Introduction to Probability and Statistics

Sec 2: Probability

Running example

A fair die is rolled once. The possible outcomes are:

$$\{1, 2, 3, 4, 5, 6\}$$

We will see how to approach and answer questions such as:

- What is the probability of rolling 4?
- What is the probability of rolling an odd number?

...and more involved/interesting questions!

Preliminary Definitions

Let us start with defining some important concepts. We'll put them into context in the next slide.

- **Experiment**: an act whose outcome *cannot* be predicted with certainty (until after the experiment is run).
- **Sample space**: the set of all possible outcomes of an experiment.
- **Sample point**: Each possible outcome of an experiment.
- **Event**: A collection of one or more possible outcomes of the experiment. An event is a subset of the sample space.

An event is something that *may* happen as a result of performing the experiment (*e.g.*, getting an even number when rolling a die).

Preliminary Definitions

Examples

1. Rolling a die and recording the outcome.
   - Experiment: rolling a die.
   - Sample space: $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$.
   - Events: Rolling a 3; Rolling an even number; ...
     $\mathcal{A} = \{3\}$ $\quad$ $\mathcal{A} = \{2, 4, 6\}$
2. Rolling two dice and recording their sum.
   - Experiment: rolling two dice.
   - Sample space: $\mathcal{S} = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.
   - Events: Sum is greater than 8; Sum is divisible by 3; ...
     $\mathcal{A} = \{9, 10, 11, 12\}$ $\quad$ $\mathcal{A} = \{3, 6, 9, 12\}$

Preliminary Definitions

One more example

3. Rolling two dice and recording each of the two outcomes.
   - Experiment: rolling two dice.
   - Sample space: $\mathcal{S} = \{(1,1),(1,2),\dots,(2,1),\dots(6,6)\}$.
   - Events: Think of some!

- An event $\mathcal{A}$ occurs if any one of the elements in $\mathcal{A}$ occur.
- Example
  Event $=$ "rolling an even number" during a single die roll.
  If we roll 4, then we say that the event occured.
  Same if we had rolled 2, or 6.

Probabilities

- What we'd like to do is assigning weights to each event, in an "consistent" way.

- The weights are probabilities

  $\longrightarrow$ the more likely an event, the higher its probability.

- For example, rolling an even number from a die ($\mathcal{A} = \{2, 4, 6\}$) should have a higher probability than rolling exactly a 2 ($\mathcal{A} = \{2\}$).

1. Let's assign a probability to each elementary outcome
2. Then compute the probability of an event by summing up the probabilities of all outcomes which form that event.

Probability of Elementary Outcomes

First Two Important Conditions

If p_i is the probability of elementary outcome i, and S is the sample space, then:

1. $0 \leq p_i \leq 1 \quad \forall\, i \in \mathcal{S}$
2. $\mathbb{P}(\mathcal{S}) = 1.$

So, the probability of a single outcome is always between 0 and 1, and the probabilities of all outcomes must sum up to 1:

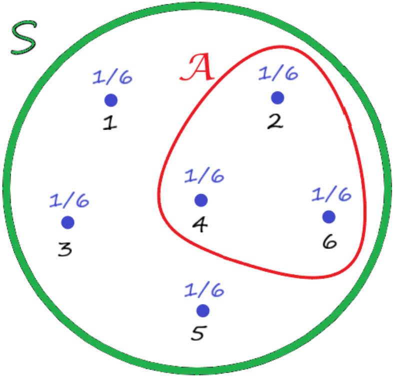
$$\sum_{i \in \mathcal{S}} p_i = 1\,.$$

That's to say, the "weight" of the whole sample space is 1.

Example 1: Fair Die

In the case of the fair die, all outcomes are equally likely:

$$p_1 = \frac{1}{6}, \quad p_2 = \frac{1}{6}, \quad p_3 = \frac{1}{6}, \quad p_4 = \frac{1}{6}, \quad p_5 = \frac{1}{6}, \quad p_6 = \frac{1}{6}.$$



- Probability of rolling an even number:

$$\mathcal{A} = \{2, 4, 6\}$$

$$\Downarrow$$

$$\mathbb{P}(\mathcal{A}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

Case of *Uniform* probability

The previous example illustrates an intuitive concept.

