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Toric geometry I

Today Begin toric geometry

> We'll start by understanding some group schemes associated to groups. In fact, it will be useful to work more generally with monoids.

> This can be phrased in a more hands-on way, but we'll aim to phrase things in a way so that we can verbatim do the same thing in the homotopical setting.

Monoid rings & diagonalizable monoid schemes

Recall If $\mathcal{C}$ is a category with finite products, a **monoid object** in $\mathcal{C}$ is an object $M \in \mathcal{C}$ equipped with a multiplication $m: M \times M \longrightarrow M$ and a unit map $u: 1_{\mathcal{C}} \longrightarrow M$ satisfying the usual axioms, which can be expressed in terms of commutative diagrams.

a property "better"

> M is a **group** if the shear maps

$$M \times M \xrightarrow{(pr_1, m)} M \times M \qquad M \times M \xrightarrow{(m, pr_2)} M \times M$$

$$\text{"}(x,y) \longmapsto (x, xy)\text{"} \qquad \text{"}(x,y) \longmapsto (xy, y)\text{"}$$

are isomorphisms

- Equivalently, there is a map $i: M \longrightarrow M$ that is both a left and right sided inverse to multiplication. It is then automatic that i is an isomorphism and unique.

an additional structure "worse"

More generally If $(\mathcal{C}, \otimes, \mathbb{1})$ is a symmetric monoidal category, then we have the notion of a **(commutative) algebra** in $\mathcal{C}$, which is an object $X \in \mathcal{C}$ together with maps

$$m: X \otimes X \longrightarrow X \quad \text{and} \quad u: \mathbb{1} \longrightarrow X$$

making the obvious associativity, unitality (and commutativity) diagrams commute

- Dually, we can talk about **(commutative) coalgebras** in $\mathcal{C}$, which are just commutative algebras in $(\mathcal{C}^{op}, \otimes, \mathbb{1})$

- A **commutative bialgebra** in $\mathcal{C}$ is an object together with both the structures of a commutative algebra and a commutative coalgebra

- We write $\mathrm{CAlg}(\mathcal{C})$, $\mathrm{cCAlg}(\mathcal{C})$, $\mathrm{bCAlg}(\mathcal{C})$ for the categories of commutative algebras, coalgebras, and bialgebras in $\mathcal{C}$

Ex.

(1) If $\mathcal{C}$ has finite products, $\mathrm{CAlg}(\mathcal{C}, \times) = \mathrm{CMon}(\mathcal{C})$

(2) For a ring A, $\mathrm{CAlg}(\mathrm{Mod}_A^{\heartsuit}, \otimes_A) = \left\{\begin{array}{c}\text{commutative} \\ A\text{-algebras}\end{array}\right\}$ in the usual sense

$$= \left\{\begin{array}{l}\text{Commutative rings} \\ B \text{ with a hom } A \to B\end{array}\right\}$$

Ntn. For a ring A, let's write

$$A\langle - \rangle : \mathrm{Set} \longrightarrow \mathrm{Mod}_A^{\heartsuit}$$

for the free A-module functor

Observe. $A\langle - \rangle$ is symmetric monoidal, in particular

(1) $A\langle pt \rangle \cong A$

(2) $A\langle S \times T \rangle \cong A\langle S \rangle \otimes_A A\langle T \rangle$

(3) $A\langle - \rangle$ preserves (commutative) algebras coalgebras

Explicitly If M is a monoid, then $A\langle M\rangle$ is naturally an associative A-algebra with multiplication

$$A\langle M\rangle \otimes_A \langle M\rangle \cong A\langle M\times M\rangle \xrightarrow{A\langle m\rangle} A\langle M\rangle$$

and unit

$$\eta: A \cong A\langle pt\rangle \xrightarrow{A\langle u\rangle} A\langle M\rangle$$

$$1 \longmapsto \text{unit for } A\langle M\rangle$$

> If M is commutative, this A-algebra is commutative.

