CHAPTER 2
INTRODUCTION TO OPTIMIZATION AND LINEAR PROGRAMMING

1. Introduction

. Linear programming (LP) is a mathematical tool for solving optimization problems with constraints. Dr. George Dantzig, who is considered the father of LP, developed the simplex method for solving linear programs in 1947. Since then, LP has been applied in agriculture, banking, education, forestry, military, manufacturing, petroleum, telecommunication, transportation, and many other industries.

. The major components of a linear program are:

(1) Decision variable: An alternative available to the decision maker in terms of amount of input or output.

(2) Objective function: A mathematical expression consisting of decision variables and coefficients that represents the goal of a linear program.

(3) Constraint: A mathematical expression consisting of decision variables and coefficients that restricts the alternatives available to the decision maker. There are two types of constraint: structural and non-negativity.

2. Formulation of Linear Programs

. **Example 2.1:** Beijing Toy Company manufactures two types of toys: car and train. A toy car sells for \$6 and uses 1 hour of machine time as well as 2 pounds of raw material. A toy train sells for \$9 and requires 1 hour of machine time as well as 4 pounds of raw material. There are 8 hours of machine time and 24 pounds of raw material available per day. Set up a linear program that may be used to determine the quantity of each type of toy to be made to maximize the company's daily total revenue.

[Solution] The key information on the problem is summarized in the following table:

Toy	Machine	Material	Unit price
Car	1	2	6
Train	1	4	9
Availability	8	24	

(1) Decision variables:

x = Number of toy cars to be made
y = Number of toy trains to be made

(2) Objective function:

$Z = f(x, y) = 6x + 9y$

(3) Constraints:

(a) Structural:

Machine time availability: $x + y \leq 8$
Raw material availability: $2x + 4y \leq 24$

(b) Non-negativity:

$x \geq 0$ and $y \geq 0$ or $x, y \geq 0$

A complete LP model for the product mix problem is presented below:

$$\begin{aligned}
\text{Maximize } Z = \; & 6x + 9y \\
\text{subject to:} \quad & x + y \leq 8 \\
& 2x + 4y \leq 24 \\
& x,\ y \geq 0
\end{aligned}$$

- **Example 2.2:** A farmer in the Hebei Province raises pigs and he needs to mix a pig feed from two grains. Each pound of Grain A costs $16 and contains 3 units of Vitamin C as well as 6 units of Vitamin D. Each pound of Grain B costs $12 and contains 5 units on Vitamin C as well as 2 units of Vitamin D. In order for the pigs to be healthy, the feed must contain at least 15 units of Vitamin C and at least 18 units of Vitamin D. Develop an LP model that may be used to determine the amount of each grain to be mixed to minimize the total cost of the pig feed.

[Solution] The key information on the problem is summarized in the following table:

Vitamin	Grain A	Grain B	Minimum requirement
C	3	5	15
D	6	2	18
Cost per pound	16	12	

(1) Decision variables:

a = Number of pounds of Grain A to be mixed
b = Number of pounds of Grain B to be mixed

(2) Objective function:

$Z = f(a, b) = 16a + 12b$

(3) Constraints:

(c) Structural:

Vitamin C requirement: $3a + 5b \geq 15$
Vitamin D requirement: $6a + 2b \geq 18$

(d) Non-negativity:

$a \geq 0$ and $b \geq 0$ or $a, b \geq 0$

A complete LP model for the pig feed problem is presented below:

Let a = Number of pounds of Grain A to be mixed
b = Number of pounds of Grain B to be mixed

$$\begin{aligned}
\text{Minimize } Z = \; & 16a + 12b \\
\text{subject to:} \quad & 3a + 5b \geq 15 \\
& 6a + 2b \geq 18 \\
& a,\ b \geq 0
\end{aligned}$$

- Once a problem is formulated as a linear program, it can be solved by using (1) the graphical method (including the iso-line approach and the corner-point approach), (2) the simplex method, or (3) computer software. Each of these is discussed below.

3. Graphical Method of Solving Linear Programs

- The graphical method provides a visual portrayal of many of the important concepts of a linear program on the two-dimensional plane. However, it is only applicable to problems that involve at most two decision variables.

- The following steps should be followed in applying the graphical method:

  (1) Plot the constraints one at a time.

  (2) Graph the feasible region.

  (3) (i) Plot at least two iso-profit (or iso-cost) lines to identify the direction in which the objective function value is increasing (or decreasing), or

  (ii) Find the coordinates of the corner points and compute the objective function value at each of them.

  (4) Determine the optimal solution and the optimal objective function value.

  (5) Draw conclusion and provide interpretation.

- **Example 2.3:** Consider the linear program formulated in Example 2.1, which is reproduced below:

$$
\begin{array}{lrrcr}
\text{Maximize } Z = & 6x + & 9y & & \\
\text{subject to:} & x + & y & \leq & 8 \\
 & 2x + & 4y & \leq & 24 \\
 & x\ , & y & \geq & 0
\end{array}
$$

  (1) Plot the constraints and graph the feasible region for the LP.

  (2) Plot iso-revenue lines to determine the optimal product mix and the maximum daily revenue.

  (3) Repeat (2) by using the corner-point approach.

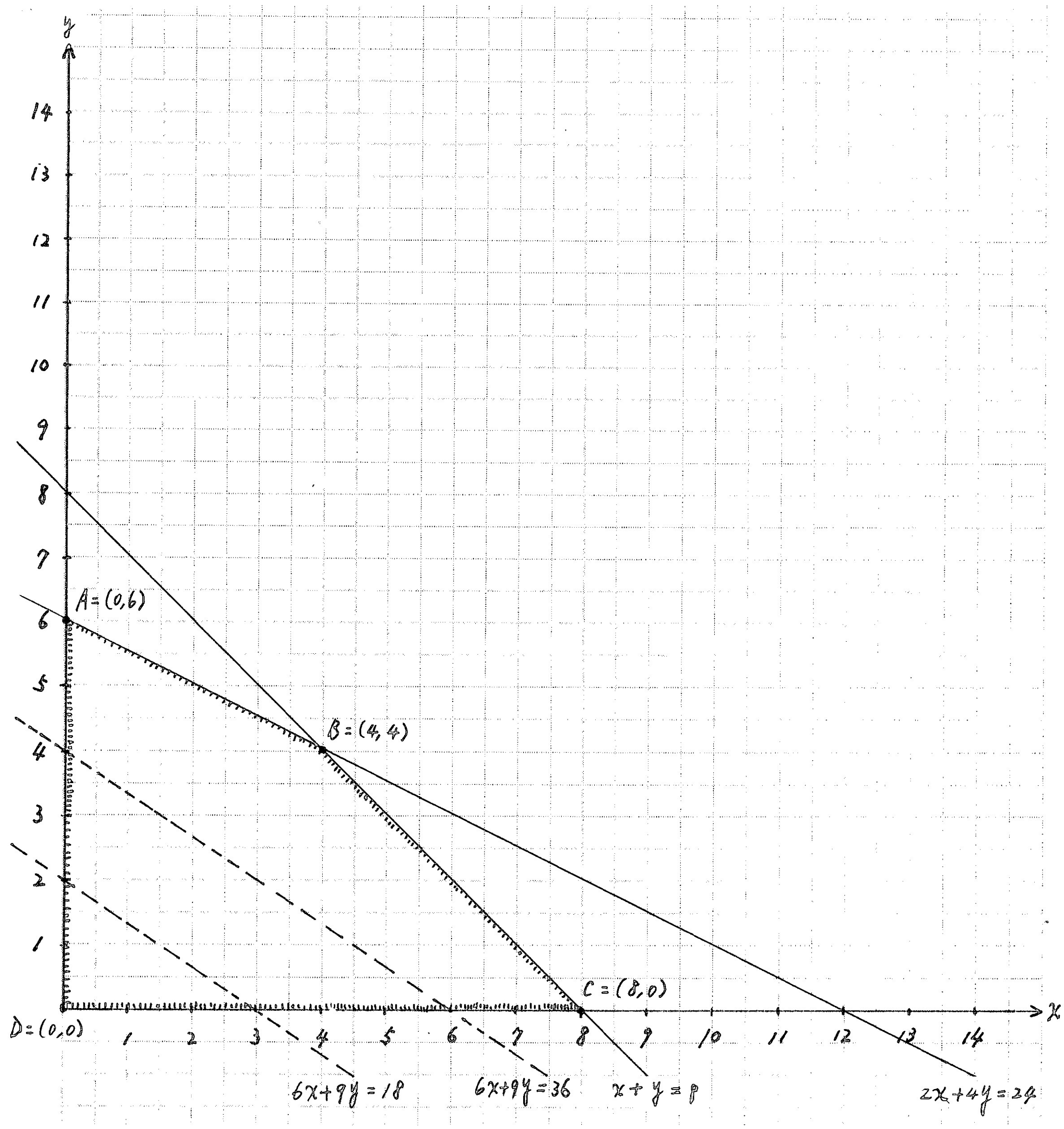
  (4) Are the results in (2) and (3) above consistent?

