01/28/2026

Toric geometry IV:

Functoriality & Smoothness

Last time Finished 'basic facts' about convex geometry.

Today Functoriality, Smoothness

Thm If (N, Σ) is a fan, then

(1) $X_\Sigma \to \mathrm{Spec}(\mathbb{Z})$ is flat, separated, and of finite presentation. Moreover, X_Σ is of relative dimension $\mathrm{rank}(N)$ over $\mathrm{Spec}(\mathbb{Z})$. (Hence the same is true after every basechange.)

(2) For every ring A and cone $\sigma \in \Sigma$, the open embedding

$$X_{\sigma, A} \hookrightarrow X_{\Sigma, A}$$

is dense. In particular, there is a dense embedding of the torus $T_{N,A}$.

(3) There is a natural $T_{N,A}$-action on $X_{\Sigma,A}$ making each embedding $X_{\sigma,A} \hookrightarrow X_{\Sigma,A}$ equivariant for the torus action.

Functoriality

Def. Let Fan denote the category with

(0) Objects: pairs (N, Σ) of a lattice N and a fan Σ in N.

(1) A morphism $(N, \Sigma) \longrightarrow (N', \Sigma')$ is a morphism of lattices $\phi: N \longrightarrow N'$ such that if $\sigma \in \Sigma$, then $\phi_{\mathbb{R}}(\sigma)$ is contained in a cone $\sigma' \in \Sigma'$

Note In the situation of (1), the assignment

$$\begin{aligned} \tilde{\phi}: \Sigma &\longrightarrow \Sigma' \\ \sigma &\longmapsto \min\{\sigma' \in \Sigma' \mid \sigma' \supset \phi_{\mathbb{R}}(\sigma)\} \end{aligned}$$

is a map of posets

Def Let $\mathrm{Sch}^{\mathbb{T}}$ be the category with:

(0) Objects: triples $(T, X, a: T \times X \longrightarrow X)$ of an algebraic torus T, a scheme X, and a T-action $a: T \times X \longrightarrow X$

(1) A morphism from (T, X, a) to (T', X', a') is a

pair of a morphism of group schemes $\phi: T \to T'$ and a morphism $f: X \to X'$ making the square

$$\begin{tikzcd} T \times X \arrow[r, "a"] \arrow[d, "\phi \times f"'] & X \arrow[d, "f"] \\ T' \times X' \arrow[r, "a'"'] & X' \end{tikzcd}$$

commute

Observation The assignment $(N, \Sigma) \mapsto X_\Sigma$ defines a functor

$$\mathrm{Fan} \longrightarrow \mathrm{Sch}^{\mathbb{I}}$$

If $\phi: (N, \Sigma) \to (N, \Sigma')$ is a morphism of fans, the induced morphism

$$X_\Sigma = \operatorname*{colim}_{\sigma \in \Sigma} X_\sigma \longrightarrow \operatorname*{colim}_{\sigma' \in \Sigma'} X_{\sigma'} = X_{\Sigma'}$$

is induced by the map of posets $\tilde{\phi}: \Sigma \to \Sigma'$ and the universal property of the colimit

Exercise If (N, Σ) and (N', Σ') are fans, let $\Sigma \times \Sigma'$ be the fan in $N \times N'$ consisting of products $\sigma \times \sigma'$ for $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$. Then

$$X_{\Sigma \times \Sigma'} \cong X_\Sigma \times X_{\Sigma'}$$

The functor of points of X_σ

Recollection. Let X be a scheme. The **functor of points** of X is the presheaf

$$\begin{aligned} X(-) : \mathrm{Sch}^{op} &\longrightarrow \mathrm{Set} \\ T &\longmapsto \mathrm{Hom}_{\mathrm{Sch}}(T, X) = X(T) \end{aligned}$$

'T-points of X'

represented by X. The functor of points is determined by its restriction to $\mathrm{Aff}^{op} \simeq \mathrm{CAlg}$, so we often refer to

$$\begin{aligned} X(-) : \mathrm{CAlg} &\longrightarrow \mathrm{Set} \\ A &\longmapsto X(A) := \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(A), X) \end{aligned}$$

as the functor of points. If $X = \mathrm{Spec}(B)$ is affine, then

$$X(A) = \mathrm{Hom}_{\mathrm{CAlg}}(B, A).$$

Ex. The functor of points of $\mathbb{P}^n_{\mathbb{Z}}$ is

$$\mathbb{P}^n_{\mathbb{Z}}(T) = \left\{ \begin{array}{l} \mathcal{L} \text{ line bundle on } T \\ + (s_0,\dots,s_n) : \mathcal{O}_T^{\oplus(n+1)} \twoheadrightarrow \mathcal{L} \\ \text{surjection on stalks} \end{array} \right\} \Big/ \cong$$

Recall $\mathbb{Z}\langle - \rangle : \mathrm{CMon} \longrightarrow \mathrm{CAlg}$ is left adjoint to the forgetful functor $A \longmapsto (A, \cdot)$.

