Law of Quadratic Reciprocity

Let p, q be distinct odd primes
Define the **Legendre Symbol** as

$$\left(\frac{p}{q}\right) = \begin{cases} 1 & \text{if } x^2 \equiv p \bmod q \text{ for some } x \in \mathbb{Z} \\ -1 & \text{otherwise.} \end{cases}$$

Then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

Law of Quadratic Reciprocity

Let p, q be distinct odd primes.
$pRq \to x^2 \equiv p \bmod q$, for some $x \in \mathbb{Z}$
$pNq \to x^2 \not\equiv p \bmod q$, for any $x \in \mathbb{Z}$

Case I
If $p = 1 + 4m$, for some $m \in \mathbb{Z}$,
then either pRq and qRp, or pNq and qNp.

Case II
If $p = 3 + 4m$, $q = 3 + 4n$ for some $m, n \in \mathbb{Z}$,
then either pRq and qNp, or pNq and qRp.

- *Carl Friedrich Gauss (1801)*