  [Solution] (1) The graph for the problem is shown the next page.

  (2) It is seen that the objective function will attain its maximum value at point B. Solving the simultaneous equations of $x + y = 8$ and $2x + 4y = 24$, we find that the optimal solution is $(x^*, y^*) = (4, 4)$ and the optimal objective function value is $6x^* + 9y^* = 6(4) + 9(4) = 60$. In other words, the company should produce 4 toy cars and 4 toy trains to maximize its daily total revenue at $60.

  (3) The coordinates of the four corner points along with the corresponding objective function values are calculated as follows:

  $A = (0, 6)$: $Z = 6x + 9y = 6(0) + 9(6) = 54$ (we need to solve the simultaneous equations of $x = 0$ and $2x + 4y = 24$ to find $x = 0$ and $y = 6$)


y
x
A = (0,6)
B = (4,4)
C = (8,0)
D = (0,0)
6x+9y=18
6x+9y=36
x + y = 8
2x+4y=24

$B = (4, 4)$: $Z = 6x + 9y = 6(4) + 9(4) = 60$ (we need to solve the simultaneous equations of $x + y = 8$ and $2x + 4y = 24$ to find $x = 4$ and $y = 4$)

$C = (8, 0)$: $Z = 6x + 9y = 6(8) + 9(0) = 48$ (we need to solve the simultaneous equations of $y = 0$ and $x + y = 8$ to find $x = 8$ and $y = 0$)

$D = (0, 0)$: $Z = 6x + 9y = 6(0) + 9(0) = 0$ (the coordinates of the origin is $x = 0$ and $y = 0$).

Since $60 > 54 > 48 > 0$, the optimal solution is $(x^*, y^*) = (4, 4)$ and the optimal objective function value is $Z^* = 60$. In other words, the company should produce 4 toy cars and 4 toy trains to maximize its daily total revenue at $60.

(4) Yes, they are consistent.

. Remarks: It can be shown that the optimal solution to a linear program always occurs at one of the corners of the feasible region. Consequently, it suffices to evaluate and compare the objective function values at the corner points to solve the LP. This is the so-called corner-point approach.

. **Example 2.4:** Consider the linear program formulated in Example 2.2, which is reproduced below:

$$\begin{array}{llcl}
\text{Maximize } Z = 16a + 12b & & & \\
\text{subject to:} & 3a + 5b & \geq & 15 \\
 & 6a + 2b & \geq & 18 \\
 & a\ ,\quad b & \geq & 0
\end{array}$$

(1) Plot the constraints and graph the feasible region for the LP.

(2) Plot iso-cost lines to determine the optimal amounts of grains to mix and the minimum total cost of the pig feed.

(3) Repeat (2) by using the corner-point approach.

(4) Are the results in (2) and (3) above identical?

[Solution] (1) The graph for the problem is shown on the next page.

(2) It is seen that the objective function will attain its maximum value at point B. Solving the simultaneous equations of $3a + 5b = 15$ and $6a + 2b = 18$, we find that the optimal solution is $(a^*, b^*) = (2.5, 1.5)$ and the optimal objective function value is $Z^* = 16a^* + 12b^* = 16(2.5) + 12(1.5) = 58$. In other words, the farmer should mix 2.5 pounds of Grains A and 1.5 pounds of Grain B to minimize the total pig feed cost at $58.

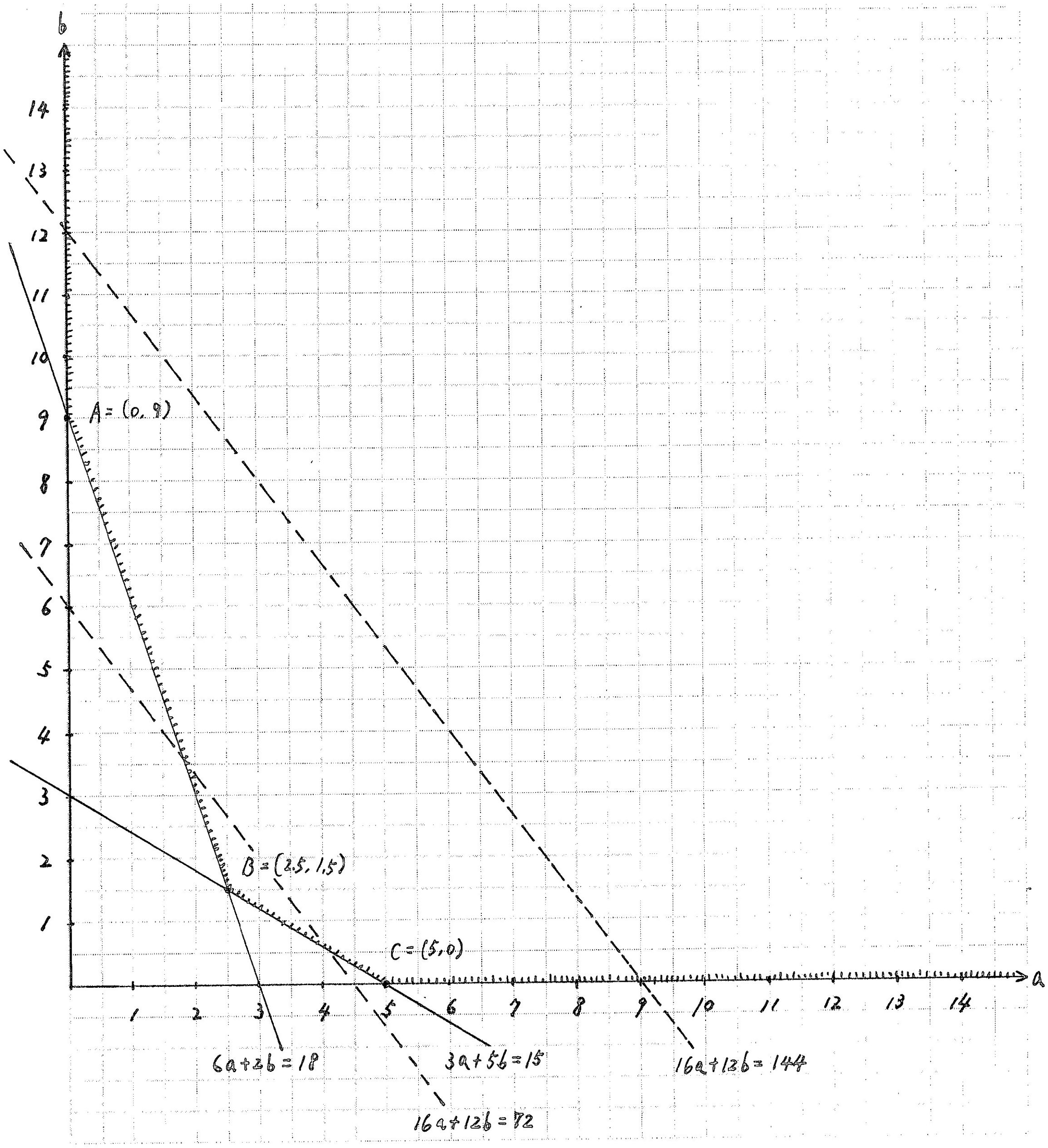
(3) The coordinates of the three corner points along with the corresponding objective function values are calculated as follows:

