MDI210 Numerical analysis and continuous optimisation

Week 1 - Linear systems

Peter Brown

Télécom Paris
peter.brown@telecom-paris.fr

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Introduction

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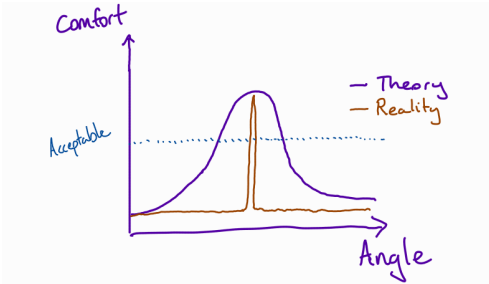

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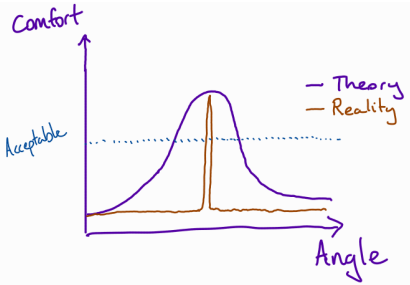
- How do we optimize?
- Efficient algorithms?
- Guaranteed optima?



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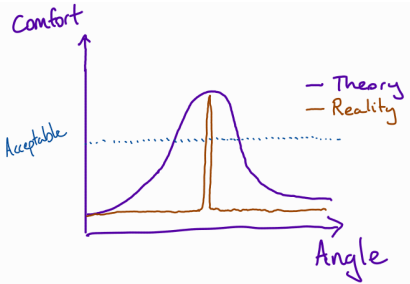

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Course outline:

- Linear systems / eigenvalues / linear programming
- Optimization of continuous functions / constraints / convex functions

Notation

Given an $m \times n$ matrix $A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$ we define

- *Transpose*: $A^t = (a_{ji})$.
- *Adjoint*: $A^* = (\overline{a_{ji}})$ (where $\overline{x}$ is the complex conjugate of x)

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A real/complex square matrix A is:

- *symmetric* if $A^t = A$
- *normal* if $A^t A = A A^t$
- *orthogonal* if $A^t A = A A^t = \mathbb{I}$

Hermitian if $A^* = A$

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The *spectrum* of a matrix A is the set of its eigenvalues. The *spectral radius* is then

$$\rho(A) := \max\{|\lambda| \ : \ \lambda \in \text{spectrum}(A)\} \tag{1}$$

A motivating example

We're interested in solving linear systems given A, b such that $Ax = b$ find x.

Example

We have a linear system

$$\begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 32 \\ 23 \\ 33 \\ 31 \end{pmatrix} \qquad \text{Solution:} \quad x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{2}$$

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Potential numerical problems:

- Floating point errors – finite precision / real data is noisy
- Truncation errors – a useful algorithm must stop at a finite time

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Question

Suppose $Ax = b$. If $A' \approx A$, $b' \approx b$ and $A'x' = b'$ then is $x' \approx x$?

Norms and distance

Definition (Norm)

Let V be a vector space. Then $\|\cdot\| : V \to \mathbb{R}$ is a norm if

1. $\|x\| \geq 0 \qquad \forall x \in V$ (non-negative)
2. $\|x\| = 0 \iff x = 0$ (positive definite)
3. $\|\alpha x\| = |\alpha|\|x\| \quad \forall x \in V$ and scalars α. (absolute homogeneity)
4. $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$ (triangle inequality)

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- $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ (1-norm)
- $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ (2-norm / Euclidean norm)
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More generally we have a p-norm for $p \in [1, \infty)$ defined as $\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$.

