

ACCQ206 exercises – Week 2

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1. Which states from the following family are entangled?

$$|\psi(\theta)\rangle := \cos(\theta)|00\rangle + \sin(\theta)|11\rangle \quad 0 \leq \theta \leq \pi/4$$

2. Let $\{|\psi_i\rangle\}_i$ be a set of states and let $p(i)$ be any probability distribution. Prove that for $\rho = \sum_i p(i) |\psi_i\rangle\langle\psi_i|$ we have

(a) $\text{Tr}[\rho] = 1$

(b) $\rho \geq 0$

Also show that a density operator ρ is pure iff $\text{Tr}[\rho^2] = 1$.

3. (a) Prove that any single qubit state ρ (density matrix) can be written as

$$\rho = \frac{I + n_x X + n_y Y + n_z Z}{2}$$

with $n_x, n_y, n_z \in \mathbb{R}$ and $n_x^2 + n_y^2 + n_z^2 \leq 1$.

(b) Show that a state is pure iff $n_x^2 + n_y^2 + n_z^2 = 1$.

4. (a) Prove that the trace (defined as $\text{Tr}[X] = \sum_i \langle i|X|i\rangle$ with $\{|i\rangle\}$ being the computational basis) is cyclic, i.e.,

$$\text{Tr}[XY] = \text{Tr}[YX].$$

(b) Use this to prove that the trace is basis independent. I.e., $\text{Tr}[X] = \sum_i \langle v_i|X|v_i\rangle$ for any orthonormal basis $\{|v_i\rangle\}_i$.

5. Let $\{P, I - P\}$ be any qubit projective measurement with $P \neq 0$. Show that

$$\langle\Phi_{ij}|(P \otimes I)|\Phi_{ij}\rangle = \frac{1}{2} \tag{1}$$

for all i, j . Where $|\Phi_{ij}\rangle$ are the four Bell-states.

6. Let ρ_A be a density matrix on some system A . We call a bipartite state $|\psi\rangle_{AB}$ a *purification* of ρ_A if $\text{Tr}_B[|\psi\rangle\langle\psi|_{AB}] = \rho_A$. I.e., there exists some additional system B and a joint state on the joint system AB such that ρ_A is the marginal of that state.

Show that purifications always exist. [Hint: consider the spectral decomposition of ρ .]

7. For this question we introduce a generalisation of a quantum measurement called a POVM. A POVM is a set of operators $\{M_x\}_x$ such that $M_x \geq 0$ (positive semidefinite) and $\sum_x M_x = I$. I.e., a POVM is a measurement but we drop the requirement of projectiveness.

We know that we can't distinguish states perfectly if they are not orthogonal, e.g., $|0\rangle$ and $|+\rangle$. But can we make it so we never make a mistake? Define a three outcome qubit POVM $\{M_0, M_+, M_{fail}\}$ that allows you to distinguish $|0\rangle$ from $|+\rangle$ sometimes but crucially you never make a misidentification. That is

$$\langle 0|M_0|0\rangle > 0, \quad \langle +|M_+|+\rangle > 0$$

and

$$\langle 0 | M_+ | 0 \rangle = 0 \quad \langle + | M_0 | + \rangle = 0.$$

8. For any pure bipartite state $|\psi\rangle_{AB}$ there exists orthonormal bases $\{|v_i\rangle_A\}_i$ for system A and $\{|w_i\rangle_B\}_i$ for system B and $\lambda_i \geq 0$ with $\sum_i \lambda_i = 1$ such that

$$|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |v_i\rangle_A \otimes |w_i\rangle_B. \quad (2)$$

This form is known as the Schmidt decomposition.

Let $|\psi\rangle$ be a state on a bipartite system AB . Show that $|\psi\rangle$ is a product state iff ρ_A and ρ_B are pure states. (Hint: use the Schmidt decomposition – see the bonus material in the notes.)

9. (Tsirelson's bound – Harder)

Alice and Bob play the CHSH game. For convenience we label the inputs as $x, y \in \{0, 1\}$ and the outputs as $a, b \in \{+1, -1\}$. The winning condition then becomes $(-1)^{xy} = ab$. Let $|\psi\rangle$ be the quantum state shared by Alice and Bob. Let Alice's projective measurement on input x be $\{A_{+1|x}, A_{-1|x}\}$ and let Bob's projective measurement on input y be $\{B_{+1|y}, B_{-1|y}\}$. Finally define the observables (expectation operators) $A_x = A_{+1|x} - A_{-1|x}$ and $B_y = B_{+1|y} - B_{-1|y}$.

- (a) Show that for any fixed (x, y) the expected value of the product ab is given by

$$\langle \psi | A_x \otimes B_y | \psi \rangle.$$

- (b) Let $K = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$. Show that the expected winning probability is

$$\frac{1}{2} + \frac{1}{8} \langle \psi | K | \psi \rangle.$$

- (c) Show that

$$K^2 = 4I - [A_0, A_1] \otimes [B_0, B_1]$$

where $[X, Y] = XY - YX$.

- (d) Show that $\langle \psi | K | \psi \rangle \leq 2\sqrt{2}$. What does this say about the maximum winning probability for the CHSH game? (Hint: begin with the Cauchy-Schwarz inequality to bound $\langle \psi | K | \psi \rangle$ in terms of $\langle \psi | K^2 | \psi \rangle$).