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Toric geometry I

Today. Begin toric geometry.

> We'll start by understanding some group schemes associated to groups. In fact, it will be useful to work more generally with monoids.

> This can be phrased in a more hands-on way, but we'll aim to phrase things in a way so that we can verbatim do the same thing in the homotopical setting.

Monoid rings & diagonalizable monoid schemes

Recall. If $\mathcal{C}$ is a category with finite products, a **monoid object** in $\mathcal{C}$ is an object $M \in \mathcal{C}$ equipped with a multiplication $m: M \times M \to M$ and a unit map $u: 1_\mathcal{C} \to M$ satisfying the usual axioms, which can be expressed in terms of commutative diagrams.

$\swarrow$ a property "better"

> M is **a group** if the shear maps

$$M \times M \xrightarrow{(pr_1, m)} M \times M \qquad M \times M \xrightarrow{(m, pr_2)} M \times M$$

$$\text{"} (x,y) \longmapsto (x, xy) \text{"} \qquad \text{"} (x,y) \longmapsto (xy, y) \text{"}$$

are isomorphisms.

- Equivalently, there is a map $i: M \to M$ that is both a left and right sided inverse to multiplication. It is then automatic that i is an isomorphism and unique.

$\curvearrowright$ an additional structure "worse"

More generally. If $(e, \otimes, 1)$ is a symmetric monoidal category, then we have the notion of a (commutative) algebra in e, which is an object $X \in e$ together with maps

$$m: X \otimes X \longrightarrow X \qquad \text{and} \qquad u: 1 \longrightarrow X$$

making the obvious associativity, unitality (and commutativity) diagrams commute.

> Dually, we can talk about (commutative) coalgebras in e, which are just commutative algebras in $(e^{op}, \otimes, 1)$

> A **commutative bialgebra** in e is an object together with both the structures of a commutative algebra and a commutative coalgebra.

> We write $CAlg(e)$, $cCAlg(e)$, $bCAlg(e)$ for the categories of commutative algebras, coalgebras, and bialgebras in e

Ex.

(1) If $\mathcal{C}$ has finite products, $\mathrm{CAlg}(\mathcal{C}, \times) = \mathrm{CMon}(\mathcal{C})$

(2) For a ring A, $\mathrm{CAlg}(\mathrm{Mod}_A^{\heartsuit}, \otimes_A) = \left\{ \begin{array}{c} \text{commutative} \\ A\text{-algebras} \end{array} \right\}$ in the usual sense

$$= \left\{ \begin{array}{l} \text{commutative rings} \\ B \text{ with a hom } A \to B \end{array} \right\}$$

Ntn. For a ring A, let's write

$$A\langle - \rangle \quad \mathrm{Set} \longrightarrow \mathrm{Mod}_A^{\heartsuit}$$

for the free A-module functor
Observe $A\langle - \rangle$ is symmetric monoidal, in particular

(1) $A\langle \mathrm{pt} \rangle \cong A$

(2) $A\langle S \times T \rangle \cong A\langle S \rangle \underset{A}{\otimes} A\langle T \rangle$

(3) $A\langle - \rangle$ preserves (commutative) algebras, coalgebras

Explicitly. If M is a monoid, then $A\langle M \rangle$ is naturally an associative A-algebra with multiplication

$$A\langle M \rangle \underset{A}{\otimes} \langle M \rangle \cong A\langle M \times M \rangle \xrightarrow{A\langle m \rangle} A\langle M \rangle$$

and unit

$$\eta : A \cong A\langle pt \rangle \xrightarrow{A\langle u \rangle} A\langle M \rangle$$
$$1 \longmapsto \text{unit for } A\langle M \rangle$$

> If M is commutative, this A-algebra is commutative.

> If G is a group, then the inverse $\iota : G \xrightarrow{\sim} G$ provides an additional operation $A\langle \iota \rangle : A\langle G \rangle \xrightarrow{\sim} A\langle G \rangle$

Upshot $A\langle - \rangle$ extends to a functor

$$A\langle - \rangle : \mathsf{CMon} \longrightarrow \mathsf{CAlg}_A$$

> In fact, this functor is left adjoint to the forgetful functor
$$\mathsf{CAlg}_A \longrightarrow \mathsf{CMon}$$
$$B \longmapsto (B, \cdot, 1)$$

Note. This doesn't give $\text{Spec}(A\langle M\rangle)$ a multiplication!

