# Autoregressive models

= realized population density

= direct density-dependence

= delayed density-dependence

Example, the Ricker-model:

See Box 2.1, p. 17 in ([Ranta *et al.* 2006](#_ENREF_12))

Writingand,we obtain:

## One trophic level

### Log-linear

#### First-order auto-regressive model, AR(1)

Let = abundance, and = log(abundance)

, log(reproductive rate), is then defined as:

Assume that the recruitment at time *t* depends linearly on the density in the previous time step, *t-1*:

Replace Rt:

Move everything in time one step:

Collect terms and rearrange:

#### Second order auto-regressive model, AR(2)

By using the same logic, we can develop a model where reproductive rate depends on both direct and delayed density-dependence:

Reproductive rate is again:

Now, let’s assume that reproductive rate depends also on density x at *t-1*. Then

Replace Rt:

Move everything in time:

Collect terms and rearrange:

The AR(2) model is usually expressed as (assuming that residuals are normally distributed with zero mean, and standard deviation σ):

or

where *μ* is the mean of.

Note relation between carry-over effects and delayed density-dependence.

#### Relationship between intercept a0 of an autoregressive model and the mean μ of time series x:

(Exemplified here for an AR2-process)

The intercept is therefore .

More general

Estimating parameters

Deviation from mean(s)

Intercept(s)

#### Solving a pth order linear difference equation

Good discussion in ([Edelstein-Keshet 2005, chapter 1](#_ENREF_3" \o "Edelstein-Keshet, 2005 #2961)) with respect to biological systems, and ([Hamilton 1994, section 1.2, p. 7](#_ENREF_5" \o "Hamilton, 1994 #2967)) for a general approach for time series data.

Substitute . This is the crucial step to understand. This is the solution to a first-order linear difference equation, and this solution is tested also for higher-order systems.

Cancel out the common term :

More generally

Which has two solutions

The solutions are *eigenvalues*. If different solutions are known, then any linear combination of these is again a solution (the principle of linear superposition).

Since and are solutions to , a general solution is

where andare arbitrary scalars. andare determined by solving the linear algebraic equations.

#### Solving the linear second-order AR(2)-process

See also ([Hamilton 1994, equations 1.2.12 - 1.2.15](#_ENREF_5" \o "Hamilton, 1994 #2967))

The AR(2)-model can be written in matrix form.

Writing

Setting up a system of two linear equations

which is equivalent to

Replace

`

Set determinant = 0:

Calculate determinant and set to zero:

(recall that the determinant for a 2x2 matrix is calculated as)

Which gives us

Rearranging

Which has two roots

The solutions are called *eigenvalues*. The dynamics of the system is characterized by the magnitude of the eigenvalues. If the discriminant is negative, the eigenvalues are given by a pair of complex conjugates of the form:

If the eigenvalues are complex, the dynamics of the system will show oscillatory behavior with

real part and imaginary part .

Written in polar coordinate form

(modulus)

(Kendall 1945)

Period

k-contours

Jenkins & Watts, peak of spectral density

### Non-linear, Royama & Kaitala

Parameters Rm, a0, a1, a2

In canonical form, by setting and:

([Kaitala *et al.* 1996a](#_ENREF_8))wrote this model in terms of

([Kaitala *et al.* 1996a](#_ENREF_8)) showed that this model is equivalent to equation 2.20b in ([Royama 1992:62](#_ENREF_13)).

*Algebra*

Simplifying the expression

*Step 1*

*Step 2*

### Non-linear, Delayed Ricker

Parameters r, a1, a2

Model explored by ([Ranta](#_ENREF_12" \o "Ranta, 2006 #2979) *[et al.](#_ENREF_12" \o "Ranta, 2006 #2979)* [2006](#_ENREF_12" \o "Ranta, 2006 #2979)). Check Box 2.4 & 2.5 and p. 22, section 2.11

Box 2.7 p. 35, generalization of equation 2.9

### Non-linear, perturbation model

The non-linear perturbation model is an extension of the non-linear model proposed by ([Royama 1992](#_ENREF_13)) and used e.g. by ([Kaitala *et al.* 1996a](#_ENREF_8)). But instead of letting environmental stochasticity enter as additive noise, the process is subjected to environmental perturbation, *u*, occurring with frequency *p*. The effect of the random perturbations is to reduce population productivity at random intervals. This model has been shown to generate cycles similar to the observed population fluctuations of Finnish grouse ([Kaitala *et al.* 1996a](#_ENREF_8), [Lindström *et al.* 1999](#_ENREF_10)).

is a random survival factor at time t. Different distributions for have been used. For instance, ([Kaitala *et al.* 1996a](#_ENREF_8), [b](#_ENREF_9)) modeled survival factors occurring with frequency *p* drawn from a uniform random distribution, .

Ludwig *et al.* ([2006](#_ENREF_11))studied the effects of perturbation frequency *p* and severity of perturbations *(1-u)* on probability of cyclic dynamics and cycle length. In their study, a normal distribution was used to generate random perturbations. The distribution was truncated to , covering the parameter range

### Nonlinear, SETAR/TAR

### Time-varying parameters

### Additional covariates

Such as climate, additive or interactive effects

## Two trophic levels

Gompertz model

### Interspecific competition

For two species, xx and yy:

Predation

## Seasonal models

2 seasons

2 seasons, time-varying parameters

Annual, relative length of winter

## Variations on a theme

ARIMA(p,d,q)

p = autoregressive part

d = integrated part

q = moving average part

The most commonly used model for an AR(2)-process is

and this model was used e.g. by Bjørnstad ([1995](#_ENREF_1)) in their seminal paper on geographic gradients in Fennoscandian small rodents. This is an ARIMA-model of order ARIMA(2,0,0). If the time series is non-stationary, either with respect to the mean or variance, a common solution is to take the first differences of , turning the model into an ARIMA(2,1,0). From an ecological viewpoint, taking first order differences of a time series is equal to the population growth rate.

([Framstad *et al.* 1997](#_ENREF_4))

([Brocklebank & Dickey 2003, p. 113](#_ENREF_2)) wrote the AR(2)model in terms of differences and the lagged term

My version.

Backshift

([Hansen *et al.* 1999a](#_ENREF_6), [Hansen *et al.* 1999b](#_ENREF_7))

## Programming notes

arima.sim uses a linear filter. The linear filter for an AR(2)-model is specified by

filter(Z, c(1+a1, a2), method=”recursive”)+ mu

where Z is a vector with innovations, generated as random white noise drawn from a specified distribution, usually a normal distribution so that. is the mean of the series.

This is equal to a recursive for-loop of the form

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Hamilton, J. D. 1994: *Time Series Analysis*. - - - Princeton University Press, Princeton, NJ.

Hansen, T. F., Stenseth, N. C. & Henttonen, H. 1999a: Multiannual vole cycles and population regulation during long winters: an analysis of seasonal density dependence. - - - *American Naturalist* 154: 129-139.

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