- If all sample points are equally likely (**uniform distibution**), then the probability of an event $\mathcal{A}$ may be simply computed as:

$$\mathbb{P}(\mathcal{A}) = \frac{\text{number of favorable outcomes}}{\text{number of all possible outcomes}} = \frac{|\mathcal{A}|}{|\mathcal{S}|}$$

- The symbol $|\cdot|$ is used to indicate the cardinality of a set (how many elements the set has).
- In the previous case ($\mathcal{A} = \{2, 4, 6\}$) we had indeed:

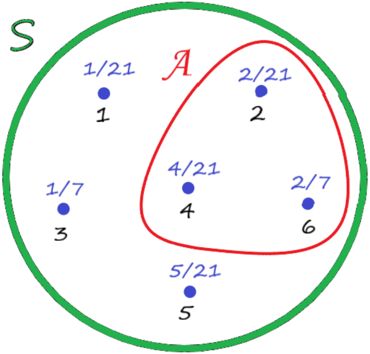
$$\mathbb{P}(\mathcal{A}) = \frac{|\mathcal{A}|}{|\mathcal{S}|} = \frac{3}{6} = \frac{1}{2}.$$

Example 2: Loaded Die

A die has been *loaded* so that the probability that side i comes up is proportional to i.

- What is the probability of each elementary outcome $1, 2, \ldots, 6$?

$$p_1 = \frac{1}{21}, \quad p_2 = \frac{2}{21}, \quad p_3 = \frac{3}{21}, \quad p_4 = \frac{4}{21}, \quad p_5 = \frac{5}{21}, \quad p_6 = \frac{6}{21}.$$



- Probability of rolling an even number:

$$\mathcal{A} = \{2, 4, 6\}$$

$$\Downarrow$$

$$\mathbb{P}(\mathcal{A}) = \frac{2}{21} + \frac{4}{21} + \frac{2}{7} = \frac{4}{7}$$

Set notation and operations

As the previous examples illustrate, we always represent an event as a set.

"Combining" events (e.g., asking that two events happen at the same time, or that at least one of the two happens) translates into "combining" sets.

Intersection

The intersection of two events $\mathcal{A}$ and $\mathcal{B}$ is the event that occurs if both $\mathcal{A}$ and $\mathcal{B}$ occur. Denoted $\mathcal{A} \cap \mathcal{B}$.

Union

The union of two events $\mathcal{A}$ and $\mathcal{B}$ is the event that occurs if either $\mathcal{A}$ or $\mathcal{B}$ (or both) occur. Denoted $\mathcal{A} \cup \mathcal{B}$.

Example

In the classical case of rolling a die, consider:

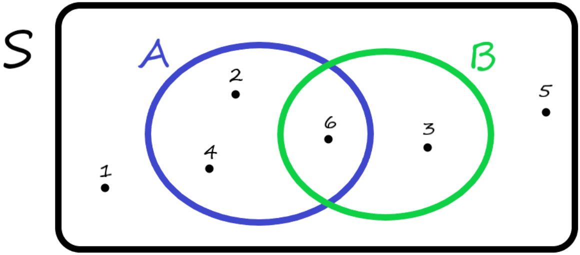
- $\mathcal{A}$: an even number is rolled.
- $\mathcal{B}$: a multiple of 3 is rolled.

Then:

$$\mathcal{A} = \{2, 4, 6\}, \quad \mathcal{B} = \{3, 6\}$$

and so:

$$\mathcal{A} \cap \mathcal{B} = \{6\}, \quad \mathcal{A} \cup \mathcal{B} = \{2, 3, 4, 6\}.$$



Mutually Exclusive Events

- Two events $\mathcal{A}$ and $\mathcal{B}$ are **mutually exclusive** if they cannot occur at the same time.

- Mathematically, two events are mutually exclusive if:

$$\mathcal{A} \cap \mathcal{B} = \emptyset.$$

- **Die Rolling Example**: Rolling a 3 ($\mathcal{A} = \{3\}$) and rolling an even number ($\mathcal{B} = \{2, 4, 6\}$) are mutually exclusive events: they can't happen simultaneously.

Probability Axioms

The following three axioms are at the basis of all probability theory. At this point, hopefully, the just look three intuitive conditions to ask for.