$A = (0, 9)$: $Z = 16a + 12b = 16(0) + 12(9) = 108$ (we need to solve the simultaneous equations of $a = 0$ and $6a + 2b = 18$ to find $a = 0$ and $b = 9$)

$B = (2.5, 1.5)$: $Z = 16a + 12b = 16(2.5) + 12(1.5) = 58$ (we need to solve the simultaneous equations of $3a + 5b = 15$ and $6a + 2b = 18$ to find $a = 2.5$ and $b = 1.5$)

$C = (5, 0)$: $Z = 16a + 12b = 16(5) + 12(0) = 80$ (we need to solve the simultaneous equations of $b = 0$ and $3a + 5b = 15$ to find $a = 5$ and $b = 0$)

Since $58 < 80 < 108$, the optimal solution is $(a^*, b^*) = (2.5, 1.5)$ and the optimal objective function value is $Z^* = 58$. In other words, the farmer should mix 2.5 pounds of Grain A with 1.5 pounds of Grain B to minimize the total pig feed cost at $58.


b
a
A = (0, 9)
B = (2.5, 1.5)
C = (5, 0)
6a+2b=18
3a+5b=15
16a+12b=144
16a+12b=72

(4) Yes, they are identical.

4. Simplex Method of Solving Linear Programs

. In contrast to the graphical method, the simplex method can be used to solve linear programs involving any number of decision variables. It consists of a series of simple but tedious tableau iterations. While most people rely on computer when applying the simplex method, some familiarity with the manual computations will be helpful in understanding the technique.

. In implementing the simplex algorithm, even the slightest calculating error can easily distort the final result. Hence, it would be best to work with numbers in fractional form.

. In solving a linear program using the simplex method, the non-negativity constraint is not considered explicitly. The steps involved for a maximization LP are:

(1) Setting up the initial simplex tableau.

(a) Convert each constraint into an equality:

"≤": Add a slack variable.

"≥": Subtract a surplus variable and then add an artificial variable.

"=": Add an artificial variable.

(b) Rewrite each resulting equality from the conversion in (a) above to include in it all of the new slack, surplus, and/or artificial variables that are missing with a coefficient of 0.

(c) Rewrite the objective function to include all of the new slack, surplus, and artificial variables with respective coefficients of 0, 0, and -M (M for a minimization problem), where M represents an extremely large positive number.

(d) Arrange the objective coefficients and the constraint coefficients into a tableau.

(e) Compute the numbers in the "Z_j" row. (This can be done by multiplying the numbers in the j^{th} column with their counterparts in the left-most or coefficient ("C_j") column and then adding up the results.)

(f) Compute the numbers in the "Z_j - C_j" row. (The "C_j" column shows the coefficients of the variables in solution ("VIS") column.)

(2) Testing the current solution for optimality.

If all the numbers in the "Z_j - C_j" row are either positive (negative for a minimization problem) or zero, the current solution is optimal; otherwise, the following steps should be followed before an improved solution is derived:

(a) The column with the most negative (positive for a minimization problem) number in the "Z_j - C_j" is termed the pivot column. A tie can be broken in an arbitrary way.

(b) Divide each number in the right-most or solution quantity ("SQ") column by the corresponding number in the pivot column that is positive. The row with the smallest ratio is termed the pivot row.

(c) The number at the intersection of the pivot column and the pivot row is termed the pivot number.

(3) Developing the improved solution.

(a) Derive new numbers in the pivot row by dividing the entire row by the pivot number.

(b) Update each of the remaining rows by performing the following operation:

New number = Old number - (Old number in the pivot column x Corresponding new number in the pivot row)

(c) Compute new numbers in the "Z_j" row.

(d) Compute new numbers in the "Z_j - C_j" row.

(4) Go to (2)

- **Example 2.5:** Reconsider the LP set up in Example 2.1, which was solved by using the graphical method in Example 2.3 and is reproduced below. Use the simplex method to solve it.

$$
\begin{array}{ll}
\text{Maximize } Z = & 6x + 9y \\
\text{subject to:} & x + y \leq 8 \\
& 2x + 4y \leq 24 \\
& x\ ,\ y \geq 0
\end{array}
$$

[Solution] The LP is modified as follows:

$$
\begin{array}{lrrrrcl}
Z = & 6x + & 9y + & 0u + & 0v & & \\
& x + & y + & u + & 0v & = & 8 \\
& 2x + & 4y + & 0u + & v & = & 24
\end{array}
$$

The initial simplex tableau is as follows:

C_j		6	9	0	0		
	VIS	x	y	u	v	SQ	
0	u	1	1	1	0	8	8/1 = 8
0	v	2	<u>4</u>	0	1	24	24/4 = 6
Z_j		0	0	0	0	0	
Z_j - C_j		-6	**-9**	0	0		

The current solution (x, y) = (0, 0) is not optimal since -6 < 0 and -9 < 0. Given that -9 is the most negative number in the "Z_j - C_j" row and 6 < 8, we see that the pivot column is "y" ("y" is the entering variable), the pivot row is "v" ("v" is the leaving variable), and the pivot number is 4. The new simplex tableau follows:

C_j		6	9	0	0		
VIS		x	y	u	v	SQ	
0	u	<u>1/2</u>	0	1	-1/4	2	2/(1/2) = 4
9	y	1/2	1	0	1/4	6	6/(1/2) = 12
Z_j		9/2	9	0	9/4	54	
Z_j - C_j		**-3/2**	0	0	9/4		

The current solution (x, y) = (0, 6) is not optimal since -3/2 < 0. Given that -3/2 is the only negative number in the "Z_j - C_j" row and 4 < 12, the pivot column is "x" ("x" is the entering variable), the pivot row is "u" ("u" is the leaving variable), and the pivot number is 1/2. The new simplex tableau follows:

C_j		6	9	0	0	
VIS		x	y	u	v	SQ
6	x	1	0	2	-1/2	4
9	y	0	1	-1	1/2	4
Z_j		6	9	3	3/2	60
Z_j - C_j		0	0	3	3/2	

The current solution (x, y) = (4, 4) is optimal since all the numbers in the "Z_j - C_j" row are positive or zero. In conclusion, the optimal solution is (x^*, y^*) = (4, 4) and the optimal objective function value is Z^* = 60. (Note that these are the same as those found in Example 2.3.)

