CHAPTER 5
NETWORK MODELING

1. Introduction

. Many decision problems can be depicted as a graph consisting of junction points interconnected by line segments. The points may represent geographic locations (such as cities) or physical facilities (such as warehouses); the lines may portray routes, pipelines, or activity relationships. Such a graphical representation is called a network.

. Networks are important tools of management science. Not only can networks be used to model a wide variety of important managerial problems, they can often be solved more efficiently than other models of the same problem. Four network models will be described in this chapter: minimal spanning tree problems, maximal flow problems, shortest route problems, and transportation problems.

2. Minimal Spanning Trees

. Many practical problems can be modeled as a network consisting of nodes and branches. For example, computer systems at various locations must be joined by wires to establish a communication network. Communities in the rapid transit system must be connected by subway lines. In situations such as these, the objective is normally to find the minimal spanning tree, i.e., a collection of branches in the network that will connect all nodes without forming any loop or cycle so that the total branch length is minimized.

. Given a network, the minimal spanning tree can be found with the greedy algorithm. The steps involved in the algorithm are as follows:

1. Beginning with any node, connect this node to the nearest node.

2. If all the nodes are connected, then the minimal spanning tree has been found; stop. Otherwise, go to Step 3.

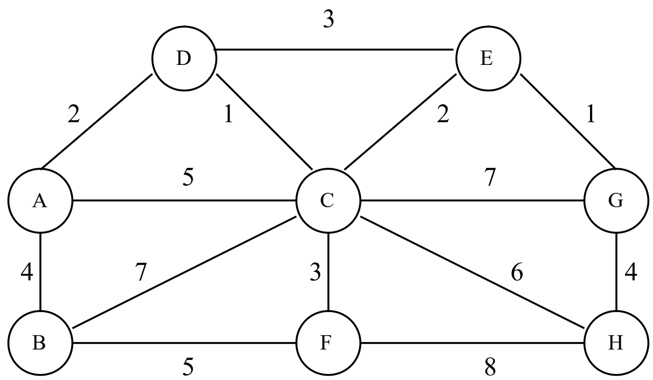
3. Among the nodes that are not connected, select the one that is the least distant from any of the connected nodes. Join this unconnected node to the nearest connected node. Go to Step 2.

. Remark: In selecting the nearest node, ties can be broken arbitrarily.

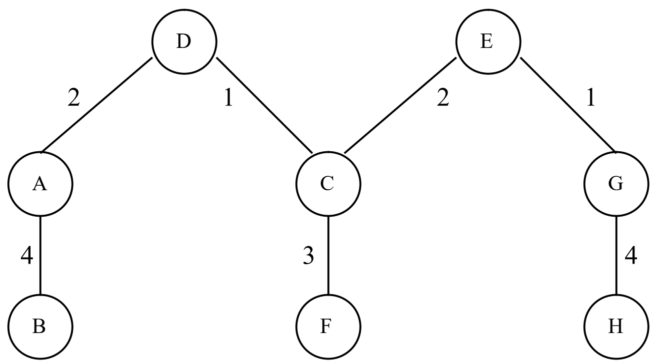
. **Example 5.1:** Prestige Builders, Inc., has set aside a portion of its exclusive housing development, Sunrise, for an equestrian/jogging park. The park will have a system of trails between the entrance and the various sites, with each trail consisting of an equestrian path and a separate jogging track.

The proposed trail system is shown below, where location A is the entrance into the park and sites B through H are various rest stations located at scenic points in the park. The numbers on the network give the distances of the winding trails in miles.

Water and sewer pipes must be installed under the trails so as to connect all locations (including the park entrance). Since installation is both expensive and disturbing to the natural environment, Prestige wants to connect all sites with the minimum number of miles of pipe. Use the greedy algorithm to solve the problem for the company.



[Solution] Applying the greedy algorithm to the network yields the following minimal spanning tree:



The minimum number of miles of water and sewer pipes is 4 + 2 + 1 + 3 + 2 + 1 + 4 = 17 miles.