Given a sample space $\mathcal{S}$, a *probability distribution* $\mathbb{P}$ on $\mathcal{S}$ must satisfy the following three conditions:

1. $\mathbb{P}(\mathcal{A}) \geq 0$ for any event $\mathcal{A} \subseteq \mathcal{S}$.
2. $\mathbb{P}(S) = 1$.
3. If $\mathcal{A}$ and $\mathcal{B}$ are mutually exclusive, then $\mathbb{P}(\mathcal{A} \cup \mathcal{B}) = \mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{B})$.

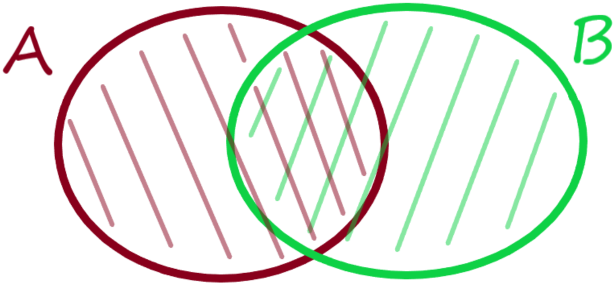
Additive law: probability of the Union

The third axiom allows to compute the probability of the union $\mathcal{A} \cup \mathcal{B}$, **only when** $\mathcal{A}$ and $\mathcal{B}$ are mutually exclusive events. What if they are not?

Additive law

The probability of the union of two events $\mathcal{A}$ and $\mathcal{B}$ is:

$$\mathbb{P}(\mathcal{A} \cup \mathcal{B}) = \mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{B}) - \mathbb{P}(\mathcal{A} \cap \mathcal{B})$$



Complementary events

For any event $\mathcal{A} \subset \mathcal{S}$, we can consider the *complementary* event: the event formed precisely by those elements which are not in $\mathcal{A}$.

Complement

The complement of an event $\mathcal{A}$ is the event that $\mathcal{A}$ does not occur. It is formed by all those elements not in $\mathcal{A}$. Denoted: $\mathcal{A}'$.

- Die Rolling Example: if $\mathcal{A}$ is the event "an even number is rolled", then $\mathcal{A}'$ is the event "an odd number is rolled":

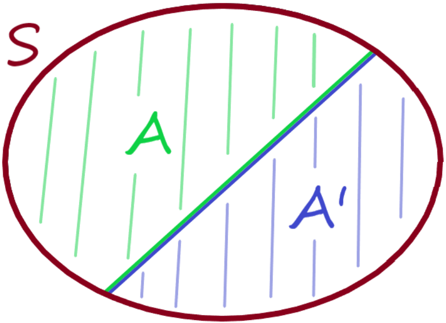
$$\mathcal{A} = \{2, 4, 6\} \quad \mathcal{A}' = \{1, 3, 5\}.$$

Formula for the Complement

- All the sample points in $\mathcal{S}$ are either in $\mathcal{A}$ or $\mathcal{A}'$ (so $\mathcal{A} \cup \mathcal{A}' = \mathcal{S}$) and no sample point can be in both ($\mathcal{A} \cap \mathcal{A}' = \emptyset$).

- Thus, from axioms 2 and 3, we get:

$$\mathbb{P}(\mathcal{A}') = 1 - \mathbb{P}(\mathcal{A}).$$



In particular, we also get:

$$\begin{aligned}\mathbb{P}(\emptyset) &= 1 - \mathbb{P}(\mathcal{S}) \\ &= 1 - 1 \\ &= 0\end{aligned}$$

Example: Dice

Two fair dice are rolled. Event $\mathcal{A}$ is that we observe a 5. Event $\mathcal{B}$ is that the dice sum to 9. Calculate:

- $\mathbb{P}(\mathcal{A} \cap \mathcal{B})$ and $\mathbb{P}(\mathcal{A} \cup \mathcal{B})$.
- $\mathbb{P}(\mathcal{A}')$.

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Conditional Probability

(and related concepts)

Conditional Probability

Sometimes, when evaluating the probability of an event, we are aware of extra information that is relevant to assess that probability.

