CHAPTER 6
INTEGER LINEAR PROGRAMMING

1. Introduction

- One of the requirements of linear programming (LP) is divisibility; namely, each decision variable must be able to take on any continuous value in the optimal solution. While this divisibility assumption does not present serious difficulty in many practical problems (e.g., 1.6 machine hours, 42.5 pounds of sugar, and 6.38 tons of steel), it is totally unrealistic and unacceptable in certain decision problems. For instance, an optimal solution calling for 1.29 dams on a river system has no practical meaning. Similarly, an automobile dealer cannot stock 6.67 station wagons. In cases such as these, it is desirable to identify the optimal integer values of the decision variables. Integer programming deals with mathematical approaches for doing that.

- There are three basic types of integer linear programming (ILP) problems, i.e., problems involving linear objective function and linear constraints and requiring integer solutions:

  (1) All-integer linear programs (AILP) - Problems in which all the decision variables are required to take on an integer value.

  (2) Mixed-integer linear programs (MILP) - Problems in which some, but not all, of the decision variables are required to take on an integer value.

  (3) Zero-one (or binary) linear programs (ZOLP) - Problems in which all the decision variables are required to take on a value of zero or one only.

2. Formulation of Integer Linear Programming Problems

- **Example 6.1:** The advertising agency promoting the new Rapido sports car wants to get the best possible exposure for the product within the available $200,000 budget. To do so, the agency must decide how much to spend on its two most effective media: evening television spots and large magazine ads. Each television spot costs $20,000 and a magazine ad involves a $5,000 expenditure. It has been estimated that 400,000 people will be reached with each television spot and 150,000 will be reached with a magazine ad.

  Dawn Shaw, the agency director, knows from experience that it is important to use both media. In this way, the advertising will reach the broadest spectrum of potential Rapido customers. As a result, she decides to contract for at least 4, but no more than 12, television spots and a minimum of 6 magazine ads.

  (1) If fractional spots and ads can be purchased, develop a mathematical program that can be used to determine the media combination with the maximum total audience exposure.

  (2) Repeat (1) if it is impossible to purchase fractional spots or ads from the media.

  [Solution] (1) Let T = Number of evening TV spots to be purchased
  M = Number of large magazine ads to be purchased

  A linear program for the problem is presented below:

$$
\begin{array}{lrrcr}
\text{Maximize } Z = & 400{,}000T + & 150{,}000M & & \\
\text{subject to} & 20{,}000T + & 5{,}000M & \leq & 200{,}000 \\
& T & & \geq & 4 \\
& T & & \leq & 12 \\
& & M & \geq & 6 \\
& T\ , & M & \geq & 0
\end{array}
$$

(2) An integer linear program for the problem is presented below:

$$
\begin{array}{lrrcrl}
\text{Maximize } Z = & 400{,}000T + & 150{,}000M & & & \\
\text{subject to} & 20{,}000T + & 5{,}000M & \leq & 200{,}000 & \\
& T & & \geq & 4 & \\
& T & & \leq & 12 & \\
& & M & \geq & 6 & \\
& T\ , & M & \geq & 0 & \text{and integer}
\end{array}
$$

- **Example 6.2:** A Boston wholesaler has been caught in the middle of a strike by independent truckers. Meat is normally supplied from a packinghouse in Chicago by refrigerated trucks. The strike, however, has created a meat crisis that can be alleviated only by air freight. The wholesaler has chartered a plane and wishes to determine the amount of beef and pork to have shipped so that the profit contribution is maximized subject to volume and weight constraints of the cargo airplane. Relevant information has been summarized below. Set up an ILP for the problem.

Meat	Profit ($/container)	Volume (cu. yd./container)	Weight (lb./container)
Beef	700	5	2,000
Pork	1,000	4	5,000
Availability	24	13,000	

[Solution] Let x = Number of containers of beef to be shipped
y = Number of containers of pork to be shipped

An AILP for the problem is presented below:

$$
\begin{array}{lrrcrl}
\text{Maximize } Z = & 700x + & 1{,}000y & & & \\
\text{subject to} & 5x + & 4y & \leq & 24 & \\
& 2{,}000x + & 5{,}000y & \leq & 13{,}000 & \\
& x\ , & y & \geq & 0 & \text{and integer}
\end{array}
$$

- **Example 6.3:** Judy Jones, a young widow, has $100,000 to invest. She wants her investments to provide her with the maximum annual interest and dividend income possible. On the advice of her financial counselor, she has chosen two stock offerings and an interest-bearing brokerage account. Each share of stock A, at $30.375 per share, has an anticipated annual dividend of $4.2. Stock B, costing $15.625 a share, has an annual dividend of $2.85. Interest on the brokerage account is currently 12 percent paid annually.