> If G is a group, then the inverse $\iota\; G \xrightarrow{\sim} G$ provides an additional operation $A\langle\iota\rangle\; A\langle G\rangle \xrightarrow{\sim} A\langle G\rangle$

Upshot $A\langle -\rangle$ extends to a functor

$$A\langle -\rangle : \mathrm{CMon} \longrightarrow \mathrm{CAlg}_A$$

> In fact, this functor is left adjoint to the forgetful functor

$$\mathrm{CAlg}_A \longrightarrow \mathrm{CMon}$$

$$B \longmapsto (B, \cdot, 1)$$

Note This doesn't give $\operatorname{Spec}(A\langle M\rangle)$ a multiplication!

- For that, we need maps

$$\operatorname{Spec}(A\langle M\rangle) \times_A \operatorname{Spec}(A\langle M\rangle) \longrightarrow \operatorname{Spec}(A\langle M\rangle)$$
$$A\langle M\rangle \otimes_A A\langle M\rangle \longleftarrow A\langle M\rangle$$

$$\operatorname{Spec}(A) \longrightarrow \operatorname{Spec}(A\langle M\rangle)$$
$$A \longleftarrow A\langle M\rangle$$

  Satisfying the usual commutativity, associativity, and unitality axioms

- That is, we need to give $A\langle M\rangle$ the additional structure of a commutative coalgebra

- This actually comes for free

Coalgebra Structure Let $\mathcal{C}$ be a category with finite products and terminal object 1. Then every object $X \in \mathcal{C}$ is naturally a commutative coalgebra with

(1) Comultiplication the diagonal $\Delta : X \longrightarrow X \times X$

(2) counit the unique map $X \longrightarrow 1$

In fact, the forgetful functor

$$cCAlg(\mathcal{C}, \times) \longrightarrow \mathcal{C}$$

is an equivalence of categories with inverse

$$(X, X \xrightarrow{\Delta} X \times X, X \to 1) \longleftarrow\!\shortmid X$$

Consequence. The functor $A\langle - \rangle : CMon \longrightarrow CAlg_A$ actually refines to a functor

$$A\langle - \rangle : CMon \longrightarrow bCAlg_A.$$

> Explicitly, the maps

$$A\langle M\rangle \xrightarrow{A\langle\Delta\rangle} A\langle M\times M\rangle \simeq A\langle M\rangle \otimes_A A\langle M\rangle$$
$$m \longmapsto m\otimes m \quad \text{A-linear extension}$$

and

$$\varepsilon: A\langle M\rangle \longrightarrow A\langle pt\rangle \simeq A$$
$$m \longmapsto 1 \quad \text{A-linear extension}$$

give $A\langle M\rangle$ a commutative coalgebra structure

> From our discussion so far, we see:

Lem. If M is a commutative monoid, then $\mathrm{Spec}(A\langle M\rangle)$ is a commutative monoid in A-schemes. If M is a group, then $\mathrm{Spec}(A\langle M\rangle)$ is a commutative A-group scheme.
Here, the structures are:
(1) multiplication: $\mathrm{Spec}(\Delta)$
(2) unit: $\mathrm{Spec}(\varepsilon)$
(3) [for groups] inverse: $\mathrm{Spec}(i)$

Def We call $\mathrm{Spec}(A\langle M\rangle)$ the **diagonalizable** A-monoid (or A-group if M is a group) scheme associated to M.

> often $\mathrm{Spec}(A\langle M\rangle)$ is denoted $D_A(M)$

Remark Often the group ring $A\langle G\rangle$ is denoted $A[G]$, but this conflicts with the typical notation for polynomial rings.

$\mathbb{A}^1$ and $\mathbb{G}_m$

> Let's unpack this abstraction for the simplest monoids

Observe For the monoid $(\mathbb{N},+)$, we have

$$A\langle\mathbb{N}\rangle \simeq \bigoplus_{d\geq 0} \underbrace{At^d}_{\substack{\text{d-th weight space} \\ \text{for the operator } t}}$$

> The ring structure comes from the addition on $\mathbb{N}$ and extends

$$at^d \cdot bt^e = abt^{d+e}.$$

- So $A\langle\mathbb{N}\rangle \simeq A[t]$ and $\mathrm{Spec}(A\langle\mathbb{N}\rangle) \simeq \mathbb{A}^1_A$