Example

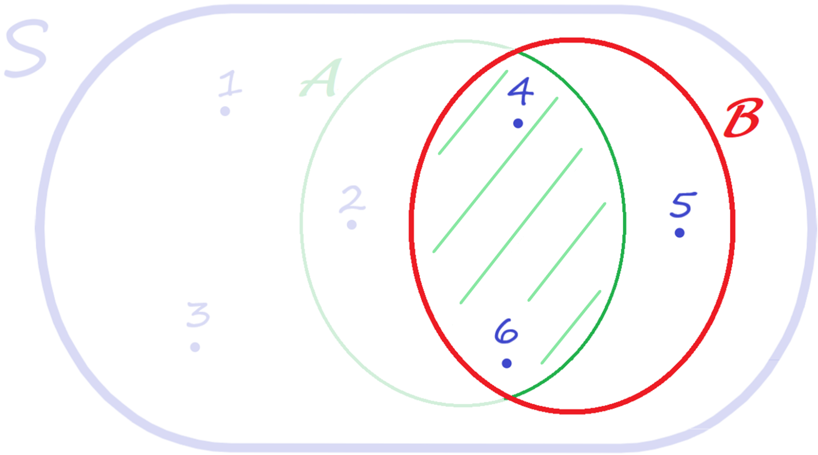
- Suppose I roll a die, you don't see the outcome. What's the probability that I rolled an even number? $1/2$
- Now suppose that I reveal to you that the number I rolled is strictly greater than 3. What's now your probability (your level of confidence) that the number I rolled is even? $2/3$

By adding the info that the rolled number is greater than 3, I have in fact restricted the actual possibilities from the original sample space

$$S = \{1, 2, 3, 4, 5, 6\} \text{ to } \underline{\text{only}} \text{ the set } \mathcal{B} = \{4, 5, 6\}.$$

- In $\mathcal{B}$, two of the three numbers are even.

Conditional Probability: Visualisation



$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \frac{2}{3}\,\mathbb{P}(\mathcal{B})$$

Conditional Probability

- Given two events $\mathcal{A}$ and $\mathcal{B}$, we define the **conditional probability of $\mathcal{A}$ given $\mathcal{B}$** as:

$$\mathbb{P}(\mathcal{A}|\mathcal{B}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{B})}.$$

- It is a measure of how likely $\mathcal{A}$ is to happen, when we know that $\mathcal{B}$ has happened.

- Note: we need $\mathbb{P}(\mathcal{B}) > 0$ in order for $\mathbb{P}(\mathcal{A}|\mathcal{B})$ to be well defined.

Question

How does the conditional probability formula read, if you multiply both sides by $\mathbb{P}(\mathcal{B})$?

Multiplication Rule

Given two events $\mathcal{A}$ and $\mathcal{B}$, we can write:

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}|\mathcal{B})\, \mathbb{P}(\mathcal{B})$$

or

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{B}|\mathcal{A})\, \mathbb{P}(\mathcal{A})\,.$$

- Sometimes, this is referred to as the **multiplication rule** of probability.
- However, observe that it is nothing new:
  It simply follows from the definition of conditional probability, by just rearranging terms (multiply by the denominator).

cond. probability ⇄ multiplication rule

Multiplication Rule: Example

Suppose I live outside the centre, and come here daily either by walking or by taking a bike.

- If it rains, I take a bike with probability 80%, and walk with pb 20%
- Otherwise, I walk with probability 80% and take a bike with pb 20%.

Weather forecasts say it will rain tomorrow with probability 40%.

- ❖ What's the probability that it will rain AND I will take a bike?

 $\mathcal{A}$: Tomorrow it rains

 $\mathcal{B}$: I take the bike.

We know that:

$$\mathbb{P}(\mathcal{A}) = 0.4 \quad \text{and} \quad \mathbb{P}(\mathcal{B}|\mathcal{A}) = 0.8.$$

Thus

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{B}|\mathcal{A})\,\mathbb{P}(\mathcal{A}) = 0.8 \times 0.4 = 0.32\,.$$

Independence

Two events $\mathcal{A}$ and $\mathcal{B}$ are said to be **independent** if knowing that one has occurred does not alter the probability that the other has occurred.

Mathematically:

$$\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A}) \quad \text{or} \quad \mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{B}).$$

- Now recall the multiplication rule. If we know that $\mathcal{A}$ and $\mathcal{B}$ are independent, then we can replace $\mathbb{P}(\mathcal{A}|\mathcal{B})$ with simply $\mathbb{P}(\mathcal{A})$. So...
- If $\mathcal{A}$ and $\mathcal{B}$ are independent events, then:

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A})\,\mathbb{P}(\mathcal{B})\,.$$

- The converse is also true, *i.e.*, if $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A})\,\mathbb{P}(\mathcal{B})$ then the events $\mathcal{A}$ and $\mathcal{B}$ are independent.

