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Toric geometry III

Last time More on cones, fans, general toric schemes

Today Finish up 'basic facts' about convex geometry

Recall from last time

Gordan's Lemma If $\sigma \subset N_{\mathbb{R}}$ is a rational convex polyhedral cone, then the monoid $S_\sigma = \sigma^\vee \cap M$ is finitely generated

$\Rightarrow \mathbb{Z}\langle S_\sigma \rangle$ is of finite type $\Leftrightarrow X_\sigma \longrightarrow \operatorname{Spec}(\mathbb{Z})$ is of finite presentation.

Key Lemma Let V be a fd vector space and $\sigma \subset V$ a convex polyhedral cone

(1) We have $v_0 \notin \sigma$ if and only if there exists $u_0 \in \sigma^\vee$ with

$$\langle u_0, v_0 \rangle < 0$$

(2) We have $(\sigma^\vee)^\vee = \sigma$.

Def. For V a fd $\mathbb{R}$-vector space and $u \in V^\vee$, the **supporting hyperplane** of u is

$$u^\perp := \{ v \in V \mid \langle u, v \rangle = 0 \}$$

A subset $\tau \subset \sigma$ is of a convex poly cone σ is a **face** of σ if there is a $u \in \sigma^\vee$ such that

$$\tau = \sigma \cap u^\perp .$$

> We explained why intersections of faces are faces

More on convex geometry

Lemma. If $\sigma \subset V$ is a convex polyhedral cone, $\tau_1 \subset \sigma$ is a face, and $\tau_2 \subset \tau_1$ is a face of τ_1, then τ_2 is a face of σ.

> We'll leave the proof as an exercise. In fact, you can prove this by showing the following stronger statement:

Exercise In the setting of the Lemma, if $\tau_1 = \sigma \cap u_1^\perp$ and $\tau_2 = \tau_1 \cap u_2^\perp$, then there is a real number $r \gg 0$ such that $u_2 + r u_1 \in \sigma^\vee$ and

$$\tau_2 = \sigma \cap (u_2 + r u_2)$$

Def A proper face τ of σ is a **facet** if τ is of codim 1.

Lemma.

(1) Any proper face is contained in a facet.

(2) Any proper face is the intersection of the faces (or facets) that contain it

Lemma. If a convex polyhedral cone $\sigma \subset V$ spans V, then

$$\underbrace{\partial\sigma}_{\text{boundary in the sense of topology}} = \bigcup_{\substack{\tau \subset \sigma \\ \text{facet}}} \tau = \bigcup_{\substack{\tau < \sigma \\ \text{proper face}}} \tau .$$

Observe. If a conv poly cone $\sigma \subset V$ spans V and $\tau \subset \sigma$ is a facet, then there is a $u \in \sigma^\vee$ unique up to multiplication by a positive scalar such that $\tau = \sigma \cap u^\perp$.

> We'll denote such a functional by u_τ and call it an equation for the hyperplane spanned by τ.

Lemma If a convex polyhedral cone $\sigma \subset V$ spans V and $\sigma \neq V$, then

$$\sigma = \bigcap_{\substack{\tau \subset \sigma \\ \text{facet}}} H_\tau ,$$

where $H_\tau = \{ v \in V \mid \langle u_\tau, v \rangle \geq 0 \}$.

Proof

By the duality theorem,

$$\sigma \subset \bigcap_{\substack{\tau \subset \sigma \\ \text{facet}}} H_\tau$$

Assume, for the sake of contradiction, that $v \in V$ is in the intersection, but not in σ. Let v' be in the interior of σ, and let

$$w = \partial\sigma \cap \left(\begin{array}{c} \text{line segment} \\ \text{between } v \text{ and } v' \end{array} \right)$$

$$= \begin{array}{l} \text{last point in } \sigma \text{ on the line} \\ \text{between } v \text{ and } v' \end{array}$$

Then w is contained in a facet $\tau \subset \sigma$. Hence

$$\langle u_\tau, v' \rangle > 0 \quad \text{and} \quad \langle u_\tau, w \rangle = 0$$

So $\langle u_\tau, v \rangle < 0$, which is a contradiction $\square$

Algorithm for finding generators of $\sigma^\vee$. Let $\sigma = \mathrm{Cone}(v_1, \dots, v_k)$ be of dimension n. Then for each set of $(n-1)$ independent vectors among $v_1, \dots, v_k$, solve for $u \in V^\vee$ anihilating this set

> If neither u nor $-u$ is ≥ 0 on all $v_1, \dots, v_k$, discard it.