> The multiplication comes from the diagonal:

$$\begin{array}{ccc} A[t] & \longrightarrow & A[t_1] \otimes_A A[t_2] \simeq A[t_1,t_2] \\ t & \longmapsto & t_1 t_2 \end{array}$$

Observe. For the group $G = \mathbb{Z}$, we have

$$A\langle \mathbb{Z} \rangle \simeq A[t^{\pm 1}]$$

- The inverse is $A[t^{\pm 1}] \xrightarrow{\sim} A[t^{\pm 1}]$
  $t \longmapsto t^{-1}$

Def. The **multiplicative group** is

$$\mathbb{G}_m = \mathrm{Spec}(A\langle \mathbb{Z} \rangle) \simeq \mathbb{A}^1 \setminus 0$$

- For a scheme S, we write

$$\mathbb{G}_{m,S} = \mathbb{G}_m \times S$$

- $\mathbb{G}_{m,S}$ is a group object in Sch_S

Remark. A more classical variant of the Fourier transform stated last time is that there is an equivalence of categories

$$\begin{pmatrix} \text{affine schemes} \\ \text{with a } \mathbb{G}_m\text{-action} \end{pmatrix} \simeq \begin{pmatrix} \mathbb{Z}\text{-graded} \\ \text{rings} \end{pmatrix}^{op}.$$

> A $\mathbb{G}_m$-action is **contracting** if it extends to an action of $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}\langle\mathbb{N}\rangle)$

$$\begin{matrix} \mathbb{G}_m\text{-action} \\ \text{is contracting} \end{matrix} \iff \begin{matrix} \text{nonnegatively} \\ \text{graded} \end{matrix}$$

> For nonnegatively graded rings S_* with corresponding affine scheme X with a $\mathbb{G}_m$-action,

$$\operatorname{Proj}(S_*) \simeq (X \setminus X^{\mathbb{G}_m}) / \mathbb{G}_m$$

$$= (\operatorname{Spec}(S_*) \setminus V(S_+)) / \mathbb{G}_m$$

$S_+ = \bigoplus_{d>0} S_d$ irrelevant ideal

Lattices

Recall. A **lattice** is a finitely generated free abelian group.

> Write $\mathrm{Latt} \subset \mathrm{Ab}$ for the full subcategory spanned by the lattices.

> **Duality**. The functor

$$(-)^{\vee} : \mathrm{Latt} \longrightarrow \mathrm{Latt}^{op}$$
$$N \longmapsto \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$$

is an equivalence of categories. It is its own inverse.

Example. If N is a lattice of rank d, then a choice of basis for N provides an isomorphism $N \simeq \mathbb{Z}^d$, hence an isomorphism

$$\mathrm{Spec}(\mathbb{Z}\langle N \rangle) \simeq (\mathbb{G}_m)^{\times d}$$

> Hence $\mathrm{Spec}(\mathbb{Z}\langle N\rangle)$ is an algebraic torus of rank d.

Goal. Given a lattice N and some combinatorial data in the vector space $N \otimes_{\mathbb{Z}} \mathbb{R}$, build an interesting scheme.

Standard Notation

- N is a lattice
- $M := N^{\vee}$ is the dual lattice
- $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. We'll also write $V = N_{\mathbb{R}}$ and $V^{\vee} = M_{\mathbb{R}}$
- $\langle -,- \rangle : M \otimes_{\mathbb{Z}} N \longrightarrow \mathbb{Z}$ the evaluation pairing induced by

$$\begin{aligned} M \times N &\longrightarrow \mathbb{Z} \\ (m,n) &\longmapsto m(n) \end{aligned}$$

Def Let $v_1, \ldots, v_n \in N_{\mathbb{R}}$. The **convex polyhedral cone** generated by $v_1, \ldots, v_n$ is