- **Remark:** Each simplex tableau solution corresponds to a corner point of the feasible solution space in the graphical method. For instance, the sequence of solutions (x, y) obtained in Example 2.5 above is (0, 0) -> (0, 6) -> (4, 4), which corresponds to the sequence of points D -> A -> B one on one in Example 2.3.

- **Example 2.6:** Reconsider the LP set up in Example 2.2, which was solved by using the graphical method in Example 2.4 and is reproduced below. Use the simplex method to solve it

$$
\begin{aligned}
\text{Minimize } Z &= 16a + 12b \\
\text{subject to: } \quad & 3a + 5b \geq 15 \\
& 6a + 2b \geq 18 \\
& a\ ,\ b \geq 0
\end{aligned}
$$

[Solution] The LP is modified as follows:

$$
\begin{aligned}
Z = \ & 16a + 12b + 0c + Md + 0e + Mf \\
& 3a + 5b - c + d + 0e + 0f = 15 \\
& 6a + 2b + 0c + 0d - e + f = 18
\end{aligned}
$$

The initial simplex tableau is as follows:

C_j		16	12	0	M	0	M		
	VIS	a	b	c	d	e	f	SQ	
M	d	3	5	-1	1	0	0	15	15/3 = 5
M	f	<u>6</u>	2	0	0	-1	1	18	18/6 = 3
Z_j		9M	7M	-M	M	-M	M	33M	
$Z_j - C_j$		**9M-16**	7M-12	-M	0	-M	0		

The current solution (a, b) = (0, 0) is not optimal since 9M - 16 > 0 and 7M - 12 > 0. Given that 9M - 16 is the most positive number in the "Z_j - C_j" row and 3 < 5, the pivot column is "a" ("a" is the entering variable), the pivot row is "f" ("f" is the leaving variable), and the pivot number is 6. The new simplex tableau follows:

C_j		16	12	0	M	0	M		
	VIS	a	b	c	d	e	f	SQ	
M	d	0	<u>4</u>	-1	<u>1</u>	1/2	-1/2	6	6/4 = 1.5
16	a	1	1/3	0	0	-1/6	1/6	3	3/(1/3) = 9
Z_j		16	4M+16/3	-M	M	M/2-16/6	-M/2+16/6	6M+48	
$Z_j - C_j$		0	**4M-20/3**	-M	0	M/2-16/6	-3M/2+16/6		

The current solution (a, b) = (3, 0) is not optimal since 4M - 20/3 > 0 and M/2 - 16/6 > 0. Given that 4M - 20/3 is the most positive number in the "Z_j - C_j" row and 1.5 < 9, the pivot column is "b" ("b" is the entering variable), the pivot row is "d" ("d" is the leaving variable), and the pivot number is 4. The new simplex tableau follows:

C_j		16	12	0	M	0	M	
	VIS	a	b	c	d	e	f	SQ
12	b	0	1	-1/4	1/4	1/8	-1/8	1.5
16	a	1	0	1/12	-1/12	-5/24	5/24	2.5
Z_j		16	12	-20/12	20/12	-44/24	44/24	58
$Z_j - C_j$		0	0	-20/12	-M+20/12	-44/24	-M+44/24	

The current solution (a, b) = (2.5, 1.5) is optimal since all the numbers in the "Z_j - C_j" row are negative or zero. In conclusion, the optimal solution is (a^*, b^*) = (2.5, 1.5) and the optimal objective function value is Z^* = 58. (Note that these are the same as those found in Example 2.4.)

5. Computer Solution of Linear Programs

. A large number of software packages are available for solving linear programs very efficiently. In what follows, two of them are applied to the product-mix problem and the pig feed problem discussed previously: Excel Solver and POM/QM for Windows.

. Before running any computer program to solve a linear program, it is important to ensure that the LP is in standard form, i.e., only constants (no decision variables) are allowed to appear at the right-hand sides of the constraints.

- **Example 2.7:** Reconsider the linear program developed in Example 2.1. Use Solver to solve the LP. Be sure to include a copy of the Answer Report and interpret the key results.

[Solution] Note that the LP is already is already in standard form. The Answer Report from Solver is displayed below:

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$A$15	Z	60	60

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$15	x	4	4	Contin
$C$15	y	4	4	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$17	LHS	8	$D$17<=$F$17	Binding	0
$D$18	LHS	24	$D$18<=$F$18	Binding	0

It is seen that the optimal solution is $(x^*, y^*) = (4, 4)$ and the optimal objective function value is $Z^* = 60$. In other words, the company should produce 4 toy cars and 4 toy trains to maximize its daily total revenue at $60.

- **Example 2.8:** Reconsider the linear program developed in Example 2.2. Use Solver to solve the LP. Be sure to include a copy of the Answer Report and interpret the key results.

[Solution] Note that the LP is already in standard form. The Answer Report from Solver is as follows:

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$L$15	Z	0	58

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$M$15	a	0	2.5	Contin
$N$15	b	0	1.5	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$O$17	<= LHS	15	$O$17>=$Q$17	Binding	0
$O$18	<= LHS	18	$O$18>=$Q$18	Binding	0

It is seen that the optimal solution is $(a^*, b^*) = (2.5, 1.5)$ and the optimal objective function value is $Z^* = 58$. In other words, the farmer should mix 2.5 pounds of Grain A with 1.5 pounds of Grain B to minimize the total cost of the pig feed at $58.

6. Additional Exercises on Solving Linear Programs

- **Example 2.9:** Top Speed Bicycle Company manufactures a line of ten-speed bicycles. The firm has plants in Baton Rouge and Omaha. Its major warehouses are located in New York, Chicago, and Los Angeles. The unit transportation costs, the plant capacities, and the market sales requirements are summarized below:

	NY	CH	LA	Capacity
BR	\$2	\$3	\$5	20,000
OM	3	1	4	15,000
Requirement	12,000	7,000	16,000	

Develop an LP model for determining the shipments of bicycles at minimum cost.

