CPSC 340 Assignment 5 (due Monday March 18 at 11:55pm)

We are providing solutions because supervised learning is easier than unsupervised learning, and so we think having solutions available can help you learn. However, the solution file is meant for you alone and we do not give permission to share these solution files with anyone. Both distributing solution files to other people or using solution files provided to you by other people are considered academic misconduct. Please see UBC's policy on this topic if you are not familiar with it:
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Instructions

Rubric: {mechanics:5}

IMPORTANT!!! Before proceeding, please carefully read the general homework instructions at `https://www.cs.ubc.ca/~fwood/CS340/homework/`. The above 5 points are for following the submission instructions. You can ignore the words "mechanics", "reasoning", etc.

We use blue to highlight the deliverables that you must answer/do/submit with the assignment.

1 Kernel Logistic Regresion

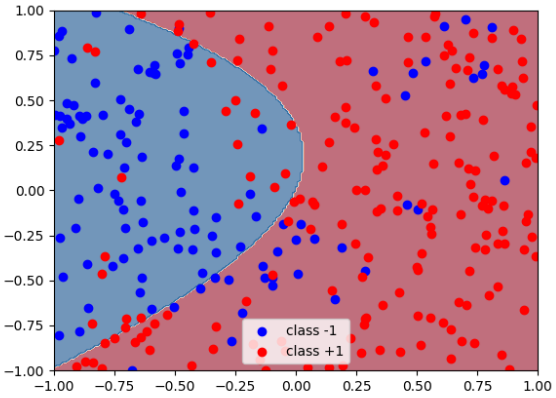
If you run `python main.py -q 1` it will load a synthetic 2D data set, split it into train/validation sets, and then perform regular logistic regression and kernel logistic regression (both without an intercept term, for simplicity). You'll observe that the error values and plots generated look the same since the kernel being used is the linear kernel (i.e., the kernel corresponding to no change of basis).

1.1 Implementing kernels

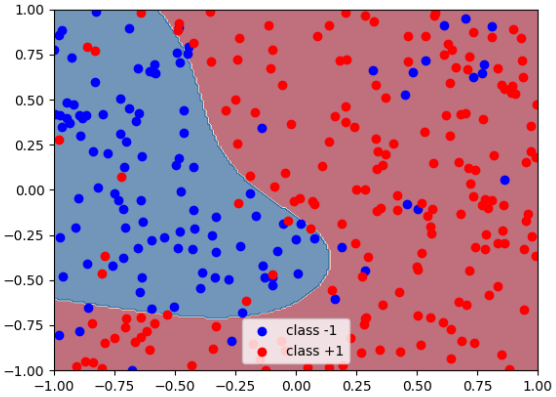
Rubric: {code:5}

Implement the polynomial kernel and the RBF kernel for logistic regression. Report your training/validation errors and submit the plots generated for each case. You should use the kernel hyperparameters $p = 2$ and $\sigma = 0.5$ respectively, and $\lambda = 0.01$ for the regularization strength.

Answer: For polynomial, the training error is 0.183 and the validation error is 0.17. Image below:



For RBF, we will accept two versions of the code, with or without the $\frac{1}{\sqrt{2\pi\sigma^2}}$ prefactor (they are equivalent, but change the scaling of the regularization, so will give different results for fixed λ). If this term is included, the training error is 0.127 and the validation error is 0.1. Image below:



If this term is not included, the training error is also 0.127 but the validation error is 0.09. The figure looks almost exactly the same.