Ex. If S is a commutative monoid and $X = \mathrm{Spec}(\mathbb{Z}\langle S \rangle)$, then for any ring A,

$$\begin{aligned} X(A) &= \mathrm{Hom}_{\mathrm{CAlg}}(\mathbb{Z}\langle S \rangle, A) \\ &= \mathrm{Hom}_{\mathrm{CMon}}(S, (A, \cdot)) \end{aligned}$$

Ex Let N be a lattice and $\sigma \subset N_{\mathbb{R}}$ a cone. Then X_σ has a natural $\mathbb{Z}$-point $x_\sigma : \operatorname{Spec}(\mathbb{Z}) \longrightarrow X_\sigma$ corresponding to the homomorphism of commutative monoids

$$\mathbb{1}_{\sigma^\perp} : S_\sigma = \sigma^\vee \cap M \longrightarrow (\mathbb{Z}, \cdot)$$

given by

$$u \longmapsto \begin{cases} 1, & u \in \sigma^\perp \\ 0, & u \notin \sigma^\perp \end{cases}$$

To see that this is a monoid hom, note that $\sigma^\perp$ is a face of the cone $\sigma^\vee$. Hence if $u+v \in \sigma^\perp$, then $u \in \sigma^\perp$ (as explained last time). So either all of $u, v,$ and $u+v$ are in $\sigma^\perp$, or none are. Hence

$$\mathbb{1}_{\sigma^\perp}(u+v) = \mathbb{1}_{\sigma^\perp}(u)\, \mathbb{1}_{\sigma^\perp}(v)$$

holds.

Smoothness

Recall. The property of a morphism of schemes $f: X \to Y$ being **smooth** is local on the source (as well as the target). There are many equivalent definitions. One is that $f: X \to Y$ is smooth if and only if

(1) f is flat and of finite presentation

(2) For every geometric point $\bar{y} \to Y$, the geometric fiber $X_{\bar{y}}$ is regular

Recall that a noetherian local ring (A, m) is <u>regular</u> if

$$\dim(A_m) = \dim_{\kappa(m)} T_m^\vee A,$$

i.e., the zariski cotangent space $T_m^\vee A = m/m^2$ has the expected dimension. A scheme X is <u>regular</u> if X is locally noetherian and the local rings $\mathcal{O}_{X,x}$ at closed points are regular.

In particular A toric scheme X_Σ is smooth over $\mathrm{Spec}(\mathbb{Z})$ if and only if all X_σ for $\sigma \in \Sigma$ are smooth over $\mathrm{Spec}(\mathbb{Z})$

Proposition Let N be a lattice of rank n and $\sigma \subset N_{\mathbb{R}}$ a cone. Then the following are equivalent:

(1) The scheme X_σ is smooth over $\mathrm{Spec}(\mathbb{Z})$ (hence the same is true after every basechange)

(2) There are generators for σ (as a cone) that can be extended to a basis for the lattice N

(3) There exists an isomorphism

$$X_\sigma \cong \mathbb{A}^d_{\mathbb{Z}} \times (\mathbb{G}_m)^{\times(n-d)}$$

Here, $d = \dim(\sigma)$

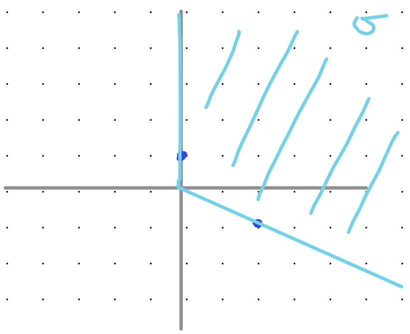
Def. Let N be a lattice

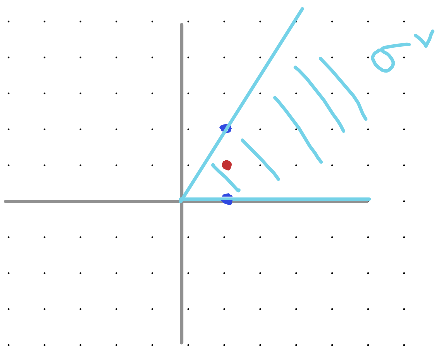
(1) A cone $\sigma \subset N_{\mathbb{R}}$ is **smooth** if there are generators for σ (as a cone) that can be extended to a basis for the lattice N

(2) A fan Σ in N is **smooth** if every cone in Σ is smooth

> Before proving the proposition, let's give an example demonstrating the necessity of (2).