[Solution] Let BN = Number of bicycles to be shipped from Baton Rouge to New York
BC = Number of bicycles to be shipped from Baton Rouge to Chicago
BL = Number of bicycles to be shipped from Baton Rouge to Los Angeles
ON = Number of bicycles to be shipped from Omaha to New York
OC = Number of bicycles to be shipped from Omaha to Chicago
OL = Number of bicycles to be shipped from Omaha to Los Angeles

$$
\begin{array}{llr}
\text{Minimize } Z = & 2BN + 3BC + 5BL + 3ON + OC + 4OL & \\
\text{subject to:} & BN + BC + BL & \leq 20{,}000 \\
& ON + OC + OL & \leq 15{,}000 \\
& BN + ON & \geq 12{,}000 \\
& BC + OC & \geq 7{,}000 \\
& BL + OL & \geq 16{,}000 \\
& BN\ , \quad \ldots\ , \quad OL & \geq 0
\end{array}
$$

- **Example 2.10:** The famous restaurant Oriental Express in Richardson, TX, is open 24 hours a day. Waiters report for duty at 3:00 A.M., 7:00 A.M., 11:00 A.M., 3:00 P.M., 7:00 P.M., or 11:00 P.M., and each works an 8-hour shift. The following table shows the minimum number of waiters needed during each of the six periods into which the day is divided:

Period	Time	Number
1	3:00 A.M. - 7:00 A.M.	3
2	7:00 A.M. - 11:00 A.M.	8
3	11:00 A.M. - 3:00 P.M.	12
4	3:00 P.M. - 7:00 P.M.	9
5	7:00 P.M. - 11:00 P.M.	16
6	11:00 P.M. - 3:00 A.M.	4

Set up an LP that can be used to determine the minimum number of waiters required for one day's operations of the restaurant.

[Solution] Let x_i be the number of waiters reporting for duty at the beginning of period i, i = 1, 2, ..., 6. An LP model for the scheduling problem follows:

$$
\begin{array}{llllllll}
\text{Minimize } Z = & x_1 + & x_2 + & x_3 + & x_4 + & x_5 + & x_6 & \\
\text{subject to:} & x_1 + & & & & & x_6 & \geq 3 \\
& x_1 + & x_2 & & & & & \geq 8 \\
& & x_2 + & x_3 & & & & \geq 12 \\
& & & x_3 + & x_4 & & & \geq 9 \\
& & & & x_4 + & x_5 & & \geq 16 \\
& & & & & x_5 + & x_6 & \geq 4 \\
& x_1 , & x_2 , & x_3 , & x_4 , & x_5 , & x_6 & \geq 0
\end{array}
$$

. **Example 2.11:** A small fabrication firm makes three basic types of components (A, B, and C) for use by other companies. Each component is processed on three machines. The processing times and machine capacities are shown in the following table.

Processing time (hours/unit)

Component	Machine 1	Machine 2	Machine 3
A	0.25	0.10	0.05
B	0.20	0.15	0.10
C	0.10	0.05	0.15
Capacity (hours)	1,600	1,400	1,500

Each component contains a different amount of each of two basic raw materials. Raw material 1 costs \$0.20 per ounce whereas raw material 2 costs \$0.35 per ounce. There are 200,000 ounces of raw material 1 and 85,000 ounces of raw material 2 available. Relevant information on material requirements has been collected and summarized below:

Component	Raw material 1 (oz/unit)	Raw material 2 (oz/unit)
A	32	12
B	26	16
C	16	9

The unit selling prices of components A, B, and C are, respectively, \$40, \$28, and \$24. Assume that the company must make at least 1,200 units of component B, labor costs are negligible, and the objective is to maximize the total profit. Formulate a linear programming model for the problem.

[Solution] Let A = Number of type A components to be made
B = Number of type B components to be made
C = Number of type C components to be made

The unit profit of type A component is 40 - 0.20(32) - 0.35(12) = \$29.4. Those for the other two types of component can be found in the same way. An appropriate LP model follows:

$$
\begin{array}{llllll}
\text{Maximize } Z = & 29.40A + & 17.20B + & 17.65C & & \\
\text{subject to:} & 0.25A + & 0.20B + & 0.10C & \leq & 1,600 \\
& 0.10A + & 0.15B + & 0.05C & \leq & 1,400 \\
& 0.05A + & 0.10B + & 0.15C & \leq & 1,500 \\
& 32A + & 26B + & 16C & \leq & 200,000 \\
& 12A + & 16B + & 9C & \leq & 85,000 \\
& & B & & \geq & 1,200 \\
& A , & B , & C & \geq & 0
\end{array}
$$

. **Example 2.12:** Big Bucks Mutual Funds, Inc., located in Harrisburg, PA, just obtained \$200,000 by converting industrial bonds to cash and is now looking for other opportunities for these funds. Considering Big Bucks' current investments, the firm's top financial analyst recommends that all new investments should be made in the oil industry, steel industry, or government bonds. Specifically, the analyst has identified five investment opportunities and projected their annual rates of return. The investments and rates of return are shown in the following table:

Investment	Projected rate of return
Atlantic Oil	7.3%
Pacific Oil	10.2%
Midwest Steel	6.1%
Harbor Steel	8.5%
Government bonds	9.6%

Management of Big Bucks has imposed the following constraints:

(1) Neither industry (oil or steel) should receive more than 50% of the total amount of money available.

(2) Government bonds should be at least 60% of the total steel industry investment.

(3) The investment in Pacific Oil cannot be more than 20% of the total oil industry investment.

Set up an LP model for the portfolio selection problem so that Big Bucks' projected return is maximized.

[Solution] Let A = Amount of money (in dollars) invested in Atlantic Oil
P = Amount of money (in dollars) invested in Pacific Oil
M = Amount of money (in dollars) invested in Midwest Steel
H = Amount of money (in dollars) invested in Huber Steel
G = Amount of money (in dollars) invested in government bonds

An LP formulation for the problem is presented below:

$$
\begin{array}{lrrrrrcr}
\text{Maximize } Z = & 0.073A + & 0.102P + & 0.061M + & 0.085H + & 0.096G & & \\
\text{subject to:} & A + & P + & M + & H + & G & \le & 200{,}000 \\
 & A + & P & & & & \le & 100{,}000 \\
 & & & M + & H & & \le & 100{,}000 \\
 & & & & & G & \ge & 0.6(M + H) \\
 & & P & & & & \le & 0.2(A + P) \\
 & A, & P, & M, & H, & G & \ge & 0
\end{array}
$$

or in the following standard form:

$$
\begin{array}{lrrrrrcr}
\text{Maximize } Z = & 0.073A + & 0.102P + & 0.061M + & 0.085H + & 0.096G & & \\
\text{subject to:} & A + & P + & M + & H + & G & \le & 200{,}000 \\
 & A + & P & & & & \le & 100{,}000 \\
 & & & M + & H & & \le & 100{,}000 \\
 & & & -0.6M\ - & 0.6H + & G & \ge & 0 \\
 & -0.2A + & 0.8P & & & & \le & 0 \\
 & A, & P, & M, & H, & G & \ge & 0
\end{array}
$$

. **Example 2.13:** International Paper Mills, Inc., operates plants in Lewiston, ME, and Hanover, NH, in addition to owning warehouses in Albany, NY, and Storrs, CT. Retail outlets are located in Boson, MA, New York, NY, and Philadelphia, PA. Typically, finished products are transported from the plants to the warehouses. The requirements at the retail outlets are met by shipments from the warehouses. Relevant information about the supplies and demands has been summarized below:

Plant	Capacity (units)
Lewiston	300
Hanover	100

Retail outlet	Demand (units)
Boston	150
New York	100
Philadelphia	150

The unit transportation costs (in dollars) for shipments from the two plants to the two warehouses and from the two warehouses to the three retail outlets are as follows:

	Albany	Storrs	Boston	New York	Philadelphia
Lewiston	7	5	-	-	-
Hanover	3	4	-	-	-
Albany	-	-	8	5	7
Storrs	-	-	5	6	10

(1) Set up a linear program for the transshipment problem.

