01 / 12 / 2026

Overview

Plan

> Course info
> Overview of what we want to prove & topics

Math 614 Course Information

Topic Mirror symmetry for toric varieties

Instructor Peter Haine
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Lecture Time & location 12:00 - 1:20 MW in KAP 265

Office hour W 2pm if that works for everyone

Grading

- 50% typing up lecture notes
  - Due 1 week after the lecture
  - Let's determine a rotating schedule now
  - Peter will type the first two lectures as a model
  - An overleaf has already been shared - let me know if you want GitHub access

> 50% a presentation on a topic related to the course, but not covered
  - In the last couple weeks of class
  - Discuss with me about what's appropriate

Overview

> The results will be about **toric varieties**. We'll discuss these during the first part of the course

- What we need to know now is that an n-dimensional toric variety (over $\mathbb{C}$, say) is an n-dimensional $\mathbb{C}$-scheme with an action of an n-dimensional algebraic torus
$$T = (\mathbb{G}_m)^{\times n}$$

- Toric varieties are really combinatorial in nature: they can be defined over any ring, or even ring spectrum

- Excellent reference: Fulton, "Introduction to toric varieties"

Examples

(1) $(\mathbb{G}_m)^{\times n}$ with its natural self-action

(2) $\mathbb{A}^n$ with its natural $(\mathbb{G}_m)^{\times n}$-action

(3) $\mathbb{P}^n$ with its natural $(\mathbb{G}_m)^{\times n}$-action

(4) $\mathbb{P}^1 \times \mathbb{P}^1$ with the coordinatewise $\mathbb{G}_m \times \mathbb{G}_m$-action

Non-example If $\overbrace{(d,n) \neq (1,n) \text{ or } (n-1,n)}^{\text{Projective space}}$, then the Grassmannian $Gr_{d,n}$ of d-planes in n-space is not a toric variety

> The starting place for our work is the following relationship between toric varieties and functions on $\mathbb{R}^n$:

Thm (Morelli, 1993). Let X be an n-dimensional smooth projective toric variety over $\mathbb{C}$. There is an injective ring homomorphism

$$\underbrace{K_0^T(X)}_{\substack{\text{Grothendieck group of} \\ T\text{-equivariant vector bundles} \\ \text{on } X}} \longrightarrow \left\{ \begin{array}{c} \text{"constructible" functions} \\ \mathbb{R}^n \longrightarrow \mathbb{Z} \end{array} \right\}$$

> Here, the right-hand side is the subgroup of all functions $f: \mathbb{R}^n \longrightarrow \mathbb{Z}$ generated by the indicator functions

$$\mathbb{1}_P(x) = \begin{cases} 1, & x \in P \\ 0, & x \notin P \end{cases}$$

where P is a polyhedron

- The **Minkowski sum** of polyhedra P and Q is

$$P + Q = \{x + y \mid x \in P,\ y \in Q\}$$

> The right-hand side is a ring with
  $+$ = addition of functions $(f+g)(x) = f(x) + g(x)$
  $\cdot$ = **convolution** this is the bilinear extension of the operation

$$\mathbb{1}_P * \mathbb{1}_Q := \mathbb{1}_{P+Q}$$

Moreover, the image of this map can be explicitly identified, and depends on X (not usl $n = \dim(X)$)

Remark For those already familiar with the relationship between ample line bundles and their moment polytopes, Morelli's result is a generalization: the K-theory class of an ample line bundle is sent to the indicator function of its moment polytope.

Categorification

- The group $K_0^T(X)$ arises from the category of T-equiv. vector bundles on X
  - or even better the ∞-category of T-equivariant perfect complexes on X
  - or even better the ∞-category of T-equivariant quasicoherent sheaves on X

- So one would hope that Morelli's theorem can be obtained by applying K_0 to a functor

$$\left\{\begin{array}{l}\text{T-equivariant}\\ \text{qcoh sheaves on } X\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{Some kind}\\ \text{of sheaves on } \mathbb{R}^n\end{array}\right\}.$$

"Implicitly derived" Notation Given a scheme or stack Y, we'll write

$QCoh(Y)$ = derived ∞-category of quasicoherent sheaves on Y (often denoted $D(Y)$ or $D_{qcoh}(Y)$)

$QCoh(Y)^{\heartsuit}$:= abelian category of quasicoherent sheaves on Y

> For a ring A,

Mod_A = derived ∞-category of A-modules (typically denoted $D(A)$)

$Mod_A^{\heartsuit}$:= abelian category of A-modules

Theorem (Fang-Liu-Treumann-Zaslow, ~2010) Let X be a smooth projective n-dimensional toric variety over $\mathbb{C}$. There is a natural, fully faithful, symmetric monoidal left adjoint functor

$$\kappa: \underbrace{QCoh(X/T)}_{\text{(1)}} \hookrightarrow \underbrace{Sh(\mathbb{R}^n, Mod_{\mathbb{C}})}_{\text{(2)}}$$

(1): quasicoherent sheaves on the quotient stack X/T = T-equivariant quasicoherent sheaves, usual $\otimes$

(2): here the tensor product is given by convolution:

$$F * G := add_!(pr_1^*(F) \otimes pr_2^*(G))$$

$add: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ addition

$pr_1, pr_2: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ projections

If $P, Q \subset \mathbb{R}^n$ are polyhedral open subsets then $\mathbb{1}_P * \mathbb{1}_Q \simeq \mathbb{1}_{P+Q}[-n]$, $\mathbb{1}_U = !$-ext of const sheaf on U w/ value $\mathbb{C}$

Moreover, the image of κ can be explicitly identified as those sheaves that are constructible (= locally constant along a stratification) and have microsupport contained in an explicit Lagrangian $\Lambda_X \subset T^*\mathbb{R}^n$ depending on X.