Example: Corrosion

In an engineering lab, the independence between corrosion and the correct functioning of a machine are investigated.

	Functioning	Malfunctioning
Corroded	0.2	0.4
Not corroded	0.3	0.1

Are the events "being corroded" and "correctly functioning" independent? See handwritten scanned solution.

Note. The numbers in the table have the following meaning:
20% of the machines are corroded and functioning, 30% not corroded and functioning, etc.

Example: Corrosion

Now imagine that the proportion of machines being corroded/functioning was as follows.

	Functioning	Malfunctioning
Corroded	0.1	0.3
Not corroded	0.15	0.45

✍ Are the events "being corroded" and "correctly functioning" independent?

Mutual Exclusivity and Independence

- Suppose $\mathcal{A}$ and $\mathcal{B}$ are mutually exclusive events:

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = 0.$$

  Then also the conditional probability of $\mathcal{A}$ given $\mathcal{B}$ must be zero (and viceversa):

$$\mathbb{P}(\mathcal{A}|\mathcal{B}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{B})} = 0.$$

  This is very intuitive, since we know from the mutual exclusivity that if $\mathcal{B}$ has occurred, then $\mathcal{A}$ has not.

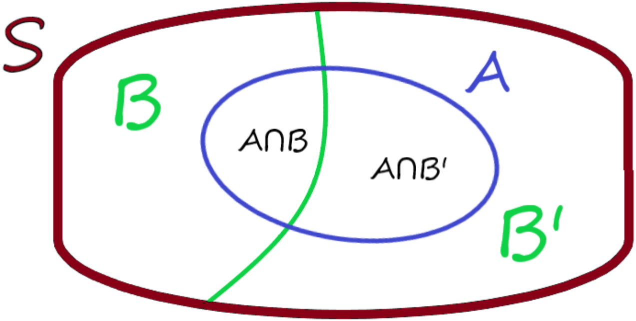
- This implies that mutually exclusive events are NOT independent.

$$\mathbb{P}(\mathcal{A}) \neq \mathbb{P}(\mathcal{A}|\mathcal{B}).$$

Very Useful Identity

Suppose we have two events, $\mathcal{A}$ and $\mathcal{B}$. I can write $\mathcal{A}$ as the union of:

- ◇ The part of $\mathcal{A}$ lying inside $\mathcal{B}$
- ◇ The part of $\mathcal{A}$ lying inside the complement of $\mathcal{B}$.



$$\mathcal{A} = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{B}')$$

Very Useful Identity

The identity we just saw is at the set level:

$$\mathcal{A} = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{B}')$$

If we move to the probability level, thanks to axiom 3 (axiom on disjoint events), we get:

$$\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{A} \cap \mathcal{B}) + \mathbb{P}(\mathcal{A} \cap \mathcal{B}')$$
$$\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{A}|\mathcal{B})\,\mathbb{P}(\mathcal{B}) + \mathbb{P}(\mathcal{A}|\mathcal{B}')\,\mathbb{P}(\mathcal{B}')$$

- Why is this useful? See next slide $\hookrightarrow$.

Bayes' Theorem

Given two events $\mathcal{A}$ and $\mathcal{B}$ (with both $\mathbb{P}(\mathcal{A}), \mathbb{P}(\mathcal{B}) > 0$), then:

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A}|\mathcal{B})\,\mathbb{P}(\mathcal{B})}{\mathbb{P}(\mathcal{A}|\mathcal{B})\,\mathbb{P}(\mathcal{B}) + \mathbb{P}(\mathcal{A}|\mathcal{B}')\,\mathbb{P}(\mathcal{B}')}.$$

- Note that the numerator and denominator are just $\mathbb{P}(\mathcal{A} \cap \mathcal{B})$ and $\mathbb{P}(\mathcal{A})$, respectively.
- While the formula may look intimidating, it is actually **very** useful.
- Bayes'theorem allows us to "invert" probabilities.
  It gives us a way of computing the probability of $\mathcal{B}$ given $\mathcal{A}$, when we know the probability of $\mathcal{A}$ given $\mathcal{B}$ and given $\mathcal{B}'$.

