ECE 434 Biophotonics Project

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A Project Report on optical trapping
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Perturbation Theory

There are only a handful of potentials for which the Schrödinger equation can be solved exactly. For the vast majority of potentials, we must resort to perturbation theory. The idea is to start with a simple potential for which the Schrödinger equation can be solved exactly, and then add a small perturbation to the potential. The Schrödinger equation can then be solved iteratively, with each term in the series representing a higher order correction to the wavefunction.

In general, time-independent perturbation theory is used when the Hamiltonian can be written as the sum of two terms, $\mathcal{H} = \mathcal{H}^0 + \mathcal{H}'$, where $\mathcal{H}^0$ is the Hamiltonian for a system for which the Schrödinger equation can be solved exactly, and $\mathcal{H}'$ is a small perturbation.

For lasers, time-dependent perturbation theory is necessary. For example, the perturbation could be a sinusoidal electric field, which is used to model the interaction between light and matter. For time-dependent perturbation theory of a two-level system with two states that are eigenstates of the unperturbed Hamiltonian,

$$\hat{\mathcal{H}}^0 \psi_a = E_a \psi_a, \quad \hat{\mathcal{H}}^0 \psi_b = E_b \psi_b$$

The time evolution of the system is thus

$$\Psi(t) = c_a \psi_a e^{-iE_a t/\hbar} + c_b \psi_b e^{-iE_b t/\hbar}$$

Then, the perturbation is "turned on". The result is that the coefficients now depend on time

$$\Psi(t) = c_a(t) \psi_a e^{-iE_a t/\hbar} + c_b(t) \psi_b e^{-iE_b t/\hbar}$$

The desired quantities are $c_a(t)$ and $c_b(t)$. These can be found by solving the time-dependent Schrödinger equation with the perturbed Hamiltonian.

$$\hat{\mathcal{H}}\Psi = i\hbar \frac{\partial \Psi}{\partial t}, \ \hat{\mathcal{H}} = \hat{\mathcal{H}}^0 + \hat{\mathcal{H}}'(t)$$

Solving this equation is not trivial, and is certainly beyond the scope of this paper. Please refer to the references for more information. However, because the states are orthogonal, the trick of taking the inner product of each state can be used. The essential result is

$$\dot{c}_a = -\frac{i}{\hbar} \mathcal{H}'_{ab} e^{-i\omega_0 t} c_b, \quad \dot{c}_b = -\frac{i}{\hbar} \mathcal{H}'_{ba} e^{i\omega_0 t} c_a \tag{1}$$

Where

$$\omega_0 \equiv \frac{E_b - E_a}{\hbar}$$

So, this says that the probability of transitioning from state a to state b is proportional to the matrix element of the perturbation between the two states.

Thus far, the coefficients c are exact. However, to calculate them exactly requires infinitely higher order perturbations. Generally, the first order perturbation is sufficient, and will be examined for the rest of the paper.

Sinusoidal Perturbations

Since electromagnetic radiation is sinusoidal, it is extremely useful to look at sinusoidal perturbations. Consider the Hamiltonian

$$\hat{\mathcal{H}}'(\vec{r},t) = V(\vec{r})\cos(\omega t)$$

This will have matrix element

$$\mathcal{H}'_{ab} = V_{ab}\cos(\omega t)$$

Where

$$V_{ab} = \langle \psi_a | V(\vec{r}) | \psi_b \rangle$$

Plugging this result into (1) gives

$$\begin{aligned} c_b(t) &\approx -\frac{i}{\hbar} V_{ba} \int_0^t dt' \ \cos(\omega t') e^{i\omega_0 t''} \\ &= -\frac{V_{ba}}{2\hbar}\left[\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega} + \frac{e^{i(\omega_0+\omega)t}-1}{\omega_0-\omega}\right] \end{aligned}$$

It is obvious that for frequencies far from the resonance frequency, the transition probability will be quite small. So, it is useful to only look at frequencies close to the resonant frequencies, and neglect the first term. This means that the transition probability for a sinusoidal perturbation is given by

$$P_{a\to b}(t) = |c_b(t)|^2 \approx \frac{|V_{ab}|^2}{\hbar}\frac{\sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}$$

Einstein's A and B Coefficients

Then derive Einstein's A and B coeffs

Fermi's Golden Rule

Ultimately get Fermi's golden rule and lasers

Finally, use above to get that we have a laser with Gaussian beam output profile

Theory of Laser Tweezers

The Ray Optics Model

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The Electric Dipole Model

$$\mathbf{F} = (\mathbf{p} \ \cdot \nabla)\mathbf{E} + \frac{d\mathbf{p}}{dt}\times\mathbf{B}$$

$$\mathbf{F} = \alpha\left[(\mathbf{E}\cdot\nabla)\mathbf{E} + \frac{d\mathbf{E}}{dt}\times\mathbf{B}\right]$$

$$\mathbf{F} = \alpha\left[\frac{1}{2}\nabla\mathbf{E}^2 + \frac{d}{dt}(\mathbf{E}\times\mathbf{B})\right]$$

$$\mathbf{F} = \frac{1}{2}\alpha\nabla E^2$$

$$\mathbf{F}_{\text{scat}}(\mathbf{r}) = \frac{k^4\alpha^2}{6\pi c n^3 \varepsilon_0^2}\mathbf{I}(\mathbf{r})\hat{z} = \frac{8\pi n_0 k^4 a^6}{3c}\left(\frac{m^2-1}{m^2+2}\right)^2\mathbf{I}(\mathbf{r})\hat{z}$$

Harmonic Potential Approximation

$$\nabla \mathbf{E}_{\text{AC Stark}} = \frac{3\pi c^2 \Gamma \mu}{2\omega_0^3 \delta} \mathbf{I}(\mathbf{r}, \mathbf{z})$$

$$I(r,z) = I_0 \left(\frac{\omega_0}{\omega(z)} \right)^2 \; e^{-\dfrac{2r^2}{\omega^2(z)}}$$

$$\omega(z) = \omega_0 \sqrt{1 + \left(\frac{z}{Z_R} \right)^2}$$

$$Z_R = \frac{\pi \omega_0^2}{\lambda}$$

$$P_0 = \frac{1}{2} \pi I_0 \omega_0^2$$

$$\frac{1}{2!} \frac{\partial^2 I}{\partial z^2}\bigg|_{r,z=0} z^2 = \frac{2P_0 \lambda^2}{\pi^3 \omega_0^6} z^2 = \frac{1}{2} m \omega_z^2 z^2$$

$$\frac{1}{2!} \frac{\partial^2 I}{\partial r^2}\bigg|_{r,z=0} r^2 = \frac{4P_0}{\pi \omega_0^4} r^2 = \frac{1}{2} m \omega_r^2 r^2$$

$$\omega_r = \sqrt{\frac{8P_0}{\pi m \omega_0^4}}$$

$$\omega_z = \sqrt{\frac{4P_0 \lambda^2}{m \pi^3 \omega_0^6}}$$

$$\frac{\omega_r}{\omega_z} = \sqrt{2} \frac{\omega_0 \pi}{\lambda}$$

Applications

[1]

References

[1] C. R. K. Jr., “Density of States: 2D, 1D, and 0D.” Department of Electrical and Computer Engineering, Georgia Institute of Technology, 2005. Accesed at , [Accessed 19-02-2024].