> Otherwise, either u or $-u$ is taken to be a generator.

- If the $k-1$ vectors generate a facet, this vector will be the u_τ from earlier.

Farkas' Theorem (stated last time) If $\sigma \subset N_{\mathbb{R}}$ is a convex polyhedral cone, then $\sigma^\vee \subset M_{\mathbb{R}}$ is too. Moreover, if σ is rational, then $\sigma^\vee$ is also rational.

Observation If $\sigma \subset V$ is a convex polyhedral cone and $\tau \subset \sigma$ is a face, then

(*) If $x, y \in \sigma$ and $x+y \in \tau$, then $x, y \in \tau$

Indeed, if $\tau = \sigma \cap u^\perp$ for $u \in \sigma^\vee$, and $x+y \in \tau$, this means that

$$0 = \langle u, x+y \rangle = \langle u, x \rangle + \langle u, y \rangle.$$

Since $u \in \sigma^\vee$, we have $\langle u, x \rangle, \langle u, y \rangle \geq 0$. Hence

$$\langle u, x \rangle = \langle u, y \rangle = 0.$$

That is, we have $x, y \in \sigma \cap u^\perp = \tau$.

Exercise (an intrinsic description of faces) Let $\sigma \subset V$ be a convex polyhedral cone. Prove that a subset $\tau \subset \sigma$ is a face if and only if (*) is satisfied.

Remark. The above is the correct definition of a face for a general convex cone in an $\mathbb{R}$-vector space. Even thought it is more intrinsic, the hyperplane definition is very useful, as we'll soon see

Duality for faces

Notation For a convex polyhedral cone σ, let $\mathrm{Face}(\sigma)$ denote the poset of faces of σ ordered by inclusion.

Proposition Let $\sigma \subset V$ be a rational convex polyhedral cone.

(1) If τ is a face of σ, then $\sigma^\vee \cap \tau^\perp$ is a face of $\sigma^\vee$ and

$$\dim(\tau) + \dim(\sigma^\vee \cap \tau^\perp) = \dim(\sigma)$$

(2) The map of posets

$$\begin{aligned} \mathrm{Face}(\sigma)^{op} &\longrightarrow \mathrm{Face}(\sigma^\vee) \\ \tau &\longmapsto \sigma^\vee \cap \tau^\perp \end{aligned}$$

is an isomorphism

The dense torus embedding

Proposition. Let $\sigma \subset N_{\mathbb{R}}$ be a rational convex polyhedral cone. Then:

(1) For every face $\tau \subset \sigma$, there is a $u \in S_\sigma = \sigma^\vee \cap M$ such that $\tau = \sigma \cap u^\perp$. Moreover, τ is rational.

(2) If $\tau = \sigma \cap u^\perp$ for $u \in S_\sigma$, then

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u).$$

Recollection. Let B be a ring. If $g \in B$ is a nonzerodivisor, then

$$\mathrm{Spec}(B_g) \simeq D(g) \hookrightarrow \mathrm{Spec}(B)$$

is a dense open

Corollary. Let $\sigma \subset N_{\mathbb{R}}$ be a rational convex polyhedral cone and $\tau \subset \sigma$ a face. Then for any ring A, the map

$$A\langle S_\sigma \rangle \longrightarrow A\langle S_\tau \rangle$$

is the localization at a single element $u \in S_\sigma \subset A\langle S_\sigma \rangle$. In particular, the induced morphism of schemes

$$X_{\tau,A} = \operatorname{Spec}(A\langle S_\tau \rangle) \longrightarrow \operatorname{Spec}(A\langle S_\sigma \rangle) = X_{\sigma,A}$$

is a dense open immersion

Proof.
By the previous result, $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$. That is, S_τ is obtained from S_σ by inverting the single element u. It follows immediately from the universal properties of the monoid algebra and localization that $A\langle S_\tau \rangle$ is obtained from $A\langle S_\sigma \rangle$ by inverting $u \in S_\sigma$

To conclude, note that since $S_\sigma \subset M$ is a submonoid and M is a lattice, every element of S_σ is a nonzerodivisor in $A\langle S_\sigma \rangle$. $\square$

Reminder of why we care For any strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, the cone point $0 \subset \sigma$ is a face. Note that $0^\vee = M_{\mathbb{R}}$, so $S_0 = M$.

- The previous result says that

$$(\mathbb{G}_m)^{\times(\operatorname{rank} M)} \simeq \operatorname{Spec}(A\langle M \rangle) \hookrightarrow \operatorname{Spec}(A\langle S_\sigma \rangle)$$

is a dense open immersion.

