Notes to Ilir on the Turing machine

Péter

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1 The problem

I am returning to the seemingly innocent problem of proving the Spill Bound property in induction, which I am finding much nastier than originally imagined. I recall that I missed the following problem: the head slides away very far on the dirt, comes back T^* time later to leave an island in a burst, then repeats this many times. This way, the dirt seems to be capable of spilling out without bound. The solution I am proposing is to add a parameter λ corresponding to the level, require Q to be exponentially times bigger than λ, and add an axiom that passing over an area $2^{\Omega(\lambda)}$ times will clean it. Here, still many details must be worked out, but I feel that this has more chance of succeeding.

2 Proposed new axioms

Let us have a parameter λ (for "level"). It may not be necessary, as the cell size B is also a function of the level, but the way things depend on the level seems simpler than how they would depend on the cell size.

We will fix some constants c_i. Consider a noise-free time interval J, and a space interval I.

Spill Bound The Spill Bound axiom bounds the amount by which the dirt can spill out of I during J, by $2B$.

Dwell Cleaning The Dwell Cleaning axiom stays essentially how it was. For a constant

$$c_{\text{dwell}} < \gamma, \tag{2.1}$$

if the head stays in an interval of size $c_{\mathrm{dwell}}B$ for time γT then the interval becomes clean.

Pass Cleaning If the head passes an interval in a noise-free way

$$\pi(\lambda) = 2^{\sigma_{\mathrm{pass}}\lambda}$$

times (constant σ_{pass} to be determined) then the interval becomes clean.

Attack Cleaning If the head exits and enters a clean area from the same side twice, then the area will be extended.

⟪P: Do we need yet this following paragraph?⟫ For this, we must make sure in simulation of a transition at cell x of M^*, that the head will not be captured at $x + B^*$ during the computation of the transition that makes it turn back to the left. To achieve this, the last turn in the simulation will go (significantly) closer to the colony end $x + B^*$ than the previous turns. This way, capturing followed by a return that does not trigger a rebuild will only be possible if the computation of the simulated transition in the colony of x terminates first.

3 Sketch of the inductive proof

In the inductive construction, we will set $Q = 2^{\Omega(\lambda)}$, $\lambda^* = \lambda + 1$.

Lemma 3.1 *(The constants $c_1, c_2, c_3 > 0$ will be specified later.) Consider an interval I of size $|I| \geq c_1\pi(\lambda)$ that will be passed by $n \geq c_2\pi(\lambda)$ times by the head in a burst-free way during some time interval J. There could be some other times in J when the head is subject to a burst inside I. Assume that the total number of these times is $\leq c_3 n$. (We don't count the times when the head enters and leaves I without any burst.) Then there is some time during J when I becomes clean.*

Proof sketch. In this sketch I am ignoring the corrections having to do with the limited spilling effects. It is sufficient to conside the case $n = c_2\pi(\lambda)$, since otherwise we can subdivide the n passes into segments of size $c_2\pi(\lambda)$, and one of them must contain fewer than $c_3c_2\pi(\lambda)$ bursts.

Let us introduce a new *virtual* time counting. We only count the times in J when the head either passes I or is subject to a burst in J. Let this virtual time interval be denoted by K: we can assume it starts at 0.

There is a subset C of the interval I, of size $\geq |I| - c_3 c_2 \beta\pi(\lambda)$, in which no burst occurs at all. Then at some virtual time $u_0 = O(\beta\pi(\lambda))$, the set C becomes clean. We will actually take for C the larger set of points in which no burst occurred until u_0. From this time on, we are looking at the shrinking of the set $I \setminus C$ in the space-virtual-time rectangle $I \times K$.

Given a space-virtual-time segment $[a, b) \times u$, let us mount on it a triangle $T \subset I \times K$ which at time $u + j$ covers the interval

$$[a + (j/4)B, v - (j/4)B) .$$

thus T reaches to virtual time $v = u + 2(b - a)/B$. We will call $v - u$ the *height* of T, denoted by $|T|$. If the complement of $[a, b)$ is clean at virtual time u and no burst occurs until virtual time v, then the Attack Cleaning property implies that the dirt is confined to T and thus disappears by virtual time v. (In the special case when $[a, b)$ reaches to the end of the interval I, we extend the base beyond the end, creating a rectangle (with the same slopes) at most twice as large whose tip reaches the end as well.)

Let us now mount a triangle over each interval of the set $I \setminus C$ at virtual time 0, creating a set $\mathcal{T}_0$ of disjoint triangles.

If no burst occurs in J then the Attack Cleaning property implies that the dirt is confined to these triangles. Now, every time a burst affects the head, it may create a dirt interval size $[x, x + \beta B)$, at some virtual time u. Let us mount a triangle over each such $[x, x + \beta B) \times \{u\}$. Let us add all these triangles to the set $\mathcal{T}_0$ to get a set of triangles $\mathcal{T}$. These are not necessarily disjoint anymore.

