Q1: Which of the following states are entangled?

$$|\psi(\theta)\rangle := \cos(\theta)|00\rangle + \sin(\theta)|11\rangle \qquad 0 \leq \theta \leq \pi/4$$

Sol^n

For $\theta = 0$, $|\psi(0)\rangle = |00\rangle = |0\rangle \otimes |0\rangle$ so not entangled.

Otherwise, let $|v\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|w\rangle = \gamma|0\rangle + \delta|1\rangle$.

Then $|v\rangle \otimes |w\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$.

Thus we need

$$\alpha\gamma = \cos(\theta)$$
$$\alpha\delta = 0$$
$$\beta\gamma = 0$$
$$\beta\delta = \sin(\theta)$$

if the state is not entangled. But $\alpha\delta = 0 \implies \alpha = 0$ or $\delta = 0$. If $\alpha = 0$ we need $\cos(\theta) = 0$ and if $\delta = 0$ we need $\sin(\theta) = 0$. But this is not possible for $0 < \theta \leq \pi/4$. Thus $|\psi(\theta)\rangle$ is entangled for all $\theta \in (0, \pi/4]$ ☒

Q2 Let $\{|\psi_i\rangle\}$ be a set of states and p_i be a probability distribution. Prove that for $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ we have

a) $Tr(\rho) = 1$

b) $\rho \geq 0$.

Also show ρ is pure iff $Tr(\rho^2) = 1$.

Sol^n

linearity

Compute in ONB containing $|\psi_i\rangle$.

(a) $$Tr(\rho) = \sum_i p_i \, Tr\left[|\psi_i\rangle\langle\psi_i|\right] = \sum_i p_i = 1$$

Probability distribution

(b) We need ρ is Hermitian and $\langle x|\rho|x\rangle \geq 0 \;\; \forall\, |x\rangle \in \mathcal{H}$.

Hermitian is clear $\rho^\dagger = \sum_i p_i \left(|\psi_i\rangle\langle\psi_i|\right)^\dagger = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho$.

And

$$\langle x|\rho|x\rangle = \sum_i p_i \langle x|\psi_i\rangle\langle\psi_i|x\rangle = \sum_i p_i |\langle x|\psi_i\rangle|^2 \geq 0.$$

For purity recall ρ is pure when $\rho = |\psi\rangle\langle\psi|$ for some state $|\psi\rangle$. Then $\rho^2 = |\psi\rangle\langle\psi|$ thus $Tr(\rho^2) = 1$ when ρ is pure. Now suppose $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ and $Tr(\rho^2) = 1$.
By the spectral theorem we can assume that $\{|\psi_i\rangle\}_i$ form an orthonormal basis. Then

$$\rho^2 = \sum_{i,j} p_i p_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|$$
$$= \sum_i p_i^2 |\psi_i\rangle\langle\psi_i|.$$

So $Tr(\rho^2) = \sum_i p_i^2$. But we know $\sum_i p_i = 1$ and so $\sum_i p_i^2 = 1$ only when $p_i = 1$ for some i and $p_j = 0 \ \forall j \neq i$. In that case $\rho = |\psi_i\rangle\langle\psi_i|$ and is pure. $\boxtimes$

Q3 a) Prove that any qubit state ρ can be written as

$$\rho = \frac{\mathbb{1} + n_x X + n_y Y + n_z Z}{2}$$

with $n_x, n_y, n_z \in \mathbb{R}$ and $n_x^2 + n_y^2 + n_z^2 \leq 1$.

