CHAPTER 13
WAITING LINES

1. Introduction

- Waiting in line is one of the most common occurrences of everyday life. All of us are familiar with waiting in line for course registration, to check out at a supermarket, to make a deposit at a bank, for service at a hamburger stand, to pay at a toll bridge, for service at a gas station, and in numerous other situations. Waiting lines, often referred to as queues, are not restricted to humans. They can also consist of automobiles, machines, jobs, or other units awaiting services. For instance, airplanes wait for clearance to take off or land. Computer programs wait for their turn to be run.

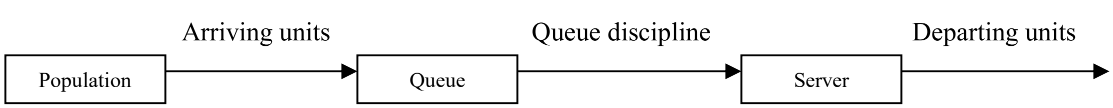
- Queuing theory can be traced back to the classic work of A. K. Erlang, a Danish telephone engineer, who studied the fluctuating demands for telephone facilities and associated service delays during the period from 1909 to 1920. Since then, thousands of articles and numerous books have been written on the subject. And, as with many other management science models, the potential for the applications of queuing theory is unlimited.

- Essentially, a queue forms whenever demand exceeds supply, i.e., whenever arriving customers cannot receive immediate service due to limited service capacity. The basic objective in most queuing models is to achieve a balance in the trade-off between the cost of increasing service capacity and that of lengthening customer waiting time.

- Unlike such optimization techniques as LP, DP, and ILP, the minimum total cost or the maximum total profit is not always sought in a queuing model. Rather, the aim is normally to determine various characteristics of the queuing system, such as the average waiting time and the average length of the waiting line. These are then used for subsequent cost/profit analysis.

2. Structure of a Queuing System

- In a typical queuing situation, customers arrive at a service system, enter a waiting line, receive service, and then leave. The process is illustrated in the following diagram:



Single-server queuing system

- The major components of a queuing model include population, arrival process, queue capacity, queue discipline, number of servers, and service process. In queuing literature, it is common to use the Kendall notation to describe a queuing system. Specifically, a queuing model is represented by (a/b/c d/e/f), where

  a = Distribution of inter-arrival times

  b = Distribution of service times

  c = Number of (parallel) servers

d = Queue discipline

e = Queue capacity

f = Population size

. Some of the commonly-used symbols in the Kendall notation are listed below:

Parameter	Notation	Value
a	M	Poisson arrivals
b	E_k	Erlang service times
	M	Exponential service times
	G	General service times
c	k	k server(s)
d	FCFS	First come, first served
	LCFS	Last come, first served
e	∞	Infinite queue capacity
	n	Queue capacity of n customers
f	∞	Infinite population
	n	Population of n customers

For example, we could have such queuing models as (M/M/1 FCFS/∞/∞), (M/G/2 FCFS/∞/8), or (M/E_k/4 FCFS/7/∞).

3. Review of Some Probability Distributions

. The Poisson distribution - A discrete random variable X is said to be Poisson distributed if

$$P_x = \frac{e^{-\lambda}\lambda^x}{x!}, \; x = 0, 1, 2 \ldots$$

where

P_x = Probability that the event X = x (i.e., X takes on the value x) will occur during a time interval

λ = Average number of occurrences of an event during the time interval

e = Base of the natural logarithm, which is equal to 2.71828...

. **Example 13.1:** Patients arrive at a physician's office at a rate of 6 per hour on the average. Under the assumption of Poisson arrivals,

(1) what is the probability that 4 patients will arrive during a given hour?

(2) what is the probability that no patients will arrive during a given hour?