Example: Rare Disease Test

Suppose a rare disease affects 1 person in every 1000 of the population. Fortunately, a diagnostic medical test exists for the disease. It is a good test, in that:

- If you have the disease, the test will be positive 95% of the times
- If you do not have the disease, the test will be negative 99% of the times.

If a patient tests positive for the disease, what is the probability that they actually have the disease?

Solution: On the board.

Partitioning a Set

In Bayes' theorem, we split the sample space S into two sets, $\mathcal{B}$ and $\mathcal{B}'$:

$$S = \mathcal{B} \cup \mathcal{B}', \qquad \text{where } \mathcal{B} \cap \mathcal{B}' = \emptyset\,.$$

More generally, we could "split" $\mathcal{S}$ into more than two sets.

<u>Def</u>: A **partition** of a set $\mathcal{S}$ divides $\mathcal{S}$ into k non-empty subsets $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$, such that:

- $B_1 \cup \mathcal{B}_2 \cup \ldots \cup \mathcal{B}_k = \mathcal{S}$
- $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for every $i \neq j \in \{1, \ldots, k\}$.

Partition: Examples

Examples of partitions of $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ are:

- $\mathcal{B}_1 = \{1, 2\}$, $\mathcal{B}_2 = \{3, 4\}$, $\mathcal{B}_3 = \{5, 6\}$.
- $\mathcal{B}_1 = \{1, 4, 6\}$, $\mathcal{B}_2 = \{2, 3, 5\}$
- $\mathcal{B}_1 = \{3\}$, $\mathcal{B}_2 = \{1, 6\}$, $\mathcal{B}_3 = \{4\}$, $\mathcal{B}_4 = \{2, 5\}$

Examples of subsets that do **not** form a partition are:

$$\ldots\ldots$$

Law of Total Probability

If $\mathcal{B}_1, \ldots, \mathcal{B}_k$ form a partition of $\mathcal{S}$, then at the **set level**:

$$\mathcal{A} \;=\; (\mathcal{A} \cap \mathcal{B}_1) \cup (\mathcal{A} \cap \mathcal{B}_2) \cup \ldots \cup (\mathcal{A} \cap \mathcal{B}_k)\,.$$

At the **probability level**:

$$\mathbb{P}(\mathcal{A}) \;=\; \mathbb{P}(\mathcal{A} \cap \mathcal{B}_1) + \mathbb{P}(\mathcal{A} \cap \mathcal{B}_2) + \ldots + \mathbb{P}(\mathcal{A} \cap \mathcal{B}_k)\,,$$

or

$$\boxed{\mathbb{P}(\mathcal{A}) \;=\; \sum_{i=1}^{k} \mathbb{P}(\mathcal{A} | \mathcal{B}_i)\, \mathbb{P}(\mathcal{B}_i)}$$

Combinatorics

(Counting ways of selecting/ordering items)

Counting Rules: Overview

In the final part of this unit, we want to find methodical ways of answering questions such as:

- You are leaving for holidays. You have 8 t-shirts available and want to take 4 with you: how many possibilities you have?
- A meal deal at a restaurant (starter+main+dessert) allows you to choose from 7 starters, 4 mains, 5 desserts. How many full meals can you create?

In other words, we want to be able to count all possible ways in which a task can be performed.

Multiplicative Counting Rules

Multiplicative counting rule

Suppose we have k sets, with:

- n_1 elements in the first set
- n_2 elements in the second

  ⋮

- n_k elements in the k^{th} set.

If we wish to take a sample of size k consisting of ONE element from each set, the number of ways this sample can be formed is:

$$n_1 \times n_2 \times \ldots \times n_k$$

Multiplicative Counting Rules: Examples

- A password consists of 2 letters followed by 4 digits. How many possible passwords are there?

$$\begin{aligned} \#\ \text{passwords} &= 26 \times 26 \times 10 \times 10 \times 10 \times 10 \\ &= 6,760,000 \end{aligned}$$

- Restaurant example: 7 starters, 4 mains, 5 desserts. You can create

$$7 \times 4 \times 5 = 140$$

different full meals.

Permutations of a Set

After knowing you have won your scholarship for AIMS, you want to call 5 friends to tell them: Amina, Bongani, Chike, Dalia and Esi.
In how many different orders can you do so?