> The functor κ is referred to as the **coherent-constructible correspondence**.

Remark Actually, FLTZ proved the small ∞-categories version of this result, and formulated things in more classical language.

> But there are important advantages to working with large ∞-categories

> Since toric varieties are combinatorial in nature, one would expect the FLTZ theorem to hold over any base ring
- In homotopy theory, there is an even deeper base ring than $\mathbb{Z}$: the sphere spectrum $\mathbb{S}$

Thm (Bai-Hu). The FLTZ theorem is true over any connective ring spectrum.

- Bai and Hu's proof is similar to the FLTZ proof, but there are a number of points where the FLTZ proof is deficient and doesn't just work over the sphere.

Main Goal of the course Explain the proof of this toric mirror symmetry result over the sphere, with as many details as we can.

Topics we'll need to cover

- Toric geometry
- How to work with ∞-categories
- Spectra & spectral algebraic geometry
- Equivariant sheaves
- Constructible sheaves
- The six functor formalism on topological spaces

Non-equivariant version

> Morelli also proved a nonequivariant version, relating $K_0(X)$ to constructible functions on $\mathbb{R}^n/\mathbb{Z}^n$

> FLTZ were unable to categorify this

- Later, Kuwagaki (2016), and the paper is fairly involved.

> The problem is basically entirely that they work with small ∞-categories
  - It follows from compatibility with the Fourier transform

Thm There is an equivalence of ∞-categories

$$\mathrm{QCoh}(\underbrace{*/(\mathbb{G}_m)^{\times n}}_{B((\mathbb{G}_m)^{\times n})}) \simeq \mathrm{Sh}(\overbrace{\mathbb{Z}^n}^{\text{as a set}}, \mathrm{Sp}) \simeq \mathrm{Fun}(\mathbb{Z}^n, \mathrm{Sp}).$$

Moreover, it is symmetric monoidal where the LHS is given the usual $\otimes$ and the RHS is given the convolution product.

Compatibility with the Fourier Transform. If X is a smooth projective n-dim toric variety $/ \mathbb{S}$, then there is a a commutative square

$$\begin{tikzcd}
\mathrm{QCoh}(BT) \arrow[r, "\sim"] \arrow[d, "\pi^*"'] & \mathrm{Sh}(\mathbb{Z}^n; \mathrm{Sp}) \arrow[d, hook, "i_!"] \\
\underbrace{\mathrm{QCoh}(X/T)}_{\text{usual } \otimes} \arrow[r, hook, "\kappa"'] & \underbrace{\mathrm{Sh}(\mathbb{R}^n; \mathrm{Sp})}_{\text{convolution}}
\end{tikzcd}$$

$i : \mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ inclusion

$i_!$ extension by zero

of symmetric monoidal functors

> Bai-Hu observed that the following is then an easy corollary.

Cor. There is a fully faithful symmetric monoidal left adj

$$\bar{\kappa} : \mathrm{QCoh}(X) \hookrightarrow \mathrm{Sh}(\mathbb{R}^n / \mathbb{Z}^n; \mathrm{Sp}).$$

Moreover, the image of $\bar{\kappa}$ can be explicitly identified as those sheaves satisfying a microsupport condition depending on X.

Some Examples

Example (Beilinson's description of $\mathrm{QCoh}(\mathbb{P}^1)$) There is an equivalence

$$\mathrm{QCoh}(\mathbb{P}^1_S) \simeq \mathrm{Fun}\left(0 \underset{x_1}{\overset{x_0}{\rightrightarrows}} 1, \mathrm{Sp}\right)$$

$$\begin{tikzcd}
\mathcal{O} \arrow[d, bend right, "x_0"'] \arrow[d, bend left, "x_1"] & & 0 \arrow[ll, mapsto] \arrow[d, bend right, "x_0"'] \arrow[d, bend left, "x_1"] \\
\mathcal{O}(1) & & 1 \arrow[ll, mapsto]
\end{tikzcd}$$

> More generally, let Q_n be the category generated by the graph

$$\begin{tikzcd}
0 \arrow[r, bend left, "x_0"] \arrow[r, bend right, "x_n"'] & 1 \arrow[r, bend left, "x_0"] \arrow[r, bend right, "x_n"'] & \cdots \arrow[r, bend left, "x_0"] \arrow[r, bend right, "x_n"'] & n
\end{tikzcd}$$

with relations $x_i x_j = x_j x_i$ for all i, j. Then

$$\mathrm{QCoh}(\mathbb{P}^n_S) \simeq \mathrm{Fun}(Q_n, \mathrm{Sp}).$$

> The proof strategy will actually be to prove an affine version of the result and glue these together. Here is a prototypical version

Example (Simpson's description of filtered objects). There is an equivalence of symmetric monoidal ∞-categories

$$\mathrm{QCoh}(\mathbb{A}^1_{\mathbb{S}}/\mathbb{G}_m) \simeq \mathrm{Fun}((\mathbb{Z}, \leq)^{op}, \mathrm{Sp})$$

$$(A * B)_n := \operatorname*{colim}_{n \leq i+j} A_n \otimes B_n$$

> From the Fourier transform, it isn't difficult to see that

$$\mathrm{QCoh}(\mathbb{A}^1_{\mathbb{S}}/\mathbb{G}_m) \simeq \left\{ \begin{matrix} \text{graded} \\ \mathbb{S}[t]\text{-modules} \end{matrix} \right\}$$

> Under this identification, informally, the equivalence sends a filtered spectrum $\mathrm{fil}^* X$ to

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{fil}^i X \cdot t^{-i}.$$