(2) How should the LP model formulated in (1) above be modified if it is possible to ship products from the Hanover plant directly to the Philadelphia retail outlet at $9 per unit and from the Boston retail outlet to the New York retail outlet at $ 4 per unit?

[Solution] (1) Let LA = Number of units of the product to be shipped from Lewiston to Albany
LS = Number of units of the product to be shipped from Lewiston to Storrs
HA = Number of units of the product to be shipped from Hanover to Albany
HS = Number of units of the product to be shipped from Hanover to Storrs
AB = Number of units of the product to be shipped from Albany to Boston
AN = Number of units of the product to be shipped from Albany to New York
AP = Number of units of the product to be shipped from Albany to Philadelphia
SB = Number of units of the product to be shipped from Storrs to Boston
SN = Number of units of the product to be shipped from Storrs to New York
SP = Number of units of the product to be shipped from Storrs to Philadelphia

An LP model for the transshipment problem is displayed below:

$$
\begin{array}{llllllll}
\text{Minimize } Z = & 7LA + & 5LS + & 3HA + & 4HS + & 8AB + & & \\
& 5AN + & 7AP + & 5SB + & 6SN + & 10SP & & \\
\text{subject to:} & LA + & LS & & & & \le & 300 \\
& HA + & HS & & & & \le & 100 \\
& LA + & HA & & & & = & AB + AN + AP \\
& LS + & HS & & & & = & SB + SN + SP \\
& AB + & SB & & & & \ge & 150 \\
& AN + & SN & & & & \ge & 100 \\
& AP + & SP & & & & \ge & 150 \\
& LA\ , & LS\ , & & \ldots\ , & SP & \ge & 0
\end{array}
$$

or in the following standard form:

$$\begin{array}{llllllcr}
\text{Minimize } Z = & 7LA + & 5LS + & 3HA + & 4HS + & 8AB + & & \\
 & 5AN + & 7AP + & 5SB + & 6SN + & 10SP & & \\
\text{subject to:} & LA + & LS & & & & \le & 300 \\
 & HA + & HS & & & & \le & 100 \\
 & LA + & HA - & AB - & AN - & AP & = & 0 \\
 & LS + & HS - & SB - & SN - & SP & = & 0 \\
 & AB + & SB & & & & \ge & 150 \\
 & AN + & SN & & & & \ge & 100 \\
 & AP + & SP & & & & \ge & 150 \\
 & LA\ , & LS\ , & & \ldots\ , & SP & \ge & 0
\end{array}$$

(2) It is necessary to define two additional variables:

HP = Number of units of the product to be shipped from Hanover to Philadelphia
BN = Number of units of the product to be shipped from Boston to New York

The modified LP is:

$$\begin{array}{llllllcr}
\text{Minimize } Z = & 7LA + & 5LS + & 3HA + & 4HS + & 8AB + & & \\
 & 5AN + & 7AP + & 5SB + & 6SN + & 10SP + & & \\
 & 9HP + & 4BN & & & & & \\
\text{subject to:} & LA + & LS & & & & \le & 300 \\
 & HA + & HS + & HP & & & \le & 100 \\
 & LA + & HA - & AB - & AN - & AP & = & 0 \\
 & LS + & HS - & SB - & SN - & SP & = & 0 \\
 & AB + & SB - & BN & & & \ge & 150 \\
 & AN + & SN + & BN & & & \ge & 100 \\
 & AP + & SP + & HP & & & \ge & 150 \\
 & LA\ , & LS\ , & & \ldots\ , & SP & \ge & 0
\end{array}$$

Example 2.14: Consider the following linear program:

$$\begin{array}{lrrcr}
\text{Minimize } Z = & 1{,}200E + & 2{,}400F & & \\
\text{subject to:} & E\ - & 3F & \le & 0 \\
 & E & & \le & 8 \\
 & 10E + & 15F & \ge & 90 \\
 & -5E + & 7.5F & \le & 45 \\
 & E\ , & F & \ge & 0
\end{array}$$

(1) Plot the constraints and graph the feasible region for the LP.

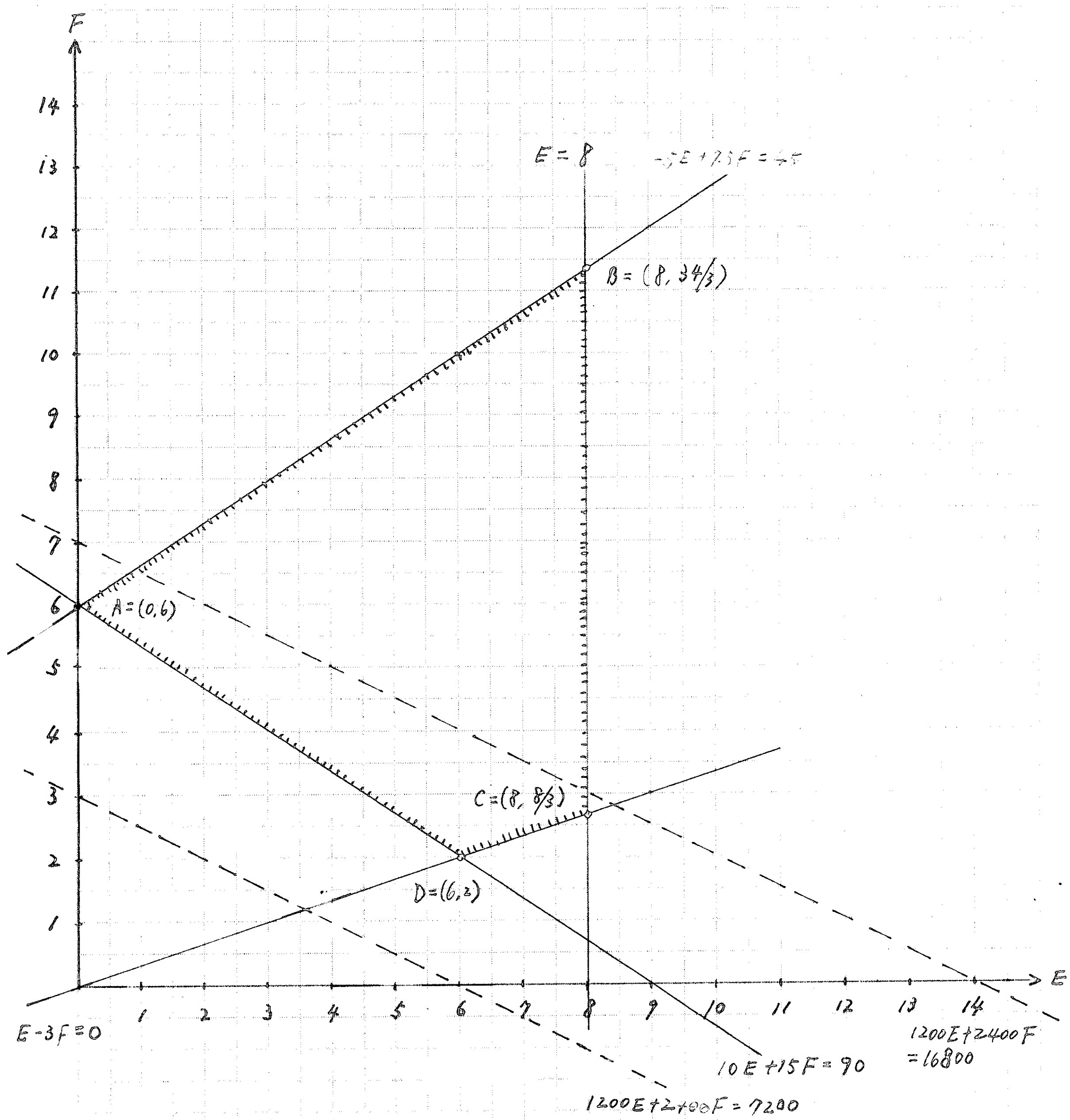
(2) Plot iso-cost lines to determine the optimal solution and the minimum total cost.

