I think of a vector space, V, as being defined by a map.

$$V : E \rightarrow \mathbb{C}^n \tag{1}$$

each $\mathbf{v} \in V$, the tuple $c \in \mathbb{C}$ used to represent v on E.

To determine the components of reprentation, i.e., the tuple, c, corresponding to a given vector, v, we can define a function, $v(e)$, which maps from a member, $e \in E$, to a number $c \in \mathbb{C}$.

$$v : e \rightarrow c \qquad\qquad e \in E \quad c \in \mathbb{C} \quad v \in V \tag{2}$$

To see why, consider a vector $\mathbf{v}$ defined on a basis E

$$\mathbf{v} = \sum_{i=1}^{dim(E)} f(\mathbf{e}_i)\mathbf{e}_i = \sum_{i=1}^{dim(E)} v_i\mathbf{e}_i \tag{3}$$

In other words, v_i, is just shorthand for $f(\mathbf{e}_i)$. The f which determines the component of the vector is a linear function defined on V. These functions form a space all of their own, denoted V^*, and referred to as the dual space of V.

As an example take a 3-dimensional vector, $\mathbf{w}$, represented on the Cartesian basis, $E = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$:

$$\mathbf{w} = g(\mathbf{x})\mathbf{x} + g(\mathbf{y})\mathbf{y} + g(\mathbf{z})\mathbf{z} = w_x\mathbf{x} + w_y\mathbf{y} + w_z\mathbf{z} \tag{4}$$

note the coeffients $\{w_x, w_y, w_z\} \in \mathbb{C}^3$, although if we are talking about Cartesian vectors we assume the values of these coefficients to be real, so the tuple is in $\mathbb{R}^3$.

Usually we think of $f(e_i)$, as being a projection of vector $\mathbf{v}$ represented on , onto the onto the basis functions $\mathbf{e}$. Essentially, the dual, $v*$, of vector v is defined as the tuple of functions which map from the members of the basis, E, to the coefficients used to represent v on this basis.

A vector operator is one which that transforms like a vector. For example, if we have a vector operator $a \in A$, we can define it on an basis B, were the members of $\hat{b}$ are themselves operators, e.g., for a Cartesian vector operator

$$\hat{\mathbf{a}} = a_x\hat{b}_x + a_y\hat{b}_y + a_z\hat{b}_z \tag{5}$$

here $\{a_i\}_{i=x,y,z}$ are just coefficients. A rotation, $\hat{R}$, of the vector operator $\hat{\mathbf{a}}$

$$\hat{\mathbf{a}}' = \hat{R}\hat{\mathbf{a}} \tag{6}$$

can be effected by a transformation of the components $\{a_i\}_{i=x,y,z}$

$$\hat{a}_{i'} = \sum_{i=x,y,z} R_{i'i}a_i\hat{b}_i \tag{7}$$

where R_{ij} is are the components of the operator used to effect rotations on the Cartesian basis $\{x, y, z\}$. In the above case, the operators $\{b_i\}_{i=x,y,z}$ form a vector space which is isomorphic to the Cartesian space $\{x, y, z\}$.

Now we can move onto the definition of tensors. A tensor, T, of type (r, s), on a vector space V is a C-valued function on

$$V \times V \times ... \times V \times V^* \times V^* \times ... \times V^T \tag{8}$$

this is supposed to be a product space formed from r products of V, and s products of V^* (the dual of V). The tensor is linear in each argument, i.e.,

$$T(v_1 + cw, v_2, ..., v_r, f_1, ..., f_s) = T(v_1, v_2, ..., v_r, f_1, ..., f_s) + cT(w, v_2, ..., v_r, f_1, ..., f_s) \tag{9}$$

Here $v_i \in V$ and $f_i \in V^*$. In other words, the tensor is multilinear function defined on a space formed from r vectors and s dual vectors. Linear operators can be viewed as tensors of type $(1, 1)$, as they take one vector and one dual vector,

$$A(f, v) = f(Av) = A_{fv} \tag{10}$$

Here A is acting on v to return a vector, v_{out}, in V, then f, the dual vector acts on this vector to return the component, A_{fv}, of the tensor associated with inputs f and v.

As with the previous definition for vectors, we can write a definition of a tensor represented on a basis formed from the product of a the Cartesian basis $\{e_x, e_y, e_z\}$ and it's dual (which is also $\{e^x, e^y, e^z\}$ as the Cartesian space is self dual). This product space has members $\{e^x e_x, e^x e_y, e^x e_z, e^y e_x, e^y e_y, e^y e_z, e^z e_x, e^z e_y, e^z e_z\}$, hence we may write the tensor as tensor as

$$A(\mathbf{f}, \mathbf{q}) = \sum_{i,j} A_i^j e^i e_j \tag{11}$$

An interesting thing to note is sometimes product spaces,can be decomposed into a sum of other spaces, e.g.,

$$\bigotimes_{i=1}^{N} V_i = \bigoplus_{j=1} W_j \tag{12}$$

a common example is the decomposition of spaces formed from the product of angular momenta. For example, the product space $\frac{1}{2} \times \frac{1}{2}$, formed from the product of the Hilbert spaces associated with two particles each with angular momenta $j_z = \frac{1}{2}$ can be decomsed into a sum of two 1-particle Hilberts spaces, 0 and 1. The former being a spinless particle, whilst the latter is a particle with spin-1. This relation is commonly written

