01/21/2026

Toric geometry II

Last time (co)algebras, monoid rings, lattices & cones, affine toric schemes.

Today More on cones, fans, general toric schemes.

Recall If V is a fd $\mathbb{R}$-vector space

(1) A subset $\sigma \subset V$ is a **convex polyhedral cone** if there exist $v_1, \dots, v_k \in V$ such that

$$\sigma = \{ r_1 v_1 + \dots + r_k v_k \mid r_i \geq 0 \}$$

(2) $\sigma \subset V$ is **strongly convex** if σ doesn't contain any 1-dim linear subspaces.

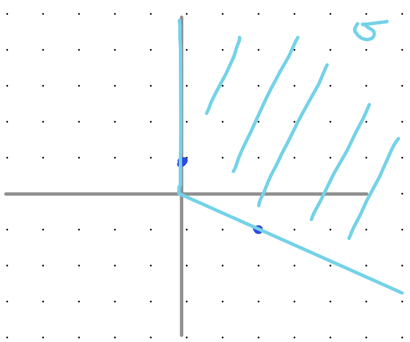
(3) $\sigma^\vee := \{ u \in V^\vee \mid \text{for all } v \in \sigma,\ \langle u, v \rangle \geq 0 \}$

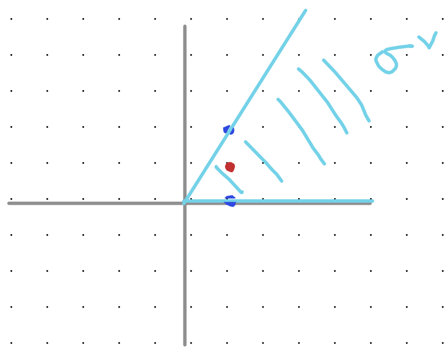
(4) If N is a lattice, a convex polyhedral cone $\sigma \subset N_{\mathbb{R}}$ is **rational** if σ is generated by elements of N.

We're interested in the monoid algebra of $S_\sigma = \sigma^\vee \cap M$.

> Let's start with an interesting example of an affine toric scheme based off an example from last time

Example. Let $N = \mathbb{Z}^2$ and $\sigma = \mathrm{cone}(e_2, 2e_1 - e_2)$



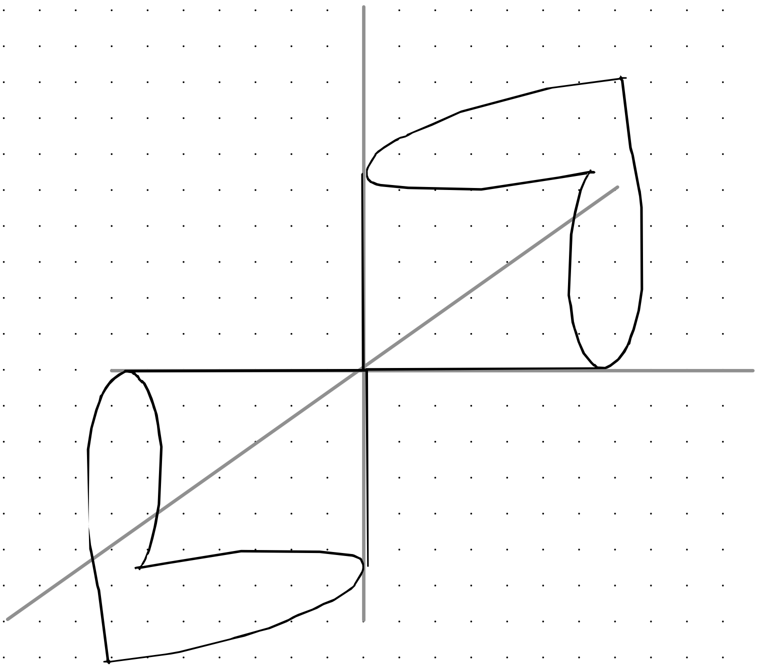


Then $\sigma^\vee = \mathrm{cone}(e_1^*, e_1^* + 2e_2^*)$. However, $S_\sigma = \sigma^\vee \cap M$ has an additional monoid generator $e_1^* + e_2^*$. So

$$\mathbb{Z}\langle S_\sigma \rangle = \mathbb{Z}[x_1, x_1x_2, x_1x_2^2]$$

$$\simeq \mathbb{Z}[x, y, z] / \langle y^2 - xz \rangle .$$

Thus X_σ is a quadratic cone.