  For security, Judy wants at least 30% of her total investment in the brokerage account. She is also not too sure of stock B, but her adviser has persuaded her by recommending that she buy at least twice as many shares of stock A as stock B. Judy must now decide how many shares of each stock to purchase and how much to put in the brokerage account. Set up an ILP for the problem.

[Solution] Let A = Number of stock A shares to be purchased
B = Number of stock B shares to be purchased
C = Amount (in dollars) to be invested in the brokerage account

An MILP for the problem is presented below:

$$\begin{array}{lrrrcrl}
\text{Maximize } Z = & 4.2A + & 2.85B + & 0.12C & & & \\
\text{subject to} & 30.375A + & 15.625B + & C & \leq & 100{,}000 & \\
& \multicolumn{3}{r}{0.3(30.375A + 15.625B + C)} & \leq & C & \\
& A & & & \geq & 2B & \\
& A\ , & B & & \geq & 0 & \text{and integer} \\
& & & C & \geq & 0 &
\end{array}$$

or

$$\begin{array}{lrrrcrl}
\text{Maximize } Z = & 4.2A + & 2.85B + & 0.12C & & & \\
\text{subject to} & 30.375A + & 15.625B + & C & \leq & 100{,}000 & \\
& 9.1125A + & 4.6875B - & 0.7C & \leq & 0 & \\
& A - & 2B & & \geq & 0 & \\
& A\ , & B & & \geq & 0 & \text{and integer} \\
& & & C & \geq & 0 &
\end{array}$$

. **Example 6.4:** Each of three jobs must be uniquely assigned to one of the three machines available. The following table contains the costs for all possible matchings of jobs and machines. Set up a mathematical program to minimize the total production cost.

		Machine		
		1	2	3
Job	1	30	33	41
	2	57	24	29
	3	39	22	62

[Solution] Let x_{ij} = 1 if job i is assigned to machine j and 0 otherwise, i = 1, 2, 3; j = 1, 2, 3. A ZOLP for the problem is presented below:

$$\begin{array}{llrrrcl}
\text{Minimize} & Z = & 30x_{11} + & 33x_{12} + & 41x_{13} + & & \\
& & 57x_{21} + & 24x_{22} + & 29x_{23} + & & \\
& & 39x_{31} + & 22x_{32} + & 62x_{33} & & \\
\text{subject to} & & x_{11} + & x_{12} + & x_{13} & = & 1 \\
& & x_{21} + & x_{22} + & x_{23} & = & 1 \\
& & x_{31} + & x_{32} + & x_{33} & = & 1 \\
& & x_{11} + & x_{21} + & x_{31} & = & 1 \\
& & x_{12} + & x_{22} + & x_{32} & = & 1 \\
& & x_{13} + & x_{23} + & x_{33} & = & 1 \\
& & x_{11}\ , & \ldots\ , & x_{33} & = & 0 \ \text{ or } 1
\end{array}$$

3. Solving Integer Linear Programs

. Various approaches have been developed to solve integer linear programs. These include, but are not limited to:

(1) Complete enumeration (or brute force) approach: Enumerate all the feasible integer solutions and select the one which yields the best objective function value. This approach is practical only for small problems with relatively few decision variables.

(2) Rounding approach: Ignore the integer restriction(s) and solve the problem as a linear program. The LP solution is then rounded off to the nearest integer solution. The major drawback of this approach is that the resultant solution may differ significantly from the optimal integer solution (in terms of the objective function value) and, even worse, may be infeasible in violating one or more structural constraints.

(3) Branch-and-bound method: This is an effective search procedure that involves solving a succession of carefully formulated linear programs. An in-depth treatment of this technique will be presented later.