Norms and distance II

Norms on square matrices can be defined a vector norm (subordinate norms)

$$\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{x:\|x\|=1} \|Ax\| = \sup_{x:0<\|x\|\leq 1} \frac{\|Ax\|}{\|x\|} \tag{3}$$

These norms satisfy additional property $\|AB\| \leq \|A\|\|B\|$ (*submultiplicative*) [Exercise: proof]

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$$\|A\|_1 = \sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sqrt{\rho(A^T A)} \qquad \text{(largest singular value of } A\text{)}$$

$$\|A\|_\infty = \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

Norms and distance III

Not all matrix norms are subordinate norms

Example

A common matrix norm that is not subordinate is the *Frobenius* norm

$$\|A\|_F = \sqrt{\mathrm{Tr}\left[A^T A\right]}$$

Other examples include the *Schatten* norms $p \geq 1$

$$\|A\|_{(p)} := \left(\mathrm{Tr}\left[\left(A^T A\right)^{p/2}\right]\right)^{1/p}$$

(matrix generalization of the p-norms).

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All norms on finite dimensional spaces are equivalent! For any two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ there exist constants $c_1, c_2 > 0$ such that

$$c_1\|x\|_a \leq \|x\|_b \leq c_2\|x\|_a \qquad \forall x \tag{6}$$

Convergence doesn't depend on choice of norm!

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Take $A = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ for $a \in \mathbb{R}$. Then $\|A\|_1 = \|A\|_2 = \|A\|_\infty = 2|a|$ and so $A^k \to 0$ when $-1/2 < a < 1/2$.

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This agrees with

$$A^n = 2^n \begin{pmatrix} a^n & a^n \\ a^n & a^n \end{pmatrix}$$

Conditioning of linear systems

Back to the example

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Perturbing b we see

$$\begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} 32.1 \\ 22.9 \\ 33.1 \\ 30.9 \end{pmatrix} \qquad \text{Solution:} \quad x' = \begin{pmatrix} 9.2 \\ -12.6 \\ 4.5 \\ -1.1 \end{pmatrix} \tag{8}$$

The relative error on b is $\frac{\|b'-b\|}{\|b\|}$ of order $1/100$ but the relative error on x is of order 10! (Order 1000 increase)

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Let $\delta x = x' - x$ and $\delta b = b' - b$, then

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Then for any *subordinate norm*

$$\|\delta x\| \leq \|A^{-1}\|\|\delta b\| \tag{10}$$

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We call $\mathrm{cond}(A) := \|A\|\|A^{-1}\|$ the *conditioning of A* – relative error amplification depends on the conditioning!

For the example A we have $\mathrm{cond}_2(A) \approx 2974$.

Properties of the conditioning

Recall $\mathrm{cond}(A) := \|A\|\|A^{-1}\|$ for some *subordinate norm* $\|\cdot\|$.

Theorem

Let A be an invertible matrix. Then

1. $\mathrm{cond}(A) \geq 1$
2. $\mathrm{cond}(A) = \mathrm{cond}(A^{-1})$
3. *For $\alpha \neq 0$ we have* $\mathrm{cond}(\alpha A) = \mathrm{cond}(A)$.
4. *If U, V are orthogonal/unitary then* $\mathrm{cond}_2(UAV) = \mathrm{cond}(A)$
5. *If A is orthogonal/unitary then* $\mathrm{cond}_2(A) = 1$.

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Remark (Geometric characterization of cond_2)

Let $\theta(A)$ be the smallest angle between vectors Ax and Ay where x and y are orthonormal. Then

$$\mathrm{cond}_2(A) = \frac{1}{\tan(\theta(A)/2)} \tag{12}$$

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Let D_1, D_2 be diagonal invertible matrices satisfying

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Remark

In practice it is difficult to optimize the above so one can try to optimize the ratio

$$\frac{\max_{ij} |a'_{ij}|}{\min_{ij: a'_{ij} \neq 0} |a'_{ij}|} \qquad A' = \Delta_1 A \Delta_2$$

which can be treat as a linear program (see notes once studied LP).