Comments on the assumptions

Strong Convexity. If σ is only convex but not strongly convex, X_σ is generally degenerate. For example, if $\sigma = N_\mathbb{R}$, then $\sigma^\vee = 0$, so $S_\sigma = 0$, so $\mathbb{Z}\langle S_0 \rangle = \mathbb{Z}$

> Spec($\mathbb{Z}$) doesn't have a dense torus embedding

> The reason we care about rationality is to guarentee that our schemes are of finite type

Gordan's Lemma. If $\sigma \subset N_\mathbb{R}$ is a rational convex polyhedral cone, then the monoid $S_\sigma = \sigma^\vee \cap M$ is finitely generated.

> For this, we need something we'll explain later

Farkas' Theorem. If $\sigma \subset N_\mathbb{R}$ is a convex polyhedral cone, then so is $\sigma^\vee$. Moreover, if σ is rational, then so is $\sigma^\vee$.

Proof of Gordan assuming Farkas.

Let $u_1, \dots, u_s \in \sigma^\vee \cap M$ be vectors that generate $\sigma^\vee$ as a cone. Consider the simplex

$$K := \{ t_1u_1 + \cdots + t_su_s \mid t_1, \dots, t_s \in [0,1] \}.$$

Since K is compact and M is discrete, $K \cap M$ is finite. We claim that $K \cap M$ generates $\sigma^\vee \cap M$ as a monoid. If $u \in \sigma^\vee \cap M$, we can write

$$u = r_1u_1 + \cdots + r_su_s \quad \text{with } r_1, \dots, r_s \geq 0.$$

Write each r_i uniquely as $r_i = m_i + t_i$ where $m_i \in \mathbb{N}$ and $t_i \in [0,1)$. Then

$$\underset{\in M}{u} = \underbrace{(m_1u_1 + \cdots + m_su_s)}_{\substack{\text{each } u_i \text{ is in } M \cap K \\ + \text{ sum is in } M}} \Longrightarrow \; + \underbrace{(t_1u_1 + \cdots + t_su_s)}_{\in M \cap K}$$

$\square$

Cor If $\sigma \subset N_{\mathbb{R}}$ is a rational convex polyhedral cone, then $\mathbb{Z}\langle S_\sigma \rangle$ is of finite type. Hence X_σ is a finite type $\mathbb{Z}$-scheme.

Preview of construction of general toric varieties

> We'll need to precisely formulate a few definitions and a number of results

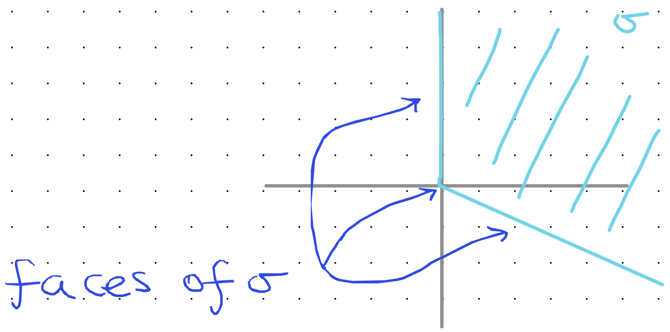
Def Let N be a lattice. A **fan** in N is a set of rational strongly convex polyhedral cones in $N_{\mathbb{R}}$ such that

(1) If $\sigma \in \Sigma$, then each **face** of σ is also in Σ

intuitive, but we need a def.

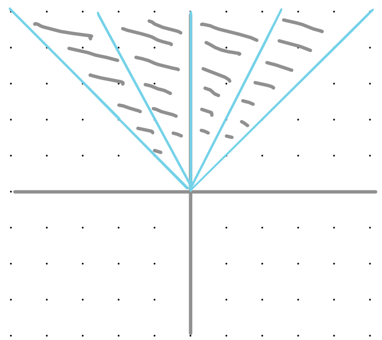
(2) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of both σ and τ.

We regard Σ as a poset ordered by inclusion.

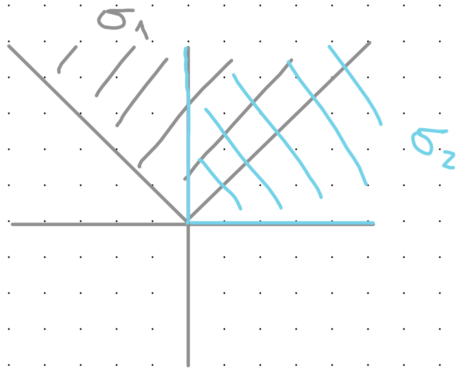


The faces of σ are the two rays and the origin (a 0-dim face).