. **Example 6.5:** Use (1) the complete enumeration method and (2) the rounding approach to solve the following all-integer linear program (AILP):

$$\begin{array}{llrcrl} \text{Maximize} & Z = 20x + 9y & & & & \\ \text{subject} & 7x + 4y & \leq & 12.6 & & \\ & x, \quad y & \geq & 0 & \text{and integer} \end{array}$$

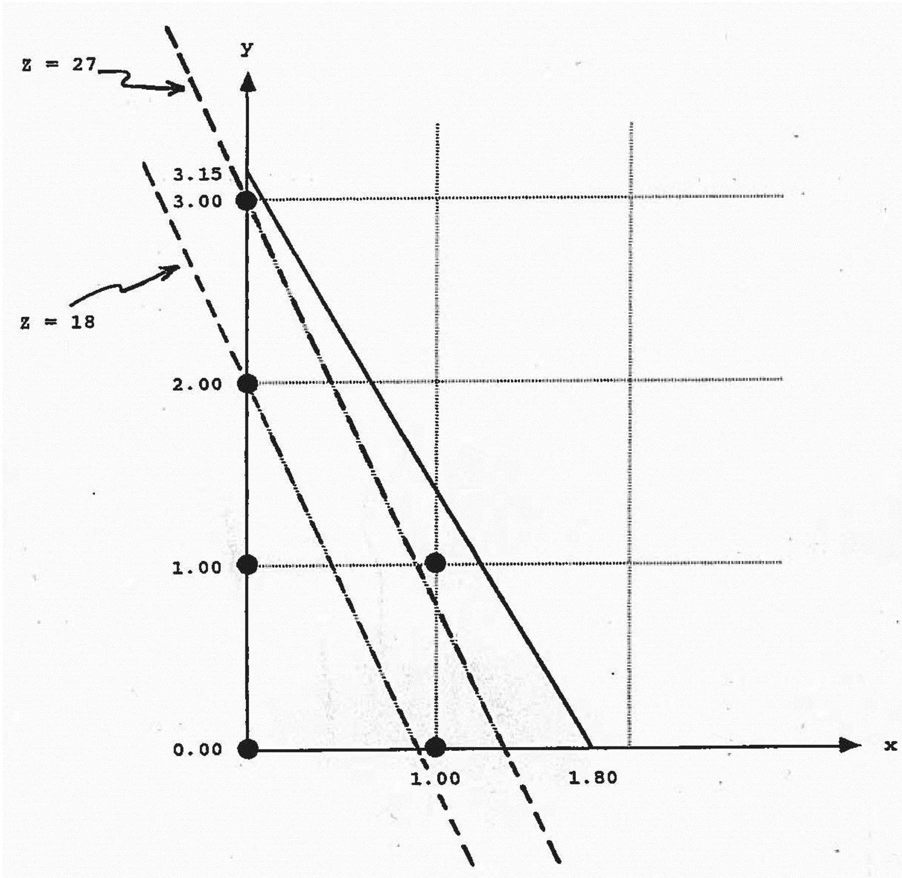
[Solution] (1) There are 6 feasible integer solutions to the problem:

(x, y)	Z
(0, 0)	20(0) + 9(0) = 0
(0, 1)	20(0) + 9(1) = 9
(0, 2)	20(0) + 9(2) = 18
(0, 3)	20(0) + 9(3) = 27
(1, 0)	20(1) + 9(0) = 20
(1, 1)	20(1) + 9(1) = 29

Since 29 > 27 > 20 > 18 > 9 > 0, $(x^*, y^*) = (1, 1)$ and $Z^* = 29$.

(2) Ignoring the integer constraint, the optimal LP solution to the problem is $(x^*, y^*) = (1.8, 0)$. The nearest integer solution to (1.8, 0) is (x, y) = (2, 0), which is infeasible since the structural constraint is violated, i.e., 7(2) + 4(0) = 14 > 12.6. If the solution is rounded off to (x, y) = (1, 0), which is feasible since 7(1) + 4(0) = 7 ≤ 12.6. However, we have Z = 20(1) + 9(0) = 20 and this represents a (29 - 20)/29 = 31% deviation from optimality.

. Remark: See the following exhibit for a graphical approach to the above problem.



4. Branch-and-Bound Method

. The branch-and-bound method was initially developed by A. H. Land and A. G. Doig in 1960. It was then further studied by J. D. C. Little and other researchers. The technique is very useful for solving ILPs.

. The branch-and-bound method is based on two basic properties of a maximization (minimization) integer linear programming problem:

(1) Z^* for the ILP is always less (greater) than or equal to the Z^* for the problem without the integer constraint(s), which is called the LP relaxation of the ILP.