> For that, we need maps

$$\text{Spec}(A\langle M\rangle) \times_A \text{Spec}(A\langle M\rangle) \longrightarrow \text{Spec}(A\langle M\rangle)$$

$$A\langle M\rangle \otimes_A A\langle M\rangle \longleftarrow A\langle M\rangle$$

$$\text{Spec}(A) \longrightarrow \text{Spec}(A\langle M\rangle)$$

$$A \longleftarrow A\langle M\rangle$$

Satisfying the usual commutativity, associativity, and unitality axioms

> That is, we need to give $A\langle M\rangle$ the additional structure of a commutative coalgebra

> This actually comes for free

Coalgebra Structure. Let $\mathcal{C}$ be a category with finite products and terminal object 1. Then every object $X \in \mathcal{C}$ is naturally a commutative coalgebra with

(1) Comultiplication the diagonal $\Delta : X \longrightarrow X \times X$

(2) Counit the unique map $X \longrightarrow 1$

In fact, the forgetful functor

$$c CAlg (\mathcal{C}, x) \longrightarrow \mathcal{C}$$

is an equivalence of categories with inverse

$$(X, X \xrightarrow{\Delta} X \times X, X \to 1) \longleftarrow X$$

Consequence. The functor $A \langle - \rangle : CMon \longrightarrow CAlg_A$ actually refines to a functor

$$A \langle - \rangle : CMon \longrightarrow bCAlg_A$$

> Explicitly, the maps

$$A\langle M\rangle \xrightarrow{\;A\langle\Delta\rangle\;} A\langle M\times M\rangle \simeq A\langle M\rangle \underset{A}{\otimes} A\langle M\rangle$$

$$m \longmapsto m\otimes m \quad A\text{-linear extension}$$

and

$$\varepsilon: A\langle M\rangle \longrightarrow A\langle pt\rangle \simeq A$$

$$m \longmapsto 1 \quad A\text{-linear extension}$$

give $A\langle M\rangle$ a commutative coalgebra structure

> From our discussion so far, we see

Lem. If M is a commutative monoid, then $\mathrm{Spec}(A\langle M\rangle)$ is a commutative monoid in A-schemes. If M is a group, then $\mathrm{Spec}(A\langle M\rangle)$ is a commutative A-group scheme. Here, the structures are

(1) multiplication $\mathrm{Spec}(\Delta)$

(2) unit $\mathrm{Spec}(\varepsilon)$

(3) [for groups] inverse: $\mathrm{Spec}(i)$

Def. We call $\text{Spec}(A\langle M\rangle)$ the **diagonalizable** A-monoid (or A-group if M is a group) scheme associated to M.

> Often $\text{Spec}(A\langle M\rangle)$ is denoted $D_A(M)$.

Remark. Often the group ring $A\langle G\rangle$ is denoted $A[G]$, but this conflicts with the typical notation for polynomial rings.

$\mathbb{A}^1$ and $\mathbb{G}_m$

> Let's unpack this abstraction for the simplest monoids.

Observe. For the monoid $(\mathbb{N}, +)$, we have

$$A\langle \mathbb{N}\rangle \cong \bigoplus_{d \geq 0} At^d$$

$\underbrace{\qquad}$ d-th weight space
for the operator t

> The ring structure comes from the addition on $\mathbb{N}$ and extends

$$at^d \cdot bt^e = abt^{d+e}$$

- So $A\langle \mathbb{N}\rangle \cong A[t]$ and $\mathrm{Spec}(A\langle \mathbb{N}\rangle) \cong \mathbb{A}^1_A$

> The multiplication comes from the diagonal:

$$A[t] \xrightarrow{\hspace{4cm}} A[t_1] \otimes_A A[t_2] \cong A[t_1, t_2]$$

$$t \longmapsto t_1 t_2$$

Observe. For the group $G = \mathbb{Z}$, we have
$$A\langle\mathbb{Z}\rangle \cong A[t^{\pm 1}]$$

> The inverse is $A[t^{\pm 1}] \xrightarrow{\sim} A[t^{\pm 1}]$
$$t \longmapsto t^{-1}$$

Def. The multiplicative group is
$$\mathbb{G}_m = \mathrm{Spec}(A\langle\mathbb{Z}\rangle) \cong \mathbb{A}^1 \smallsetminus 0$$

> For a scheme S, we write
$$\mathbb{G}_{m,S} = \mathbb{G}_m \times S$$

- $\mathbb{G}_{m,S}$ is a group object in Sch_S

Remark. A more classical variant of the Fourier transform stated last time is that there is an equivalence of categories

$$\begin{pmatrix} \text{affine schemes} \\ \text{with a } \mathbb{G}_m\text{-action} \end{pmatrix} \cong \begin{pmatrix} \mathbb{Z}\text{-graded} \\ \text{rings} \end{pmatrix}^{op}.$$