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \tag{13}$$

The reason we care about this is that if the space on which a tensor is defined can be decomposed in this manner, then an equivalent decomposition must exist for the tensor. Consequently, an example of such a tensor decomposition might be

$$A = B + C \tag{14}$$

where A is a rank 2 tensor, and B and C are both rank one tensors. This has important applications in Socnini's theory. Before moving on to see how this is applied, let us first consider tensor operators, which like the vector operators mentioned above possess the feature that they transform as tensors do. The difference is that the spaces, V, and,

V^*, are themselves comprised of operators. for example, we might represent a tensor of (1,1),as

$$\hat{A} = \sum_{i,j} A_i^j \hat{s}^i \hat{s}_j \tag{15}$$

where the $\hat{s}_{i,j}$ are operators which act on a single vector (or dual vector). Tensor operators transform in much the same way as normal tensors

$$\hat{A}' = \hat{R}\hat{A} = \sum_{i,i',j,j'} R_{ii'} R_{jj'} A_{i'}^{j'} \hat{b}^i \hat{b}_j \tag{16}$$

The interesting thing here is that instead of transforming the coefficients of the tensor operator directly, we could in principle transform the basis in terms of which the tensor is defined. For example,

$$\hat{A}' = \hat{R}\hat{A} = \sum_{i,i',j,j'} A_i^j R_{ii'} R_{jj'} \hat{b}^{i'} \hat{b}_{j'}. \tag{17}$$

This approach can be taken one step further if the aswell as begin represented on a product space, the tensor operator is formed from the product of two vector operators, e.g.,

$$\hat{A} = \hat{v} \otimes \hat{w} \tag{18}$$

where each of these vectors may be represented on a basis

$$\hat{\mathbf{v}} = v_x \hat{b}_x + v_y \hat{b}_y + v_z \hat{b}_z \tag{19}$$

$$\hat{\mathbf{w}} = w_x \hat{b}_x + w_y \hat{b}_y + w_z \hat{b}_z \tag{20}$$

In this case it is possible to effect a transformation of the tensor operator by transforming each one of the vector operators from which it is composed, e.g.,

$$\hat{A}' = \hat{R}\hat{A} = \hat{U}_v \hat{U}_w \hat{v} \otimes \hat{w} \tag{21}$$

Therefore, we could transform the tensor operator by performing a transformation on the bases used to represent each of the vector operators from which it is composed. This comes in very useful in going from the standard definition of magnetic resonance tensors, to the definition laid out by Soncini.

An example of the above is an operator $\hat{A} = \hat{s}^{(1)} \otimes \hat{s}^{(2)}$, which is composed of two spin operators, one acting on vectors defining particle 1, the other acting on vectors defining particle 2. When each of these spin operators is represented in a space formed by a Kramers doublet we can expand them out in a sequence of Pauli Matrices, e.g.,

$$[\hat{s}_1]_{\{\phi,\bar{\phi}\}} = \sum_i^{x,y,z} c_i \sigma_i \tag{22}$$

The product basis used for representing the tensor is formed from the Kronecker product of each of these basis, e.g., one member, $\sigma_{ij}^{(2e)}$, of this product basis is given by

$$\sigma_{ij}^{(two)} = \sigma_i \otimes \sigma_j \tag{23}$$

Definition of these product matrices can be found in the spin_mats.pdf.

This basis is not particularly nice; it contains 9 distinct matrices (16 if we were to also include the identity in our basis). Furthermore, it is formed from 4 matrices, when typically we know that two interacting spins suggests a triplet and a singlet, suggesting a $1\times 1 matrix, and 3\times 3$ matrix.

and it is impossible to decompose this space into a may be possible to rexpress it as a sum of other,lower ra tensors, defined on different spaces. For example, we might have

$$A(f, v) = B(x) + D(y) \tag{24}$$

In a similar vein, we could define a vector operator, $\tilde{A}$, which takes two vectors. with coefficients $\{\tilde{a}_i\}_{i=x,y,z}$. Now define a member of $a \in A$ represented on the space of all complex, Hermtian, 2×2 matrices, $H_2(\mathbf{C})$, which we shall suggestively write

$$[a]_{H_{2\times 2}} = \begin{bmatrix} a_z + c & a_x + ia_y \\ a_x - ia_y & -a_z + c \end{bmatrix} \tag{25}$$

So the components of a representation, $[v]_E$ of a vector $v \in V$ on a basis E is

$$[v]_E = \sum f(e_i) = c_i \tag{26}$$

The map corresponding to an n-dimensional vector takes use from one space, E, (which can be thought of as the basis), to the space of all n-tuples of complex numbers, $\mathbb{C}^n$, which can be thought of as the coefficients of the vectors. So

$$v(e^i) \to v_i \tag{27}$$

For reasons of simplicity

$$v(e_i) = \tag{28}$$