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Question

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Let A be diagonalizable ($P^{-1}AP = \mathrm{diag}(\lambda_i)$). Let $\|\cdot\|$ be a matrix norm so that $\|\mathrm{diag}(\delta_i)\| = \max_i |\delta_i|$.
Then for any matrix δA we have

$$\mathrm{spectrum}(A + \delta A) \subset \bigcup_{i=1}^{n} D_i \tag{14}$$

where

$$D_i = \{z \in \mathbb{C} : |z - \lambda_i| \leq \mathrm{cond}(P)\|\delta A\|\}. \tag{15}$$

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We define the conditioning of A for the eigenvalue problem to be

$$\Gamma(A) = \min\{\mathrm{cond}(P) : P^{-1}AP \text{ is diagonal}\} \tag{16}$$

Solving linear systems

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First let's think about a special case – *triangular matrices*.

Upper triangular A

Suppose A is an upper triangular matrix. Then we have the linear equations

$$\begin{aligned} a_{1,1}x_1 + \cdots + a_{1,n-1}x_{n-1} + a_{1,n}x_n &= b_1 \\ &\cdots \\ a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n &= b_{n-1} \\ a_{n,n}x_n &= b_n \end{aligned}$$

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We can then use *backwards substitution*:

$$\begin{aligned} x_n &= \frac{b_n}{a_{n,n}} \\ x_{n-1} &= \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} \\ &\cdots \end{aligned}$$

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We have $n(n-1)/2$ additions, $n(n-1)/2$ multiplications and n divisions.

Upper triangular A – efficiency

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For comparison, computing the inverse is of order $O(n^3)$ operations.

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Let $P \in \mathbb{R}^{n \times n}$ be an invertible matrix. Then

$$\{x : Ax = b\} = \{x : PAx = Pb\} \tag{18}$$

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Gaussian Elimination: Use a sequence of invertible operators $P_1, \ldots, P_k$ such that

$$P_n \ldots P_2 P_1 A = U \tag{19}$$

where U is upper triangular. Then solve

$$Ux = P_n \ldots P_2 P_1 b\,. \tag{20}$$

Gaussian elimination

The *elementary row operations* (all invertible) are

- Interchange any two rows.
- Multiply a row by a constant.
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The algorithm:

1. For $j \in \{1, \ldots n\}$ repeat steps 2-4:
2. Choose a non-zero element (*pivot*) from column j below and including $a_{j,j}$.
3. If not row j then swap chosen elements row with row j.
4. Use linear combinations of rows to set all $a_{ij} = 0$ for $i > j$.

Gauss method – example

Consider the linear system

$$\begin{aligned} 2x_1 &+ x_2 - 3x_3 = 5 \\ 4x_1 &+ x_2 + 5x_3 = -1 \\ 10x_1 &- 7x_2 + 13x_3 = -3 \end{aligned}$$

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Step 1.2: Choose pivot.

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Step 1.3: Swap chosen pivot row so pivot on diagonal. (done)

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Consider the linear system

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Step 1.4: Use row sums to remove below elements.

$$R_2 \mapsto R_2 - 2R_1 \qquad \text{and} \qquad R_3 \mapsto R_3 - 5R_1$$

Gauss method – example

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Step 2.2: Move to diagonal (done)

Gauss method – example

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Step 2.3: Remove zeros below pivot

$$R_3 \mapsto R_3 - 12R_2$$

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Gauss method – example

Now we're left with an upper triangular linear system

$$\begin{aligned} 2x_1 &+ x_2 - 3x_3 = 5 \\ 0 &- x_2 + 11x_3 = -11 \\ 0 &- 0 - 104x_3 = 104 \end{aligned}$$

which we can compute like before!

Applying backward substitution we get

$$x_3 = -1, \qquad x_2 = 0, \qquad x_1 = 1$$

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Remark

Whenever you exchange rows you flip the sign of the determinant. We can compute $\det(A)$ as

$$\det(A) = (-1)^q \times \Pi_i p_i$$

where p_i is the value of the i-th pivot.

Gauss method - choice of pivot

- *Default strategy*: Choose the term on the diagonal (or next available one).
- *Partial pivot*: Choose pivot with largest modulus in the current column.
- *Total pivot*: Choose pivot with largest modulus in the remaining residual matrix.