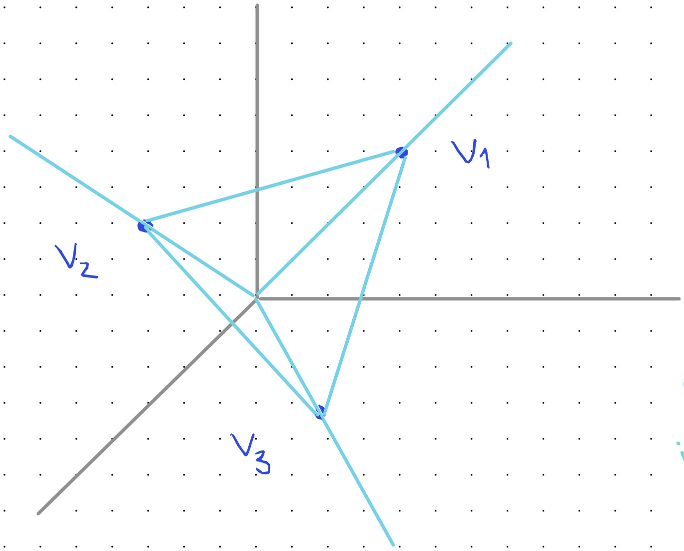
$$\operatorname{cone}(v_1, \ldots, v_n) := \{ r_1 v_1 + \cdots + r_n v_n \mid r_1, \ldots, r_n \geq 0 \}.$$

We call $v_1, \ldots, v_n$ generators. A subset $\sigma \subset N_{\mathbb{R}}$ is a **convex** polyhedral cone if $\sigma = \mathrm{Cone}(v_1, \ldots, v_n)$ for vectors $v_1, \ldots, v_n \in N_{\mathbb{R}}$

> we define $\dim(\sigma) = \dim_{\mathbb{R}}(\mathbb{R}\sigma)$

vector space spanned by σ.

Picture.



$v_i = 0$ in x_i

1 else

$\mathrm{Cone}(v_1, v_2, v_3)$ is a tetrahedral cone.

Def If $\sigma \subset N_{\mathbb{R}}$ is a convex polyhedral cone

> The **dual** of σ is

$$\sigma^{\vee} = \{ u \in M_{\mathbb{R}} \mid \text{for all } v \in \sigma, \ \langle u, v \rangle \geq 0 \}$$

- Note that $\sigma^{\vee}$ is a submonoid of $M_{\mathbb{R}}$

> σ is **rational** if there are $v_1, \ldots, v_n \in N$ such that $\sigma = \mathrm{Cone}(v_1, \ldots, v_n)$

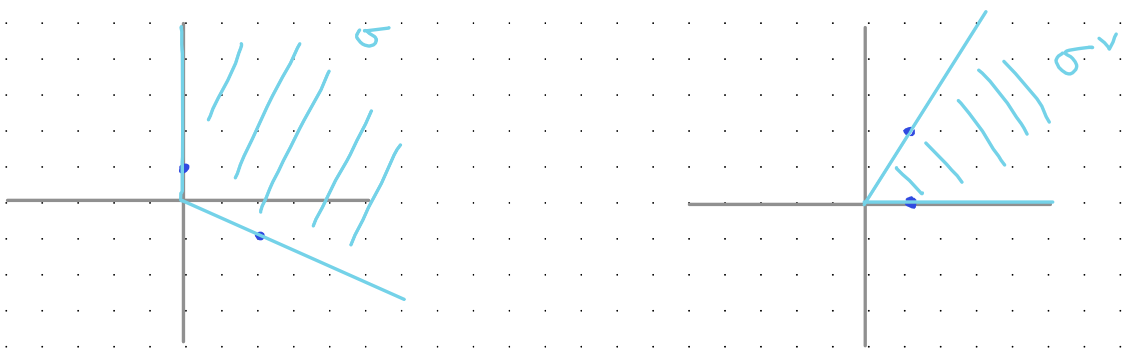
> σ is **strongly convex** if the only linear subspace σ contains is 0

> We write $S_\sigma := \sigma^{\vee} \cap M$

- Note that S_σ is a submonoid of M.

Note We'll soon explain why we'll almost exclusively be interested in rational strongly convex polyhedral cones.