(3) Use the corner-point approach to find the optimal solution as well as the minimum total cost.

(4) Are the results in (2) and (3) above consistent?

[Solution] (1) The graph for the problem is displayed on the next page.

(2) It is noted that the objective function will attain its minimum value at point D. Solving the simultaneous equations of 10E + 15F = 90 and E - 3F = 0, we find that the optimal solution is $(E^*, F^*) = (6, 2)$ and the optimal objective function value is $Z^* = 1{,}200E^* + 2{,}400F^* = 1{,}200(6) + 2{,}400(2) = 12{,}000$.


F
E
E = 8
-5E+7.5F=45
B = (8, 34/3)
A = (0,6)
C = (8, 8/3)
D = (6,2)
E-3F=0
10E+15F=90
1200E+2400F = 7200
1200E+2400F = 16800
1
2
3
4
5
6
7
8
9
10
11
12
13
14

(3) The coordinates of the four corner points along with the corresponding objective function values are calculated below:

A = (0, 6): Z = 1,200E + 2,400F = 1,200(0) + 2,400(6) = 14,400 (we need to solve the simultaneous equations of E = 0 and -5E + 7.5F = 45 to find E = 0 and F = 6.)

B = (8, 34/3): Z = 1,200E + 2,400F = 1,200(8) + 2,400(34/3) = 36,800 (we need to solve the simultaneous equations of E = 8 and -5E + 7.5F = 45 to find E = 8 and F = 34/3.)

C = (8, 8/3): Z = 1,200E + 2,400F = 1,200(8) + 2,400(8/3) = 16,000 (we need to solve the simultaneous equations of E = 8 and E - 3F = 0 to find E = 8 and F = 8/3.)

D = (6, 2): Z = 1,200E + 2,400F = 1,200(6) + 2,400(2) = 12,000 (we need to solve the simultaneous equations of E - 3F = 0 and 10E + 15F = 90 to find E = 6 and F = 2.)

Since 12,000 < 14,400 < 16,000 < 36,800 < 22,800, the optimal solution is (E^*, F^*) = (6, 2) and the optimal objective function value is Z^* = 12,000.

(4) Yes, they are.

- **Example 2.15:** Use the simplex method to solve the following linear program:

$$
\begin{array}{llcl}
\text{Minimize } Z = & 10A + 20B & & \\
\text{subject to:} & A + B & = & 50 \\
 & A & \geq & 20 \\
 & B & \leq & 40 \\
 & A\ ,\ B & \geq & 0
\end{array}
$$

[Solution] The LP is modified as follows:

$$
\begin{array}{rrrrrrrcl}
Z = & 10A + & 20B + & MC + & 0D + & ME + & 0F & & \\
 & A + & B + & C + & 0D + & 0E + & 0F & = & 50 \\
 & A + & 0B + & 0C - & D + & E + & 0F & = & 20 \\
 & 0A + & B + & 0C + & 0D + & 0E + & F & = & 40
\end{array}
$$

The initial simplex tableau is as follows:

C_j		10	20	M	0	M	0		
	VIS	A	B	C	D	E	F	SQ	
M	C	1	1	1	0	0	0	50	50/1 = 50
M	E	<u>1</u>	0	0	-1	1	0	20	20/1 = 20
0	F	0	1	0	0	0	1	40	
Z_j		2M	M	M	-M	M	0	70M	
$Z_j - C_j$		**2M-10**	M-20	0	-M	0	0		

The current solution (A, B) = (0, 0) is not optimal since 2M - 10 > 0 and M - 20 > 0. Given that 2M - 10 is the most positive number in the "Z_j - C_j" row and 20 < 50, the pivot column is "A" ("A" is the entering variable), the pivot row is "E" ("E" is the leaving variable), and the pivot number is 1. The new simplex tableau follows:

C_j		10	20	M	0	M	0		
	VIS	A	B	C	D	E	F	SQ	
M	C	0	1	1	<u>1</u>	-1	0	30	30/1 = 30
10	A	1	0	0	-1	1	0	20	
0	F	0	1	0	0	0	1	40	
Z_j		10	M	M	M-10	-M+10	0	30M+200	
$Z_j - C_j$		0	M-20	0	**M-10**	-2M+10	0		

The current solution (A, B) = (20, 0) is not optimal since M - 20 > 0 and M - 10 > 0. Given that M - 10 is the most positive number in the "Z_j - C_j" row and 30 is the only positive ratio, the pivot column is "D" ("D" is the entering variable), the pivot row is "C" ("C" is the leaving variable), and the pivot number is 1. The new simplex tableau follows:

C_j		10	20	M	0	M	0	
	VIS	A	B	C	D	E	F	SQ
0	D	0	1	1	1	-1	0	30
10	A	1	1	1	0	0	0	50
0	F	0	1	0	0	0	1	40
Z_j		10	10	10	0	0	0	500
$Z_j - C_j$		0	-10	10-M	0	-M	0	

The current solution (A, B) = (50, 0) is optimal since all the numbers in the "Z_j - C_j" row are negative or zero. In conclusion, the optimal solution is (A^*, B^*) = (50, 0) and the optimal objective function value is Z^* = 500.

. **Example 2.16:** Use the simplex method to solve the linear program below:

$$
\begin{array}{llllll}
\text{Maximize } Z = & 12y1 + & 6y2 & & \\
\text{subject to:} & 2y1 + & 4y2 & \leq & 16 \\
 & 5y1 + & 3y2 & \geq & 15 \\
 & y1 & & \leq & 5 \\
 & y1\ , & y2 & \geq & 0
\end{array}
$$

[Solution] The LP is modified as follows:

$$
\begin{array}{lllllllll}
Z = & 12y1 + & 6y2 + & 0y3 + & 0y4\ - & My5 + & 0y6 & & \\
 & 2y1 + & 4y2 + & y3 + & 0y4 + & 0y5 + & 0y6 & = & 16 \\
 & 5y1 + & 3y2 + & 0y3\ - & y4 + & y5 + & 0y6 & = & 15 \\
 & y_1 + & 0y2 + & 0y3 + & 0y4 + & 0y5 + & y6 & = & 5
\end{array}
$$

The initial simplex tableau is as follows:

C_j		12	6	0	0	-M	0	
	VIS	y1	y2	y3	y4	y5	y6	SQ	
0	y3	2	4	1	0	0	0	16	16/2 = 8
-M	y5	5	3	0	-1	1	0	15	15/5 = 3
0	y6	1	0	0	0	0	1	5	5/1 = 5
Z_j		-5M	-3M	0	M	-M	0	-15M	
$Z_j - C_j$		**-5M-12**	-3M-6	0	M	0	0		