Sol^n

Can show that $\{\mathbb{1}, X, Y, Z\}$ form an orthogonal basis with respect to the inner-product $\langle R, S\rangle = Tr[R^\dagger S]$ for the Hilbert space of 2x2 matrices with elements in $\mathbb{C}$. Thus we can always write

$$\rho = \frac{1}{2}(n_0 \mathbb{1} + n_x X + n_y Y + n_z Z)$$

for some $n_0, n_x, n_y, n_z \in \mathbb{C}$.
Now we need $Tr[\rho] = 1$, as $Tr[X] = Tr(Y) = Tr(Z) = 0$
$\Rightarrow$ that $n_0 = 1$.
Secondly we need ρ to be Hermitian (as it is positive semidefinite), thus $\rho = \rho^\dagger$. This implies (noting X, Y, Z are all Hermitian)

$$\bar{n}_x X + \bar{n}_y Y + \bar{n}_z Z = n_x X + n_y Y + n_z Z.$$

where $\bar{\alpha}$ denote the complex conjugate of α.
As $\mathbb{1}, X, Y, Z$ are an orthogonal basis this implies that

$$\bar{n}_i = n_i \quad \Rightarrow \quad n_x, n_y, n_z \in \mathbb{R}$$

Thus we arrive at $\rho = \frac{1}{2}\begin{pmatrix} 1+n_z & n_x - in_y \\ n_x + in_y & 1-n_z \end{pmatrix}$

Finally for $\rho \geq 0$ we require the eigenvalues of ρ to be non-negative. We find the eigenvalues of the above matrix to be

$$\left\{ \frac{1}{2}\left(1 \pm \sqrt{n_x^2+n_y^2+n_z^2}\right) \right\}$$

Thus we need $n_x^2 + n_y^2 + n_z^2 \leq 1$

b) Show ρ is pure iff $n_x^2 + n_y^2 + n_z^2 = 1$.

Sol^n Note that ρ is pure iff $Tr(\rho^2) = 1$.

We have $Tr(\rho^2) = \frac{1}{2}(1 + n_x^2 + n_y^2 + n_z^2)$

Thus $Tr(\rho^2) = 1 \iff n_x^2 + n_y^2 + n_z^2 = 1$ ☒

Q4 a) Prove the trace is cyclic, i.e. $Tr(XY) = Tr(YX)$.

Sol^n

Write $X = \sum_{ij} x_{ij} |i\rangle\langle j|$ and $Y = \sum_{ij} y_{ij} |i\rangle\langle j|$.

Then $XY = \sum_{ijk} x_{ij} y_{jk} |i\rangle\langle k|$

$YX = \sum_{ijk} y_{ij} x_{jk} |i\rangle\langle k|$

Thus $Tr[XY] = \sum_{ij} x_{ij} y_{ji} = \sum_{ij} y_{ij} x_{ji} = Tr(YX)$.

b) Use this to prove trace is Basis independent i.e.,

$Tr(X) = \sum_i \langle v_i | X | v_i \rangle$ for any orthonormal basis $\{|v_i\rangle\}_i$.

Sol^n

Let $U = \sum_i |v_i\rangle\langle i|$, then U is unitary and we have

$$Tr[X] = Tr(UU^\dagger X) = Tr(U^\dagger X U) = \sum_i \langle i | U^\dagger X U | i \rangle$$
$$= \sum_i \langle i | \left(\sum_j |j\rangle\langle v_j|\right) X \left(\sum_k |v_k\rangle\langle k|\right) | i \rangle$$
$$= \sum_i \langle i | i \rangle\langle v_i | X | v_i \rangle\langle i | i \rangle$$
$$= \sum_i \langle v_i | X | v_i \rangle$$ ☒

Small error in question sheet

5) Let $\{P, \mathbb{1} - P\}$ be any qubit projective measurement with $P \neq 0, \mathbb{1}$
Show that $\langle \Phi_{ij} | (P \otimes \mathbb{1}) | \Phi_{ij} \rangle = \frac{1}{2}$.