3. Maximal Flow Problems

. The maximal flow problem in a network setting can take on a variety of forms in practice. For instance, the network might represent a system of roads and highways and the objective might be to maximize the rate of flow of vehicles through the traffic system. Other examples of maximal flow networks include systems of airline routes, distribution systems, and pipelines that carry natural gas, oil, or water.

. Consider a network in which each branch has a capacity in each direction that indicates the maximum flow possible through the branch in that direction. The Ford-Fulkerson method can be used to determine the maximal flow through the network. The steps involved in the method are as follows:

1. Find a path leading from the beginning node (or source) to the ending node (or sink) of the network in which each branch has a nonzero capacity. If no such path exists, then the maximal flow problem has been solved; go to Step 3. Otherwise, go to Step 2.

2. Determine the minimum of the branch capacities on the path just identified in Step 1. Subtract that amount from each of the branch capacities on the path in the same direction and then add the same amount to each of the branch capacities on the same path in the reverse direction. Go to Step 1.

3. The maximal flow through the network can be determined in either of the following two ways:

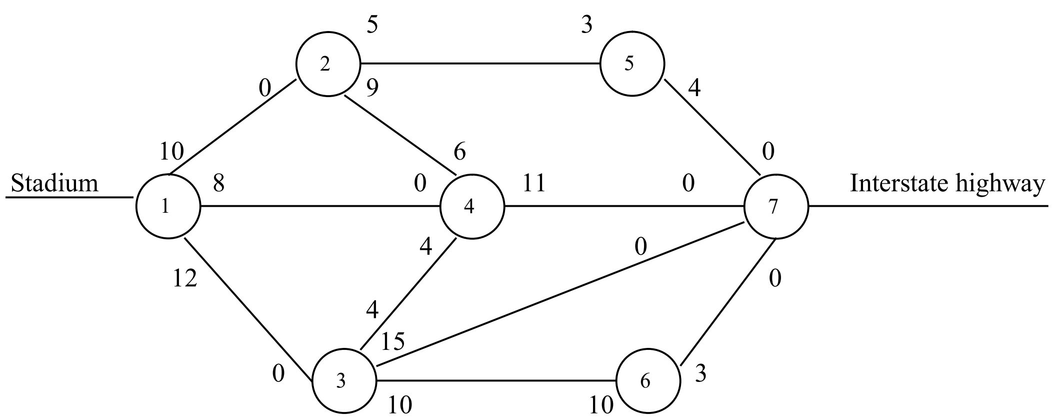
(1) Subtract the total flow capacity out of the source after the implementation of the algorithm from that before the implementation.

(2) Subtract the total flow capacity into the sink before the implementation of the algorithm from that after the implementation.

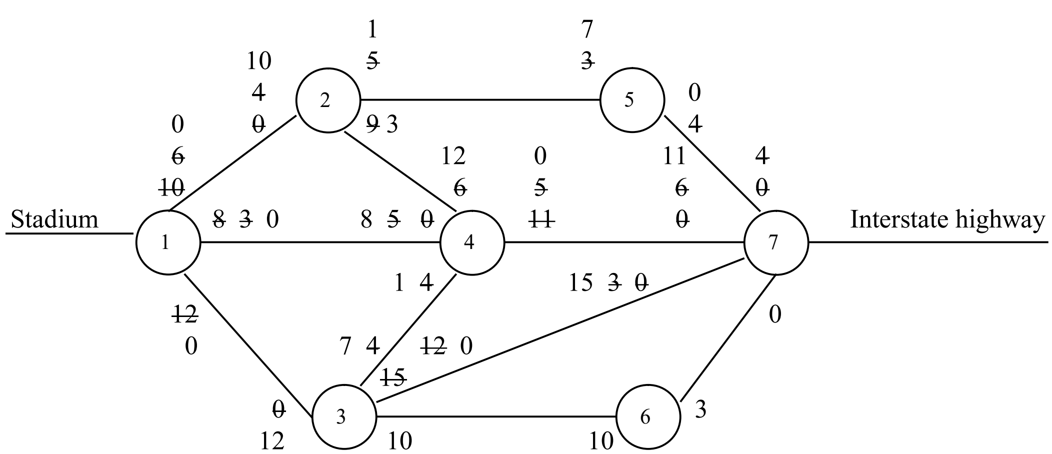
. **Example 5.2:** A new football stadium is being planned for Tran State University located in Meadowtown, Transylvania. The stadium is to be built on land north of the campus. Due to the influx of alumni on football weekends, major traffic problems are expected after each football game.