The current solution (y1, y2) = (0, 0) is not optimal since -5M - 12 < 0 and -3M - 6 < 0. Given that -5M - 12 is the most negative number in the "Z_j - C_j" row and 3 < 5 < 8, the pivot column is "y1" ("y1" is the entering variable), the pivot row is "y5" ("y5" is the leaving variable), and the pivot number is 5. The new simplex tableau is presented below:

C_j		12	6	0	0	-M	0		
	VIS	y1	y2	y3	y4	y5	y6	SQ	
0	y3	0	14/5	1	2/5	-2/5	0	10	10/(2/5) = 25
12	y1	1	3/5	0	-1/5	1/5	0	3	
0	y6	0	-3/5	0	1/5	-1/5	1	2	2/(1/5) = 10
Z_j		12	36/5	0	-12/5	12/5	0	36	
$Z_j - C_j$		0	6/5	0	**-12/5**	M+12/5	0		

The current solution (y1, y2) = (3, 0) is not optimal since -12/5 < 0. Given that -12/5 is the only negative number in the "Z_j - C_j" row and 10 < 25, the pivot column is "y4" ("y4" is the entering variable), the pivot row is "y6" ("y6" is the leaving variable), and the pivot number is 1/5. The new simplex tableau follows:

C_j		12	6	0	0	-M	0		
	VIS	y1	y2	y3	y4	y5	y6	SQ	
0	y3	0	4	1	0	0	-2	6	6/4 = 1.5
12	y1	1	0	0	0	0	1	5	
0	y4	0	-3	0	1	-1	5	10	
Z_j		12	0	0	0	0	12	60	
$Z_j - C_j$		0	**-6**	0	0	M	12		

The current solution (y1, y2) = (5, 0) is not optimal since -6 < 0. Given that -6 is the only negative number in the "Z_j - C_j" row and 1.5 is the only positive ratio, the pivot column is "y2" (y2 is the entering variable), the pivot row is "y3" (y3 is the leaving variable), and the pivot number is 4. The new simplex tableau follows:

C_j			12	6	0	0	-M	0		
	VIS		y1	y2	y3	y4	y5	y6		SQ
6	y2		0	1	1/4	0	0	-1/2		3/2
12	y1		1	0	0	0	0	1		5
0	y4		0	-3	3/4	1	-1	7/2		29/2
Z_j			12	6	3/2	0	0	9		69
$Z_j - C_j$			0	0	3/2	0	M	9		

The current solution (y1, y2) = (5, 1.5) is optimal since all the numbers in the "Z_j - C_j" row are positive or zero. In conclusion, the optimal solution is ($y1^*$, $y2^*$) = (5, 1.5) and the optimal objective function value is Z^* = 69.

- **Example 2.17:** Run Solver to solve the LP for the personnel scheduling problem formulated in Example 2.10. Be sure to include a copy of the Answer Report and interpret the key results.

[Solution] The Answer Report is shown below:

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$U$9	x1 + x2 ≥ 8 Z	0	31

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$V$9	x1 + x2 ≥ 8 x1	0	3	Contin
$W$9	x1 + x2 ≥ 8 x2	0	5	Contin
$X$9	x1 + x2 ≥ 8 x3	0	7	Contin
$Y$9	x1 + x2 ≥ 8 x4	0	12	Contin
$Z$9	x1 + x2 ≥ 8 x5	0	4	Contin
$AA$9	x1 + x2 ≥ 8 x6	0	0	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$AB$11	x3 + x4 ≥ 9 LHS	3	$AB$11>=$AD$11	Binding	0
$AB$12	x4 + x5 ≥ 16 LHS	8	$AB$12>=$AD$12	Binding	0
$AB$13	x5 + x6 ≥ 4 LHS	12	$AB$13>=$AD$13	Binding	0
$AB$14	x1, ..., x6 ≥ 0 LHS	19	$AB$14>=$AD$14	Not Binding	10
$AB$15	LHS	16	$AB$15>=$AD$15	Binding	0
$AB$16	LHS LHS	4	$AB$16>=$AD$16	Binding	0

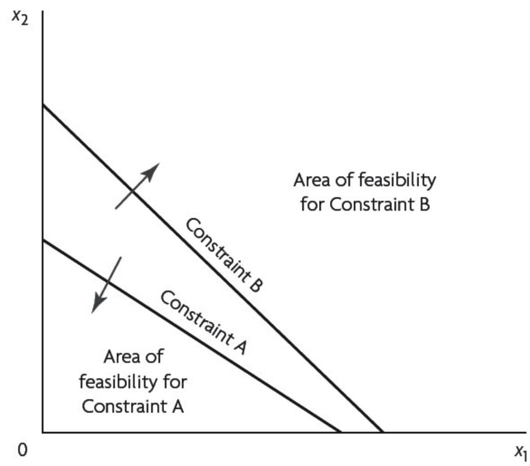
It is seen that the optimal solution is (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*) = (3, 5, 7, 12, 4, 0) and the optimal objective function value is Z^* = 31. In other words, Oriental Express should use the following schedule to minimize the total number of waiters needed at 31:

Period	Time	Number of waiters scheduled
1	3:00 A.M. - 7:00 A.M.	3
2	7:00 A.M. - 11:00 A.M.	5
3	11:00 A.M. - 3:00 P.M.	7
4	3:00 P.M. - 7:00 P.M.	12
5	7:00 P.M. - 11:00 P.M.	4
6	11:00 P.M. - 3:00 A.M.	0

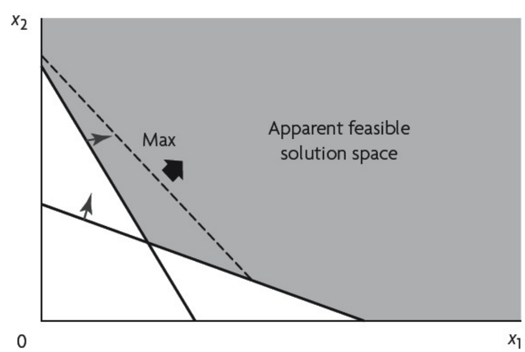
7. Special Linear Programs

. There are four special types of linear programs in which either a regular optimal solution does not exist or one of the constraints is redundant. Each of them is graphically illustrated below ((Stevenson and Ozgur, 2007):

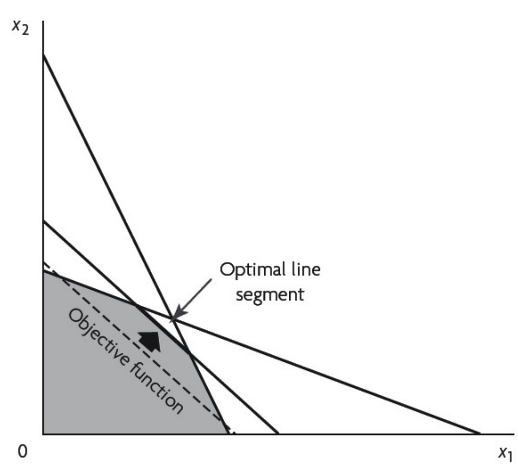
(1) No optimal solution -



(2) Unbounded optimal solution -



(3) Multiple optimal solutions -



(4) Redundant constraint -