega}$. Then given any $A-$connection form $\hat{\omega}$ we can induce an ordinary connection form,

$$\omega = \hat{\omega} + \hat{\tau}\mathring{\omega}. \tag{33}$$

[9] This is an abuse of notation really we mean a map $\Gamma(P)$, but choose ρ for simplicity.

[10] For those unfamiliar with the theorem, it has many variants in functional analysis. In general, it proves the existence and uniqueness of a natural isometric isomorphism between a space and its dual using an inner product on the space. That is, the inner product action on an element of the space defines a unique element of the dual space.

where $\hat{\tau} = \hat{\omega}\iota + 1_L$ is the reduced kernel endomorphism of the $A-$connection $\hat{\omega}$. Under pullback (precomposition) by ι the ordinariness of this connection is immediate. That is, $\iota^*\omega = -1_L$ so $\tau = 0$. Being an ordinary connection it corresponds to a splitting, so we can assign a connection $\nabla : \Gamma T\mathcal{M} \to A$ to the right splitting map. What about the curvature of this induced ordinary $A-$connection? Well with the theory we have developed so far we must first look at covariant derivative associated to the connection.

Let Θ, $\hat{\Theta}$ and $\mathring{\Theta}$ in $\mathrm{End}(A)$ be the covariant derivatives associated to ω, $\hat{\omega}$ and $\mathring{\omega}$ respectively. The ordinary connections, the background covariant derivative $\mathring{\Theta}$ and the induced covariant derivative are defined by their respective splittings as in (30),

$$\mathring{\Theta} = \mathring{\nabla}\rho = 1_A + \iota\mathring{\omega}, \tag{34}$$
$$\Theta = \nabla\rho = 1_A + \iota\omega; \tag{35}$$

However, we cannot define the $A-$connection using a splitting, so instead we define the $A-$connection covariant derivative, $\hat{\Theta}$ by the analogous notion without requiring it be a projection (as $\mathring{\Theta}$ and Θ clearly are since $\mathring{\nabla}$ and ∇ are right-inverses of ρ), i.e. we lose the first equality from the above,

$$\hat{\Theta} = 1_A + \iota\hat{\omega}. \tag{36}$$

To see this is not a projection simply square it,

$$\begin{aligned}\hat{\Theta}^2 &= \hat{\Theta}(1_A + \iota\hat{\omega}) = \hat{\Theta} + \iota\hat{\omega} + \iota\hat{\omega}\iota\hat{\omega} \\ &= \hat{\Theta} + \iota(1_L + \hat{\omega}\iota)\hat{\omega} = \hat{\Theta} + \iota\tau\hat{\omega} \\ &= \hat{\Theta} + \tau\hat{\omega}\end{aligned}$$

The last equality follows since $\tau \in \mathrm{End}(L)$. Thus one can quite easily see that a $A-$covariant derivative is a projection only if $\tau = 0$, i.e. the connection is ordinary. Combining (34)–(36) with (33) gives

$$\begin{aligned}\hat{\Theta} &= 1_A + \iota(\omega - \tau\mathring{\omega}) = \Theta - \iota\tau\mathring{\omega} = \Theta - \iota\tau\iota\mathring{\omega} \\ &= \Theta + \iota\tau - \iota\tau\mathring{\Theta}\end{aligned} \tag{37}$$

With these in hand we now construct the curvature of the $A-$connection $\hat{F} \in \Omega^2(A, L)$. Let us choose the kernel representation on A as in Example 9 (we will adapt our notation for d_∇ to $\hat{d}$ since we have not defined it by a splitting) and use the following construction to reach an alternative form of the curvature,

Definition 30. *The graded Lie bracket* $[A, B] \in \Omega^{a+b}(A, L)$ *for* $A \in \Omega^a(A, L)$ *and* $B \in \Omega^b(A, L)$ *is defined as,*

$$[A, B](\mathfrak{X}_1, ..., \mathfrak{X}_{a+b}) = \frac{1}{a!b!} \sum_{\sigma \in \mathfrak{S}_{\mathfrak{a}+\mathfrak{b}}} (-1)^\sigma [A(\mathfrak{X}_1, ..., \mathfrak{X}_a), B(\mathfrak{X}_{a+1}, ..., \mathfrak{X}_{a+b})],$$

where $\mathfrak{X}_i \in A$.