The current street configuration with the traffic flows (in hundreds of cars per hour) through each road section in both directions is shown below. Note that some streets are transformed into one-way streets for the short period after each game with police officers directing traffic.

The University Planning Board wants to know the maximum number of cars that can travel from the stadium to the interstate highway after each football game. Use the Ford-Fulkerson method to find the maximal hourly traffic flow through the street network for the Planning Board.



[Solution] An application of the Ford-Fulkerson algorithm below shows that the maximal traffic flow through the street network is (10 + 8 + 12) - (0 + 0 + 0) = (4 + 11 + 15 + 0) - (0 + 0 + 0 + 0) = 30 or 3,000 cars per hour.



4. Shortest Route Problems

- Given a network in which each branch has an associated length, the objective of the shortest route problem is to find the shortest path from a specified node (usually node 1) to any of the other nodes. This problem encompasses a broad range of important practical applications, including transportation, delivery, and equipment replacement.

- Dijkstra's algorithm is an efficient node-labeling procedure for solving shortest route problems. In implementing the method, each node that has been scanned receives a tentative label (d, i) or a permanent label [d, i], where d is the cumulative distance between node 1 and the node in question and i is the preceding node from which the current node is reached.

- Dijkstra's algorithm involves the following steps:

  1. Assign the permanent label [0, 1] to node 1.

  2. Assign the tentative label (d, 1) to each of the nodes adjacent to node 1, where d is the distance between node 1 and the adjacent node.

  3. Among all the tentatively labeled nodes, let node j be the one with the smallest distance d. Node j then becomes permanently labeled. If all the nodes are permanently labeled, the process is complete; go to Step 5. Otherwise, go to Step 4.

  4. Consider each non-permanently labeled node k which is adjacent to node j.

     (1) If node k has received a tentative label (d, i), let s be the sum of the distance d at node j and the distance between nodes j and k. If s is less than the distance d at node k, then replace (d, i) with (s, j).

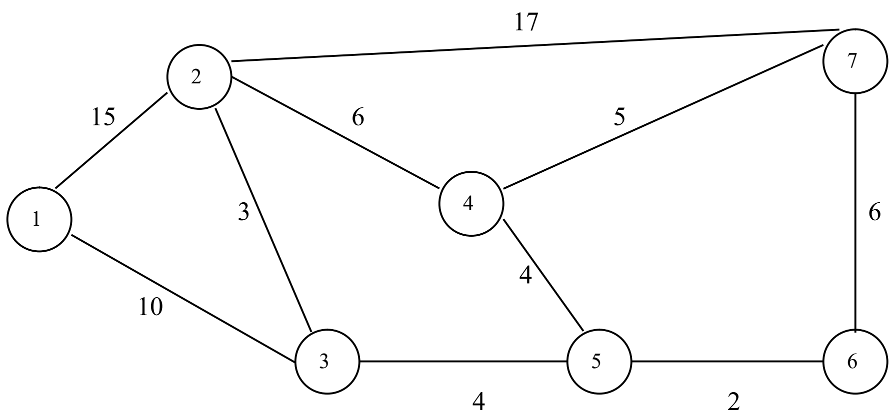
     (2) If node k has not been labeled, let s be the sum of the distance d at node j and the distance between nodes j and k. Assign (s, j) to node k. Go to Step 3.