Picture Let's consider $\sigma = \text{cone}(e_2, 2e_1 - e_2)$



Def If $\sigma \subset N_{\mathbb{R}}$ is a rational convex polyhedral cone, the **affine toric scheme** associated to σ is

$$X_\sigma := \text{Spec}(\mathbb{Z}\langle \underbrace{S_\sigma}_{\sigma^\vee \cap M} \rangle)$$

Observe The inclusion of monoids $S_\sigma \hookrightarrow M$ induces a map of monoid schemes

$$T := \text{Spec}(\mathbb{Z}\langle M \rangle) \longrightarrow X_\sigma .$$

In particular, X_σ has a natural action of the torus T.

How to get $\mathbb{P}^1$: Consider $N = \mathbb{Z}$, so $N_{\mathbb{R}} = \mathbb{R}$ and the cones $\sigma_- = \mathbb{R}_{\leq 0}$, $\sigma_0 = \{0\}$, $\sigma_+ = \mathbb{R}_{\geq 0}$

> Then $(\mathbb{R}_{\leq 0})^\vee = \mathbb{R}_{\leq 0}$, $\{0\}^\vee = \mathbb{R}$, $(\mathbb{R}_{\geq 0})^\vee = \mathbb{R}_{\geq 0}$

> $M \cap \mathbb{R}_{\leq 0} = \mathbb{Z}_{\leq 0}$, $M \cap \mathbb{R} = \mathbb{Z}$, $M \cap \mathbb{R}_{\geq 0} = \mathbb{Z}_{\geq 0}$

$$\mathbb{Z}_{\leq 0} \hookrightarrow \mathbb{Z} \hookleftarrow \mathbb{Z}_{\geq 0}$$

- Taking group algebras we get inclusions

$$\mathbb{Z}[t^{-1}] \hookrightarrow \mathbb{Z}[t^{\pm 1}] \hookleftarrow \mathbb{Z}[t]$$

> On spectra, we get a diagram of open immersions

$$\begin{tikzcd}
\mathbb{A}^1 = \mathbb{P}^1 \setminus 0 \arrow[d, equal] & \mathbb{G}_m \arrow[d, equal] & \mathbb{P}^1 \setminus \infty = \mathbb{A}^1 \arrow[d, equal] \\
\mathrm{Spec}(\mathbb{Z}[t^{-1}]) & \mathrm{Spec}(\mathbb{Z}[t^{\pm 1}]) \arrow[l, hook'] \arrow[r, hook] & \mathrm{Spec}(\mathbb{Z}[t])
\end{tikzcd}$$

- The pushout is just $\mathbb{P}^1$.

Next Time We'll generalize this to construct a toric scheme from a collection of cones in $N_{\mathbb{R}}$ with some closure properties

Observe. If $\mathcal{C}$ is a category with finite products, then every map $f: Y \to X$ exhibits Y as a comodule over X with coaction map the graph

$$Y \xrightarrow{gr_f = (f, id)} X \times Y.$$

Consequence. If $M' \xrightarrow{\phi} M$ is a monoid homomorphism, then $A\langle M' \rangle$ is an $A\langle M \rangle$-comodule via

$$\begin{array}{ccc} A\langle M' \rangle & \xrightarrow{A\langle gr_\phi \rangle} & A\langle M \times M' \rangle \simeq A\langle M \rangle \otimes_A A\langle M' \rangle \\ m' & \longmapsto & \phi(m') \otimes m' \end{array}$$

> So $\mathrm{Spec}(A\langle M' \rangle)$ has a natural

Basic facts about cones

Lemma. Let $\sigma \subset N_{\mathbb{R}}$ be a convex polyhedral cone.

(1) If $v_0 \notin \sigma$, then there exists a $u_0 \in \sigma^\vee$ with $\langle u_0, v_0 \rangle < 0$.

(2) We have $(\sigma^\vee)^\vee = \sigma$

Gordon's Lemma. If $\sigma \subset N_{\mathbb{R}}$ is a **rational** convex polyhedral cone, then the monoid $S_\sigma = \sigma^\vee \cap M$ is finitely generated. In particular, $\mathbb{Z}\langle S_\sigma \rangle$ is a finite type $\mathbb{Z}$-algebra.