Ex A fan in $\mathbb{R}^2$ demonstrating the reason for the name



Remark Condition (2) rules out pictures like



Note The assignment $\sigma \mapsto S_\sigma$ defines a functor

$$\Sigma^{op} \longrightarrow \mathrm{CMon}(\mathrm{Set})$$

Composing with the monoid algebra functor and Spec, we obtain a functor

$$\begin{aligned} X_\bullet : \Sigma &\longrightarrow \mathrm{CMon}(\mathrm{Sch}) \\ \sigma &\longmapsto X_\sigma = \mathrm{Spec}(\mathbb{Z}\langle S_\sigma \rangle). \end{aligned}$$

> Note that $0 \in \Sigma$ and $S_\sigma = M$ is a lattice, so X_0 is a torus

> As with the $\mathbb{P}^1$ example from last time, we want to take the colimit of this diagram

$$X_\Sigma = \operatorname*{colim}_{\sigma \in \Sigma} X_\sigma \quad \text{toric scheme associated to } \Sigma$$

- In order to do this, we want each $X_\tau \to X_\sigma$ to be an open immersion (so the colimit exists).

- In fact, we want each $X_\tau \to X_\sigma$ to be a **dense** open immersion, so that there is a dense embedding

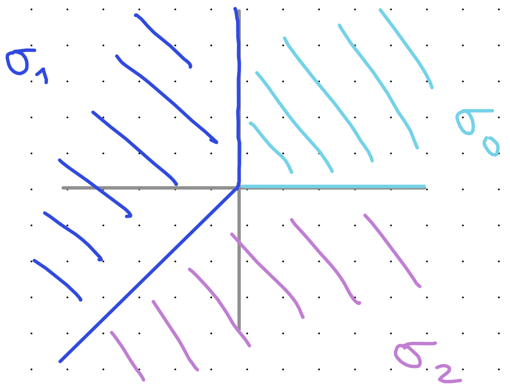
$$\text{torus } X_0 \longrightarrow X_\sigma$$

of monoid schemes. I.e., a dense embedding of the torus X_0 so that the natural X_0-action on itself extends to an action on X_σ. This will guarentee that X_Σ has the same structure.

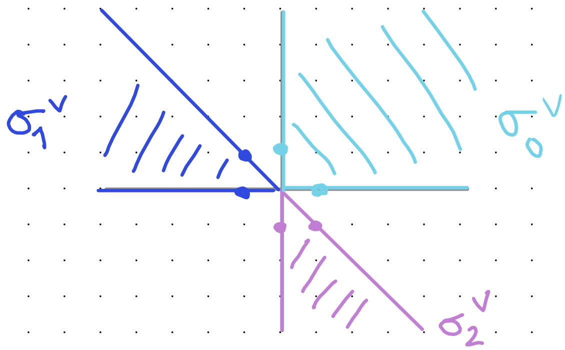
> This will all follow from some basic facts about cones/convex geometry.

Constructing $\mathbb{P}^2$

Example Let $N = \mathbb{Z}^2$ and consider the fan containing the cones



and all of their faces. The dual cones are



Dots denote generators of the monoid S_σ

> So as subrings of $\mathbb{Z}\langle M\rangle = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$, we have

$$\mathbb{Z}\langle S_{\sigma_0}\rangle \simeq \mathbb{Z}[x,y] \qquad \mathbb{Z}\langle S_{\sigma_1}\rangle \simeq \mathbb{Z}[x^{-1}, x^{-1}y]$$

$$\mathbb{Z}\langle S_{\sigma_2}\rangle \simeq \mathbb{Z}[xy^{-1}, y^{-1}].$$

> The Spectra of these rings give the standard cover of $\mathbb{P}^2$ by 3 copies of $\mathbb{A}^2$. Unwinding the definitions, one can check that

$$X_\Sigma \simeq \mathbb{P}^2.$$

More Generally Consider the fan Σ in $\mathbb{Z}^n$ whose cones are the cones on all subsets of cardinality $\leq n$ of the $n+1$ vectors

$$e_1, e_2, \dots, e_n, \quad -(e_1 + \cdots + e_n)$$

Then $X_\Sigma \simeq \mathbb{P}^n$.