(2) Z^* at each node is the upper (lower) bound for the entire branch of descendant nodes.

. **Example 6.6:** Telfa Corporation manufactures tables and chairs. A table requires 1 hour of labor and 9 sq bd ft of wood, and a chair requires 1 hour of labor and 5 sq bd ft of wood. Currently, 6 hours of labor and 45 sq bd ft of wood are available. Each table contributes \$8 to profit, and each chair contributes \$5 to profit.

(1) Formulate the problem as an AILP to maximize Telfa's total profit.

(2) Solve the AILP in (1) using the branch-and-bound method.

[Solution] (1) Let x_1 = Number of tables to be manufactured

x_2 = Number of chairs to be manufactured

An AILP for the problem is presented below:

$$
\begin{array}{llrcl}
\text{Maximize } Z = & 8x_1 + 5x_2 & & & \\
\text{subject to} & x_1 + x_2 & \leq & 6 & \\
 & 9x_1 + 5x_2 & \leq & 45 & \\
 & x_1, \quad x_2 & \geq & 0 & \text{and integer}
\end{array}
$$

(2) Initially, LB = $-\infty$. We begin with subproblem 1, the LP relaxation of the ILP:

$$
\begin{array}{llrc}
\text{Maximize } Z = & 8x_1 + 5x_2 & & \\
\text{subject to} & x_1 + x_2 & \leq & 6 \\
 & 9x_1 + 5x_2 & \leq & 45 \\
 & x_1, \quad x_2 & \geq & 0
\end{array}
$$

The optimal solution is $(x_1^*, x_2^*) = (3.75, 2.25)$ and $Z^* = 41.25$. Since this is not an all-integer solution and both of x_1^* and x_2^* are fractional, we can branch on either x_1 or x_2. Suppose we branch on x_1 to create the following two subproblems:

subproblem 2 = subproblem 1 + "$x_1 \geq 4$"
subproblem 3 = subproblem 1 + "$x_1 \leq 3$"

We can choose to solve any of the two new subproblems. Suppose subproblem 2 is solved. The optimal solution is $(x_1^*, x_2^*) = (4, 1.8)$ and $Z^* = 41$. Since this is not an all-integer solution and only x_2^* is fractional, we branch on x_2 to create the following two new subproblems:

subproblem 4 = subproblem 2 + "$x_2 \geq 2$"
subproblem 5 = subproblem 2 + "$x_2 \leq 1$"

Subproblems 3, 4, and 5 are now unsolved. We arbitrarily choose to solve subproblem 4. Since the problem is infeasible, it is **pruned** and no further branching will be pursued there. We then arbitrarily choose to solve subproblem 5. The optimal solution is $(x_1^*, x_2^*) = (4.44, 1)$ and $Z^* = 40.56$. Since this is not an all-integer solution, we branch on x_1 to create the following two new subproblems:

subproblem 6 = subproblem 5 + "$x_1 \geq 5$"
subproblem 7 = subproblem 5 + "$x_1 \leq 4$"

Subproblems 3, 6, and 7 are now unsolved. We arbitrarily choose to solve subproblem 7. The optimal solution is $(x_1^*, x_2^*) = (4, 1)$ and $Z^* = 37$. Since this is an all-integer solution, it is a candidate for the optimal solution to the AILP. It follows that no further branching will be pursued there and LB = 37.