Choosing larger modulus reduces numerical instability due to divisions but is more expensive.

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Example (Choice of pivot)

Consider the system (solution $(x_1, x_2) = (1.0001, 0.9999)$)

$$\begin{aligned} 10^{-4}x_1 + \quad x_2 &= \quad 1 \\ x_1 + \quad x_2 &= \quad 2 \end{aligned}$$

After *default pivoting* we have

$$x_2 = \frac{2 - 10^4}{1 - 10^4} = \frac{9998}{9999} \approx_{\text{lose 1 digit}} 1 \implies x_1 = 0$$

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After *partial pivoting* we have

$$x_2 = \frac{1 - 2 \times 10^{-4}}{1 - 10^{-4}} = \frac{0.9998}{0.9999} \approx_{\text{lose 1 digit}} 1 \implies x_1 = 1$$

Gauss-Jordan method – computing inverses

We don't have to solve a single linear system

We can solve n systems at the same time and compute the inverse!

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So we solve 3 systems simultaneously

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Step 1.2: Remove the elements below pivot

$$\begin{array}{rcrcrcc|c|c} -2x_1 & + & 4x_2 & - & 19x_3 & = & 0 & 0 & 1 \\ & & & + & \frac{1}{2}x_3 & = & 0 & 1 & \frac{1}{2} \\ & - & x_2 & + & \frac{9}{2}x_3 & = & 1 & 0 & \frac{1}{2} \end{array}$$

$$R_2 \mapsto R_2 + \frac{1}{2}R_1 \qquad R_3 \mapsto R_3 + \frac{1}{2}R_1$$

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$$\begin{array}{rcrcrcl|c|c}
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Step 2.2: Remove coefficients above and below the pivot

$$\begin{array}{rcrcrcl|c|c} -2x_1 & & & - & \frac{2}{9}x_2 & = & \frac{38}{9} & \frac{28}{9} & 0 \\ & + & \frac{9}{2}x_3 & - & x_2 & = & 1 & \frac{1}{2} & 0 \\ & & & + & \frac{1}{9}x_2 & = & -\frac{1}{9} & \frac{4}{9} & 1 \end{array}$$

$$R_1 \mapsto R_1 + \frac{38}{9}R_2 \qquad R_3 \mapsto R_3 9 - \frac{1}{9}R_2$$

Gauss-Jordan method – computing inverses

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-2x_1 & & & - & \frac{2}{9}x_2 & = & \frac{38}{9} & \frac{28}{9} & 0 \\
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Step 3.2: Remove coefficients above the pivot

$$\begin{array}{rcrcrcr|c|c} -2x_1 & & & & & = & 4 & 4 & 2 \\ & + & \frac{9}{2}x_3 & & & = & 0 & \frac{9}{2} & 9 \\ & & & + & \frac{1}{9}x_2 & = & -\frac{1}{9} & \frac{4}{9} & 1 \end{array}$$

$$R_1 \mapsto R_1 + 2R_3 \qquad R_2 \mapsto R_2 + 9R_3$$

Gauss-Jordan method – computing inverses

Now we're almost done

$$\begin{array}{rcrcrcl|c|c} -2x_1 & & & & & = & 4 & 4 & 2 \\ & + & \frac{9}{2}x_3 & & & = & 0 & \frac{9}{2} & 9 \\ & & & + & \frac{1}{9}x_2 & = & -\frac{1}{9} & \frac{4}{9} & 1 \end{array}$$

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Step 4: Solution

$$\begin{array}{rcr|c|c} x_1 & = & -2 & -1 & -2 \\ x_2 & = & -1 & 9 & 4 \\ x_3 & = & 0 & 2 & 1 \end{array}$$

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$$\begin{array}{rcr|r|r} x_1 & = & -2 & -1 & -2 \\ x_2 & = & -1 & 9 & 4 \\ x_3 & = & 0 & 2 & 1 \end{array}$$

and so

$$A^{-1} = \begin{pmatrix} -2 & -1 & -2 \\ -1 & 9 & 4 \\ 0 & 2 & 1 \end{pmatrix}$$

The LU decomposition

Suppose every pivot in the Gauss method was on the diagonal.