Using this definition we can recast the curvature $\hat{F} \in \Omega^2(A, L)$ defined as in (26)

$$\iota\hat{F}(\mathfrak{X}, \mathfrak{Y}) = [\hat{\Theta}(\mathfrak{X}), \hat{\Theta}(\mathfrak{Y})] - \hat{\Theta}([\mathfrak{X}, \mathfrak{Y}]) \tag{38}$$

in the familiar form,

Proposition 5. *The **Cartan structure equation** for the $A-$connection $\hat{\omega}$ is,*

$$\hat{F} = \hat{d}\hat{\omega} + \frac{1}{2}[\hat{\omega}, \hat{\omega}]. \tag{39}$$

Proof. Let $\mathfrak{X}, \mathfrak{Y} \in A$ then,

$$\begin{aligned}
\hat{d}\hat{\omega}(\mathfrak{X}, \mathfrak{Y}) &= \mathfrak{X} \cdot \hat{\omega}(\mathfrak{Y}) - \mathfrak{Y} \cdot \hat{\omega}(\mathfrak{X}) - \hat{\omega}([\mathfrak{X}, \mathfrak{Y}]) \\
[\hat{\omega}, \hat{\omega}](\mathfrak{X}, \mathfrak{Y}) &= 2[\hat{\omega}(\mathfrak{X}), \hat{\omega}(\mathfrak{Y})] \\
\implies \iota\hat{F}(\mathfrak{X}, \mathfrak{Y}) &= [\mathfrak{X}, \hat{\omega}(\mathfrak{Y})] - [\mathfrak{Y}, \hat{\omega}(\mathfrak{X})] + \iota[\hat{\omega}(\mathfrak{X}), \hat{\omega}(\mathfrak{Y})] - \iota\hat{\omega}([\mathfrak{X}, \mathfrak{Y}])
\end{aligned}$$

Then applying (36) to (38) gives the result since ι is an injective Lie algebra morphism. □

We could use the decomposition equations for $\hat{\Theta}$ and $\hat{\omega}$ defined using the reduced kernel endomorphism τ to try decompose the curvature into background and induced curvatures, and some admixture of these. Indeed this is how we proceed but first we introduce analogous notions of covariant derivative and curvature for the inherently non-geometric quantity τ, this will allow us to represent the curvature $\hat{F}$ is a particularly nice form.

Definition 31. *The analogous notion of curvature of τ is known as **algebraic curvature** $R_\tau : L \times L \to L$,*

$$R_\tau(\ell, \tilde{\ell}) = [\tau(\ell), \tau(\tilde{\ell})] - \tau([\ell, \tilde{\ell}]) \tag{40}$$

This is a curvature in the sense that it measures the obstruction to τ being a Lie algebra homomorphism.

Then we use a covariant derivative on the $\mathrm{End}(L)$ bundle, $\mathcal{D} : \Gamma T\mathcal{M} \times \mathrm{End}(L) \to \mathrm{End}(L)$ and define its action on τ as,

$$\mathcal{D}_X\tau(\ell) = [\nabla_X, \tau(\ell)] - \tau([\mathring{\nabla}_X, \ell]). \tag{41}$$

Thus $\mathcal{D}\tau \in \Omega^1(\mathcal{M}, \mathrm{End}(L))$ so pulling back by the anchor will give $\rho^*\mathcal{D}\tau \in \Omega^1(A, \mathrm{End}(L))$. Then we can define a bracket operation resulting in a two form $\rho^*\mathcal{D}\tau \bullet \mathring{\omega} \in \Omega^2(A, L)$,

$$(\rho^*\mathcal{D}\tau \bullet \mathring{\omega})(\mathfrak{X}, \mathfrak{Y}) = (\rho^*\mathcal{D}\tau(\mathfrak{X}))(\mathring{\omega}(\mathfrak{Y})) - (\rho^*\mathcal{D}\tau(\mathfrak{Y}))(\mathring{\omega}(\mathfrak{X})) \tag{42}$$

This leads to the following important result about a decomposition of the curvature of a general transitive Lie algebroid connection,

Theorem 3. *The $2-$form curvature $\hat{F} \in \Omega^2(A, L)$, associated to a general $A-$connection $\hat{\omega} \in \Omega^1(A, L)$—$A$ a transitive Lie algebroid and L the Lie algebra bundle which is the kernel of its anchor—is*

$$\hat{F} = \rho^*(F - \tau\mathring{F}) - \rho^*\mathcal{D}\tau \bullet \mathring{\omega} + \mathring{\omega}^* R_\tau \tag{43}$$