Sol^n One can verify that $|\Phi_{ij}\rangle = (\mathbb{1} \otimes X^j Z^i)|\Phi_{00}\rangle$

with $|\Phi_{00}\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$ where $M^0 = \mathbb{1}$ for any M.

$$\begin{aligned}
\text{So } \langle \Phi_{ij} | (P \otimes \mathbb{1}) | \Phi_{ij} \rangle &= \langle \Phi_{00} | (\mathbb{1} \otimes Z^i X^j)(P \otimes \mathbb{1})(\mathbb{1} \otimes X^j Z^i) | \Phi_{00} \rangle \\
&= \langle \Phi_{00} | (P \otimes Z^i X^j X^j Z^i) | \Phi_{00} \rangle \\
&= \langle \Phi_{00} | (P \otimes \mathbb{1}) | \Phi_{00} \rangle
\end{aligned}$$

where last line follows from $Z^2 = X^2 = \mathbb{1}$.

So we only need to show the result for $|\Phi_{00}\rangle$. Let $P = |v\rangle\langle v|$ for some state $|v\rangle$ (this is general form of projector that is not $\mathbb{1}$ or 0 on qubit)

$$\begin{aligned}
\text{Then } \langle \Phi_{00} | (|v\rangle\langle v| \otimes \mathbb{1}) | \Phi_{00} \rangle &= \tfrac{1}{2}(\langle 00| + \langle 11|)(|v\rangle\langle v| \otimes \mathbb{1})(|00\rangle + |11\rangle) \\
&= \tfrac{1}{2}(\langle 0|v\rangle\langle v|0\rangle + \langle 1|v\rangle\langle v|1\rangle) \\
&= \tfrac{1}{2}
\end{aligned}$$

The last equality follows from the fact that $\exists\ \alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$ and $|v\rangle = \alpha|0\rangle + \beta|1\rangle$ as P is a projector so $|v\rangle$ is a quantum state. $\boxtimes$

<u>2nd method (using partial trace)</u>

A faster method is to note that if we trace over the second qubit in $|\Phi_{ij}\rangle\langle\Phi_{ij}|$ we get $\mathrm{Tr}_2[|\Phi_{ij}\rangle\langle\Phi_{ij}|] = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \mathbb{1}/2$.

And so

$$\begin{aligned}
\langle \Phi_{ij} | (P \otimes \mathbb{1}) | \Phi_{ij} \rangle &= \mathrm{Tr}[(P \otimes \mathbb{1})|\Phi_{ij}\rangle\langle\Phi_{ij}|] = \mathrm{Tr}[\mathrm{Tr}_2[(P \otimes \mathbb{1})|\Phi_{ij}\rangle\langle\Phi_{ij}|]] \\
&= \mathrm{Tr}[P\, \mathrm{Tr}_2[|\Phi_{ij}\rangle\langle\Phi_{ij}|]] \\
&= \mathrm{Tr}[P \cdot \mathbb{1}/2] \\
&= \tfrac{1}{2}\mathrm{Tr}[P] = \tfrac{1}{2} \quad \text{as } P = |v\rangle\langle v|. \quad \boxtimes
\end{aligned}$$

6) Let ρ_A be a density matrix on A and let $|\psi\rangle_{AB}$ be a pure state on the joint system AB. We say $|\psi\rangle_{AB}$ is a purification of ρ_A if

$$\rho_A = \mathrm{Tr}_B\left[|\psi\rangle\langle\psi|_{AB}\right].$$

Show that purifications always exist.

Sol^n

By spectral decomposition $\exists$ an orthonormal basis $\{|v_i\rangle\}_i$ for A and $\lambda_i \geq 0$ such that

$$\rho_A = \sum_i \lambda_i |v_i\rangle\langle v_i|.$$

Let B be a system isomorphic to A and define

$$|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\, |v_i\rangle_A \otimes |v_i\rangle$$

Then we have $|\psi\rangle\langle\psi|_{AB} = \sum_{ij} \sqrt{\lambda_i\lambda_j}\, |v_i\rangle\langle v_j| \otimes |v_i\rangle\langle v_j|$

and

$$\mathrm{Tr}_B\left[|\psi\rangle\langle\psi|_{AB}\right] = \sum_{ij} \sqrt{\lambda_i\lambda_j}\, |v_i\rangle\langle v_j| \cdot \mathrm{Tr}\left[|v_i\rangle\langle v_j|\right]$$