  5. The shortest route from node 1 to a given node can be found by starting at the given node and moving to its preceding node. The backward movement continues through the network until node 1 is reached.

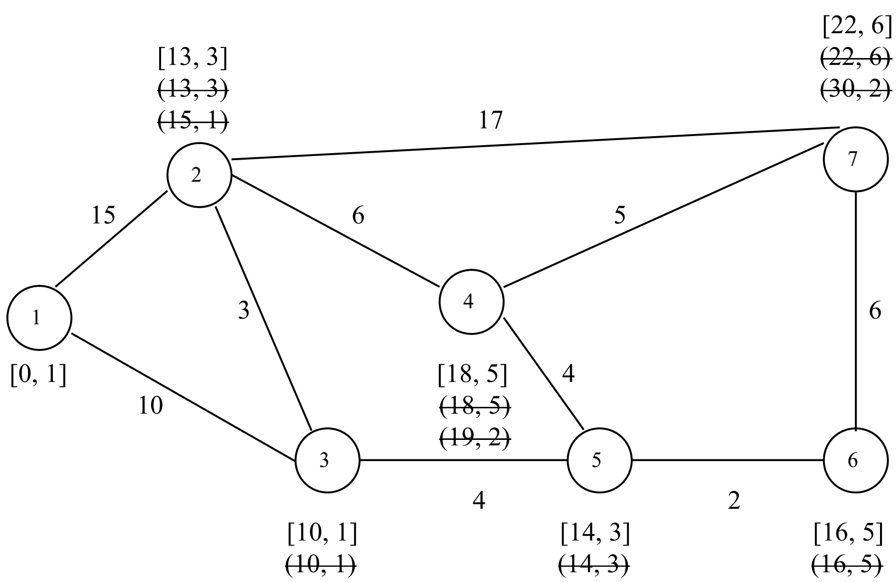
- Remark: Dijkstra's algorithm applies only to networks where all the branch lengths are nonnegative.

- **Example 5.3:** Gorman Construction Company has several construction projects located throughout a three-county area. Construction sites are sometimes located as far as 50 miles from Gorman's main office. With multiple daily trips carrying personnel, equipment, and supplies to and from the construction locations, the costs associated with the transportation activities are substantial. For any given construction site, the travel alternatives between the site and the office can be depicted by a network of roads, streets, and highways.

  The network shown below describes the travel alternatives to and from six of Gorman's newest construction sites (nodes 2, 3, 4, 5, 6, and 7), where the distances are in miles. Gorman would like to determine the route or path that will minimize the total travel distance from the office (node 1) to each site. Apply Dijkstra's algorithm to solve the shortest route problem for the construction company



[Solution] An application of Dijkstra's algorithm yields the following:



The shortest paths from Gorman's office to the six construction sites:

Site	Shortest route	Distance (miles)
2	1-3-2	13
3	1-3	10
4	1-3-5-4	18
5	1-3-5	14
6	1-3-5-6	16
7	1-3-5-6-7	22

5. Transportation Problems

- Transportation deals with the movement of raw materials, component parts, final products, or services from a set of supply points (e.g., plants) to a set of demand points (e.g., markets). The objective is to schedule shipments in such a way that the total transportation cost is minimized.

- The transportation method is a mathematical technique that can be applied to identify the lowest cost plan for shipping goods from origins to destinations. It can also be used to determine where to locate a facility. Information needed to apply the model is usually arranged into a transportation matrix (or tableau).