If two virtual triangles T_1, T_2 intersect, and T be the smallest virtual triangle containing both, then it is easy to see that $|T| \leq |T_1| + |T_2|$. We will denote $T = T_1 + T_2$.

Let us now perform the following operation that will create a disjoint set of triangles. We start with $\mathcal{T}$ and if we find two intersecting triangles in it, then we replace them with their sum. We repeat this until the remaining set $\mathcal{T}'$ consists of disjoint triangles. By the above remark $\sum_{T \in \mathcal{T}} |T| = \sum_{T \in \mathcal{T}'} |T|$. The Attack Cleaning property will show that the complement of these triangles is clean. The sum of their heights (ignoring the boundary effect, which brings in at most a factor 2) is at most $2c_3 c_2 \beta\pi(\lambda)$. If c_3 is small compared to c_1 and $c_2\beta$, then therefore there will be virtual times in K not intersecting with any element of $\mathcal{T}'$. At the corresponding real times, the interval I is clean. □

Lemma 3.2 *Let $[a, b)$ be a clean interval of size $\geq 6QB$. If the head passes it without a burst then $[a + 2B, b - 2B)$ becomes clean for M^*.*

The factors $6, 2$ will probably need to be changed.

Proof sketch. The machine, passing the interval will either find it healthy, in which case the normal operation of the simulation makes it clean for M^*, or completes a rebuilding operation. □

In this lemma the constants will probably change.

Lemma 3.3 *Consider an interval K of size $3QB$ and a time interval J in which no burst of M^* occurs. If $c_2\pi(\lambda)$ bursts occur in K during J then at some time in J the interval K becomes clean in M^*.*

Proof sketch. For

$$d = c_{\text{dwell}}/3$$

(where c_{dwell} was introduced in (2.1)), both positive and negative i, and $j = 0, \dots, d-1$, let

$$K_{ij} = x + 3QB\,[di + j, di + j + 1)\,,\ K_i = \bigcup_{j=0}^{d-1} K_{ij} = x + 3QB\,[di, d(i+1))\,,$$

so $K = K_{00}$. Consider the sweeps corresponding to the $c_2\pi(\lambda)$ bursts in K. If a sweep does not exit K on either side then Dwell Cleaning and Attack Cleaning of M^* becomes applicable. Assume this does not happen; then either half of these sweeps exits K on the left, or half of them exits it on the right. Without loss of generality assume that they pass on the right. Then they will have to pass the whole interval K_0 (otherwise again Dwell Cleaning and Attack Cleaning of M^* applies), and thus each interval $K_{0,j}$ gets at least this many passes. So, the interval K_0 gets $(c_2/2)\pi(\lambda)$ noise-free passes. Also, none of the K_{0j} becomes clean for M^* during the first half of these passes, since then the Attack Cleaning property would clean K_{00} as well during the remaining passes.

Now, for $i = 1, 2, \dots$ we will show that there is an i' with $|i'| \leq i$ such that each $K_{i'j}$ gets at least $c_2 2^{i-1}\pi(\lambda)$ burst-free overpasses during J, and $K_{i'j}$ does not become clean during the first half of these. This clearly must break down somewhere, leading to a contradiction.

Suppose that the statement was proved up to some i, we will prove it for $i+1$. In order for the interval $K_{i'j}$ not to become clean for M^*, by Lemmas 3.1, 3.2 the total number of bursts happening in it and belonging to sweeps not passing it must be at least $c_3 c_2 2^{i-1}\pi(\lambda)$. This is true of all j, so the total number of bursts whose sweeps do not pass $K_{i'}$ is at least $dc_3 c_2 2^{i-1}\pi(\lambda)$. If a sweep does not exit $K_{i'}$ on either side then Dwell Cleaning becomes applicable, so either half of these sweeps exits $K_{i'}$ on the left, or half of them exits it on the right. If they pass on the right then let $(i+1)' = i'+1$, otherwise $(i+1)' = i'-1$. Then they will have to pass the whole interval $K_{(i+1)'}$ (again, otherwise Dwell Cleaning applies), and thus each interval $K_{(i+1)',j}$ gets at least this many passes. Now choosing d (and thus c_{dwell}) large enough to get $dc_3 > 2$ finishes the proof. □

3.1 Spill bound

Consider the neighboring intervals I, K separated by point x, where I is clean for M^*. We want to limit the amount that K is going to spill over to I during a time interval J that the head spends in K and is free from M^*-noise.

Spill into I can occur in two ways: one, as regulated by the Spill Bound axiom for level λ, the other way is via bursts near the boundary x. The first way is limited by $2B$, so the important one to consider are the bursts. In order to extend the boundary to the right by QB, we need Q/β bursts near x. This can be excluded using Lemma 3.3.

3.2 Dwell Cleaning, Attack Cleaning

The proof these properties should go similarly to before.

3.3 Pass Cleaning

The proof relies on Lemma 3.3, and then Lemmas 3.1 and 3.2.