> A $\mathbb{G}_m$-action is **contracting** if it extends to an action of $\mathbb{A}^1_\mathbb{Z} = \mathrm{Spec}\,(\mathbb{Z}\langle\mathbb{N}\rangle)$

$$\begin{array}{ccc} \mathbb{G}_m\text{-action} & \Longleftrightarrow & \text{nonnegatively} \\ \text{is contracting} & & \text{graded} \end{array}$$

> For nonnegatively graded rings S_* with corresponding affine scheme X with a $\mathbb{G}_m$-action,

$$\mathrm{Proj}(S_*) \cong \left(X \smallsetminus X^{\mathbb{G}_m}\right)/\mathbb{G}_m$$
$$= \left(\mathrm{Spec}(S_*) \smallsetminus V(S_+)\right)/\mathbb{G}_m$$

$$S_+ = \bigoplus_{d > 0} S_d$$

irrelevant ideal

Lattices

Recall. A lattice is a finitely generated free abelian group

> Write Latt $\subset$ Ab for the full subcategory spanned by the lattices
> Duality. The functor

$$(-)^{\vee} : \text{Latt} \longrightarrow \text{Latt}^{op}$$

$$N \longmapsto \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$$

is an equivalence of categories. It is its own inverse.

Example. If N is a lattice of rank d, then a choice of basis for N provides an isomorphism $N \simeq \mathbb{Z}^d$, hence an isomorphism

$$\text{Spec}(\mathbb{Z}\langle N \rangle) \simeq (\mathbb{G}_m)^{\times d}$$

> Hence $\text{Spec}(\mathbb{Z}\langle N \rangle)$ is an algebraic torus of rank d.

Goal. Given a lattice N and some combinatorial data in the vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$, build an interesting scheme.

Standard Notation.
- N is a lattice
- $M := N^{\vee}$ is the dual lattice
- $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. We'll also write $V = N_{\mathbb{R}}$ and $V^{\vee} = M_{\mathbb{R}}$.

- $\langle -, - \rangle : M \otimes_{\mathbb{Z}} N \longrightarrow \mathbb{Z}$ the evaluation pairing induced by

$$M \times N \longrightarrow \mathbb{Z}$$
$$(m, n) \longmapsto m(n)$$

Def. Let $v_1, \ldots, v_n \in N_{\mathbb{R}}$. The **convex polyhedral cone** generated by $v_1, \ldots, v_n$ is
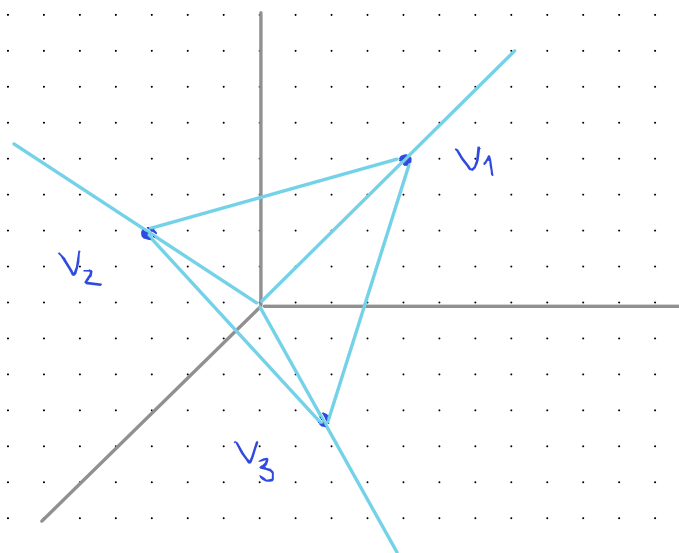
$$\mathrm{cone}(v_1, \ldots, v_n) := \{ r_1 v_1 + \cdots + r_n v_n \mid r_1, \ldots, r_n \geq 0 \}.$$

We call $v_1, \ldots, v_n$ generators. A subset $\sigma \subset N_{\mathbb{R}}$ is a convex

polyhedral cone if $\sigma = \text{Cone}(v_1, ..., v_n)$ for vectors $v_1, ..., v_n \in N_{\mathbb{R}}$

> We define $\dim(\sigma) = \dim_{\mathbb{R}}(\mathbb{R}\sigma)$

$\uparrow$ vector space spanned by σ.

Picture.

$v_i = 0$ in x_i

$\quad 1$ else

V_1

V_2

V_3

Cone(v_1, v_2, v_3)

is a tetrahedral cone.