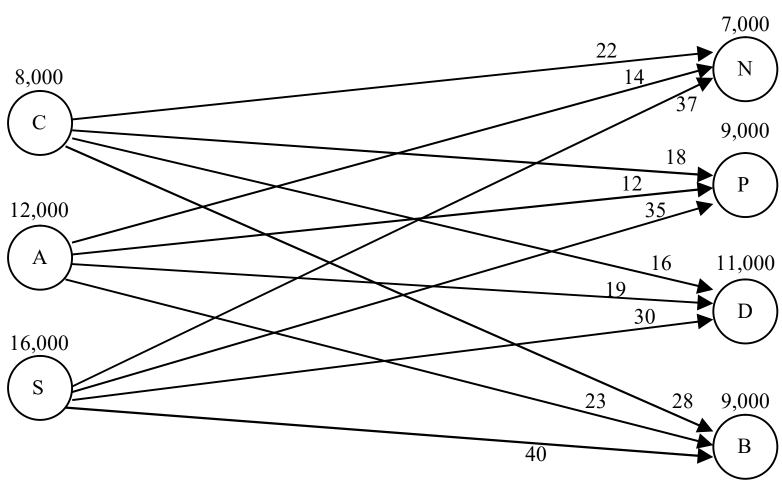
- **Example 5.4:** Supercola Corporation bottles cola in three plants located in Chicago, Atlanta, and Seattle with monthly capacities of 8,000, 12,000, and 16,000 cases, respectively. The cola is shipped from these plants to four distributors located in Newark, Pittsburgh, Detroit, and Boston with monthly requirements of 7,000, 9,000, 11,000, and 9,000, respectively. Relevant cost data are summarized in the following table:

From	To	Cost ($/case)
Chicago	Newark	22
	Pittsburgh	18
	Detroit	16
	Boston	28
Atlanta	Newark	14
	Pittsburgh	12
	Detroit	19
	Boston	23
Seattle	Newark	37
	Pittsburgh	35
	Detroit	30
	Boston	40

(1) Develop a network diagram of the transportation problem.

(2) Draw up a transportation matrix for the problem.

[Solution] (1) The transportation network is as follows:



(2) The transportation matrix is shown below:

	N	P	D	B	Supply
C	22	18	16	28	8,000
A	14	12	19	23	12,000
S	37	35	30	40	16,000
Demand	7,000	9,000	11,000	9,000	36,000 36,000

- To apply the transportation method to solve a problem, the procedure described below should be followed:

  (1) Develop an initial feasible solution.

  (2) Test the current solution for optimality. If all the evaluations are zero or positive, the current solution is optimal; stop. Otherwise, go to (3).

  (3) Obtain an improved solution and then go to (2).

2. Development of an Initial Feasible Solution

- Several approaches have been developed for obtaining an initial feasible solution (IFS) to a transportation problem: (1) the northwest-corner method, (2) the intuitive (or minimum-cell-cost) method, and (3) Vogel's approximation method (VAM). In what follows, we will examine only the first method.

- **Example 5.5:** Use the northwest-corner method to obtain an IFS to the transportation problem described in Example 5.4 and compute the total cost for the IFS.

  [Solution] The northwest-corner IFS is presented below:

	N	P	D	B	Supply
C	22 7,000	18 1,000	16	28	8,000
A	14	12 8,000	19 4,000	23	12,000
S	37	35	30 7,000	40 9,000	16,000
Demand	7,000	9,000	11,000	9,000	36,000 36,000

From	To	Shipment
C	N	7,000
	P	1,000
A	P	8,000
	D	4,000
S	D	7,000
	B	9,000

The total transportation cost is 22(7,000) + 18(1,000) + ... + 40(9,000) = $914,000.

3. Optimality Test and Improvement of a Solution

. Once a feasible solution is on hand, two methods are available for testing and, if necessary, improving the current solution: (1) the stepping-stone method and (2) the modified distribution (MODI) method. Our focus will be on the second method in this section.

. **Example 5.6:** Consider the following transportation problem:

	1	2	3	Supply
1	$\boxed{5}$	$\boxed{8}$	$\boxed{2}$	100
2	$\boxed{4}$	$\boxed{9}$	$\boxed{5}$	400
3	$\boxed{6}$	$\boxed{7}$	$\boxed{3}$	500
Demand	300	500	200	1,000 1,000

Beginning with the northwest-corner IFS, use the MODI method for cell evaluation to find the optimal solution and the minimum total cost.