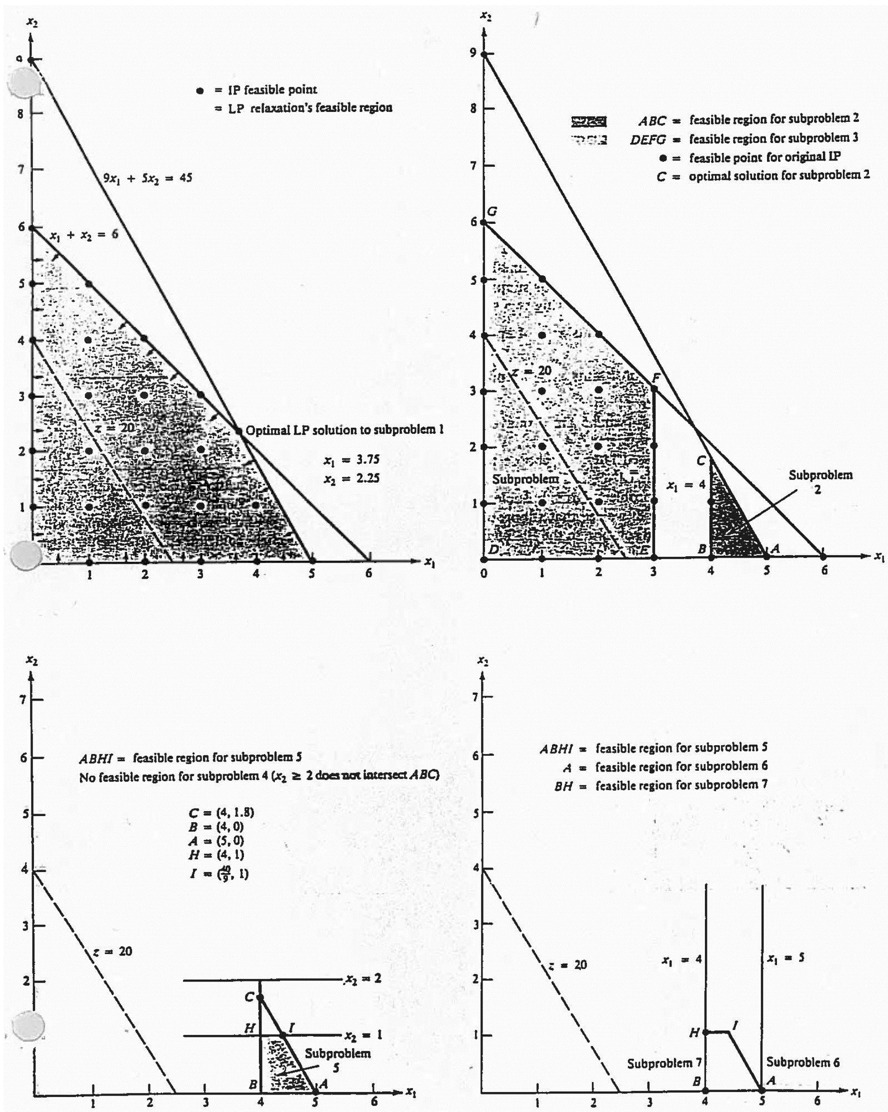
Now the only remaining unsolved problems are subproblems 3 and 6. Suppose we choose to solve subproblem 6. The optimal solution is $(x_1^*, x_2^*) = (5, 0)$ and $Z^* = 40$. Since this is an all-integer solution and 40 > 37 = LB, subproblem 7 with $Z^* = 37$ is pruned and LB = 40. No further branching will be pursued there.

Finally, subproblem 3 is solved and the optimal solution is $(x_1^*, x_2^*) = (3, 3)$ with $Z^* = 39$.

Since this is an all-integer solution and 39 < 40 = LB, subproblem 3 is pruned. No further branching will be pursued there.

In conclusion, the optimal solution to the AILP is $(x_1^*, x_2^*) = (5, 0)$ and the optimal objective function value is $Z^* = 40$ = LB. Namely, Telfa should manufacture 5 tables and no chairs to realize the maximum total profit of $40.

- **Remark:** A graphical representation of the branching and bounding process for the problem is shown below:



- **Example 6.7:** Solve the following AILP using the branch-and-bound method:

$$
\begin{array}{llrrcrl}
\text{Maximize} & Z = & 3x + & 5y & & & \\
\text{subject to} & & 2x + & 4y & \leq & 25 & \\
& & x & & \leq & 8 & \\
& & & 2y & \leq & 10 & \\
& & x\,, & y & \geq & 0 & \text{and integer}
\end{array}
$$

[Solution] In conclusion, the optimal solution is $(x^*, y^*) = (8, 2)$ and the optimal objective function value is $Z^* = 34$.

- **Example 6.8:** Use the branch-and-bound method to solve the following AILP:

$$
\begin{array}{llrrrcrl}
\text{Minimize} & Z = & 4x + & 3y + & 5z & & & \\
\text{subject to} & & 2x - & 2y + & 4z & \geq & 7 & \\
& & 2x + & 4y - & 2z & \geq & 5 & \\
& & x\,, & y\,, & z & \geq & 0 & \text{and integer}
\end{array}
$$

[Solution] In conclusion, the optimal solution is $(x^*, y^*, z^*) = (4, 0, 0)$ and the optimal objective function value is $Z^* = 16$.