- So we see that X_σ has a dense embedding of a torus that extends to an action on all of X_σ.

Intersections of cones

Lemma. If $\sigma, \sigma' \subset N_{\mathbb{R}}$ are convex polyhedral cones and $\sigma \cap \sigma'$ is a face of both, then there is a $u \in \sigma^\vee \cap (-\sigma')^\vee$ such that

$$\sigma \cap \sigma' = \sigma \cap u^\perp = \sigma' \cap u^\perp$$

Moreover, if σ and σ' are rational, then u can be chosen to be in $M \cap \sigma^\vee \cap (-\sigma')^\vee$.

Proposition. If $\sigma, \sigma' \subset N_{\mathbb{R}}$ are rational convex polyhedral cones and $\sigma \cap \sigma'$ is a face of both, then

$$S_{\sigma \cap \sigma'} = S_\sigma + S_{\sigma'} \leftarrow \text{submonoid of } M \text{ generated by } S_\sigma \text{ and } S_{\sigma'}$$

Corollary If $\sigma, \sigma' \subset N_{\mathbb{R}}$ are strongly convex rational polyhedral cones such that $\sigma \cap \sigma'$ is a face of both, then the induced morphism $X_{\sigma\cap\sigma'} \longrightarrow X_\sigma \times X_{\sigma'}$ is a closed immersion.

Proof

Equivalently, we need to show that the ring homomorphism

$$\mathbb{Z}\langle S_\sigma \times S_{\sigma'}\rangle \simeq \mathbb{Z}\langle S_\sigma\rangle \otimes_{\mathbb{Z}} \mathbb{Z}\langle S_{\sigma'}\rangle \longrightarrow \mathbb{Z}\langle S_{\sigma\cap\sigma'}\rangle$$

is surjective. By the previous result, $S_{\sigma\cap\sigma'} = S_\sigma + S_{\sigma'}$ and this map is induced by the surjective addition map

$$S_\sigma \times S_{\sigma'} \xrightarrow{+} S_{\sigma\cap\sigma'}.$$

$\square$

Summary

Convention From now on, a **cone** in $N_{\mathbb{R}}$ means a **rational strongly convex polyhedral cone**

If Σ is a fan in N, we write

$$T_N := X_0 = \mathrm{Spec}(\mathbb{Z}\langle M\rangle) \quad \text{or } T \text{ if } N \text{ is clear}$$

We've constructed a diagram

$$X_\bullet : \Sigma \longrightarrow \mathrm{CMon}(\mathrm{Sch})$$

of commutative monoid schemes in which each $X_\tau \longrightarrow X_\sigma$ is a dense open embedding.

> We then define $X_\Sigma := \operatorname*{Colim}_{\sigma \in \Sigma} X_\sigma$.

- Since the underlying space of this colimit is the colimit in Top, this scheme has a dense open embedding of the torus $T = X_0$ and a T-action extending the natural one.

> Each X_σ is an open subscheme of X_Σ, and

$$X_\sigma \cap X_{\sigma'} = X_{\sigma\cap\sigma'}.$$

Lemma Let (N,Σ) be a fan Then $X_\Sigma \longrightarrow \mathrm{Spec}(\mathbb{Z})$ is of finite presentation, flat, and separated (Hence the same is true after every basechange.

Proof

Since flatness and finite presentation are local on the source, it suffices to show that for each $\sigma\in\Sigma$, the affine toric scheme X_σ is flat and of finite presentation. We already saw the finite presentation last time. For flatness, by definition $X_\sigma = \mathrm{Spec}(\mathbb{Z}\langle S_\sigma\rangle)$ So we need to see that the $\mathbb{Z}$-module $\mathbb{Z}\langle S_\sigma\rangle$ is flat This is clear. as a module, $\mathbb{Z}\langle S_\sigma\rangle$ is free

For separatedness, we need to show that the diagonal

$$\Delta : X_\Sigma \longrightarrow X_\Sigma \times X_\Sigma$$

is a closed immersion. Since $X_\Sigma \times X_\Sigma$ admits an open cover by $(X_\sigma \times X_\tau)_{(\sigma,\tau)\in\Sigma\times\Sigma}$ and $\Delta^{-1}(X_\sigma \times X_\tau) = X_\sigma \cap X_\tau = X_{\sigma\cap\tau}$, it suffices to see that the natural morphism

$$X_{\sigma\cap\tau} \longrightarrow X_\sigma \times X_\tau$$

is a closed immersion. We saw this earlier. $\square$