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Consider the matrix P_k that does the linear combinations for column k, it takes the form

$$P_k = \begin{pmatrix} 1 & 0 & & & & & \\ 0 & 1 & 0 & & & & \\ & 0 & \ddots & \ddots & & & \\ & & \ddots & 1 & 0 & & \\ & & & -\frac{a_{k+1}}{a_k} & \ddots & & \\ & & & \vdots & & 1 & 0 \\ & & & -\frac{a_n}{a_k} & & 0 & 1 \end{pmatrix} = I - v_k e_k^T$$

I.e., an identity matrix with coefficients under the k-th diagonal.

The LU decomposition

Therefore

$$P_n \dots P_1 A = U$$

where $P_1, \dots, P_n$ are *lower triangular* and U is *upper triangular*.

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One can show P_k^{-1} is also lower triangular , therefore

$$\begin{aligned} A &= P_1^{-1} \dots P_n^{-1} U \\ &= LU \end{aligned} \tag{21}$$

I.e., we can decompose A into the product of a lower triangular and upper triangular matrix!

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What if the pivots are not on the diagonal?

The LU decomposition

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Question

What if the pivots are not on the diagonal?

Can rearrange rows/cols of A so all pivots on diagonal so

$$QAR = LU$$

The LU decomposition

Theorem

Let $A \in \mathbb{R}^{n \times n}$ be an invertible square matrix such that for $1 \leq k \leq n$ the submatrix

$$A_k = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}$$

is invertible. Then the Gauss method can be performed with all pivots on the diagonal and hence the LU decomposition exists for A.

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Remark

The LU decomposition can be computed when trying to solve more than one linear system. Once you have the *LU* decomposition you need to solve two triangular systems per original system

$$Ly = b \qquad \text{then} \qquad Ux = y$$

Cholesky Decomposition

A matrix A is *positive definite* if it is symmetric and

$$x^T Ax > 0, \qquad \forall x \neq 0 \tag{21}$$

For positive definite matrices we have a more efficient LU decomposition with $L = U^T$.

Example

We have $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, then $x^T Ax = x_1^2 + (x_1 + x_2)^2 > 0$ and so A is positive definite.

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Theorem (Cholesky decomposition)

Let $A \in \mathbb{R}^{n \times n}$ be positive definite. Then there exists a lower triangular matrix $B \in \mathbb{R}^{n \times n}$ such that $A = BB^T$.

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Let $A \in \mathbb{R}^{n \times n}$ be positive definite. Then there exists a lower triangular matrix $B \in \mathbb{R}^{n \times n}$ such that $A = BB^T$.

Proof.

By induction... □

Cholesky Decomposition – Proof

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Base case $n = 1$ is trivial. Consider general positive definite A and partition as

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where $W \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{n-1 \times n-1}$. By direct calculation we have

$$A = \begin{pmatrix} a_{11} & W^T \\ W & C \end{pmatrix} = \begin{pmatrix} \sqrt{a_{11}} & 0 \\ W/\sqrt{a_{11}} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C - WW^T/a_{11} \end{pmatrix} \begin{pmatrix} \sqrt{a_{11}} & W^T/\sqrt{a_{11}} \\ 0 & I \end{pmatrix}$$

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The middle matrix is block diagonal and positive definite therefore $C - WW^T/a_{11} = LL^T$ (Cholesky decomposition) by induction assumption. So

$$\begin{aligned} A &= \begin{pmatrix} \sqrt{a_{11}} & 0 \\ W/\sqrt{a_{11}} & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & LL^T \end{pmatrix} \begin{pmatrix} \sqrt{a_{11}} & W^T/\sqrt{a_{11}} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{a_{11}} & 0 \\ W/\sqrt{a_{11}} & L \end{pmatrix} \begin{pmatrix} \sqrt{a_{11}} & W^T/\sqrt{a_{11}} \\ 0 & L^T \end{pmatrix} \end{aligned}$$