- **Example 6.9:** Use the branch-and-bound method to solve the following MILP:

$$
\begin{array}{llrrcrl}
\text{Maximize} & Z = & 2x + & y & & & \\
\text{subject to} & & 5x + & 2y & \leq & 8 & \\
& & x + & y & \leq & 3 & \\
& & x\,, & & \geq & 0 & \text{and integer} \\
& & & y & \geq & 0 &
\end{array}
$$

[Solution] In conclusion, the optimal solution is $(x^*, y^*) = (1, 1.5)$ and the optimal objective function value is $Z^* = 3.5$.

- **Example 6.10:** Use Solver to solve the ZOLP formulated for the assignment problem in Example 6.4.

[Solution] The result from Solver shows that $(x_{11}^*, x_{12}^*, x_{13}^*, x_{21}^*, x_{22}^*, x_{23}^*, x_{31}^*, x_{32}^*, x_{33}^*) = (1, 0, 0, 0, 0, 1, 0, 1, 0)$ and $Z^* = 81$. In other words, jobs 1, 2, and 3 should be assigned to, respectively, machines 1, 3, and 2 to minimize the total production cost at $81.

- **Example 6.11:** Solve the AILP in Example 6.8 using Solver. Is the optimal solution identical to that in Example 6.8 based on the branch-and-bound method?

[Solution] Running Solver to solve the AILP, we find an optimal solution of $(x^*, y^*, z^*) = (4, 0, 0)$ and the minimum objective function value is $Z^* = 16$. This is identical to what obtained in Example 6.8.

- **Example 6.12:** Solve the MILP in Example 6.9 using Solver. Is the optimal solution identical to that in Example 6.9 based on the branch-and-bound method?

[Solution] Running Solver to solve the MILP, we find an optimal solution of $(x^*, y^*) = (1, 1.5)$ and the maximum objective function value is $Z^* = 3.5$. This is identical to what obtained in Example 6.9.

5. Applications of Binary Variables in Integer Linear Programming

- Many real-world problems require some or all of the decision variables to take on the value of 0 or 1 only. These decision variables are called binary variables. Assignment problem, capital budgeting problem, and knapsack problem are good examples of this type.

- **Example 6.13:** A scout is packing a knapsack for a camping trip. The scout is considering taking 10 different items on the trip. Their values for the trip and weights are shown in the accompanying table. The maximum weight allowed each scout is 15 pounds. Furthermore, there should be at least 1 food item packed in the knapsack and not more than 1 item from among the camera, television, and radio.

  (1) Set up an ILP for the problem to maximize the total value.

  (2) Suppose it is desirable that the dried meat be taken only if the burner is taken. How can this additional restriction be incorporated into the model formulated in (1)?

Item	Value (points)	Weight (pounds)
Compass/watch	300	0.15
Jar of peanut butter	200	0.50
Portable burner	75	4.00
Canteen	225	2.50
Sleeping bag	250	5.00
Dried meat	125	2.00
Fishing gear	75	4.00
Movie camera	25	1.25
Portable television	10	2.00
Radio	50	0.80

[Solution] (1) Let item 1 = compass/watch, item 2 = jar of peanut butter, ..., item 10 = radio. Further let $x_i = 1$ if item i is packed in the knapsack and 0 otherwise, i = 1, 2, ..., 10. A ZOLP for the knapsack problem is presented below:

$$
\begin{array}{lrrrrl}
\text{Maximize } Z = & 300x_1 + & 200x_2 + & 75x_3 + & & \\
& 225x_4 + & 250x_5 + & 125x_6 + & & \\
& 75x_7 + & 25x_8 + & 10x_9 + & & \\
& 50x_{10} & & & & \\
\text{subject to} & 0.15x_1 + & 0.5x_2 + & 4x_3 + & & \\
& 2.5x_4 + & 5x_5 + & 2.5x_6 + & & \\
& 4x_7 + & 1.2x_8 + & 2x_9 + & & \\
& 0.8x_{10} & & & \leq & 15 \\
& x_2 + & x_6 & & \geq & 1 \\
& x_8 + & x_9 + & x_{10} & \leq & 1 \\
& x_1 , & \ldots , & x_{10} & = & 0 \text{ or } 1
\end{array}
$$

(2) Add the constraint $x_6 \leq x_3$ or, equivalently, $x_3 - x_6 \geq 0$ to the formulation.