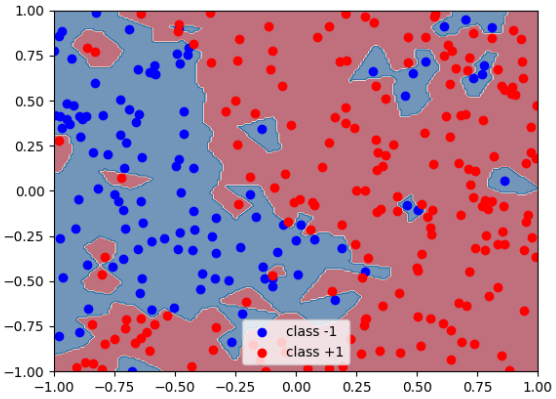
1.2 Hyperparameter search

Rubric: {code:3}

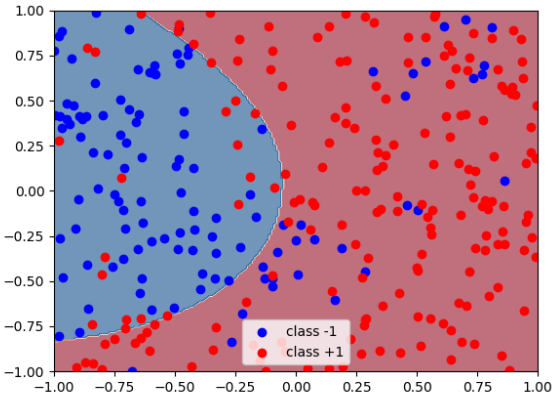
For the RBF kernel logistic regression, consider the hyperparameters values $\sigma = 10^m$ for $m = -2, -1, \ldots, 2$ and $\lambda = 10^m$ for $m = -4, -3, \ldots, 0$. In `main.py`, sweep over the possible combinations of these hyperparameter values. Report the hyperparameter values that yield the best training error and the hyperparameter values that yield the best validation error. Include the plot for each.

Note: on the job you might choose to use a tool like scikit-learn's `GridSearchCV` to implement the grid search, but here we are asking you to implement it yourself by looping over the hyperparameter values.

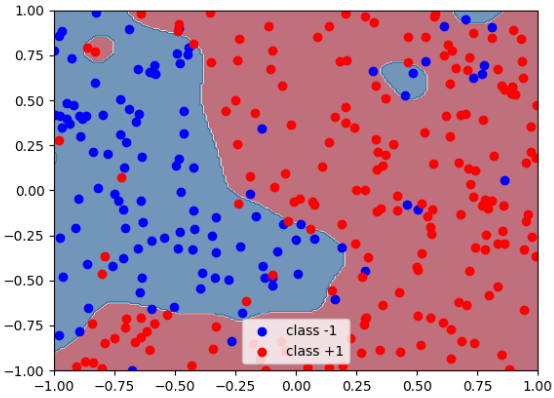
Answer: Again we'll accept the two implementations. For both implementations, the lowest training error is achieved with $\sigma = 10^{-2}$ and $\lambda = 10^{-4}$. Image below:



With the extra factor, the lowest validation error is achieved with $\sigma = 1$ and $\lambda = 0.1$. Figure below:



Without the extra factor, the lowest validation error is achieved with $\sigma = 0.1$ and $\lambda = 1$. The figure looks different; see below:



1.3 Reflection

Rubric: {reasoning:1}

Briefly discuss the best hyperparameters you found in the previous part, and their associated plots. Was the training error minimized by the values you expected, given the ways that σ and λ affect the fundamental tradeoff?

Answer: The results make sense: smaller σ leads to a more complex model, and smaller λ means less regularization; both of these should reduce training error. The validation error is minimized by larger values of these hyperparameters because the small values were overfitting. We can see in the plots that the first one is indeed a much more complex fit.

2 MAP Estimation

Rubric: {reasoning:8}

In class, we considered MAP estimation in a regression model where we assumed that:

- The likelihood $p(y_i \mid x_i, w)$ is a normal distribution with a mean of $w^T x_i$ and a variance of 1.
- The prior for each variable j, $p(w_j)$, is a normal distribution with a mean of zero and a variance of λ^{-1}.

Under these assumptions, we showed that this leads to the standard L2-regularized least squares objective function,

$$f(w) = \frac{1}{2}\|Xw - y\|^2 + \frac{\lambda}{2}\|w\|^2,$$

which is the negative log likelihood (NLL) under these assumptions (ignoring an irrelevant constant). For each of the alternate assumptions below, show how the loss function would change (simplifying as much as possible):

1. We use a Laplace likelihood with a mean of $w^T x_i$ and a scale of 1, and we use a zero-mean Gaussian prior with a variance of σ^2.

$$p(y_i \mid x_i, w) = \frac{1}{2}\exp(-|w^T x_i - y_i|), \quad p(w_j) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{w_j^2}{2\sigma^2}\right).$$

Answer:

$$f(w) = \|Xw - y\|_1 + \frac{1}{2\sigma^2}\|w\|^2.$$

2. We use a Gaussian likelihood where each datapoint has its own variance σ_i^2, and where we use a zero-mean Laplace prior with a vairance of λ^{-1}.

$$p(y_i \mid x_i, w) = \frac{1}{\sqrt{2\sigma_i^2\pi}} \exp\left(-\frac{(w^T x_i - y_i)^2}{2\sigma_i^2}\right), \quad p(w_j) = \frac{\lambda}{2}\exp(-\lambda|w_j|).$$

You can use Σ as a diagonal matrix that has the values σ_i^2 along the diagonal.