[Solution] Since $\lambda = 6$, we have

(1) $P_4 = (2.71828)^{-6}(6^4)/4! \approx 0.1339$

(2) $P_0 = (2.71828)^{-6}(6^0)/0! \approx 0.0025$

. The exponential distribution - A continuous random variable T is subject to the exponential distribution if

$P(T \leq t) = 1 - e^{-\mu t}, t \geq 0$

where

$P(T \leq t)$ = Probability that T will take on a value between and including 0 and t

$1/\mu$ = Average value that T will take on

e = Base of the natural logarithm, which is equal to 2.71828...

. **Example 13.2:** Suppose that 4 customers are serviced at a checkout counter per hour on the average. Under the assumption of exponential service times,

(1) what is the probability that the service a customer receives is completed within 12 minutes?

(2) what is the probability that the service a customer receives is completed within an amount of time between 15 and 20 minutes?

[Solution] Since $1/\mu = 1/4 = 15$, we have $\mu = 4$ and

(1) $P(T \leq 12/60) = 1 - e^{-4(12/60)} = 1 - e^{-0.8} \approx 0.5507$

(2) $P(15/60 \leq T \leq 20/60) = P(T \leq 20/60) - P(T \leq 15/60) = [1 - e^{-4(20/60)}] – [1 - e^{-4(15/60)}] \approx 0.1043$

. Remark: Consider a queuing model in which customers arrive at a system for service. If the arrivals are Poisson distributed with a mean of λ, then the inter-arrival times are exponentially distributed with a mean of $1/\lambda$. Conversely, if the service times are exponentially distributed with a mean of $1/\mu$, then the numbers of customers serviced during the time period are Poisson distributed with a mean of μ.

4. (M/M/1 FCFS/∞/∞) Model

. Assumptions:

(1) Poisson arrivals with a mean of λ

(2) Exponential service times with a mean of $1/\mu$

(3) One server

(4) FCFS queue discipline

(5) Infinite queue capacity

(6) Infinite population

(7) No reneging (i.e., leaving a queue before being served) or balking (i.e., refusing to join a queue because of its length)

(8) Steady-state situation

(9) $\mu > \lambda$

. Operating characteristics:

(1) ρ = Utilization factor, i.e., the probability that the facility is busy

(2) P_0 = Probability that the server is idle

(3) L = Average number of customers in the system

(4) L_q = Average number of customers in the queue

(5) W = Average waiting time in the system

(6) W_q = Average waiting time in the queue

(7) P_n = Probability that there are n customers in the system

(8) $P_{>n}$ = Probability that there are more than n customers in the system

(9) $P_{>t}$ = Probability that a customer is in the system longer than t time units

(10) $P_{>t'}$ = Probability that a customer is in the queue longer than t' time units

. Remark: The first six are called the primary operating characteristics.

. Results:

(1) $\rho = \lambda/\mu$

(2) $P_0 = 1 - \rho$

(3) $L = \lambda/(\mu - \lambda)$

(4) $L_q = \lambda^2/[\mu(\mu - \lambda)]$

(5) $W = 1/(\mu - \lambda)$

(6) $W_q = \lambda/[\mu(\mu - \lambda)]$

(7) $P_n = \rho^n(1 - \rho)$

(8) $P_{>n} = \rho^{n+1}$

(9) $P_{>t} = e^{(\lambda-\mu)t}$

(10) $P_{>t'} = \rho e^{(\lambda-\mu)t'}$

. Important relationships among some operating characteristics:

(1) $L = \lambda W$

(2) $L_q = \lambda W_q$

(3) $W = W_q + 1/\mu$

(4) $L = L_q + \lambda/\mu$

. **Example 13.3:** The United Bank has one drive-up teller window in operation. Cars arrive at the window in a Poisson fashion at a rate of 15 customers per hour. It takes the teller an average of 2 minutes to service each customer. Service is on an FCFS basis and times are exponentially distributed. There is an infinite calling population and an infinite queue capacity. Management is interested in the operating characteristics of the system.

(1) What is the probability that a car arriving at the window has to wait?