Kronecker delta

$$= \sum_{ij} \sqrt{\lambda_i\lambda_j}\, |v_i\rangle\langle v_j| \cdot \delta_{ij}$$

$$= \sum_i \lambda_i |v_i\rangle\langle v_i| = \rho_A \qquad \boxtimes$$

measurement without projective constraint

7) Design a 3-outcome qubit POVM $\{M_0, M_+, M_{fail}\}$ that allows us to distinguish $|+\rangle$ and $|0\rangle$ without making errors. I.e.,

$$\langle 0|M_0|0\rangle > 0 \qquad \langle +|M_+|+\rangle > 0$$

$$\langle +|M_0|+\rangle = 0 \qquad \langle 0|M_+|0\rangle = 0$$

never guess 0 when +   never guess + when 0.

Sol^n The constraint $\langle +|M_0|+\rangle = 0$ means we should use $M_0 = c_0|-\rangle\langle -|$ for some $c_0 \geq 0$. Similarly $\langle 0|M_+|0\rangle = 0 \Rightarrow M_+ = c_+|1\rangle\langle 1|$ for some $c_+ \geqslant 0$. We need $c_0, c_+ > 0$ to satisfy

$$\langle 0|M_0|0\rangle > 0 \quad \text{and} \quad \langle +|M_+|+\rangle > 0.$$

Now because $M_0 + M_+ + M_{fail} = \mathbb{1}$ we can define $M_{fail} = \mathbb{1} - M_0 - M_+$.
We just need to check that it is a valid POVM so $M_0, M_1, M_{fail} \geqslant 0$ (are positive semidefinite).

M_0 and M_+ are positive semidefinite by construction as $c_0, c_+ > 0$.
Then

$$M_{fail} = \begin{pmatrix} 1 - \frac{c_0}{2} & \frac{c_0}{2} \\ \frac{c_0}{2} & 1 - \frac{c_0}{2} - c_+ \end{pmatrix}$$

One can check this is positive semidefinite, this is equivalent to all eigenvalues are non-negative which is equivalent to

$$c_0 + c_+ + \sqrt{c_0^2 + c_+^2} \leqslant 2$$

Simplifying we have a valid POVM when all constraints are satisfied

① $c_0 > 0$ and $c_+ > 0$

② $c_0 + c_+ + \sqrt{c_0^2 + c_+^2} \leqslant 2$

Choosing for example $c_0 = c_+ = \frac{1}{2}$ we have a valid POVM

$$M_0 = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 1/4 \end{pmatrix} \quad M_+ = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad M_{fail} = \begin{pmatrix} 3/4 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

and

$$\langle 0|M_0|0\rangle = \frac{1}{4} \qquad \langle +|M_+|+\rangle = \frac{1}{4}$$

$$\langle 0|M_+|0\rangle = 0 \qquad \langle +|M_0|+\rangle = 0$$

⊠

8) Show that a bipartite state $|\psi\rangle_{AB}$ is product iff ρ_A and ρ_B are pure.

Sol^n ($\Rightarrow$)

If $|\psi\rangle = |v\rangle \otimes |w\rangle$ then $\rho_A = Tr_B\left[|v\rangle\langle v| \otimes |w\rangle\langle w|\right] = |v\rangle\langle v|$ which is pure.

Similarly $\rho_B = |w\rangle\langle w|$ is pure.

($\Leftarrow$) By the schmidt-decomposition $\exists$ an orthonormal basis $\{|v_i\rangle\}_i$ of A and $\{|w_i\rangle\}_i$ of B and nonnegative coefficients $\lambda_i \geq 0$ such that

$$|\psi\rangle = \sum_i \lambda_i |v_i\rangle|w_i\rangle$$

Thus $\rho_A = \sum_i \lambda_i |v_i\rangle\langle v_i|$ and $\rho_B = \sum_i \lambda_i |w_i\rangle\langle w_i|$.