Basic facts about cones

- We'll need a key result from convex geometry that we won't prove

Key Lemma. Let V be a fd vector space and $\sigma \subset V$ a convex polyhedral cone.
(1) We have $v_0 \notin \sigma$ if and only if there exists $u_0 \in \sigma^\vee$ with

$$\langle u_0, v_0 \rangle < 0$$

(2) We have $(\sigma^\vee)^\vee = \sigma$.

Def. For $\sigma \subset V$ a convex polyhedral cone and $u \in \sigma^\vee$, the **supporting hyperplane** of u is

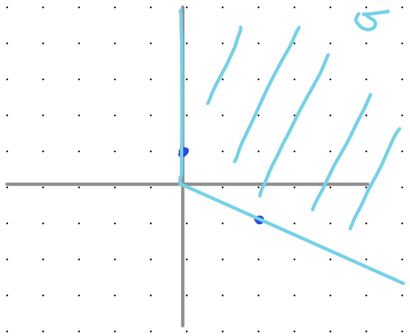
$$u^\perp := \{ v \in V \mid \langle u, v \rangle = 0 \}$$

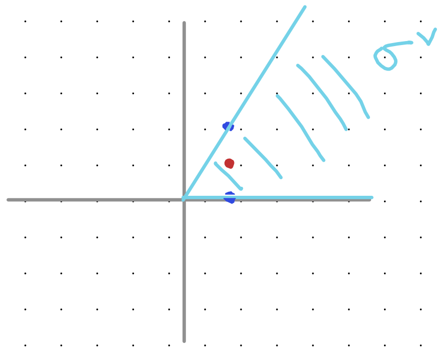
A subset $\tau \subset \sigma$ is a **face** of σ if there is a $u \in \sigma^\vee$ such that

$$\tau = \sigma \cap u^\perp.$$

> So a cone is a face of itself: $\sigma^\perp = V$, so $\sigma = \sigma \cap \sigma^\perp$.
- Otherwise we call these **proper faces**

Example To visualize this, let $N = \mathbb{Z}^2$ and $\sigma = \text{cone}(e_2, 2e_1 - e_2)$

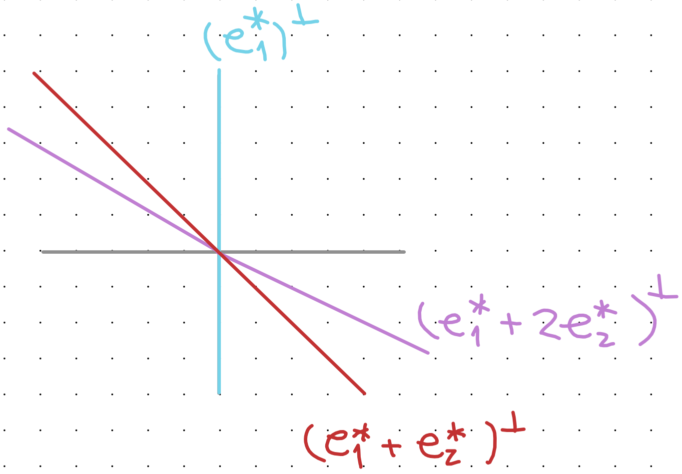




Then

$$(e_1^*)^\perp = \mathbb{R}e_2 \qquad (e_1^* + 2e_2^*)^\perp = \{(x,y) \mid x + 2y = 0\}$$

$$(e_1^* + e_2^*) = \{(x,y) \mid x + y = 0\}$$



Observe If $\sigma = \mathrm{cone}(v_1, \ldots, v_k)$ and $u \in \sigma^\vee$, then

$\sigma \cap u^\perp$ is generated by those v_i such that $\langle u, v_i \rangle = 0$

Lemma.

(1) If $\tau \subset \sigma$ is a face, then τ is also a convex polyhedral cone

(2) A convex polyhedral cone σ has finitely many faces

Lemma. If $\sigma \subset V$ is a convex polyhedral cone and $\tau_1, \tau_2 < \sigma$ are faces, then $\tau_1 \cap \tau_2$ is a face.

Proof.

Note that
$$(\sigma \cap u_1^\perp) \cap (\sigma \cap u_2^\perp) = \sigma \cap (u_1 + u_2)^\perp$$
$\square$

Lemma. If $\sigma \subset V$ is a convex polyhedral cone, $\tau_1 \subset \sigma$ is a face, and $\tau_2 < \tau_1$ is a face of τ_1, then τ_2 is a face of σ.