(2) What is the probability that the teller is idle or what is the percentage of the time the teller is idle?

(3) What is the average number of cars in the queue?

(4) What is the average number of cars in the system?

(5) How long on average does a car arriving at the window have to wait before it is serviced?

(6) How long on average does a car spend in the system?

(7) Do you think it is necessary to add another teller?

[Solution] $\lambda = 15$, $\mu = 60/2 = 30$

(1) Since $\rho = \lambda/\mu = 15/30 = 0.5$, there is a 50% chance that a car arriving at the teller window has to wait.

(2) Since $P_0 = 1 - \rho = 1 - 0.5 = 0.5$, there is a 50% chance that the teller is idle.

(3) Since $L_q = \lambda^2/[\mu(\mu - \lambda)] = (15)^2/[30(30 - 15)] = 0.5$, there are, on the average, 0.5 cars in the queue.

(4) Since $L = \lambda/(\mu - \lambda) = 15/(30 - 15) = 1$, there is, on the average, 1 car in the system.

(5) Since $W_q = \lambda/[\mu(\mu - \lambda)] = 15/[30(30 - 15)] = 1/30$ hours = 2 minutes, a car arriving at the teller window has to wait in the queue, on the average, for 2 minutes.

(6) Since $W = 1/(\mu - \lambda) = 1/(30 - 15) = 1/15$ hours = 4 minutes, a car has to spend, on the average, 4 minutes in the system.

(7) A 2-minute wait seems to be reasonable. Thus, it is not necessary to add another teller.

. **Example 13.4:** Consider the operation of a central supply room for a large office where employees pick up needed supplies. On the average, 25 employee customers withdraw supplies during each hour of normal operation. A full-time clerk is required to check persons out of central supply on a "fist come, first served" basis, primarily to assure proper accounting control of requisitioned items. Each requisition takes an average of 2 minutes. Assume that the pattern of arrivals at the checkout counter is a close approximation to a Poisson process and that the checkout times are exponentially distributed.

(1) Assuming infinite population and infinite queue capacity, compute the primary operating characteristics of the system.

(2) Suppose that the cost of the clerk's labor is $5 per hour, whereas the unproductive time of employee customers is valued at an average hourly payroll figure of $7. Ignoring the cost of the time employees spend selecting supplies off the shelves, what is the average daily total cost of checking out of the supply room?

(3) Suppose that the daily cost figure in (2) is highly excessive and management is considering speeding up the checkout process by partially automating the system. A partially automated system, which costs $150 per day in addition to the clerk's wages, enables the clerk to double his service rate. Would the partial automation be a worthwhile endeavor?

[Solution] $\lambda = 25$, $\mu = 60/2 = 30$

(1) (a) $\rho = 25/30 \approx 0.8333$

(b) $P_0 = 1 - \rho = 1 - 0.8333 = 0.1667 = 16.67\%$

(c) $L_q = \lambda^2/[\mu(\mu - \lambda)] = (25)^2/[30(30 - 25)] \approx 4.1667$

(d) $L = \lambda/(\mu - \lambda) = 25/(30 - 25) = 5$

(e) $W_q = \lambda/[\mu(\mu - \lambda)] = 25/[30(30 - 25)] = 1/6$ (hours)

(f) $W = 1/(\mu - \lambda) = 1/(30 - 25) = 1/5$ (hours)

(2) Note that the average daily total cost consists of the daily cost of the clerk and the daily queuing cost. Since the daily cost of the clerk is 5(8) = $40 and the daily queuing cost is 25(8)(1/5)(7) = $280, the average daily total cost is 40 + 280 = $320.

(3) If the partially automated system is installed, then $\lambda = 25$, $\mu = 30(2) = 60$, and $W = 1/(\mu - \lambda) = 1/(60 - 25) = 1/35$ hours. It follows that the average daily queuing cost is

25(8)(1/35)(7) = \$40 and the average daily total cost is 40 + 40 + 150 = \$230, which is smaller than \$320. Hence, the partially automated system should be installed.