Answer:

$$f(w) = \frac{1}{2}(Xw - y)^T\Sigma^{-1}(Xw - y) + \lambda\|w\|_1.$$

The first term can also be written as $\frac{1}{2}\|\Sigma^{-\frac{1}{2}}(Xw - y)\|^2$.

3. We use a (very robust) student t likelihood with a mean of $w^T x_i$ and ν degrees of freedom, and a zero-mean Gaussian prior with a variance of λ^{-1},

$$p(y_i|x_i, w) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{(w^T x_i - y_i)^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad p(w_j) = \frac{\sqrt{\lambda}}{\sqrt{2\pi}}\exp\left(-\lambda\frac{w_j^2}{2}\right).$$

where Γ is the "gamma" function (which is always non-negative).

Answer:

$$f(w) = \frac{\nu+1}{2}\sum_{i=1}^{n}\log\left(1 + \frac{(w^T x_i - y_i)^2}{\nu}\right) + \frac{\lambda}{2}\|w\|^2.$$

4. We use a Poisson-distributed likelihood (for the case where y_i represents counts), and we use a uniform prior for some constant κ,

$$p(y_i|w^T x_i) = \frac{\exp(y_i w^T x_i)\exp(-\exp(w^T x_i))}{y_i!}, \quad p(w_j) \propto \kappa.$$

(This prior is "improper" since $w \in \mathbb{R}^d$ but it doesn't integrate to 1 over this domain, but nevertheless the posterior will be a proper distribution.)

Answer:

$$f(w) = \sum_{i=1}^{n}\left(-y_i w^T x_i + \exp(w^T x_i)\right).$$

If we wanted, we could write define $v_i = \exp(w^T x_i)$ and write this in matrix notation as

$$f(w) = -y^T Xw + 1^T v.$$

3 Principal Component Analysis

Rubric: {reasoning:3}

Consider the following dataset, containing 5 examples with 2 features each:

x_1	x_2
-4	3
0	1
-2	2
4	-1
2	0

Recall that with PCA we usually assume that the PCs are normalized ($\|w\| = 1$), we need to center the data before we apply PCA, and that the direction of the first PC is the one that minimizes the orthogonal distance to all data points.

1. What is the first principal component?
2. What is the reconstruction loss (L2 norm squared) of the point (-3, 2.5)? (Show your work.)
3. What is the reconstruction loss (L2 norm squared) of the point (-3, 2)? (Show your work.)

Hint: it may help (a lot) to plot the data before you start this question.

Answer: The first variable has a mean of 0 so centering it does nothing. The mean of the second variable is 1 so the centered data looks like this:

x_1	x_2
-4	2
0	0
-2	1
4	-2
2	-1

1. We see that all the centered variables lie along the $x_2 = -0.5x_1$. One vector spanning this line would be $(2, -1)$ which has a norm of $\sqrt{5}$, and normalizing this vector gives $w_1 = (2/\sqrt{5}, -1/\sqrt{5})$ (the numbers could be expressed in other forms and could also have the opposite sign).
2. We first subtract the mean $(0, 1)$ to give $(-3, 1.5)$ and multiply by w_1,

$$z = -3 \cdot 2/\sqrt{5} + 1.5 \cdot (-1)/\sqrt{5} = -7.5/\sqrt{5}.$$

   To go back to the original space, we multiply this by w_1 and add back the means:

$$\hat{x} = -\frac{7.5}{\sqrt{5}}(2/\sqrt{5}, -1/\sqrt{5}) + (0, 1) = (-3, 1.5) + (0, 1) = (-3, 2.5),$$

   which is the same as the original point so the reconstruction loss is 0.
3. To get the low-dimensional representation, we first subtract the mean $(0, 1)$ to give $(-3, 1)$ and then multiply by w_1,

$$\begin{aligned} z &= -3 \cdot 2/\sqrt{5} + 1 \cdot (-1)/\sqrt{5} \\ &= -6/\sqrt{5} - 1/\sqrt{5} \\ &= -7/\sqrt{5}. \end{aligned}$$

   To go back to the original space, we multiply this by w_1 and add back the means:

$$\hat{x} = \frac{-7}{\sqrt{5}}(2/\sqrt{5}, -1/\sqrt{5}) + (0, 1) = (-2.8, 1.4) + (0, 1) = (-2.8, 2.4).$$

   so the reconstruction loss is

$$(-3 - (-2.8))^2 + (2 - 2.4)^2 = 0.2$$

4 PCA Generalizations

4.1 Robust PCA

Rubric: {code:10}

If you run `python main -q 4.1` the code will load a dataset X where each row contains the pixels from a single frame of a video of a highway. The demo applies PCA to this dataset and then uses this to reconstruct the original image. It then shows the following 3 images for each frame:

1. The original frame.
2. The reconstruction based on PCA.
3. A binary image showing locations where the reconstruction error is non-trivial.