[Solution] The northwest-corner IFS is presented below:

	1	2	3	Supply
1	$\boxed{5}$ 100	$\boxed{8}$	$\boxed{2}$	100
2	$\boxed{4}$ 200	$\boxed{9}$ 200	$\boxed{5}$	400
3	$\boxed{6}$	$\boxed{7}$ 300	$\boxed{3}$ 200	500
Demand	300	500	200	1,000 1,000

TC = 5(100) + 4(200) + … + 3(200) = 5,800

Evaluation:

$R1 + K1 = 5$	$R1 = 0$	$K1 = 5$
$R2 + K1 = 4$		$R2 = -1$
$R2 + K2 = 9$		$K2 = 10$
$R3 + K2 = 7$		$R3 = -3$
$R3 + K3 = 3$		$K3 = 6$

1-2: $C12 - (R1 + K2) = 8 - (0 + 10) = -2$
1-3: $C13 - (R1 + K3) = 2 - (0 + 6) = \mathbf{-4}$
2-3: $C23 - (R2 + K3) = 5 - (-1 + 6) = 0$
3-1: $C31 - (R3 + K1) = 6 - (-3 + 5) = 4$

The current solution is not optimal since -2 < 0 and -4 < 0. To improve it, 100 units should be moved to cell 1-3 since a "-" sign appears in each of cells 3-3, 2-2, and 1-1 and min {200, 200, 100} = 100. The new solution is:

	1	2	3	Supply
1	$\boxed{5}$	$\boxed{8}$	$\boxed{2}$ 100	100
2	$\boxed{4}$ 300	$\boxed{9}$ 100	$\boxed{5}$	400
3	$\boxed{6}$	$\boxed{7}$ 400	$\boxed{3}$ 100	500
Demand	300	500	200	1,000 / 1,000

TC = 5,800 - 4(100) = 5,400

Evaluation:

$R1 + K3 = 2$	$R1 = 0$	$K3 = 2$
$R2 + K1 = 4$		$K1 = 1$
$R2 + K2 = 9$		$R2 = 3$
$R3 + K2 = 7$		$K2 = 6$
$R3 + K3 = 3$		$R3 = 1$

1-1: $C11 - (R1 + K1) = 5 - (0 + 1) = 4$
1-2: $C12 - (R1 + K2) = 8 - (0 + 6) = 2$
2-3: $C23 - (R2 + K3) = 5 - (3 + 2) = 0$
3-1: $C31 - (R3 + K1) = 6 - (1 + 1) = 4$

The current solution is optimal since all the evaluations are zero or positive. In conclusion, the optimal shipping plan is:

```
From      To      Shipment
---------------------------------
1         3         100
2         1         300
          2         100
3         2         400
          3         100
---------------------------------
```

The minimum total transportation cost is $5,400.

- **Example 5.7:** Consider the northwest-corner IFS obtained in Example 5.5. Use the MODI method for cell evaluation to find the optimal solution and the minimum total transportation cost.

[Solution] The IFS is reproduced below:

	N	P	D	B	Supply
C	22 7,000	18 1,000	16	28	8,000
A	14	12 8,000	19 4,000	23	12,000
S	37	35	30 7,000	40 9,000	16,000
Demand	7,000	9,000	11,000	9,000	36,000 36,000

TC = 22(7,000) + 18(1,000) + … + 40(9,000) = 914,000

Evaluation:

R1 + K1 = 22	R1 = 0	K1 = 22
R1 + K2 = 18		K2 = 18
R2 + K2 = 12		R2 = -6
R2 + K3 = 19		K3 = 25
R3 + K3 = 30		R3 = 5
R3 + K4 = 40		K4 = 35

C-D: 16 - (0 + 25) = **-9**
C-B: 28 - (0 + 35) = -7
A-N: 14 - (-6 + 22) = -2
A-B: 23 - (-6 + 35) = -6
S-N: 37 - (5 + 22) = 10
S-P: 35 - (5 + 18) = 12