Example Let $N = \mathbb{Z}^2$ and $\sigma = \mathrm{cone}(e_2, 2e_1 - e_2)$





Then $\sigma^\vee = \text{cone}(e_1^*, e_1^* + 2e_2^*)$. We saw that X_σ is the quadratic cone

$$X_\sigma \simeq \text{Spec}\left(\mathbb{Z}[x,y,z] / \langle y^2 - xz \rangle\right),$$

which has a singularity at the origin. The problem here is exactly that e_2 and $2e_1 - e_2$ don't generate $\mathbb{Z}^2$, they generate $\mathbb{Z} \times 2\mathbb{Z}$. This forces $\sigma^\vee \cap M$ to acquire an extra generator as well as relations.

Recollection. Let N be a lattice and $N' \subset N$ a sublattice. The following are equivalent:

(1) The quotient N/N' is a lattice

(2) For $x \in N$, if there is a positive integer k such that $kx \in N'$, then $x \in N'$.

Def If S is a commutative monoid, a submonoid $S' \subset S$ is **saturated** if (2) is satisfied

Ex. If $\sigma \subset N$ is a (rational strongly convex polyhedral) cone, then $S_\sigma \subset M$ is saturated

Proof of Proposition

Clearly $(3) \Rightarrow (1)$

Let us first treat the case where σ spans $N_{\mathbb{R}}$, so $\sigma^\perp = 0$

We'll prove that $(1) \Rightarrow (2) \Rightarrow (3)$ by showing that if (1) is satisfied, then σ is generated by a basis for N, so $X_\sigma \cong \mathbb{A}^d_{\mathbb{Z}}$

Note that if X_σ is smooth, then for any separably closed field k, $X_{\sigma,k}$ is smooth. Note that x_σ basechanges to define a closed point of $X_{\sigma,k}$ corresponding to the maximal ideal

$$m = \mathrm{Ker}\left(k\langle S_\sigma \rangle \xrightarrow{k\text{-lin ext of } \mathbb{1}_{\sigma^\perp}} k \right)$$

$$= k\langle S_\sigma \setminus 0 \rangle .$$

Thus m^2 is the ideal in $k\langle S_\sigma \rangle$ generated

$$S_\sigma^{(2)} = (S_\sigma \setminus 0) + (S_\sigma \setminus 0)$$

= sums of two nonzero elements in S_σ

Thus the cotangent space m/m^2 has basis images of $(S_\sigma \setminus 0) \setminus S_\sigma^{(2)}$, i.e, elements that are not sums of two vectors.

> The smallest integral vectors along the edges of $\sigma^\vee$ are of this form

Since $X_{\sigma,k}$ is smooth, in particular regular,

$$\dim_k(m/m^2) = \dim(X_\sigma) = n$$

In particular, $\sigma^\vee$ cannot have more than n edges, and the minimal generators in $S_\sigma = \sigma^\vee \cap M$ generate S_σ as a monoid. Since S_σ generates M as a group, the minimal generators for S_σ are a basis for M. By duality σ must be generated by a basis for N, as desired.

Case where $\dim(\sigma) < \operatorname{rank}(N)$

Let

$$N_\sigma := \sigma \cap N + (-\sigma \cap N) = \text{sublattice generated by } \sigma \cap N.$$

Since σ is saturated, N_σ is also saturated. Hence

N/N_σ is a lattice of rank $n-d$. Choose a splitting

$$N \cong N_\sigma \oplus N''$$

and write $\sigma = \sigma' \oplus 0$, where σ' is a cone in N_σ. Dually, we decompose

$$M \cong \underset{\substack{\| \\ (N_\sigma)^\vee}}{M'} \oplus M''$$

Then $S_\sigma = ((\sigma')^\vee \cap M') \oplus M''$. Hence

$$X_\sigma \cong \operatorname{Spec}(\mathbb{Z}\langle(\sigma')^\vee \cap M'\rangle) \times \operatorname{Spec}(\mathbb{Z}\langle M''\rangle)$$

$$\cong X_{\sigma'} \times (\mathbb{G}_m)^{\times(n-d)}$$

Since X_σ is smooth, so is $X_{\sigma'}$. Now σ' spans $(N_\sigma)_{\mathbb{R}}$, so the first part of the argument shows that

$$X_\sigma \cong \mathbb{A}^d_{\mathbb{Z}}. \qquad \square$$