. **Example 13.5:** Refer to (3) in Example 13.4 above.

(1) What is the probability that the clerk is idle?

(2) What is the probability that there is only 1 employee in the checkout process?

(3) What is the probability that there are 2 employees in the checkout process?

(4) What is the probability that there are 3 or more employees in the checkout process?

(5) What is the probability that an employee spends more than 6 minutes in the checkout process?

(6) What is the probability that an employee has to wait for more than 3 minutes before he or she is serviced?

[Solution] If the partially automated system is installed, we have $\lambda = 25$, $\mu = 60$, and $\rho = \lambda/\mu = 25/60 \approx 0.4167$.

(1) $P_0 = 1 - \rho = 1 - 0.4167 = 0.5833$

(2) $P_1 = \rho^1(1 - \rho) = 0.4167(1 - 0.4167) \approx 0.2431$

(3) $P_2 = \rho^2(1 - \rho) = (0.4167)^2(1 - 0.4167) \approx 0.1013$

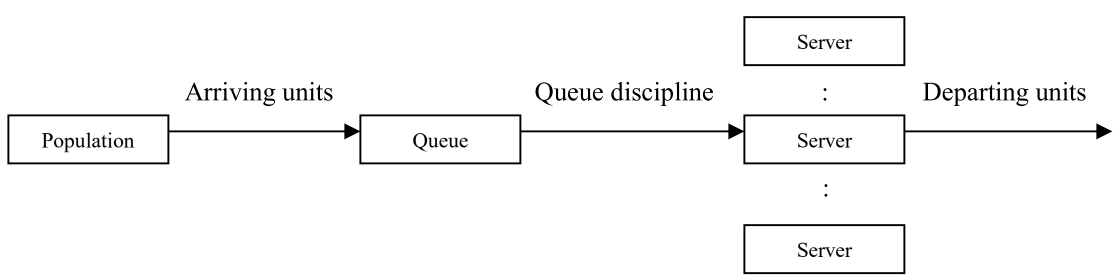
(4) $P_{>2} = \rho^{2+1} = (0.4167)^3 = 0.0723$. Alternatively, $P_{>2} = 1 - P_0 - P_1 - P_2 = 1 - 0.5833 - 0.2431 - 0.1013 = 0.0723$

(5) $P_{>6/60} = P_{>0.1} = e^{(\lambda-\mu)(0.1)} = e^{(25-60)(0.1)} \approx 0.0302$

(6) $P_{>3/60} = P_{>0.05} = \rho e^{(\lambda-\mu)(0.05)} = (0.4167)e^{(25-60)(0.05)} \approx 0.0724$

5. (M/M/k FCFS/∞/∞) Model

. The model is illustrated in the following diagram, where a single queue is formed:



Multi-server queuing system

. Assumptions:

(1) Poisson arrivals with a mean of λ

(2) Exponential service times with a mean of $1/\mu$

(3) k (k > 1) servers

(4) FCFS queue discipline

(5) Infinite queue capacity

(6) Infinite population

(7) No reneging (i.e., leaving a queue before being served) or balking (i.e., refusing to join a queue because of its length)

(8) Steady-state situation

(9) $k\mu > \lambda$

. Operating characteristics:

(1) ρ = Utilization factor, i.e., the probability that the system is busy

(2) P_0 = Probability that the system is idle

(3) L = Average number of customers in the system

(4) L_q = Average number of customers in the queue

(5) W = Average waiting time in the system

(6) W_q = Average waiting time in the queue

(7) P_n = Probability that there are n customers in the system

. Remark: The first six are called the primary operating characteristics.

