Notes to Ilir on the Turing machine

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1 The problem

I am returning to the seemingly innocent problem of proving the Spill Bound property in induction, which I am finding much nastier than originally imagined. I recall that I missed the following problem: the head slides away very far on the dirt, comes back T^* time later to leave an island in a burst, then repeats this many times. This way, the dirt seems to be capable of spilling out without bound. The solution I proposed in an earlier letter was to add a parameter λ corresponding roughly to the level, require Q to be many times bigger than λ, and add an axiom that passing over an area λ times will clean it. However, I was not able to apply this in induction, and am proposing now a significantly more complex model. Here, still many details must be worked out, but I feel that this has more chance of succeeding.

2 Proposed new axioms

Let us have a parameter λ (for "level"). There is also a new property of positions, called *level*, and written $L(x)$, taking integer and half-integer values between 0 and λ.

Let $G(\lambda)$ be some function of λ with $1 \ll G(\lambda) \ll 2^{\lambda}$. It will be chosen roughly as γ_λ where γ_k was newly introduced in the new definition of sparsity, see the long paper. The role of γ_k is to require that bursts at level k be not just one colony but γ_k colonies separated from each other.

Here are the rules for clean cells:

(L1) If a cell x has a clean neighborhood including its body and at least B on both sides of it, then $L(x) = \lambda$.

(L2) Even if the cleanness extends only to the side of x away from the head then we have already $L(x) \geq \lambda - 1/2$.

We will fix some constants c_i. Consider a noise-free time interval J, and a space interval I.

Spill Bound The old Spill Bound axiom bounded the amount by which the dirt can spill out of I during J. I may have to replace this with a weaker one, saying that it can be spilled out by at most an amount of $\beta|I|/G(\lambda)$.

Dwell Cleaning The old Dwell Cleaning axiom says the following. If $|I| \leq c_{\text{space}}B$, and the head spends a cumulative time T during J in I (possibly while entering and exiting repeatedly), then at time v, a clean point appears in I. The new one will be similar.

Pass Cleaning Let $p(t)$ be the head position at time t. Suppose $L(p(t)) < \lambda$ for all t in J, moreover during the time interval J the head covers the whole of I. Then the *average score* over I:

$$\frac{1}{|I|} \sum_{x \in I} 2^{L(x)} \tag{2.1}$$

increases by $c_{\text{incr}}(1 + 2^{-\lambda})$ for some constant $c_{\text{incr}} > 0$.

Exit Cleaning This replaces the Attack Cleaning axiom (which could stay, but this seems simpler): when the head steps on an area with $L = \lambda$, this area increases by an amount $\geq B$ in the direction where the head is coming from.

3 Sketch of the inductive proof

In the inductive construction, we will set $Q = 2^{\lambda}$, $\lambda^* = \lambda + 1$. For the hierarchy this gives $Q_k = 2^k$, which is still OK for the sparsity argument.

3.1 Definition of level

The level $L(x)$ should essentially be defined as expected. Assume that x is in direction j from the head. It has level $\lambda + 1$ if it is in a healthy area containing a colony with at least a whole colony to the left and to the right of x. If it only contains a colony in the direction j then it has level $\lambda + 1/2$. There may come some finer distinctions here. The goal is to amortize potential loss or no gain at the time we leave a clean area, towards the gain at Exit Cleaning when entering again.

3.2 Dwell Cleaning

The proof of the Dwell Cleaning property should go similarly to before.

3.3 Pass Cleaning

There are different cases to consider, but in order to understand the choice of the score function (2.1), the following example is characteristic. Let I consist of two adjacent intervals I_1 and I_2 where where $L(x) < \lambda$ on I_1 and $= \lambda$ on I_2, with $|I_2| \le QB$. Suppose that the head passes through I, first through I_1 then I_2. The average score increases by $c_{\mathrm{incr}}(1 + 2^{-(\lambda-1)})$ on I_1. The problem is that it may not increase I_2. If $|I_2|$ was significantly larger than QB then some colonies will be found/created on it, raising the average level to $\lambda+1$. The axiom needs only to consider the time interval before this happens. When it is not happening, we rely on the Exit Cleaning axiom: the interval I_2 will increase by B, thus the average score will increase by at least $2^{\lambda-1}$ from $2^{\lambda-1}$ to 2^{λ} on an interval of size B. Since $Q = 2^{\lambda}$, we increase the average score of I_2 by $\overset{*}{>} 1$.

The general case is messier, but the example shown is motivating the choice of $2^{L(x)}$ in the score function. We want the amount gained by Exit Cleaning to be comparable with the amount lost by a burst. At Exit Cleaning we gain $2^{\lambda-1}B$; even on a clean area of size QB this increases the average score by $\overset{*}{>} 1$. At a burst the level of an area of size βB may go from λ to 0, losing $2^{\lambda}\beta B$. However, given that no new colony is found/created, bursts are as far apart as $G(\lambda)QB = G(\lambda)2^{\lambda}B$, so they decrease the average score only by $\beta/G(\lambda) \ll 1$.

The role of the factor $1 + 2^{-\lambda}$ in the Pass Cleaning axiom is that even when we only apply the axiom in recursion, the bursts will take away a fraction $2^{-\lambda}$, and

$$1 + 2^{-(\lambda-1)} - 2^{-\lambda} = 1 + 2^{-\lambda}.$$

3.4 Spill bound

Consider the neighboring intervals I, K separated by point x, where K is clean for M^*. We want to limit the amount that I is going to spill over to K during a time interval that the head spends in I and is free from M^*-noise.

Essentially dirt could only spill into K via bursts near the boundary x. Between these bursts, the head would have to pass an interval of size $\geq G(\lambda)B^* = G(\lambda)2^\lambda B$, and to increase the score of I by $\stackrel{*}{>} G(\lambda)B^*$, while the burst decreased it only by $\stackrel{*}{<} \beta 2^\lambda B \stackrel{*}{<} \beta B^* \ll G(\lambda)B^*$. The total number of such times before the score of I reaches $2^{\lambda+1}$ is

$$\stackrel{*}{<} 2^{\lambda+1}|I|/G(\lambda)B^* = 2(|I|/B^*)Q/G(\lambda).$$

The size of the spill is about βB times bigger than this, that is $\stackrel{*}{<} \beta|I|/G(\lambda)$.