Proof. Using (33) to expand the Cartan structure equation (39) and applying to $\mathfrak{X}, \mathfrak{Y} \in A$ gives,

$$\begin{aligned}
\hat{F} &= \mathfrak{X} \bullet \omega(\mathfrak{Y}) - \mathfrak{Y} \bullet \omega(\mathfrak{X}) - \omega([\mathfrak{X}, \mathfrak{Y}]) - \mathfrak{X} \bullet \tau\hat{\omega}(\mathfrak{Y}) + \mathfrak{Y} \bullet \tau\hat{\omega}(\mathfrak{X}) + \tau\hat{\omega}([\mathfrak{X}, \mathfrak{Y}]) \\
&= [\omega(\mathfrak{X}), \omega(\mathfrak{Y})] + [\tau\hat{\omega}(\mathfrak{X}), \tau\hat{\omega}(\mathfrak{Y})] + \frac{1}{2}(-[\omega(\mathfrak{X}), \tau\hat{\omega}(\mathfrak{Y})] + [\omega(\mathfrak{Y}), \tau\hat{\omega}(\mathfrak{X})] \\
&- [\tau\hat{\omega}(\mathfrak{X}), \omega(\mathfrak{Y})] + [\tau\hat{\omega}(\mathfrak{Y}), \omega(\mathfrak{X})]
\end{aligned}$$

A long tedious application of (43) to $\mathfrak{X}, \mathfrak{Y} \in A$, then expanding and cancelling terms gives the same expansion as above. □

This is the curvature associated to not only ordinary connections on transitive Lie algebroids, but also connections on the Lie algebroid which are not defined by any splitting. This result is developed in [10] with a view towards developing gauge theories on Lie algebroids, which is successfully done in the paper. In 1975 [13] showed that Kaluza-Klein theory on principal bundles is equivalent to gauge theory on principal bundles. Also the splittings of the Atiyah Lie algebroid are in one to one correspondence with principal connections and thus gauge theory on principal bundles can be recast in terms of these splittings. Thus since: (i) transitive Lie algebroids are in essence a generalisation of Atiyah bundles, (ii) Lie algebroid connections generalise the splittings, and (iii) there is existing literature on differential geometry and gauge theories on transitive Lie algebroids, it may be possible to construct a Kaluza-Klein theory also, which may even be equivalent in some circumstances.

5 Conclusion and Outlook

It would be a huge leap to say that this essay has been a comprehensive review of the underlying geometry of Kaluza-Klein, but on our short journey we have covered much mathematical ground, so let us recap.

Section 3 walked through the intuition of short exact sequences, giving us some nice insight into why they might be a useful construction aiding us in the deconstruction of a metric into dimensionally reducible parts. We then employed a functorial approach to the question of when exactly does a splitting of a sequence of bundles defined over the base lead to a splitting of metric bundles, i.e. bundles whose fibers are metrics. Functorially we required additivity which amounted to (9), which is a statement on the orthogonality of the metric on the constituent spaces. We constructed a functor in (11) (in a rather contrived way), to impose this orthogonality condition and thus transform the original problematic non-additive functor into one which preserves the exact sequence structure it acts on. We constructed the Atiyah sequence (21) since splittings of this sequence correspond to principal connections on principal $G-$bundles. We then applied our construction to the Atiyah sequence for the $G-$bundle and specifically looked at a $G = U(1)$ gauge theory and showed how it reproduces the original Kaluza-Klein metric. We also seen that our construction works on any gauge theory with semi-simple compact G.

Section 4 has introduced Lie algebroids, their connection to the Atiyah bundle, and through this a connection to gauge theory exists. Then through gauge theory using the Atiyah bundle, one may postulate the existence of a link to Kaluza-Klein, as [13] showed is the case when usingprincipal $G-$bundles, however the nature of the link is unknown. Specifically, since we have a theory of differential geometry on these structures, and since they have a well developed gauge theory in terms of Lie algebroid connections, we might expect that a Kaluza-Klein theory involving the curvature of these connections exists. We have not developed this theory here but have ended the essay at a point were one can begin to consider the Lie algebroid connection curvature as a combination of curvatures on the constituent spaces in a dimensionally reducible way à la Kaluza-Klein.