If ρ_A and ρ_B are pure $\Rightarrow$ that only one λ_i is non-zero.

Thus $\rho_A = |v_i\rangle\langle v_i|$ for some i and $\rho_B = |w_i\rangle\langle w_i|$

$$\Longrightarrow \quad |\psi\rangle = |v_i\rangle \otimes |w_i\rangle \text{ is a product state.} \quad \boxtimes$$

8) (Deriving the Tsirelson bound)

Alice and Bob play the CHSH game. For convenience we let the inputs $x, y \in \{0,1\}$ and the outputs $a, b \in \{+1, -1\}$. The winning condition then becomes

$$(-1)^{xy} = ab$$

Let Alice's projective measurement on input x be $\{A_{1|x}, A_{-1|x}\}$ and Bob's projective measurement on input y be $\{B_{1|y}, B_{-1|y}\}$. Let the quantum state shared between Alice and Bob be $|\psi\rangle$

Define observables
$$A_x = A_{1|x} - A_{-1|x}$$
$$B_y = B_{1|y} - B_{-1|y}$$

a) Show that for any fixed xy the expected value of ab is given by

$$\langle\psi| A_x \otimes B_y |\psi\rangle$$

<u>Sol^n</u>

$$\begin{aligned}
\langle\psi| A_x \otimes B_y |\psi\rangle &= \langle\psi| (A_{1|x} - A_{-1|x}) \otimes (B_{1|y} - B_{-1|y}) |\psi\rangle \\
&= \langle\psi| A_{1|x} \otimes B_{1|y} |\psi\rangle - \langle\psi| A_{1|x} \otimes B_{-1|y} |\psi\rangle \\
&\quad - \langle\psi| A_{-1|x} \otimes B_{1|y} |\psi\rangle + \langle\psi| A_{-1|x} \otimes B_{-1|y} |\psi\rangle \\
&= p(a=1, b=1 | x, y) - p(a=1, b=-1 | x, y) \\
&\quad - p(a=-1, b=1 | x, y) + p(a=-1, b=-1 | x, y) \\
&= \sum_{ab} ab \, p(ab | xy) \\
&= \mathbb{E}[ab | X=x, Y=y]
\end{aligned}$$

b) Let $K = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$. Show that Alice and Bob's winning probability is

$$\frac{1}{2} + \frac{1}{8} \langle\psi| K |\psi\rangle .$$

Sol^n $\langle\psi|A_x\otimes B_y|\psi\rangle = p(11|xy) + p(-1-1|xy) - p(1,-1|xy) - p(-1,1|xy)$

As $p(1,-1|xy) = 1 - p(11|xy) - p(-1-1|xy) - p(-1,1|xy)$ we get

$$\langle\psi|A_x\otimes B_y|\psi\rangle = 2\big(p(11|xy) + p(-1-1|xy)\big) - 1$$

Performing the same argument with $p(11|xy)$ we also arrive at

$$\langle\psi|A_x\otimes B_y|\psi\rangle = 1 - 2\big(p(1,-1|xy) + p(-1,1|xy)\big)$$

Thus
$$\begin{aligned}\langle\psi|K|\psi\rangle = {} & 2\big(p(11|00) + p(-1-1|00) + p(11|01) + p(-1-1|01) \\ & + p(11|10) + p(-1-1|10) + p(1,-1|11) + p(-1,1|11)\big) \\ & - 4\end{aligned}$$

$$= 2\big(4\cdot \mathbb{P}(\text{Alice and Bob win})\big) - 4$$

$$\Rightarrow \mathbb{P}(\text{Alice and Bob win}) = \frac{1}{2} + \frac{1}{8}\langle\psi|K|\psi\rangle$$

c) Show $K^2 = 4\mathbb{1} - [A_0,A_1]\otimes[B_0,B_1]$ where $[X,Y] = XY - YX$.