- **Example 6.14:** A farmer is considering entering one of two possible new markets: dog food or cattle feed. he production processes for the two new products under consideration differ significantly. The farmer has established the following production constraints:

(1) $2x + 4y + z \geq 30$ (dog food)
(2) $10x + 8y \geq 180$ (cattle feed)

where

x = Amount (in bushels) of corn to be used
y = Amount (in bushels) of wheat to be used
z = Amount (in 100-pound bags) of horse meat to be used

Either constraint (1) or constraint (2) is applicable, depending on whether the farmer decides to produce dog food or cattle feed. Suppose that the cost of corn is \$30 per bushel, the cost of wheat is \$35 per bushel, and the cost of horse meat is \$2 per pound. Set up an ILP for the problem to minimize the total cost.

[Solution] Let M be a very large positive number (say, for example, 999,999). Further let d_i = **0** if constraint i is applicable and **1** otherwise, i = 1, 2. An ILP for the problem is presented below:

$$
\begin{array}{llrrrrrcrl}
\text{Minimize} & Z = & 30x + & 35y + & 200z + & 0d_1 + & 0d_2 & & & \\
\text{subject to} & & 2x + & 4y + & z & & & \geq & 30 - Md_1 & \\
 & & 10x + & 8y & & & & \geq & 180 - Md_2 & \\
 & & & & & d_1 + & d_2 & = & 1 & \\
 & & x\,, & y\,, & z & & & \geq & 0 & \text{and integer} \\
 & & & & & d_1\,, & d_2 & = & 0 & \text{or } 1
\end{array}
$$

or

$$
\begin{array}{llrrrrrcrl}
\text{Minimize} & Z = & 30x + & 35y + & 200z + & 0d_1 + & 0d_2 & & & \\
\text{subject to} & & 2x + & 4y + & z + & M\,d_1 & & \geq & 30 & \\
 & & 10x + & 8y + & & M\,d_2 & & \geq & 180 & \\
 & & & & & d_1 + & d_2 & = & 1 & \\
 & & x\,, & y\,, & z & & & \geq & 0 & \text{and integer} \\
 & & & & & d_1\,, & d_2 & = & 0 & \text{or } 1
\end{array}
$$

. **Example 6.15:** There are six townships (1, 2, 3, 4, 5, and 6) in Denton County of Transylvania. The county must determine how many and where to build fire stations to ensure that at least one station is within 15 minutes of driving time of each city. The times (in minutes) required to drive between the townships are shown in the accompanying table. Formulate a ZOLP for the problem and solve it using Solver.

	1	2	3	4	5	6
1	0	11	20	30	28	24
2	11	0	25	35	22	13
3	20	25	0	15	31	27
4	30	35	15	0	14	20
5	28	22	31	14	0	12
6	24	13	27	20	12	0

[Solution] Let x_i be 1 if a fire station is located in township i and 0 otherwise, i = 1, 2, …, 6. A ZOLP for the facility location problem is presented below:

$$
\begin{array}{llcccccccl}
\text{Minimize} & Z = & x_1 + & x_2 + & x_3 + & x_4 + & x_5 + & x_6 & & \\
\text{subject to} & & x_1 + & x_2 & & & & & \geq & 1 \\
 & & x_1 + & x_2 + & & & & x_6 & \geq & 1 \\
 & & & & x_3 + & x_4 & & & \geq & 1 \\
 & & & & x_3 + & x_4 + & x_5 & & \geq & 1 \\
 & & & & & x_4 + & x_5 + & x_6 & \geq & 1 \\
 & & & x_2 + & & & x_5 + & x_6 & \geq & 1 \\
 & & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & = & 0 \text{ or } 1
\end{array}
$$

Running Solver to solve the problem, we obtain an optimal solution of $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*) = (0, 1, 0, 1, 0, 0)$ and the corresponding optimal objective function value of $Z^* = 2$. Thus, Dolphin County must build at least two fire stations, one in township 2 and the other in township 4, in order to meet the 15-minute response time requirement.

- **Remark:** The problems described in Examples 6.4, 6.13, and 6.15 are known as the assignment problem, the knapsack problem, and the set covering problem, respectively.