Indeed in Section 4 we have also shown that a Lie algebroid connection splits into a geometric curvature $\rho^*(F - \tau\hat{F})$, a mixture of geometric and algebraic curvatures $\rho^*\mathcal{D}\tau \bullet \mathring{\omega}$ and a completely algebraic curvature $\mathring{\omega}^* R_\tau$. Which as mentioned in the introduction may lead one to think about applications of this theory in more generality, specifically on Courant algebroids.

5.1 Further Work

I will now list topics of further work that develop the theory of the essay in some sense. Some of the questions—mainly the ones related to string theory—may already be answered in the vast literature, however some of the questions seem to be open.

- It is clear that no Kaluza-Klein theory has yet been developed in the general case of transitive Lie algebroids, which amounts to finding a decomposition similar to (43), but which is dimensionally reducible, i.e. consists of dimensionless quantities only defined on the base.

– Maybe even more general cases of Lie algebroids can be considered. Provided one can form a short exact sequence similar to the transitive Lie algebroids it should be possible to apply this theory. A recent paper [14], has looked at Kaluza-Klein on transitive Lie algebroids when also endowed with another Euclidean product, a more restrictive case.

– Finally one could attempt to generalise all the theory in the essay to Courant algebroids, specifically to the generalised tangent bundle $T \oplus T^*$ of generalised geometry. Since on these structures vector fields and forms are on equal footing the functor from Section 3 would have to be generalised to include the second alternating power functor $\bigwedge^2$ functor (and make it additive) to incorporate this fact, so that one still ends up with a pseudo-Riemannian metric $T \oplus T^*$.

References

1. Betounes, D., 1991. Kaluza-Klein geometry. Differential Geometry and its Applications, 1(2), pp.7788. https://doi.org/10.1016/0926-2245(91)90023-3
2. Pradines, J., 1967. Théorie de Lie pour les groupoïdes différentiables. Calcul différenetiel dans la catégorie des groupoïdes infinitésimaux,C. R. Acad. Sci. Paris Sér. A-B, pp.245-248.
3. Bleecker, D., 1981. Gauge Theory and Variational Principles, Reading, Mass: Addison-Wesley Pub. Co., Advanced Book Program/World Science Division.
4. Lee, J.M. American Mathematical Society, 2009. Manifolds and differential geometry, Providence, R.I.: American Mathematical Society.
5. Mackenzie, K.C.H., 2005. General Theory of Lie Groupoids and Lie Algebroids, Cambridge: Cambridge University Press.
6. Gualtieri, M., 2011. Generalized complex geometry. Annals of Mathematics, 174(1), pp.75123. https://doi.org/http://doi.org/10.4007/annals.2011.174.1.3
7. Ellwood, I.T., 2007. NS-NS fluxes in Hitchin's generalized geometry. Journal of High Energy Physics, 2007(12), p.084. https://doi.org/10.1088/1126-6708/2007/12/084
8. Blumenhagen, R. et al., 2012. Bianchi identities for nongeometric fluxes from quasiPoisson structures to Courant algebroids. Fortschritte der Physik, 60(1112), pp.12171228. https://doi.org/10.1002/prop.201200099
9. Eisenbud, D., 2004. Commutative algebra with a view toward algebraic geometry Corr., New York, N.Y.: Springer.
10. Fournel, Lazzarini Masson, 2013. Formulation of gauge theories on transitive Lie algebroids. Journal of Geometry and Physics, 64(1), pp.174191. https://doi.org/10.1016/j.geomphys.2012.11.005
11. Vákár, M., 2011. Principal Bundles and Gauge Theories. Universiteit Utrecht. `http://philsci-archive.pitt.edu/id/eprint/10008`.
12. Dieck, T.tom., 2008. Algebraic topology, Zurich: European Mathematical Society.
13. Cho, Y.M., 1975. Higherdimensional unifications of gravitation and gauge theories. Journal of Mathematical Physics, 16(10), pp.20292035. https://doi.org/10.1063/1.522434
14. Boucetta Essoufi, 2019. The geometry of generalized CheegerGromoll metrics on the total space of transitive Euclidean Lie algebroids. Journal of Geometry and Physics, 140, pp.161177. https://doi.org/10.1016/j.geomphys.2019.02.010