. Results:

(1) $\rho = \lambda/k\mu$

(2) $$P_0 = \frac{1}{\sum_{n=0}^{k-1} \frac{(\frac{\lambda}{\mu})^n}{n!} + \frac{(\frac{\lambda}{\mu})^k}{(k-1)!}\left(\frac{\mu}{k\mu - \lambda}\right)}$$

(3) $L_q = \dfrac{(\frac{\lambda}{\mu})^k (\lambda\mu)}{(k-1)!(k\mu - \lambda)^2} (P_0)$

(4) $L = L_q + \lambda/\mu$

(5) $W_q = L_q/\lambda$

(6) $W = W_q + 1/\mu$

(7) $$P_n = \begin{cases} \dfrac{(\frac{\lambda}{\mu})^n}{k!k^{n-k}} (P_0), \text{ if } n > k \\ \\ \dfrac{(\frac{\lambda}{\mu})^n}{n!} (P_0), \text{ if } 0 \le n \le k \end{cases}$$

. **Example 13.6:** Mrs. Drake is manager of the United Bank in Example 13.3. She has noted that on Friday evenings there are still sometimes several cars in line when 2 teller windows are open from 5 to 9 p.m. Cars arrive on these nights at a rate of 54 per hour. Service time still averages 2 minutes per car at each window.

(1) Compute the operating characteristics for the 2-teller system.

(2) What is the probability that there are 5 cars in the system?

(3) What is the probability that there is 1 car in the system?

[Solution] Note that $k = 2$, $\lambda = 54$, and $\mu = 60/2 = 30$.

(1) (a) $\rho = \lambda/k\mu = 54/[2(30)] = 0.9 = 90\%$

(b) $$P_0 = \frac{1}{\Sigma \dfrac{(54/30)^n}{n!} + \dfrac{(54/30)^2}{(2-1)!} \left[\dfrac{30}{2(30) - 54}\right]} \approx 0.053$$

Alternatively, with $\lambda/\mu = 54/30 = 1.8$, we see from the table on the last page of this chapter that $P_0 = 0.053$.

(c) $$L_q = \frac{(54/30)^2(54)(30)}{(2-1)![2(30) - 54]^2} (0.053) \approx 7.6691$$

(d) $L = 7.6691 + 54/30 \approx 9.4691$

(e) $W_q = 7.6691/54 \approx 0.1420$ (hours)

(f) $W = 0.1420 + 1/30 \approx 0.1754$ (hours)

(2) Since $n = 5 > 2 = k$, $P_5 = \dfrac{(54/30)^5}{2!2^{5-2}} (0.0526) \approx 0.0621$.

(3) Since $n = 1 < 2 = k$, $P_1 = \dfrac{(54/30)^1}{1!} (0.0526) \approx 0.0947$.

. **Example 13.7:** Star Oil service station (full service only) is considering how many of its two identical pumps to staff during the night. Past experience indicates that there are 16 random arrivals per hour on the average during the 9 p.m. - 7 a.m. period. Each customer brings the station a revenue of \$15. The service time takes three minutes on the average, and follows a exponential distribution.

Long waiting lines create ill will. In addition, customers may not enter the station if they see long lines. Therefore, the management of Star Oil estimates that each customer-hour of waiting time in the service station effectively costs \$30. The operating cost of manning each pump is \$50 per hour. Suppose that the arrivals are subject to Poisson distribution. Should 1 or 2 pumps be manned during the night?

[Solution] In this problem, $\lambda = 16$ and $\mu = 60/3 = 20$. If only 1 pump is manned, $W_q = 16/[20(20 - 16)] = 0.2$ hours. The net profit during the 10-hour period = total revenue - total operating cost - total queuing (or ill-will) cost = 15(16)(10) - 50(10) - 30(16)(10)(0.2) = \$940.

If 2 pumps are manned, then $k = 2$ and $\lambda/k\mu = 16/[2(20)] = 0.4$. We see from the table on the last page of this chapter that $L_q = 0.152$. It follows that $W_q = 0.152/16 \approx 0.0095$ hours. Thus, the net profit during the 10-hour period = 15(16)(10) - 2(50)(10) - 30(16)(10)(0.0095) = \$1,354.4.