The current solution is not optimal since -9 < 0, -7 < 0, -2 < 0, and -6 < 0. To improve it, 1,000 units should be moved to cell C-D since a "-" sign appears in each of cells A-D and C-P and min {4,000, 1,000} = 1,000. The new solution is:

	N	P	D	B	Supply
C	22 7,000	18	16 1,000	28	8,000
A	14	12 9,000	19 3,000	23	12,000
S	37	35	30 7,000	40 9,000	16,000
Demand	7,000	9,000	11,000	9,000	36,000 / 36,000

TC = 914,000 - 9(1,000) = 905,000

Evaluation:

R1 + K1 = 22	R1 = 0	K1 = 22
R1 + K3 = 16		K3 = 16
R2 + K2 = 12		R2 = 3
R2 + K3 = 19		K2 = 9
R3 + K3 = 30		R3 = 14
R3 + K4 = 40		K4 = 26

C-P: 18 - (0 + 9) = 9
C-B: 28 - (0 + 26) = 2
A-N: 14 - (3 + 22) = **-11**
A-B: 23 - (3 + 26) = -6
S-N: 37 - (14 + 22) = 1
S-P: 35 - (14 + 9) = 12

The current solution is not optimal since -11 < 0 and -6 < 0. To improve it, 3,000 units should be moved to cell A-N since a "-" sign appears in each of cells C-N and A-D and min {7,000, 3,000} = 3,000. The new solution is:

	N	P	D	B	Supply
C	22 4,000	18	16 4,000	28	8,000
A	14 3,000	12 9,000	19	23	12,000
S	37	35	30 7,000	40 9,000	16,000
Demand	7,000	9,000	11,000	9,000	36,000 / 36,000

TC = 905,000 - 11(3,000) = 872,000

Evaluation:

R1 + K1 = 22	R1 = 0	K1 = 22
R1 + K3 = 16		K3 = 16
R2 + K1 = 14		R2 = -8
R2 + K2 = 12		K2 = 20
R3 + K3 = 30		R3 = 14
R3 + K4 = 40		K4 = 26

C-P: 18 - (0 + 20) = **-2**
C-B: 28 - (0 + 26) = 2
A-D: 19 - (-8 + 16) = 11
A-B: 23 - (-8 + 26) = 5
S-N: 37 - (14 + 22) = 1
S-P: 35 - (14 + 20) = 1

The current solution is not optimal since -2 < 0. To improve it, 4,000 units should be moved to cell C-P since a "-" sign appears in each of cells A-P and C-N and min {9,000, 4,000} = 4,000. The new solution is:

	N	P	D	B	Supply
C	22	18 4,000	16 4,000	28	8,000
A	14 8,000	12 5,000	19	23	12,000
S	37	35	30 7,000	40 9,000	16,000
Demand	7,000	9,000	11,000	9,000	36,000 / 36,000

TC = 872,000 - 2(4,000) = 864,000

Evaluation:

R1 + K2 = 18	R1 = 0	K2 = 18
R1 + K3 = 16		K3 = 16
R2 + K1 = 14		R2 = -6
R2 + K2 = 12		K1 = 20
R3 + K3 = 30		R3 = 14
R3 + K4 = 40		K4 = 26

C-N: 22 - (0 + 20) = 2
C-B: 28 - (0 + 26) = 2
A-D: 19 - (-6 + 16) = 9
A-B: 23 - (-6 + 26) = 3
S-N: 37 - (14 + 20) = 3
S-P: 35 - (14 + 18) = 3

The current solution is optimal since all the evaluations are zero or positive. In conclusion, the least-cost shipping scheme is:

From	To	Shipment
C	P	4,000
	D	4,000
A	N	7,000
	P	5,000
S	D	7,000
	B	9,000

The minimum total transportation cost is $864,000.