Sol^n

$A_{1|x} + A_{-1|x} = \mathbb{1}$

First note that $A_x = A_{1|x} - A_{-1|x} = 2A_{1|x} - \mathbb{1}$

So $A_x^2 = 4A_{1|x} - 4A_{1|x} + \mathbb{1} = \mathbb{1}$.

Similarly $B_y^2 = \mathbb{1}$.

Now $K = A_0\otimes(B_0+B_1) + A_1\otimes(B_0-B_1)$

So
$$\begin{aligned}K^2 = {} & A_0^2\otimes(B_0+B_1)^2 + A_0A_1\otimes(B_0+B_1)(B_0-B_1) \\ & + A_1A_0\otimes(B_0-B_1)(B_0+B_1) + A_1^2\otimes(B_0-B_1)^2\end{aligned}$$

$$\begin{aligned}= {} & \mathbb{1}\otimes(2\mathbb{1} + B_0B_1 + B_1B_0) + A_0A_1\otimes(\mathbb{1} - B_0B_1 + B_1B_0 - \mathbb{1}) \\ & + A_1A_0\otimes(\mathbb{1} - B_1B_0 + B_0B_1 - \mathbb{1}) + \mathbb{1}(2\mathbb{1} - B_0B_1 - B_1B_0)\end{aligned}$$

$$= 4\mathbb{1} + A_0A_1 \otimes (B_1B_0 - B_0B_1) + A_1A_0(-B_1B_0 + B_0B_1)$$
$$= 4\mathbb{1} - A_0A_1 \otimes [B_0, B_1] + A_1A_0 \otimes [B_0, B_1]$$
$$= 4\mathbb{1} - [A_0, A_1] \otimes [B_0, B_1]$$

d) Show $\langle\psi| K |\psi\rangle \leq 2\sqrt{2}$. (Hint: use Cauchy-Schwarz). What is the maximum winning probability for quantum stategies?

Sol^n First use Cauchy-Schwarz $|\langle v|w\rangle| \leq \|v\| \, \|w\|$

So $\langle\psi|K|\psi\rangle \leq \underbrace{\| |\psi\rangle \|}_{=1} \, \| K|\psi\rangle \|$

$$= \sqrt{\langle\psi|K^{+}K|\psi\rangle} = \sqrt{\langle\psi|K^2|\psi\rangle}$$

Now we look to bound $\langle\psi|K^2|\psi\rangle$.

By previous question $\langle\psi|K^2|\psi\rangle \leq 4 - \langle\psi|[A_0, A_1] \otimes [B_0, B_1]|\psi\rangle$

Now we use operator norm $\|X\| = \sup_{|\psi\rangle} \langle\psi|X|\psi\rangle$

$$\langle\psi|K^2|\psi\rangle \leq 4 + \|[A_0, A_1] \otimes [B_0, B_1]\|$$
$$= 4 + \|[A_0, A_1]\| \, \|[B_0, B_1]\|$$

$\|X \otimes Y\| = \|X\| \, \|Y\|$

Now $\|[A_0, A_1]\| = \|A_0A_1 - A_1A_0\| \leq \|A_0A_1\| + \|A_1A_0\|$ Triangle inequality

$\leq \|A_0\| \, \|A_1\| + \|A_1\| \, \|A_0\|$ Submultiplicativity

≤ 2 $\|A_x\| \leq 1$

For Hermitian operators $\|X\| = \sup\{|\lambda| : \lambda \text{ is eigenvalue}\}$
And A_x has eigenvalues $\{+1, -1\}$

Thus $\langle\psi|K^2|\psi\rangle \leq 8$

$\Rightarrow \langle\psi|K|\psi\rangle \leq \sqrt{8} = 2\sqrt{2}$.

Using part (b) this shows that

$$\mathbb{P}(\text{Alice and Bob win}) \leq \frac{1}{2} + \frac{1}{8} 2\sqrt{2} = \frac{1}{2} + \frac{\sqrt{2}}{4} = \cos^2(\pi/8)$$

As the bound is for an arbitrary state and measurements we can conclude that no quantum strategy can win with a higher probability