- A **permutation** of a set is any rearrangement of all its elements, in a specific order.
- The number of permutations of a set of size n is

$$n! = n\,(n-1)\,(n-2)\ldots 1\,.$$

- In other words: you can order n elements in $n!$ ("n factorial") different ways.

Examples and Factorial Computations

You can meet Amina, Bongani, Chike, Dalia and Esi in

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

different orders.

- Factorials grow **really** quickly! For example:
  - $5! = 120$
  - $8! = 40,320$
  - $11! = 39,916,800$
  - $20! =$ more than a quintillion!!! $(> 10^{18})$
- However, they simplify really nicely. For example:

$$\frac{22!}{19!} = \frac{22 \times 21 \times 20 \times 19!}{19!} = 22 \times 21 \times 20 = 9,240.$$

Permutations of Size k from a Larger Set

Suppose 60 people take part to a marathon. As usual, the fastest three win a gold/silver/bronze medal, respectively. What are all possibilities for who will win each of the three medals?

In this case we are interested in all possible ways of ordering subsets of 3 elements (the winners) from a larger set of 60 elements (all runners).

- There are 60 possibilities for the gold medal winner. For each of these, 59 possibilities for the silver medal. Then 58 for the bronze. So:

$$60 \times 59 \times 58.$$

- Recalling the "tricks" to simplify factorials, we can write that as

$$\frac{60!}{57!} = 60 \times 59 \times 58.$$

Example: ID number

A student ID number consists of 6 digits.

1. How many unique ID numbers are there?

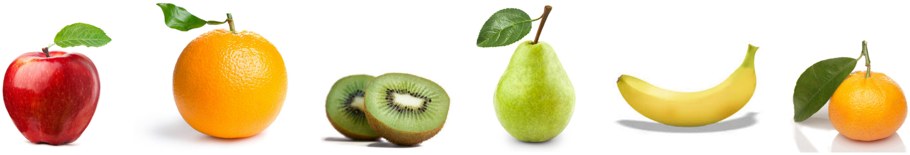
$$10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$$

2. If each digit may only appear once per ID number, how many unique ID numbers can be created?

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151,200$$

Combinations Rule: Example

Suppose to have the following six pieces of fruit in front of you:



- You can take **three** of them with you. In how many ways can you do so?

Combinations Rule: Example

You have 6 ways of choosing the first fruit, 5 for the second, 4 for the third...

$$6 \times 5 \times 4 = 120.$$

- However, in the above calculation, the same set of three fruits are counted multiple times.
- For example, the triple (apple, pear, banana) and (banana, apple, pear) are counted twice, but they should not: you took home the same three fuit in both cases.
- So: we need to divide by how many times a triple appears, *i.e.*, by 3!.

Combinations Rule: Example

So, you can choose

$$\frac{6 \times 5 \times 4}{3!} = \frac{120}{6} = 20 \tag{1}$$

different sets of 3 fruits, from the original 6.

If we call N the total number of fruits ($N = 6$) and k the number of fruits we want to take ($k = 3$), then:

- The numerator in (1) is simply the number of permutations of size k from a set of N ($\frac{N!}{(N-k)!} = 120$)
- The denominator is the number of permutations of k elements ($k! = 6$).

Combinations Rule

Given a set of N elements, the number of unordered subsets of size k which can be formed from the set is:

$$\binom{N}{k} = \frac{N!}{k!\,(N-k)!}$$

(read: N choose k).

Examples

$$\binom{6}{3} = \frac{6!}{3!\,3!} = \frac{6 \cdot 5 \cdot 4}{3!} = 5 \cdot 4 = 20$$

$$\binom{10}{4} = \frac{10!}{4!\,6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4!} = \frac{10 \cdot 9 \cdot 7}{3} = 210$$

Example: Illegal Trading

A trading manager knows that 3 out of 10 traders under her supervision are making illegal trades. If she selects 2 workers at random from the 10, what is the probability that they have both been trading illegally?

- All possible sets of 2 from the 10: $\binom{10}{2} = 45$.
- All "favorable" sets: $\binom{3}{2} = 3$.

So the probability is

$$\frac{\binom{3}{2}}{\binom{10}{2}} = \frac{3}{45} = \frac{1}{15}.$$