Cholesky Decomposition – Algorithm

$$A = \begin{pmatrix} b_{11} & 0 & \dots & 0 \\ b_{21} & b_{22} & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{21} & \dots & b_{n1} \\ 0 & b_{22} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{nn} \end{pmatrix}$$

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and so

$$a_{ij} = \sum_{k=1}^{\min\{i,j\}} b_{ik} b_{jk} = \langle (b_{ik})_k, (b_{jk})_k \rangle = a_{ji}$$

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For the first column

$$b_{11} = \sqrt{a_{11}} \qquad \text{and} \qquad b_{i1} = \frac{a_{i1}}{b_{11}} \quad \text{for } 2 \leq i \leq n$$

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$$a_{ij} = \sum_{k=1}^{\min\{i,j\}} b_{ik} b_{jk} = \langle (b_{ik})_k, (b_{jk})_k \rangle = a_{ji}$$

For the first column

$$b_{11} = \sqrt{a_{11}} \qquad \text{and} \qquad b_{i1} = \frac{a_{i1}}{b_{11}} \quad \text{for } 2 \leq i \leq n$$

For column $2 \leq j \leq n$

$$b_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} b_{jk}^2} \qquad \text{and} \qquad b_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} b_{ik} b_{jk}}{b_{jj}}$$

Cholesky Decomposition – Algorithm II

Remark (Properties)

1. Suppose we solve $Ax = b$ for a positive definite A.
   - Gaussian elimination – approx $\frac{2n^3}{3}$ operations
   - Cholesky decomposition – approx $\frac{n^3}{3}$ operations
2. If $A = BB^T$ then

$$\det(A) = \det(BB^T) = \det(B)\det(B^T) = (b_{11}b_{22}\dots b_{nn})^2$$

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Example

We have

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 2 & 3 \\ 0 & 3 & 10 \end{pmatrix}$$

$$x^T Ax = (2x_1 - x_2)^2 + (x_2 + 3x_3)^2 + x_3^2 > 0$$

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Example

We have

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 2 & 3 \\ 0 & 3 & 10 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x^T Ax = (2x_1 - x_2)^2 + (x_2 + 3x_3)^2 + x_3^2 > 0$$

Some exercises (working over $\mathbb{R}$)

1. Let V be an orthogonal matrix. Show that $\mathrm{cond}_2(V) = 1$.
2. Let $\|\cdot\|$ be a subordinate norm. Show that
$$\|Ax\| \leq \|A\|\|x\| \qquad \text{and} \qquad \|AB\| \leq \|A\|\|B\|$$
3. Let $\alpha \neq 0$ and U, V be orthogonal. Prove
$$(a)\quad \mathrm{cond}(A) \geq 1 \qquad (b)\quad \mathrm{cond}(A) = \mathrm{cond}(A^{-1})$$
$$(c)\quad \mathrm{cond}(\alpha A) = \mathrm{cond}(A) \qquad (d)\quad \mathrm{cond}_2(A) = \mathrm{cond}_2(UAV)$$
4. Let $A = \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}$ for $0 < \epsilon < 1$ and let $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
   (a) Using Gauss' method solve $Ax = b$.
   (b) Using Gauss-Jacobi method compute A^{-1}.
   (c) Compute $\mathrm{cond}_1(A)$ and $\mathrm{cond}_\infty(A)$. When is A well-conditioned? What happens at $\epsilon = 1$?
   (d) For what values of ϵ is A positive definite?
   (e) Find the LU decomposition of A.
   (f) Compute the Cholesky decomposition of A.
5. Suppose $Ax = b$ and $A'x' = b$. Show that
$$\frac{\|x' - x\|}{\|x'\|} \leq \mathrm{cond}(A)\frac{\|A' - A\|}{\|A\|}$$