In conclusion, 2 pumps should be manned during the night since 1,354.4 > 940.

TABLE 16-5 Infinite Source Values for L_q and P_0 given λ/μ and s

λ/μ	s	L_q	P_0
.15	1	.026	.850
	2	.001	.860
.20	1	.050	.800
	2	.002	.818
.25	1	.083	.750
	2	.004	.778
.30	1	.129	.700
	2	.007	.739
.35	1	.188	.650
	2	.011	.702
.40	1	.267	.600
	2	.017	.667
.45	1	.368	.550
	2	.024	.633
	3	.002	.637
.50	1	.500	.500
	2	.033	.600
	3	.003	.606
.55	1	.672	.450
	2	.045	.569
	3	.004	.576
.60	1	.900	.400
	2	.059	.538
	3	.006	.548
.65	1	1.207	.350
	2	.077	.509
	3	.008	.521
.70	1	1.633	.300
	2	.098	.481
	3	.011	.495
.75	1	2.250	.250
	2	.123	.455
	3	.015	.471
.80	1	3.200	
	2	.152	.429
	3	.019	.447
.85	1	4.817	.150
	2	.187	.404
	3	.024	.425
	4	.003	.427
90	1	8.100	.100
	2	.229	.379
	3	.030	.403
	4	.004	.406
.95	1	18.050	.050
	2	.277	.356
	3	.037	.383
	4	.005	.386
1.0	2	.333	.333
	3	.045	.364
	4	.007	.367
1.1	2	.477	.290
	3	.066	.327
	4	.011	.367
1.2	2	.675	.250
	3	.094	.294
	4	.016	.300
	5	.003	.301
1.3	2	.951	.212
	3	.130	.264
	4	.023	.271
	5	.004	.272
1.4	2	1.345	.176
	3	.177	.236
	4	.032	.245
	5	.006	.246
1.5	2	1.929	.143
	3	.237	.211
	4	.045	.221
	5	.009	.223
1.6	2	2.844	.111
	3	.313	.187
	4	.060	.199
	5	.012	.201
1.7	2	4.426	.081
	3	.409	.166
	4	.080	.180
	5	.017	.182
1.8	2	7.674	.053
	3	.532	.146
	4	.105	.162
	5	.023	.165
1.9	2	17.587	.026
	3	.688	.128
	4	.136	.145
	5	.030	.149
	6	.007	.149
2.0	3	.889	.111
	4	.174	.130
	5	.040	.134
	6	.009	.135
2.1	3	1.149	.096
	4	.220	.117
	5	.052	.121
	6	.012	.122
2.2	3	1.491	.081
	4	.277	.105
	5	.066	.109
	6	.016	.111
2.3	3	1.951	.068
	4	.346	.093
	5	.084	.099
	6	.021	.100
2.4	3	2.589	.056
	4	.431	.083
	5	.105	.089
	6	.027	.090
	7	.007	.091
2.5	3	3.511	.045
	4	.533	.074
	5	.130	.080
	6	.034	.082
	7	.009	.082
2.6	3	4.933	.035
	4	.658	.065
	5	.161	.072
	6	.043	.074
	7	.011	.074
2.7	3	7.354	.025
	4	.811	.057
	5	.198	.065
	6	.053	.067
	7	.014	.067
2.8	3	12.273	.016
	4	1.000	.050
	5	.241	.058
	6	.066	.060
	7	.018	.061
2.9	3	27.193	.008
	4	1.234	.044
	5	.293	.052
	6	.081	.054
	7	.023	.055
3.0	4	1.528	.038
	5	.354	.047
	6	.099	.049
	7	.028	.050
	8	.008	.050
3.1	4	1.902	.032
	5	.427	.042
	6	.120	.044
	7	.035	.045
	8	.010	.045
3.2	4	2.386	.027
	5	.513	.037
	6	.145	.040
	7	.043	.040
	8	.012	.041