Def If $\sigma \subset N_\mathbb{R}$ is a convex polyhedral cone

> The **dual** of σ is

$$\sigma^\vee = \{u \in M_\mathbb{R} \mid \text{for all } v \in \sigma, \ \langle u, v \rangle \geq 0\}$$

  - Note that $\sigma^\vee$ is a submonoid of $M_\mathbb{R}$
> σ is **rational** if there are $v_1, \ldots, v_n \in N$ such that
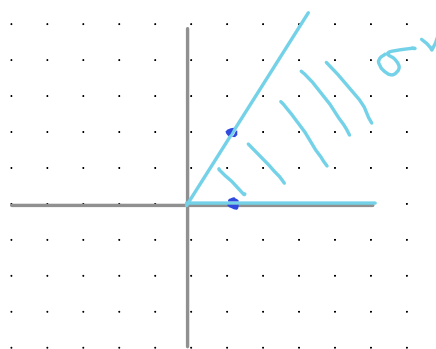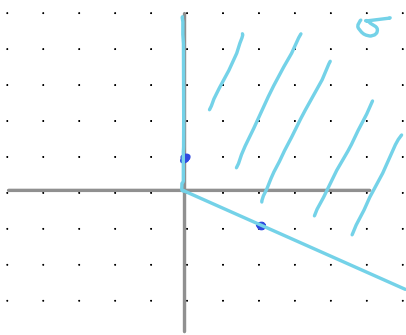 $\sigma = \text{Cone}(v_1, \ldots, v_n)$

> σ is **strongly convex** if the only linear subspace σ contains
  is 0.

> We write $S_\sigma := \sigma^\vee \cap M$
  - Note that S_σ is a submonoid of M.

Note We'll soon explain why we'll almost exclusively be interested
in rational strongly convex polyhedral cones.

Picture. Let's consider $\sigma = \text{cone}(e_2, 2e_1 - e_2)$

Def. If $\sigma \subset N_{\mathbb{R}}$ is a rational convex polyhedral cone, the affine toric scheme associated to σ is

$$X_\sigma := \text{Spec}(\mathbb{Z}\langle S_\sigma \rangle)$$

$$\underbrace{}_{\sigma^\vee \cap M}$$

Observe. The inclusion of monoids $S_\sigma \hookrightarrow M$ induces a map of monoid schemes

$$T := \text{Spec}(\mathbb{Z}\langle M \rangle) \longrightarrow X_\sigma.$$

In particular, X_σ has a natural action of the torus T.

How to get $\mathbb{P}^1$. Consider $N = \mathbb{Z}$, so $N_\mathbb{R} = \mathbb{R}$ and the cones
$\sigma_- = \mathbb{R}_{\leq 0}$, $\sigma_0 = \{0\}$, $\sigma_+ = \mathbb{R}_{\geq 0}$

> Then $(\mathbb{R}_{\leq 0})^\vee = \mathbb{R}_{\leq 0}$, $\{0\}^\vee = \mathbb{R}$, $(\mathbb{R}_{\geq 0})^\vee = \mathbb{R}_{\geq 0}$

> $M \cap \mathbb{R}_{\leq 0} = \mathbb{Z}_{\leq 0}$, $M \cap \mathbb{R} = \mathbb{Z}$, $M \cap \mathbb{R}_{\geq 0} = \mathbb{Z}_{\geq 0}$

$$\mathbb{Z}_{\leq 0} \longrightarrow \mathbb{Z} \longleftarrow \mathbb{Z}_{\geq 0}$$

- Taking group algebras we get inclusions

$$\mathbb{Z}[t^{-1}] \longrightarrow \mathbb{Z}[t^{\pm 1}] \longleftarrow \mathbb{Z}[t]$$

> On spectra, we get a diagram of open immersions

$$\mathbb{A}^1 = \mathbb{P}^1 \setminus 0 \qquad\qquad \mathbb{G}_m \qquad\qquad \mathbb{P}^1 \setminus \infty = \mathbb{A}^1$$
$$\| \qquad\qquad\qquad \| \qquad\qquad\qquad \|$$
$$\mathrm{Spec}(\mathbb{Z}[t^{-1}]) \longleftarrow \mathrm{Spec}(\mathbb{Z}[t^{\pm 1}]) \longrightarrow \mathrm{Spec}(\mathbb{Z}[t])$$

- The pushout is just $\mathbb{P}^1$.

Next Time. We'll generalize this to construct a toric scheme from a collection of cones in $N_\mathbb{R}$ with some closure properties.