Recently, latent-factor models have been proposed as a strategy for "background subtraction": trying to separate objects from their background. In this case, the background is the highway and the objects are the cars on the highway. In this demo, we see that PCA does an OK job of identifying the cars on the highway in that it does tend to identify the locations of cars. However, the results aren't great as it identifies quite a few irrelevant parts of the image as objects.

Robust PCA is a variation on PCA where we replace the L2-norm with the L1-norm,

$$f(Z, W) = \sum_{i=1}^{n}\sum_{j=1}^{d} |\langle w^j, z_i\rangle - x_{ij}|,$$

and it has recently been proposed as a more effective model for background subtraction. Complete the class *pca.RobustPCA*, that uses a smooth approximation to the absolute value to implement robust PCA. Briefly comment on the results.

Note: in its current state, *pca.RobustPCA* is just a copy of *pca.AlternativePCA*, which is why the two rows of images are identical.

Hint: most of the work has been done for you in the class *pca.AlternativePCA*. This work implements an alternating minimization approach to minimizing the (L2) PCA objective (without enforcing orthogonality). This gradient-based approach to PCA can be modified to use a smooth approximation of the L1-norm. Note that the log-sum-exp approximation to the absolute value may be hard to get working due to numerical issues, and a numerically-nicer approach is to use the "multi-quadric" approximation:

$$|\alpha| \approx \sqrt{\alpha^2 + \epsilon},$$

where ϵ controls the accuracy of the approximation (a typical value of ϵ is 0.0001).

Answer: See *pca.RobustPCA*. The results look slightly better (though admittedly not amazing).

4.2 Reflection

Rubric: {reasoning:3}

1. Briefly explain why using the L1 loss might be more suitable for this task than L2.

   Answer: L1 is "robust to outliers", which means it ignores them, or, in other words, will fail to reconstruct them. This is good, as we want to detect outliers here - and our detection criterion is failure to reconstruct parts of the image.

2. How does the number of video frames and the size of each frame relate to n, d, and/or k?

   Answer: The number of frames is n, the total number of pixels per frame (width times height) is d, and k is a hyperparameter unrelated to these things, as long as it's less than d.

3. What would the effect be of changing the threshold (see code) in terms of false positives (cars we identify that aren't really there) and false negatives (real cars that we fail to identify)?

   Answer: A higher threshold means fewer false positives but more false negatives.

5 Very-Short Answer Questions

Rubric: {reasoning:11}

1. Assuming we want to use the original features (no change of basis) in a linear model, what is an advantage of the "other" normal equations over the original normal equations?

   Answer: Likely faster if $n \ll d$.

2. In class we argued that it's possible to make a kernel version of k-means clustering. What would an advantage of kernels be in this context?

   Answer: Can find non-convex clusters.

3. In the language of loss functions and regularization, what is the difference between MLE and MAP?

   Answer: In MLE you don't have a regularizer.

4. What is the difference between a generative model and a discriminative model?

   Answer: Discriminative doesn't model probability of features X.

5. With PCA, is it possible for the loss to increase if k is increased? Briefly justify your answer.

   Answer: No, in the worse case you could always pick W to include the old W as a subspace, so the loss cannot be worse than it used to be with smaller k.

6. What does "label switching" mean in the context of PCA?

   Answer: You obtain the same model if you switch the order of the factors.

7. Why doesn't it make sense to do PCA with $k > d$?

   Answer: With $k = d$ you can already get an error of 0.

8. In terms of the matrices associated with PCA (X, W, Z, $\hat{X}$), where would an "eigenface" be stored?

   Answer: A row of W.

9. What is an advantage and a disadvantage of using stochatic gradient over SVD when doing PCA?

   Answer: Stochastic gradient could be faster for huge datasets, SVD gives you the exact answer. SVD also gives orthonormal basis vectors.

10. Which of the following step-size sequences lead to convergence of stochastic gradient to a stationary point?

    (a) $\alpha^t = 1/t^2$.

    (b) $\alpha^t = 1/t$.

    (c) $\alpha^t = 1/\sqrt{t}$.

(d) $\alpha^t = 1$.

Answer: Only 2 and 3 (with 1 it will converge but not necessarily to a stationary point, with 4 it won't converge).

11. We discussed "global" vs. "local" features for e-mail classification. What is an advantge of using global features, and what is advantage of using local features?

    Answer: Global features let you predict for new users, local features let you make personalized predictions for individual users (if you have enough data for them).